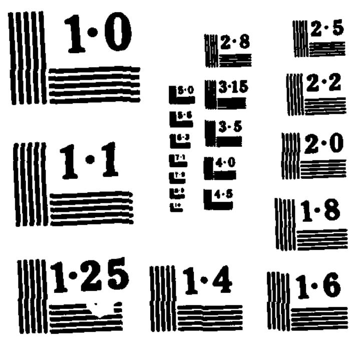

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CONVERGENCE OF CARDINAL SERIES

Carl de Boor, Klaus Höllig
and Sherman Riemenschneider

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CONVERGENCE OF CARDINAL SERIES

Carl de Boor[1], Klaus Höllig[1,2] and Sherman Riemenschneider[3]

Technical Summary Report #2834
June 1985

ABSTRACT

The result of this paper is a generalization of our characterization of the limits of multivariate cardinal splines. Let M_n denote the n-fold convolution of a compactly supported function $M \in L_2(\mathbb{R}^d)$ and denote by

$$S_n := \{ \sum_{j \in \mathbb{Z}^d} c(j) M_n(\cdot - j) : c \in \ell_2(\mathbb{Z}^d) \}$$

the span of translates of M_n. We prove that there exists a set Ω with $\mathrm{vol}_d(\Omega) = (2\pi)^d$ such that for any $f \in L_2(\mathbb{R}^d)$,

$$\mathrm{dist}(f, S_n) \to 0 \quad \text{as} \quad n \to \infty ,$$

if and only if the support of the Fourier transform of f is contained in $\bar{\Omega}$.

AMS (MOS) Subject Classifications: 41A63, 41A30, 42B99

Key Words: multivariate, cardinal series, convergence, Fourier transform

Work Unit Number 3 (Numerical Analysis and Scientific Computing)

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SIGNIFICANCE AND EXPLANATION

In recent reports [#2485, #2686] we have studied the convergence of interpolation with box-splines as their degree tends to infinity. The result stated in the abstract generalizes one of our main theorems. Although more general, the proof is quite short and the essential features of our earlier arguments become more apparent.

The responsibility for the wording and views expressed in this descriptive summary lies with MRC, and not with the authors of this report.

CONVERGENCE OF CARDINAL SERIES

Carl de Boor[1], Klaus Höllig[1,2] and Sherman Riemenschneider[3]

1. Introduction. We extract the essential features of our earlier arguments [1-4] concerning the limits of box-splines as their degree tends to infinity. Somewhat surprisingly, the resulting discussion, although covering a more general situation, is very much shorter.

We start with a compactly supported (nonzero) L_2-function M on $\mathbf{R}^d$ for which the Fourier transform

$$\hat{M}(\xi) := \int M(x) \exp(-ix\xi) dx$$

satisfies

$$|\hat{M}(\xi)| = O(|\xi|^{-1}), \quad |\xi| \to \infty. \tag{1}$$

With

$$M_n := M * \cdots * M$$

denoting the n-fold convolution of M, we consider approximation in L_2 from the *span*

$$S_n := \{ \sum_{j \in \mathbf{Z}^d} c(j) M_n(\cdot - j) \ : \ c \in l_2(\mathbf{Z}^d)\}$$

of the integer translates of M_n. We wish to characterize the class

$$S_\infty := \{f \in L_2(\mathbf{R}^d) \ : \ \lim_{n \to \infty} dist(f, S_n) = 0\}.$$

For this we introduce the set

$$\Omega := \{\xi \in \mathbf{R}^d : |\hat{M}(\xi + 2\pi j)| < |\hat{M}(\xi)|, \ j \in \mathbf{Z}^d \backslash 0\}$$

and establish the following

Proposition. Ω is a fundamental domain, i.e.

$$\Omega \cap (\Omega + 2\pi j) = \emptyset, \ j \neq 0$$
$$\cup_j (\bar{\Omega} + 2\pi j) = \mathbf{R}^d.$$

The class S_∞ consists of functions of exponential type characterized by the set Ω.

Theorem. $f \in S_\infty$ iff $supp\, \hat{f} \subset \bar{\Omega}$.

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2. Proof of the Proposition. The assumption (1) implies that for any positive C,

$$\#\{j \ : \ |\hat{M}(\xi + 2\pi j)| \geq C\} < \infty. \tag{2}$$

Let

$$D := \{\xi \in \mathbf{R}^d \ : \ \hat{M}(\xi) \neq 0\}.$$

On D, the quotient

$$a_j(\xi) := \hat{M}(\xi + 2\pi j)/\hat{M}(\xi)$$

is well defined. In particular,

$$\Omega = \{\xi \in \mathbf{R}^d \ : \ |a_j(\xi)| < 1 \text{ for } j \in \mathbf{Z}^d \backslash 0\}.$$

Lemma. For all $\xi \in \mathbf{R}^d$ there is $j \in \mathbf{Z}^d$ such that $\xi + 2\pi j \in \bar{\Omega}$.

Proof. Since $\hat{M}$ is an entire function, it is sufficient to prove this for $\xi \in D$. The set

$$J(\xi) := \{j \in \mathbf{Z}^d \ : \ |\hat{M}(\xi + 2\pi j)| = \sup_k |\hat{M}(\xi + 2\pi k)|\}$$

is finite and nonempty, by (2). Hence we are done unless $\#J(\xi') > 1$ for all ξ' in some neighborhood of ξ. In this case at least one of the real analytic functions

$$f_j - f_{j'}$$

with

$$f_k := |\hat{M}(\cdot + 2\pi k)|^2$$

vanishes on some open set, hence must vanish identically. But this implies that

$$|\hat{M}| = |\hat{M}(\cdot + 2\pi r)|$$

for $r := j - j' \neq 0$, contradicting (1).

To finish the proof of the Proposition, assume that ξ and $\xi + 2\pi j$ are both in Ω. Then, the assumption $j \neq 0$ leads to the contradiction

$$1 > 1/|\hat{M}(\xi + 2\pi j)/\hat{M}(\xi)| = |\hat{M}(\xi + 2\pi j - 2\pi j)/\hat{M}(\xi + 2\pi j)| > 1.$$

3. Proof of the Theorem. We introduce the trigonometric polynomial

$$P_n(\xi) := \sum_j M_n(j) \exp(ij\xi) = \sum_j \hat{M}(\xi + 2\pi j) = \hat{M}(\xi) \sum_j (a_j(\xi))^n$$

with the last equality holding, at least, on D. For any $j \neq 0$ and $\xi \in \Omega$,

$$|a_j(\xi)| \leq 1 - \epsilon(j, \xi) \tag{3.1}$$

for some positive $\epsilon(j, \xi)$, while, by (1) and (2),

$$|a_j(\xi)| \leq 1/(1 + C|j|) \tag{3.2}$$

for some positive C uniformly for all but finitely many j. Consequently, for $\xi \in \Omega$,

$$P_n(\xi)/\hat{M}_n(\xi) = \sum_j (a_j(\xi))^n \to 1, \ n \to \infty, \tag{4}$$

and the convergence is uniform on compact subsets Ω_1 of Ω. This shows, in particular, that, for large enough n, P_n does not vanish on such Ω_1.

(i) Assume that $f \in L_2$ and $supp\hat{f} \subset \bar{\Omega}$ and denote by χ the characteristic function of such a set Ω_1. Since Ω is a fundamental domain, we can expand $\hat{f}\chi/P_n$ in a Fourier series,

$$(\hat{f}\chi/P_n)(\xi) = \sum_j c_n(j) \exp(ij\xi), \ \xi \in \Omega,$$

with coefficients $c_n \in L_2$. This implies that

$$s_n := \sum_j c_n(j) M_n(\cdot - j) \ \in \ L_2.$$

Since $\hat{f}$ vanishes outside $\bar{\Omega}$,

$$|\hat{f} - \hat{s}_n|^2_{L_2(\mathbf{R}^d)} = |\hat{f} - \hat{s}_n|^2_{L_2(\Omega)} + \sum_{j \neq 0} |\hat{s}_n(\cdot + 2\pi j)|^2_{L^2(\Omega)}.$$

The first term is estimated by

$$|\hat{f} - \hat{s}_n|_{L_2(\Omega)} \leq |\hat{f} - \chi\hat{f}|_{L_2(\Omega)} + |\chi\hat{f} - \chi\hat{f}\hat{M}/P_n|_{L_2(\Omega)}.$$

The first norm on the right hand side is small if Ω_1 is chosen close to Ω. For fixed Ω_1, the second norm is small by (4) if n is sufficiently large.
For the terms in the sum it follows from (2) and (3) that

$$\begin{aligned} |\hat{M}_n(\cdot + 2\pi j)(\hat{f}\chi/P_n)|_{L_2(\Omega)} &= |(a_j)^n \hat{M}_n(\hat{f}\chi/P_n)|_{L_2(\Omega)} \\ &\leq (|a_j|_{L_\infty(\Omega_1)})^n |\hat{M}_n/P_n|_{L_\infty(\Omega_1)} |\hat{f}|_{L_2(\Omega_1)} \ \to \ 0, \ n \to \infty. \end{aligned}$$

(ii) Assume that $s_n = \sum_j c_n(j) M_n(\cdot - j)$ converges to f in L_2. From

$$\hat{s}_n(\xi + 2\pi j) = (a_j(\xi))^n \hat{s}_n(\xi), \ \xi \in D,$$

we see that

$$|\hat{s}_n|_{L_2(\Omega_1 + 2\pi j)} \leq (|a_j|_{L_\infty(\Omega_1)})^n \ |\hat{s}_n|_{L_2} \to 0.$$

It follows from (3) that, as an element of L_2, $\hat{f}$ vanishes off $\bar{\Omega}$.

REFERENCES

[1] C. de Boor and K. Höllig, B-splines from parallelepipeds, MRC TSR #2320 (1982), J. d'Anal. Math. **42** (1982/83), 99–115.

[2] C. de Boor, K. Höllig and S. D. Riemenschneider, Convergence of bivariate cardinal interpolation, Constr. Approx. **1** (1985), 183-193.

[3] C. de Boor, K. Höllig and S. D. Riemenschneider, Bivariate cardinal interpolation by splines on a three direction mesh, to appear in Illinois J. Math.

[4] C. de Boor, K. Höllig and S. D. Riemenschneider, The limits of multivariate cardinal splines, Proc. Conf. on Multidimensional Constructive Function Theory, W. Schempp and K. Zeller eds., Oberwolfach 1985.

[5] I. J. Schoenberg, Notes on spline functions. III, On the convergence of the interpolating cardinal splines as their degree tends to infinity, Israel J. Math. **16** (1973), 87–93.

[6] I. J. Schoenberg, *Cardinal spline interpolation*, SIAM, Philadelphia (1973).

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The result of this paper is a generalization of our characterization of the limits of multivariate cardinal splines. Let M_n denote the n-fold convolution of a compactly supported function $M \in L_2(\mathbb{R}^d)$ and denote by

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