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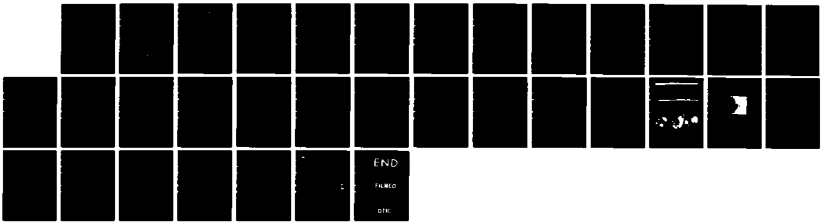

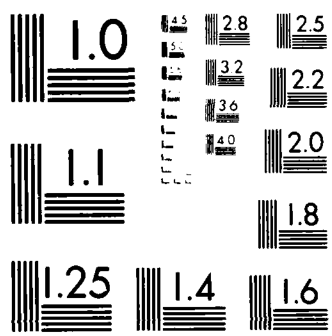

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On Stratified Vortex Motions Under Gravity

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A general class of stratified vortex flows is investigated in order to understand the vortex patterns developed in stratified late wakes. The flow profiles to be considered vary in both the radial and the axial directions. Three necessary conditions (obtained by the method of generalized progressing wave expansion) and three sufficient conditions (obtained by the integral method of complex conjugation) are discussed for stability of the present flow. Through the discussion on the stability mechanisms and the physical explanation of the earlier and the present criteria, we see that the necessary conditions for stability in fact represent a generalized state of "statically stable" distribution for the steady flow, a precondition for the Richardson criteria described by the three sufficient conditions for stability. Two of the necessary and sufficient conditions represent the criteria in their respective directions. The third ones act as a constraint to the other two. As a result, the three necessary and sufficient conditions require that the flow be stable in the centrifugal force field, in the gravitational force field, and in the pressure field that restrains the density variations in the two force fields.

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11. ABSTRACT (Continue on reverse if necessary and identify by block number)

A new Brunt-Väisälä frequency and a new Richardson number are proposed as a result of the third stability condition. The newly defined Brunt-Väisälä frequency can be viewed as a measure of the interaction of the density variation in one direction with the force field in the other direction. This Richardson number is a ratio between the interactions of the density variations with the force fields and the interactions of the velocity gradients in two directions. The former interaction is a result of the pressure restraint condition and is measured by the new Brunt-Väisälä frequency. The latter interaction determines whether the velocity gradients in the second direction strengthens or weakens the resultant shear effect. Because of the generality of the flow profiles being considered, the criteria established in this investigation are valid for a wide range of problems in oceanographic and atmospheric studies.

CONTENTS

ON STRATIFIED VORTEX MOTIONS UNDER GRAVITY

I. PURPOSE AND SCOPE

Vortex shedding in stratified fluids has been a fascinating subject in atmospheric and oceanographic studies. The "pancake" vortices developed in the late wake of a towed axisymmetric body and the vortex trails evolved in the lee of certain islands have intrigued researchers because of the coherent flow patterns generated. The existence of the organized vortex patterns implies the existence of some stability criteria that govern the generation, evolution and collapse of the flow. In this paper, we use stability analysis to examine this behavior.

General stability characteristics of a vortex flow with arbitrary density and velocity profiles varying in both the axial and the radial directions are to be investigated. Since the flow is stratified in both the axial and radial directions, two force fields, the gravitational and the centrifugal, are affected by the density variation. A constraint relation exists and requires the pressure to be balanced in both directions. This constraint is responsible for the density variations in both force fields. In addition, the variation of the velocities in both directions produces two shear layers perpendicular to each other. The shear layers are responsible for the instability mechanism that might preclude or destroy the organized vortex patterns. This paper presents an overall view of certain stratified vortex flows and criteria that govern the motion of the fluid under these circumstances.

Section 2 briefly describes the underlying flow phenomena and the influence of/on the density distribution. Section 3 presents the instability mechanisms that control the motion of the fluid. Section 4 examines the necessary and sufficient conditions for stability, the essential criteria that may govern the existence of the vortex motions. These conditions are discussed and interpreted in light of the mechanisms. Section 5 presents the interfacial conditions for discontinuous flow profiles or sharp flow layers. Finally, section 6 gives two examples with exact solutions to demonstrate the stability behavior and section 7 draws the overall conclusions.

II. INTRODUCTION

1. The Phenomena

Vortex patterns which can develop in stratified wakes in the atmosphere and the oceans have stimulated considerable interest in recent years. Vortex trails generated in the lee of certain islands as shown in Figure 1 and "pancake" vortices which can evolve in the late wakes of towed axisymmetric bodies as shown in Figure 2 are fascinating because of the organized structures reminincent of the two dimensional Kármán vortex street behind a bluff body. The stratified patterns are three-dimensional owing to their small vertical extent with respect to the generating body size.

When an axisymmetric body is towed through a stratified fluid, vortices are first shed three-dimensionally. However, the flow soon collapses in the vertical direction because the gravitational effect inhibits the vertical motion. The resultant vortex structure which can then develop in the late wake is confined within a relatively thin layer, and can appear reminiscent of the two dimensional Karman vortex street if observed from the gravitational direction.

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The inhibition of the fluid motion and the development of the horizontal flow patterns can qualitatively be examined by the motion of the fluid particles in the gravitational force field. When a sphere is towed through a stratified fluid, the fluid particles in four locations are of particular interest. Figures 3a and 3b respectively show the top view and side view of a sphere and of the four fluid particles A, B, C and D being considered. Since the fluid is stably stratified in the vertical direction, the densities of particles A and B are equal while the density of particle C is lighter than that of particle D. To create a rotational motion in three-dimensions, particles A and B, and particles C and D have to interchange their positions. The interchange of particles A and B requires no work to be done in the gravitational force field. The interchange of particles C and D, however, requires net work to be done which is equal to the increase of the potential energy at the new locations. Accordingly the vertical motion becomes suppressed by the gravitational forces and the resultant vortex motion will be confined in a relatively thin layer.

2. The Density Distribution

The density inhomogeneity plays a very subtle role in generating vortices, internal waves or other organized flow patterns in stratified fluids. These organized flow patterns exist only in stratified fluids, no matter how weakly stratified, but not in homogeneous fluids. The fluids in the atmosphere and in the ocean are very weakly stratified but are capable of supporting these organized flow patterns.

One of the mechanisms for generating the organized flow patterns is the additional restoring force in stably stratified fluids and will be discussed in the next section. Here we analyze the influence of the vertical density profile upon the flow behavior by considering the following hypothetical case.

Let a line vortex have an axis of symmetry along the z-axis of a cylindrical coordinate system (r,θ,z). The constraint equation of the pressure balance for the present flow is

$$\frac{\partial}{\partial z}(\rho_0\, r\Omega^2) + \frac{\partial}{\partial r}(\rho_0 g) = 0 \tag{2.1}$$

where ρ_0 and Ω are respectively the density and the angular velocity of the flow. Let us first assume that the density is stratified only in the axial direction, i.e., $\rho_0 = \rho_0(z)$. The constraint equation requires the angular velocity to be described by

$$\Omega = f(r)/\sqrt{\rho_0\,(z)} \tag{2.2}$$

where $f(r)$ is an arbitrary function of the radius. The steady-state pressure is now governed by

$$P(r,z) = \int r f^2(r)\; dr - g \int \rho_0\,(z)\; dz. \tag{2.3}$$

Next consider two vortices, one on top of the other, with a common axis and a common boundary located at $z = Z$. The vortex on top has a density ρ_1 and an angular velocity Ω_1 while the one below has a density ρ_2 and an angular velocity Ω_2. The pressures in each individual region are, respectively,

$$\begin{aligned} P_1 &= \int r\, f_1^2\; dr - g \int \rho_1\; dz \\ P_2 &= \int r\, f_2^2\; dr - g \int \rho_2\; dz. \end{aligned} \tag{2.4}$$

The pressure balance condition at the common boundary $z = Z$ requires that

$$P_1\,(r,Z) = P_2\,(r,Z).$$

which implies

$$f_1^2\,(r) = f_2^2\,(r)$$

or

$$\rho_1(Z)\;\Omega_1^2(r) = \rho_2(Z)\;\Omega_2^2(r) = \text{constant.} \tag{2.5}$$

In other words, if the density were restricted to be z-axis dependent only, the angular velocity would be inversely proportional to the density of the fluid. Such a restriction in turn implies that, for a statically stable density distribution of a line vortex, the rotational velocity should be large above and small below. This is apparently not the vortex pattern developed in the late wake behind a towed sphere as shown in Figure 2, and suggests that the steady-state density will have to be redistributed from its original axis-dependent distribution.

To understand the behavior of such a vortex pattern behind an axisymmetric body, one needs to first investigate the stability characteristics of a general class of flows which have their density and velocity components varying in more than one direction.

III. INSTABILITY MECHANISMS

The hydrodynamic stability of stratified parallel flows has been one of the central problems in fluid mechanics and has been studied extensively in this century. It is concerned with when and, to a lesser extent, how the flow patterns are generated, evolve, collapse, and eventually transit to turbulence. These phenomena are basically governed by two instability mechanisms, the Rayleigh-Taylor instability and the Kelvin-Helmholtz instability as to be discussed as follows.

1. Horizontal Parallel Flows

1a. *Rayleigh-Taylor Instability*

The Rayleigh-Taylor instability derives from the equilibrium or constant acceleration state of fluids with density inhomogeneity. Activated by the body force of the density variation in a force field (gravitational for example), this instability is concerned with the motion of the fluid particles along the direction of the force field. We use one simple but important example to demonstration this instability mechanism.

Consider two equal volume fluid elements within a flow regime in a Cartesian coordinate system with its z-axis pointing in the opposite direction of gravity as shown Fig. 4. The elements located at $Q_1(z_1)$ has a density equal to ρ_1 while the one at $Q_2(z_2)$ has a density equal to ρ_2, and they are both at rest. If we interchange the positions of the two elements, the work done in the gravitational force field are,

$$W_1 = -\rho_1 g (z_2 - z_1) \quad (3.1)$$

$$W_2 = \rho_2 g (z_2 - z_1) \quad (3.2)$$

where g is the gravitational constant. The total work done due to the exchange is

$$\delta W = (\rho_2 - \rho_1) g (z_2 - z_1). \quad (3.3)$$

Instability of the flow, corresponding to positive work done during the interchange, will occur if $\rho_2 > \rho_1$. Another way to view this type of instability is to examine the pressure gradient experienced by the fluid elements. If the element originally located at Q_2 is displaced to Q_1, the gravitational force experienced at its new location is $\rho_2 g$ while the prevailing pressure at Q_1 is $\rho_1 g$. The element will continue moving downward if $\rho_2 > \rho_1$, and the motion is unstable. This is a type of the Rayleigh-Taylor instability in which the motion of fluids is caused by the force field (or acceleration) and the direction of the motion is aligned with the force direction.

In the case of $\rho_1 > \rho_2$, stability of the flow, corresponding to negative work done in (3.3), is assured. This stability denotes an equilibrium state of fluids commonly known as the "statically stable" distribution of density in stratified fluids. The density gradient along the force direction is a measure of the "strength" of the stratification. The physical explanation of this measurement is given as follows:

Again consider a stably stratified fluid as in Fig. 4 with $\rho_1 > \rho_2$. If we interchange the positions of the two fluid elements, the element with density equal to ρ_1 experiences excess gravitation forces while the one with density equal to ρ_2 encounters excess bouyancy forces in their respective new locations. These forces try to restore the elements to their original equilibrium positions, and thus produce a forced oscillation of the elements. This oscillation is measured by the well-known Brunt-Väisälä frequency defined as

$$N_z = \left\{ -\frac{g}{\rho_0} \frac{d\rho_0}{dz} \right\}^{1/2} \tag{3.4}$$

where $\rho_0(z)$ is the density profile of the flow field. In practice, this frequency is used to indicate how strongly the fluid is stratified.

1b. *Kelvin-Helmholtz Instability*

The Kelvin-Helmholtz instability is triggered by the velocity gradients in the shear layer. The instability is caused by the relative horizontal motion within the flow field. We will again use the flow regime in Fig. 4 with a velocity field to demonstrate this mechanism.

The velocity field $U(z)$ being considered is parallel and unidirectional along the x-axis, and varies only in the z direction. Consider again that the fluid element originally located at Q_1 has a density equal to ρ_1 and a velocity equal to U_1 while the one at Q_2 has a density equal to ρ_2 and a velocity equal to U_2. The fluid is statically stable, i.e., $\rho_1 \geqslant \rho_2$. When the two elements are in relative motion a horizontal shear layer will be generated. The excess kinetic energy gained by these two elements may cause their interchange of position in the gravitational force field, and thus induce instability.

The kinetic energy that deviates from the mean flow is

$$\delta K = \frac{1}{2} \rho_1 U_1^2 + \frac{1}{2} \rho_2 U_2^2 - \frac{1}{2} (\rho_1 + \rho_2) \left[\frac{1}{2} (U_1 + U_2) \right]^2 \tag{3.5}$$

while the work done required to interchange the two elements in the gravitational force field is given in (3.3). If we use the differential forms such that

$$\begin{aligned} z_1 &= z & z_2 &= z + \delta z \\ \rho_1 &= \rho_0(z) & \rho_2 &= \rho_0(z) + \delta \rho_0 \\ U_1 &= U(z) & U_2 &= U(z) + \delta U \end{aligned} \tag{3.6}$$

and neglect the inertia effect of the density variations, Eqs. (3.3) and (3.5) can be written as

$$\delta W = -\delta \rho_0 \, g \, \delta z \tag{3.7}$$

$$\delta K = \frac{1}{4} \rho_0 (\delta U)^2. \tag{3.8}$$

The interchange of the two elements are impossible if $\delta W \geqslant \delta K$, i.e.,

$$-\frac{\delta \rho_0}{\delta z} g \geqslant \frac{\rho_0}{4} \left(\frac{\delta U}{\delta z} \right)^2. \tag{3.9}$$

By defining the Richardson number

$$J_z = N_z^2 / \left(\frac{dU}{dz} \right)^2 \tag{3.10}$$

to denote the ratio between the buoyancy effect and the inertia effect Eq. (3.9) can be written as

$$J_z \geqslant \frac{1}{4}. \tag{3.11}$$

This is the well-known Richardson criterion for stratified parallel flows stating that the stability of the flow is assured if the local Richardson number is greater than or equal to a quarter everywhere within the flow domain, and the precondition (or the necessary condition) for this Richardson criterion is $\frac{d\rho_0}{dz} < 0$, a statically stable density profile in the gravitational force field. The mathematical derivation of (3.11) was given by Miles (1961) and Howard (1961) using the normal mode method, and will not be repeated here.

2. Rotating Flows

The instability of rotating flows also has two origins, the centrifugal one and the shear one. The former is an instability of the Rayleigh-Taylor type while the latter is an instability of the Kelvin-Helmholtz type. To distinguish the instability of rotating flows from that of horizontal parallel flows, we will identify them as the centrifugal instability and the shear instability.

Unlike the velocity in horizontal parallel flows which is mainly responsible for the generation of the Kelvin-Helmholtz instability, the angular velocity is involved in both the centrifugal and shear instabilities because of its dual role in stability. While the angular velocity gradient generates shear which always tends to destabilize the flow, the rotation of fluid creates centrifugal forces which can either stabilize or destabilize the flow. This behavior was mathematically investigated by Fung (1983, 1984) and is physically interpreted in this section. For discussion simplicity, we consider flow profiles varying only in the radial direction. The gravitational effects will temporarily be ignored for the discussion in this section.

2a. *Centrifugal Instability*

As an instability of the Rayleigh-Taylor type, the centrifugal instability is activated by the centrifugal force field created by the rotation of fluid. Unlike the gravitational effect which exerts constant pull on fluids, the centrifugal effect depends on both the angular velocity and the position of the fluids. We will, however, apply the basic principle used in the discussion of the Rayleigh-Taylor instability in Sec. 1a to demonstrate this force effect.

Consider two equal volume fluid elements located at Q_1 and Q_2 within a flow regime in a cylindrical coordinate system with its z-axis coinciding with the axis of rotation as shown in Fig. 5. The element located at Q_1 has a density ρ_1 and as tangential velocity V_1 while the one at Q_2 has a density ρ_2 and an tangential velocity V_2. When the two particles interchange their positions, the principle of conservation of circulation requires that the element originally located at Q_1 possess a tangential velocity $(r_1 V_1)/r_2$ and the element originally located at Q_2 posses a tangential velocity $(r_2 V_2)/r_1$. The corresponding work done by these two elements in the centrifuged force field are,

$$W_1 = \frac{\rho_1}{2}\left\{\frac{V_1^2}{r_1} + \frac{r_1^2 V_1^2}{r_2^3}\right\}(r_2 - r_1) \tag{3.12}$$

$$W_2 = \frac{\rho_2}{2}\left\{\frac{V_2^2}{r_2} + \frac{r_2^2 V_2^2}{r_1^3}\right\}(r_1 - r_2). \tag{3.13}$$

Instability of the flow will occur if the total work done by the interchange of the two elements is positive, i.e.,

$$\delta W = \rho_1 r_1^2 V_1^2 - \rho_2 r_2^2 V_2^2 > 0. \tag{3.14}$$

An alternative way to look at this type of instability is to examine the pressure gradient experienced by the fluid element due to the interchange. If the element originally located at Q_1 is displaced to Q_2, the conservation of circulation demands that the element experience a centrifugal force $\rho_1(r_1 V_1/r_2)^2/r_2$ at its new location. At the same time, the prevailing pressure gradient at that location is $\rho_2 V_2^2/r_2$. The element will continue its motion outwards if the centrifugal force it experiences is greater than the prevailing pressure gradient. Equation (3.14) is then satisfied and the flow becomes unstable.

The mathematical derivation of this stability condition was given by Synge (1933) in a differential form, saying that the necessary and sufficient condition for stability of the flow subject to axisymmetric perturbations is

$$\Phi \geqslant 0 \tag{3.15}$$

where

$$\Phi = \frac{1}{\rho_0 r^3} \frac{d}{dr}[\rho_0 (rV)^2] \tag{3.16}$$

is the Rayleigh-Synge discriminant. The discriminant can be separated into two parts and the well-known Rayleigh-Synge criterion in (3.15) can be written as

$$\Phi = N_r^2 + \frac{1}{r^3}\frac{d}{dr}(rV)^2 \geqslant 0. \tag{3.17}$$

Here the "natural" oscillation frequency, reminiscent of the Brunt-Väisälä frequency in the gravitational force field, is defined as

$$N_r = \left(\frac{V^2}{\rho_0 r}\frac{d\rho_0}{dr}\right)^{1/2} \tag{3.18}$$

to measure the density variation in the radial direction. It is obvious from the above discussion on condition (3.14) that the Rayleigh-Synge discriminant is composed of two parts that affect the centrifugal balance, the variation of density in the centrifugal force field, and the conservation of circulation. The Rayleigh-Synge criterion in (3.15) is therefore a condition for centrifugal stability, a type of "statically stable" flow profile in the centrifugal force field created by the fluid rotation.

2b. *Rotating Shear Instability*

As an instability of the Kelvin-Helmholtz type, the rotating shear instability is triggered by the angular velocity gradient in the shear layer. In the present case, both the centrifugal force field and the shear layer are created by the rotating velocity itself (Fung 1983), and it is sometimes difficult to separate the two effects. In addition, the interchange of fluid elements is not restricted to be two-dimensional as in the case previously discussed. By considering a radius-dependent swirling flow, Fung & Kurzweg (1975) derived a Richardson criterion for stability of the flow subject to three-dimensional perturbations. In the absence of axial velocities, their result reduces to

$$\Phi - \frac{1}{4}\left(r\frac{d\Omega}{dr} + 4\Omega\right)^2 \geqslant 0 \tag{3.19}$$

for assured stability. Here $\Omega = V/r$ is the angular velocity. It is obvious that condition (3.19) will be violated if $\Phi < 0$ In those cases both centrifugal instability (Rayleigh-Taylor type) and rotating shear instability (Kelvin-Helmholtz type) take place. We can therefore conclude that the precondition (necessary condition) for the Richardson criterion in Eq. (3.19) is condition (3.15), a requirement for "statically stable" profiles in the the centrifugal force field. By defining the Richardson number

$$J_r = N_r^2/r^2\left(\frac{d\Omega}{dr}\right)^2 \tag{3.20}$$

to represent the ratio between the radial "buoyancy" effect and the angular shear effect, the criterion in Eq. (3.19) can be written as

$$J_r \geqslant \frac{1}{4}. \tag{3.21}$$

Thus, stability of a radius-dependent flow is assured against arbitrary small perturbations if the local Richardson number is nowhere less than a quarter within the flow domain. It is also clear that the Richardson criterion will be violated if the density is decreasing radially outwards. As a matter of fact, Fung (1983, 1984) showed that the instability of the flow will occur if the density is a decreasing function of radius. In those cases, instability of the flow is no longer restricted to that of the shear origin only.

It should be emphasized that the Richardson criteria derived for two-dimensional parallel flows (3.11) and rotating shear flows (3.21) are merely sufficient conditions for stability. Violating those conditions does not necessary lead to instability.

IV. STABILITY CRITERIA

In the previous discussion on the instability mechanisms, we were concerned with flow profiles varying in one direction. In those cases, only one force field and one shear layer exist. When the fluid is stratified in two directions, the effect of both the gravitational and the centrifugal force fields appear. The restoring forces in both the axial and radial direction not only exert their influence on their respective directions, but also interact with each other. The interaction appears as the pressure balance condition governed by the equations of motion. In addition, when the angular velocity varies in both the radial and axial directions, the shear effect produced by the velocity gradients not only exerts its influence but also posseses interactions on both the $r-\theta$ and $r-z$ planes. In other words, two force fields and two shear layers exist within the flow and interact with each other. Intuitively, the stability of the present flow should involve one Richardson criterion in the radial direction and one in the gravitational direction. However, the presence of the two force fields, the density stratification in two directions, and the two shear layers further complicates the problem. We are concerned here with the influence of these interactions upon the final stability and the resultant stability criteria.

Mathematically, one can apply the normal mode method to vortex flows with profiles varying only in one direction and obtain a set ordinary differential equations governing the characteristics of the flow. When the flow profiles varies in two directions, however, the governing equations will no longer be ordinary. Additional boundary conditions in the second directions are also required, and the resultant system becomes much more complex.

In this section, we will consider mathematically flow profiles of a more general nature to describe the organized flow patterns developed in the late wake. A general class of vortex flows with their steady state distributions depending on both the axial and radial coordinates will be investigated. Necessary conditions for stability of the flows are derived by using the method of generalized progressing wave expansion. The necessary conditions can be interpreted using the physical arguments given in the preceding discussion on the fundamental instability mechanisms. Sufficient conditions for stability will also be examined. A new Brunt-Väisälä and a new Richardson number can be defined. The former measures the interaction of the density variation in one direction with the force field in the other direction. The latter denotes the ratio between the newly defined Brunt-Väisälä frequency and the interaction of the velocity gradients in two directions.

The stratified vortex flow being considered has a density $\rho_0(r,z)$ and angular velocity $\Omega(r,z)$ in a cylindrical domain (r,θ,z) with the z-axis in the gravitational direction. The fluid is assumed to be inviscid and incompressible. To satisfy the pressure balance condition anywhere within the flow regime we need the constraint equation

$$\frac{\partial}{\partial z}(\rho_0\, r\, \Omega^2) + \frac{\partial}{\partial r}(\rho_0\, g) = 0. \tag{4.1}$$

The linearized equations for the present flow subject to small perturbations are

$$\rho_0\left\{\frac{\partial \hat{u}}{\partial t} + \Omega\frac{\partial \hat{u}}{\partial \theta} - 2\Omega\ \hat{v}\right\} - \hat{\rho}\, r\Omega^2 = -\frac{\partial \hat{p}}{\partial r} \tag{4.2}$$

$$\rho_0\left\{\frac{\partial \hat{v}}{\partial t} + \Omega\frac{\partial \hat{v}}{\partial \theta} + \left[\frac{\partial}{\partial r}(r\Omega) + \Omega\right]\hat{u} + \frac{\partial}{\partial z}(r\Omega)\,\hat{w}\right\} = -\frac{1}{r}\frac{\partial \hat{p}}{\partial \theta} \tag{4.3}$$

$$\rho_0\left\{\frac{\partial \hat{w}}{\partial t} + \Omega\frac{\partial \hat{w}}{\partial \theta}\right\} = -\frac{\partial \hat{p}}{\partial z} - \hat{\rho} g \tag{4.4}$$

$$\frac{\partial \hat{\rho}}{\partial t} + \Omega\frac{\partial \hat{\rho}}{\partial \theta} + \frac{\partial \rho_0}{\partial r}\hat{u} + \frac{\partial \rho_0}{\partial z}\hat{w} = 0 \tag{4.5}$$

$$\frac{\partial \hat{u}}{\partial r} + \frac{\hat{u}}{r} + \frac{\partial \hat{v}}{r\partial \theta} + \frac{\partial \hat{w}}{\partial z} = 0. \tag{4.6}$$

Here $\hat{u}$, $\hat{v}$, $\hat{w}$, $\hat{p}$, and $\hat{\rho}$ are respectively the perturbation velocities in the r,θ,z directions, the perturbation pressure and the perturbation density.

The boundary conditions for the system are

$$\hat{u} = 0 \quad \text{at} \quad r = R_1, R_2 \tag{4.7a}$$

and

$$\hat{w} = 0 \quad \text{at} \quad z = Z_1, Z_2 \tag{4.7b}$$

where R_1, R_2, Z_1, and Z_2 are locations of the rigid boundaries. For unbounded flows, the perturbation velocities will vanish at infinity.

1. Necessary Conditions for Stability

By transforming the linearized equations to a symmetric hyperbolic system and by using the method of generalized progressing waves expansion, Fung (1985a) derived the necessary conditions for stability of a general class of compressible flows with profiles varying in both the axial and radial directions. His results for incompressible flows reduce to the results in the present case, so that the necessary conditions for stability of the flows are

$$\Phi_r \geqslant 0 \tag{4.8a}$$

$$-\frac{g}{\rho_0}\frac{\partial \rho_0}{\partial z} \geqslant 0 \tag{4.8b}$$

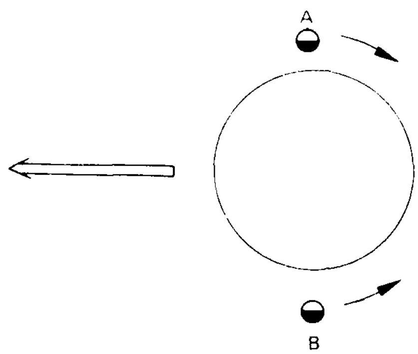
$$\left(-\frac{g}{\rho_0}\frac{\partial \rho_0}{\partial z}\right)\Phi_r \geqslant \left(\frac{g}{\rho_0}\frac{\partial \rho_0}{\partial r}\right)^2 \tag{4.8c}$$

within the flow domain. Here

$$\Phi_r = \frac{1}{\rho_0 r^3}\frac{\partial}{\partial r}\left[\rho_0\,(r^2\Omega)^2\right] \tag{4.9}$$

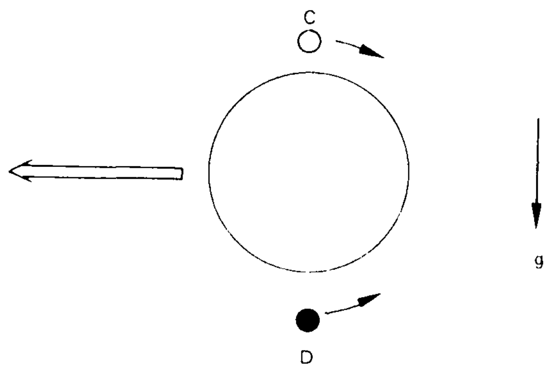
reduces to the classical Rayleigh-Synge discriminant in Eq. (3.16) if the density and the velocity of the flow depend only on the radius.

The physical meaning of these necessary conditions can also be interpreted on the basis of kinetic energy and work done during the movement of fluid particles in the centrifugal and gravitational force fields as follows.



(a)

Figure 3a — Top view of a sphere towed in a stably straified fluid



(b)

Figure 3b — Side view of a sphere towed in a stably stratified fluid

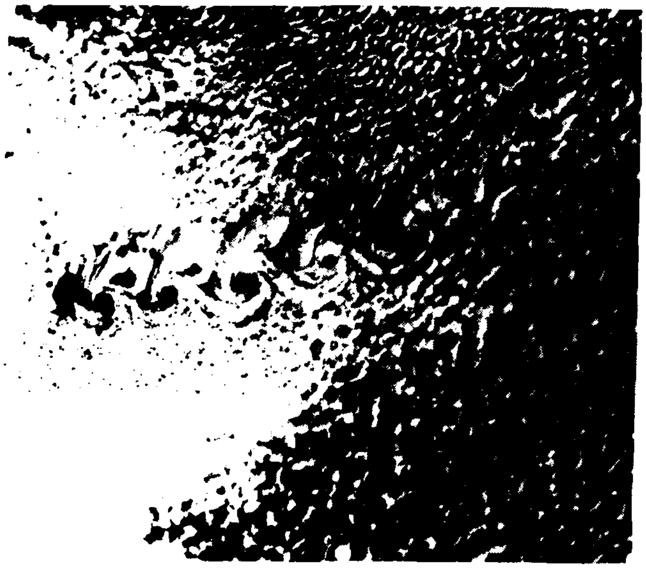


Figure 2 — A pattern of vortices in the clouds downstream from the Arctic island of Jan Mayen, east of Greenland. The vortices were formed by the wind flowing past the mountainous island. The photograph was taken from a NOAA satellite. Photo courtesy of Dr. Emil Simiu of the National Bureau of Standards.

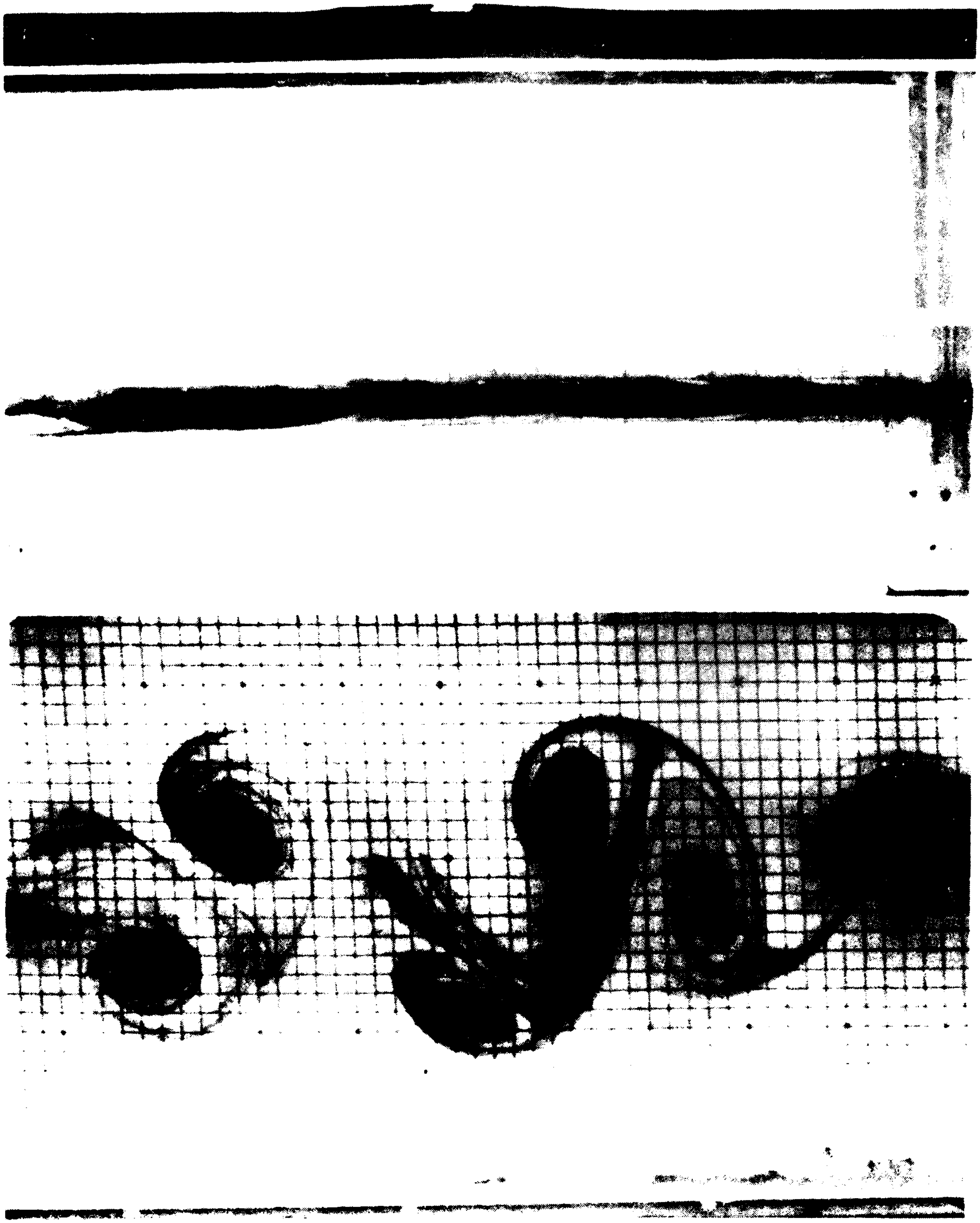


Figure 1 — Vortex structure in the wake of a sphere towed through thermally stratified water. The upper photograph shows the alternate vortex pattern in the horizontal plane at a late time, t = 285 seconds, after the passage of the sphere. The relatively narrow vertical extent of the wake is shown in the lower photograph. Photographs courtesy of Dr. Timothy Kao of Catholic University.

APPENDIX

Consider a function

$$F = aX^2 + 2bXY + cY^2 \tag{A1}$$

where X and Y are independent variables, and a,b, and c are arbitrary scalar. All the quantities in Eq. (A1) are real. The function F can be normalized into

$$F = k_1S_1^2 + k_2S_2^2 \tag{A2}$$

where

$$k_{1,2} = \frac{1}{2}[(a + c) \pm \sqrt{(a + c)^2 - 4(ac - b^2)}]. \tag{A3}$$

The function F will always be positive for all values of S_1 and S_2 if k_1 and k_2 are real and positive, i.e.,

$$a + c \geqslant 0 \tag{A4}$$

$$ac \geqslant b^2 \tag{A5}$$

Eqs. (A4) and (A5) will be satisfied if

$$a \geqslant 0 \tag{A6}$$

$$c \geqslant 0 \tag{A7}$$

$$ac \geqslant b^2 \tag{A8}$$

The function F in Eq. (A1) will always be positive if conditions (A6) through (A8) are satisfied. The conditions in Eqs. (4.8) can then be reached by matching Eq. (4.12) with (A1).

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VII. CONCLUSIONS AND DISCUSSIONS

Through a discussion of both the Rayleigh-Taylor and Kelvin-Helmholtz instabilities, we have analyzed the instability mechanisms governing a stratified vortex flow possessing two shear layers and two force fields, the centrifugal one and the gravitational one. Since the present flow distribution varies in both the axial and the radial directions, the governing equations can no longer be ordinary. To overcome this difficulty, we have used the method of generalized progressing wave expansion to obtain a set of three necessary conditions for stability. We have further used integral techniques to obtain a set of three sufficient conditions for stability. The necessary conditions represent a generalized state of statically stable flow profiles, a criterion for the instability of the Rayleigh-Taylor type. The sufficiency conditions denote a generalized form of the Richardson criterion, a ratio indicating the balance between the generalized "buoyancy" effect induced by the restoring force in the two force fields and the inertia effect produced by the Kelvin-Helmholtz instability in the shear layer. Two of the necessary and sufficient conditions for stability represent the criteria in their respective directions. The third ones, resulting from the constraint relation for a pressure balance everywhere within the flow field, act as a restraint to the other two conditions. As a result, the three necessary and sufficient conditions derived require that the flow be stable in the centrifugal force field, in the gravitational force field, and in the pressure field that restrains the density variation in the two force fields.

The newly defined Richardson number J_{rz}, greater than a quarter for guaranteed flow stability, represents a ratio between the interaction of the density variation in one direction with the force field in the other direction, and the interaction of the velocity variation in one direction with that in the other direction. The former interaction is measured by a new Brunt-Väisälä frequency N_{rz} and determines the manner in which the density should be stably stratified under the influence of both the force fields. It is a result of the pressure balance condition. The latter interaction determines whether the presence of the velocity gradients in the second direction strenghtens or weakens the resultant shear effect. As a result, the newly established Richardson criterion serves as a constraint on the other two Richardson criteria and provides a stability condition for all the combinations of density and velocity profiles varying in both the axial and radial directions.

The angular velocity of the flow plays a dual role in flow stability. The velocity itself, on one hand, interacts with the density gradient in the radial direction to create a centrifugal force field which can either stabilize or destabilize the flow. Such an interaction is measured by the Brunt-Väisälä frequency N_r to represent the restoring force in the radial direction. This is analagous to the interaction of the axial density gradient with the gravitational force field, which is measured by the Brunt-Väisälä frequency N_z to represent the restoring force in the axial direction. The angular velocity gradients, on the other hand, induce shear effects which, in general, destabilize the flow. However, the final stability depends on their interaction. For assured stability, the velocity gradients in the radial and the axial directions must be of opposite sign.

The stability criteria investigated here are for a general class of rotating flows in the gravitational force field with the velocity and density varying in both the axial and radial directions. The conditions are therefore valid for the vortex patterns in the late wake, the vortex trails on the lee of certain mountains, and also for a wide range of problems in atmospheric and oceanographic studies.

VIII. ACKNOWLEDGMENT

This author wishes to acknowledge the support of the Naval Research Laboratory.

where $\kappa = kR$. Equation (6.4) describes the stability characteristics of the present flow distribution for all the axial and azimuthal wave numbers. We will consider a particular case as an example. For $k = 0$, Eq. (6.4) can be solved for the complex frequency ω and the flow is stable if

$$\rho_2[m\Omega_2 - (m-1)\Omega_1]^2 + (1 + \frac{\rho_2}{\rho_1})[(m-1)\rho_1\Omega_1^2 - m\rho_2\Omega_2^2] \geqslant 0. \qquad (6.5)$$

The stability domain is plotted in Fig. 8 for different azimuthal wave numbers. The stability boundaries for the axisymmetric mode $\rho_2\Omega_2^2 - \rho_1\Omega_1^2 = 0$, a mode for the centrifugal instability obtained either by integrating the Rayleigh-Synge criterion across the interface or by expanding the modified Bessel functions asymptotically in Eq. (6.4), is also plotted to compare with these for the rotating shear instability. It is obvious that the stability domain for the two types of instabilities are quite different.

The second example to be examined is two uniformly rotating flows with different density, one above the other as follows:

$$\begin{aligned} \Omega(r,z) &= \Omega_1 & & \\ \rho_0(r,z) &= \rho_1 & &\text{for } z \geqslant 0 \\ \Omega(r,z) &= \Omega_2 & & \\ \rho_0(r,z) &= \rho_2 & &\text{for } z < 0 \end{aligned} \qquad (6.6)$$

where the quantities with numeric indices are constants. The gravitational effect is included in the present case. The contraint equation requires that

$$\rho_1\Omega_1^2 = \rho_2\Omega_2^2 \qquad (6.7)$$

to satisfy the pressure balance at the axial interface. The pressure and the axial velocity perturbations obtained by solving Eqs. (4.18) to (4.20) for the flow profile in Eq. (6.6) are

$$\begin{aligned} w_j &= i\frac{k}{\rho_j N_j}\left[A_j J_m(kq_j r) + B_j Y_m(kq_j r)\right]\left[C_j e^{kz} - D_j e^{-kz}\right] \\ p_j &= \left[A_j J_m(kq_j r) + B_j Y_m(kq_j r)\right]\left[C_j e^{kz} + D_j e^{-kz}\right] \qquad j = 1,2 \end{aligned} \qquad (6.8)$$

where J_m and Y_m are the Bessel functions of the first and second kinds. For bounded solutions at infinity, we require

$$C_1 = D_2 = 0. \qquad (6.9)$$

Matching both conditions (5.2) at the interface $z = 0$, we obtain a relatively simple secular relation governing the stability characteristics of the flow as follows

$$\rho_2[(m\Omega_2 - \omega)^2 - kg] + \rho_2[m\Omega_1 - \omega)^2 + kg] = 0. \qquad (6.10)$$

Solving Eq. (6.10) for the complex eigen-frequency ω, we find the flow will be stable if

$$kg(\rho_2^2 - \rho_1^2) - m^2\rho_1\rho_2(\Omega_2 - \Omega_1)^2 \geqslant 0. \qquad (6.11)$$

The first term represents the effect of the density variation in the gravitational force field. It is obvious that Eq. (6.11) will always be violated if $\rho_2 < \rho_1$, a statically unstable state of density distribution. Instability of the Rayleigh-Taylor type will take place. The second term denotes the shear effect arising from the angular velocity difference, an instability of the Kelvin-Helmholtz type which always destabilizes the flow.

The physical intepretation of Eqs. (5.2) can also be reached by similar arguments and will not be repeated here. Similar to the case of the cylindrical vortex sheet, the deformation of an interface in the axial direction induces a perturbation to the flow in two ways: the perturbation to the pressure field and the perturbation to the gravitational force field. Any discontinuity arising in densities should be included in the jump condition given in Eq. (5.2b).

Even though Eqs. (5.1) and (5.2) are derived for flows with an interface in the radial or axial direction, they are in fact valid for flow profiles with or without discontinuities.

VI. TWO ANALYTICAL EXAMPLES

In the following we will use two examples to demonstration the stability characterics described by Eqs. (4.18) to (4.20). The first one has an interface in the radial direction while the second one possesses an interface in the axial direction.

The first example to be examined is a uniformly rotating core surrounded by a potential vortex with different density. All flow quantities depend only on the radius as follows

$$\left.\begin{array}{l}\Omega(r,z) = \Omega_1 \\ \rho_0(r,z) = \rho_1\end{array}\right\} \qquad \text{for } 0 \leqslant r < R$$

$$\left.\begin{array}{l}\Omega(r,z) = \Omega_2(R/r)^2 \\ \rho_0(r,z) = \rho_2\end{array}\right\} \qquad \text{for } R \leqslant r < \infty. \tag{6.1}$$

Here the quantities with numeric indices are constants. The gravitational effect is neglected in this example.

The solutions for the flow profiles in the inner and outer regions are respectively,

$$u_1 = \frac{N_1}{r}\left\{A_1\left[\frac{2m\Omega_1}{N_1} + \frac{kqrI_m'(kqr)}{I_m(kqr)}\right]I_m(kqr) + B_1\left[\frac{2m\Omega_1}{N_1} + \frac{kqrK_m'(kqr)}{K_m(kqr)}\right]K_m(kqr)\right\}e^{\pm ikz}$$

$$p_1 = -i\,\rho_1\,(N_1^2 - 4\Omega_1^2)[A_1 I_m(kqr) + B_1\,K_m(kqr)]\,e^{\pm ikz}$$

and

$$u_2 = ik[A_2 I_m'(kr) + B_2 K_m'(kr)]\,e^{\pm ikz}$$

$$p_2 = \rho_2[m\Omega_2(R/r)^2 - \omega][A_2 I_m(kr) + B_2 K_m(kr)]\,e^{\pm ikz}$$

Here

$$N_1 = m\Omega_1 - \omega,$$

$$q = \left\{1 - \frac{4\Omega_1^2}{N_1^2}\right\}^{1/2}. \tag{6.2}$$

k is the axial wave number and the prime denotes the total derivative with respect to the arguments of the modified Bessel functions of the first and second kinds, I_m and K_m. For bounded solutions at infinity and on the axis, we require

$$A_2 = B_1 = 0. \tag{6.3}$$

Matching both conditions (5.1) at the interface $r = R$, we obtain the secular relation

$$\frac{\rho_1(N_1^2 - 4\Omega_1^2)}{\dfrac{\kappa q I_m'(\kappa q)}{I_m(\kappa q)} + \dfrac{2m\Omega_1}{N_1}}\;\frac{-\rho_2 N_2^2}{\dfrac{\kappa\,K'_m(\kappa)}{K_m(\kappa)}} = \rho_2\,\Omega_2^2 - \rho_1\Omega_1^2 \tag{6.4}$$

and

$$<p>_z + i\left(\frac{w}{N}\right)_{z=Z} <\rho_0\, g>_z = 0 \tag{5.2b}$$

where $<\Phi>_z = \Phi(Z+0) - \Phi(Z-0)$.

The physical meaning of Eqs. (5.1) and (5.2) can be explained by a simple linear perturbation method. Assume that the cylindrical vortex sheet located at the radial position R is disturbed such that the deformed interface is prescribed by

$$r = R + \hat{\eta}(r,\theta,z;t) \tag{5.3}$$

where $R >> \hat{\eta}$. Taking the total derivative of Eq. (5.3) and assuming periodic solutions in Eq. (4.17) yield

$$\eta(r,z) = -i\frac{u(r,z)}{m\Omega - \omega}. \tag{5.4}$$

Equation (5.1a) follows if no gap is allowed to exist at the disturbed interface.

The dynamic interfacial condition can also be obtained by examining the equation of motion. The steady-state form of the Euler equation of motion in the radial direction reads

$$\frac{\partial P_0}{\partial r} = \rho_0\, r\Omega^2 \tag{5.5}$$

where $P_0\,(r,z)$ is the total pressure. The steady-state total pressures inside and outside the vortex sheet are respectively

$$P_{01}(r) = \int_{R_1}^{r} \rho_{01}\,\zeta\Omega^2\, d\zeta \qquad \text{for } R_1 \leqslant r \leqslant R \tag{5.6a}$$

$$P_{02}(r) = \int_{R_1}^{R} \rho_{01}\,\zeta\Omega^2\, d\zeta + \int_{R}^{r} \rho_{02}\zeta\Omega^2\, d\zeta \qquad \text{for } R \leqslant r < \infty, \tag{5.6b}$$

where the subscripts 1 and 2 respectively denote the quantities prescribed in the inner and outer regions separated by the vortex sheet, and R_1 is a reference radial location. Let the vortex sheet be perturbed according to Eq. (5.3). The total pressure is now

$$P_j = P_{0_j} + p_j \qquad j = 1,2 \tag{5.7}$$

and should be balanced at the perturbed interface, i.e.,

$$P_1(R + \hat{\eta}) = P_2(R + \hat{\eta}). \tag{5.8}$$

Subtracting Eqs. (5.6) from Eq. (5.8) and assuming that all the quantities in the mean flow are bounded and continuous in the interval $[R, R + \hat{\eta}]$, we obtain the first order perturbation condition for dynamical balance evaluated at the undeformed interfaces as follows:

$$<p>_R + <\rho_0\, r\Omega^2>\, \hat{\eta}(R) = 0. \tag{5.9}$$

The dynamic interfacial condition (5.1b) follows if periodic solutions for the perturbation quantities in Eq. (4.17) are assumed once again.

As shown in Fig. (7), the mathematical steps adopted to derive Eq. (5.9) from Eq. (5.8) simply demonstrate a dissolution of the total pressure force acting at the *disturbed* surface of the vortex sheet (Fig. 7a) into the individual force components acting at the *steady-state* interface (Fig. 7b). As a matter of fact, Eq. (5.9) also can be reached simply by balancing all the force components acting at a differential element $\hat{\eta}_{(R)}R\ d\theta$ (per unit axial wave length) that experiences the centripetal acceleration induced by the angular velocity. This procedure of force decomposition clearly demonstrates that the deformation of the interface described by Eq. (5.3) induces a perturbation to the flow in two ways: the perturbation to the pressure field and the perturbation to the centrifugal force field arising from the azimuthal velocity. Any discontinuity arising in densities and azimuthal velocity should be included in the jump condition given by Eq. (5.1b) or (5.9).

a,b) that N_{rz} must be real preceding stability. This frequency can also be viewed as a measure of the density variation in both the force fields or as a constraint of the pressure balance condition described in (4.1). The newly defined Richardson number J_{rz} now stands for the ratio between the interaction of the density variations (in one direction) with force fields (in the other direction) and the interaction of the velocity gradients in one direction with the other direction. The latter is the interaction of the shear stresses that trigger the shear instability. By using the relation in (4.26), the stability condition in (4.24c) can be written as

$$J_{rz} \geqslant \frac{1}{2}\left\{\left(\frac{E}{F}J_r - \frac{F}{E}J_z\right)^2 + (J_r + J_z)\right\} \tag{4.24d}$$

Therefore we can immediately conclude the following two arguments for assured stability: [1] the Richardson number J_{rz} must be greater than a quarter, and [2] the velocity gradients in the radial and the axial directions must be of opposite signs. Unlike the case of two-dimensional stratified flows and of radius-dependent rotating flows in which velocity gradients always have destabilizing effects, the present flow possesses an interaction of the velocity gradients which can carry either stabilizing or destabilizing effects depending on the sign of the interaction. In other words, the velocity gradients may together upset flow stability although the flow is stable in their respective directions. Accordingly, conditions (4.24c) or (4.24d) can be viewed as a constraint on both conditions (4.24 a,b) and provides a stability constraint to all the combinations and variations of the density and velocity profiles.

V. INTERFACIAL CONDITIONS

Even though the previously discussed necessary and sufficient conditions for stability provide us with some upper bound information on stability or instability, the criteria for flows of this kind do not yield sufficient knowledge of instabilities for an arbitrary flow profile. Solutions to the governing stability equations must be obtained before the detailed instability characteristics for the particular flow profile can be observed. Unfortunately, analytical solutions in terms of well-known functions for the present vortex flow are very difficult to obtain except in the limit of adjacent layers with different but constant characteristics. Matching the solutions at the common boundary between two such flow regions therefore becomes the basis for the analysis of this limiting case. In matching these discontinuous profiles, appropriate interfacial conditions must be used.

Since two force fields exist in the present flow, two sets of interfacial conditions in the radial and axial directions are required to handle any possible discontinuities or rapid change of profiles within the flow field. They can simply be obtained by integrating the governing equations in (4.18) to (4.20) across the corresponding interfaces as follows.

Assume two interfaces located respectively at $r = R$ and $z = Z$. Integrating Eqs. (4.18) and (4.20) across the interface at $r = R$ in the radial direction yields

$$< \frac{u}{N} >_R = 0 \tag{5.1a}$$

and

$$< p >_R - i\left(\frac{u}{N}\right)_{r=R} < \rho_0 r \Omega^2 >_R = 0 \tag{5.1b}$$

where $<\Phi>_R = \Phi(R_{+0}) - \Phi(R_{-0})$. Integrating Eqs. (4.18) and (4.19) across the interface at $z = Z$ in the axial direction yields

$$< \frac{w}{N} >_z = 0 \tag{5.2a}$$

where the Richardson numbers in the radial direction, the axial direction and their interaction are, respectively, defined as

$$J_r = N_r^2/r^2 \left(\frac{\partial\Omega}{\partial r}\right)^2 \tag{4.25a}$$

$$J_z = N_z^2/r^2 \left(\frac{\partial\Omega}{\partial z}\right)^2 \tag{4.25b}$$

$$J_{rz} = -N_{rz}^2/r^2 \left(\frac{\partial\Omega}{\partial r}\right)\left(\frac{\partial\Omega}{\partial z}\right). \tag{4.25c}$$

The relation between these three numbers is

$$J_{rz} = -\frac{1}{2}\left(\frac{E}{F} J_r + \frac{F}{E} J_z\right) \tag{4.26}$$

where

$$F = \frac{r\Omega^2}{g} \tag{4.27}$$

is a Froude number that indicates the ratio between the centrifugal force field and the gravitational force field, and

$$E = \left(\frac{\partial\Omega}{\partial r}\right)/\left(\frac{\partial\Omega}{\partial z}\right) \tag{4.28}$$

is a ratio between the radial velocity gradient and the axial velocity gradient.

Equations (4.24) provide us with the stability conditions for all combinations of the velocity and density variations. Equations (4.24 a,b) require that both the Richardson numbers J_r and J_z be positive as a precondition for stability, indicating positive N_r^2 and N_z^2. This requirement implies that the density *should be stably stratified in the centrifugal* and the gravitational force fields, i.e., $\frac{\partial\rho_0}{\partial r} > 0$ and $\frac{\partial\rho_0}{\partial z} < 0$. For the flow being considered, velocity gradients exist in both the radial and axial directions, indicating that the shear instability may take place in more than one plane. Equations (4.24 a,b) therefore represent the stability condition with respect to their corresponding planes. The Richardson number J_r in criterion (4.25a) represents a ratio between the centrifugal restoring force along the radial direction and the shear effect transverse to the direction, while the Richardson number J_z in criterion (4.25b) denotes a ratio between the buoyant restoring force along the gravitational direction and the shear effect perpendicular to that direction. The interpretation of the two sufficiency conditions is the same as those in the two-dimensional stratified flows and the rotating shear flows as previously discussed. Also like the two-dimensional stratified flows, these two conditions do not restrict the density variation as long as they are satisfied. In others words, there is no limit on how strong the density is stratified in the radial and gravitational directions. In fact, for two-dimensional shear flows and radius-dependent rotating flows, the stronger the density is stratified, the more stable the flow will be. This characteristic also prevails in the present flow.

The Brunt-Väisälä frequency N_r measures the density variation along the centrifugal direction in the centrifugal force field while the frequency N_z measures the density variation along the gravitational direction in the gravitational force field. Both frequencies must first be real preceding stability, i.e., the density is stably stratified in both the radial and axial directions. While both N_r and N_z can be viewed as the interaction of the density gradient and the force along the same direction, the newly defined Brunt-Väisälä frequency N_{rz} can be viewed as a measure for the interaction of the density variation in one direction with the force field in the other direction. We can conclude from (4.23 a,b) and (4.24

$$\Phi_z = \frac{1}{\rho_0}\frac{\partial}{\partial z}(\rho_0 r\Omega^2)$$

$$\Psi_r = \frac{g}{\rho_0}\frac{\partial \rho_0}{\partial r}$$

$$\Psi_z = \frac{g}{\rho_0}\frac{\partial \rho_0}{\partial z}$$

and $N = m\Omega - \omega$ is the Doppler-shifted frequency. The boundary conditions for the system are

$$u = 0 \qquad \text{at} \qquad r = R_1, R_2 \tag{4.21a}$$

and

$$w = 0 \qquad \text{at} \qquad z = Z_1, Z_2. \tag{4.21b}$$

By using the proper transformation and the complex conjugation, Fung (1985b) derived three Richardson criteria saying that the stability of the present flow will be assured if

$$\frac{r\Omega^2}{\rho_0}\frac{\partial \rho_0}{\partial r} - \frac{r^2}{4}\left(\frac{\partial \Omega}{\partial r}\right)^2 \geqslant 0 \tag{4.22a}$$

$$-\frac{g}{\rho_0}\frac{\partial \rho_0}{\partial z} - \frac{r^2}{4}\left(\frac{\partial \Omega}{\partial z}\right)^2 \geqslant 0 \tag{4.22b}$$

and

$$\left\{\frac{r\Omega^2}{\rho_0}\frac{\partial \rho_0}{\partial r} - \frac{r^2}{4}\left(\frac{\partial \Omega}{\partial r}\right)^2\right\}\left\{-\frac{g}{\rho_0}\frac{\partial \rho_0}{\partial z} - \frac{r^2}{4}\left(\frac{\partial \Omega}{\partial z}\right)^2\right\}$$

$$\geqslant \left\{\frac{1}{2}\left(r\Omega^2\frac{\partial \rho_0}{\partial z} - g\frac{\partial \rho_0}{\partial r}\right) - \frac{r^2}{4}\left(\frac{\partial \Omega}{\partial r}\right)\left(\frac{\partial \Omega}{\partial z}\right)\right\}^2 \tag{4.22c}$$

are all satisfied throughout the flow domain. If we introduce the Brunt-Väisäilä frequencies

$$N_r^2 = \frac{r\Omega^2}{\rho_0}\frac{\partial \rho_0}{\partial r} \tag{4.23a}$$

$$N_z^2 = -\frac{g}{\rho_0}\frac{\partial \rho_0}{\partial z} \tag{4.23b}$$

and

$$N_{rz}^2 = \frac{1}{2\rho_0}\left(g\frac{\partial \rho_0}{\partial r} - r\Omega^2\frac{\partial \rho_0}{\partial z}\right) \tag{4.23c}$$

to measure the density variation in the radial direction, the axial direction and their interaction in the centrifugal and gravitational force fields, equations (4.23) can be written as

$$J_r \geqslant \frac{1}{4} \tag{4.24a}$$

$$J_z \geqslant \frac{1}{4} \tag{4.24b}$$

and

$$\left(J_r - \frac{1}{4}\right)\left(J_z - \frac{1}{4}\right) \geqslant \left(J_{rz} - \frac{1}{4}\right)^2 \tag{4.24c}$$

using the integral method employed by Fung (1983), that the first term in Eq. (4.12) is a differential representation of a stable centrifugal force field. This mechanism is contained in Eq. (4.8a) which states that the flow should be stable in the radial direction. Two parts are involved in this first term. The first part is the variation of density in the centrifugal force field. The second part is the Rayleigh discriminant representing the effect on the centrifugal force field by conservation of circulation.

If the variation in the radial direction is suppressed, Eq. (4.12) reduces to a condition representing the variation of density in the gravitational force field. This mechanism is contained in (4.8b) which states that the density profile should be statically stable in the axial direction.

The second term in Eq. (4.12) represents the interaction between the radial and axial variations of the density in the centrifugal and gravitational force fields. This coupled variation is reflected in condition (4.8c) representing a requirement for stability imposed on the simultaneous density variations in the radial and axial directions.

For potential flows, both Eqs. (4.1) and (4.8c) reduce to

$$-\frac{\partial \rho_0}{\partial r} / \frac{\partial \rho_0}{\partial z} = \frac{r\Omega^2}{g} \tag{4.16}$$

stating that the ratio between the density gradient in the radial direction and that in the axial direction should be compatible with the ratio between the centrifugal force field and the gravitational force field.

The arguments just presented for the physical mechanisms of the necessary conditions allows us to conclude that Eqs. (4.8) represent a generalized state of "statically stable" profiles for the steady flow. To secure stability for the basic flow, it is necessary that the steady-state distribution satisfy (a) the radial force balance condition, (b) the axial force balance condition, and (c) a pressure balance condition constraining the variations of density in both the centrifugal and gravitational force fields. As a result of the third constraint, Eqs. (4.8) do not represent three independent conditions. Either one of the two conditions in (4.8a) and (4.8b) will have to be automatically satisfied if the other condition and Eq. (4.8c) are fulfilled.

2. Sufficient Conditions for Stability

Since equations (4.2) to (4.6) are cyclic in θ and t, and their coefficients depend only on r and z, we can therefore introduce a solution with the form

$$\hat{f}\,(r,\theta,z;\,t) = f(r,z)\,\exp i\,(m\theta - \omega\, t). \tag{4.17}$$

Here m is the azimuthal wave number, an integer, and $\omega = \omega_r + i\omega_i$ is the complex eigenfrequency. Equations (4.2) to (4.6) can then be combined into three first order partial differential equation as follows:

$$\rho_0\, N\left\{\frac{\partial u}{\partial r} + \left[1 - \frac{m}{N}\left(r\frac{\Omega}{\partial r} + 2\Omega\right)\right]\frac{u}{r} - \frac{m}{N}\,\frac{\partial \Omega}{\partial z}\,w + \frac{\partial w}{\partial z}\right\} = i\frac{m^2}{r^2}p \tag{4.18}$$

$$\rho_0\,\{\Psi_r\, u + (N^2 + \Psi_z)\, w\} = i\, N\,\frac{\partial p}{\partial z} \tag{4.19}$$

$$\rho_0\,\{(N^2 - \Phi_r)\, u - \Phi_z w\} = i\, N\left\{\frac{\partial p}{\partial r} + \frac{2m\Omega}{Nr}p\right\}. \tag{4.20}$$

Here

$$\Phi_r = \frac{1}{\rho_0\, r^3}\frac{\partial}{\partial r}\left[\rho_0\,(r^2\Omega)^2\right]$$

Consider two fluid particles originally located at Q_1 and Q_2 within the flow regime in the $r-z$ plane as illustrated in Fig. 6. The particle at Q_1 has a density ρ_0 and a tangential velocity V while the particle at Q_2 has a density $\rho_0 + \delta\rho_0$ and a tangential velocity $V + \delta V$. Here $V = r\Omega$ and $\delta = \delta r\frac{\partial}{\partial r} + \delta z\frac{\partial}{\partial z}$. First we use the energy approach by considering the variation of the total energy as a result of a perturbation to the system. In the the steady-state, the kinetic and potential energy of the two particles are given by

$$\text{K.E.} \qquad \frac{1}{2}\{\rho_0 V^2 + (\rho_0 + \delta\rho_0)(v + \delta V)^2\} \tag{4.10}$$

and

$$\text{P.E.} \qquad \rho_0\, gz + (\rho_0 + \delta\rho_0)g(z + \delta z).$$

When the two particles interchange their positions, the kinetic and the potential energy of the perturbed system are

$$\text{K.E.} \qquad \frac{1}{2}\left\{\rho_0\left[\frac{r\ V}{r + \delta r}\right]^2 + (\rho_0 + \delta\ \rho_0)\left[\frac{(r + \delta r)(V + \delta V)}{r}\right]^2\right\} \tag{4.11}$$

and

$$\text{P.E.} \qquad \rho_0\, g(z + \delta z) + (\rho_0 + \delta\rho_0)\ gz.$$

Here the conservation of circulation has been applied to the resultant kinetic energy in (4.11). If the perturbation is small, the stability of the system requires

$$\Phi_r\ (\delta r)^2 - \left[\frac{g}{\rho_0}\frac{\partial\rho_0}{\partial r} - \frac{1}{\rho_0}\frac{\partial}{\partial z}(\rho_0\, r\Omega^2)\right](\delta r)\ (\delta z) - \frac{g}{\rho_0}\ \frac{\partial\rho_0}{\partial z}(\delta z)^2 \geqslant 0 \tag{4.12}$$

everywhere within the flow domain.

An alternative approach to observe the stability characteristics of the system is to examine the work done by the two particles in the centrifugal and gravitational force fields. When the two particles interchange their positions, the work done by the particle originally located at Q_1 is

$$W_1 = -\frac{\rho_0}{2}\left\{\frac{V^2}{r} + \frac{(r\ V)^2}{(r + \delta r)^3}\right\}\delta r + \rho_0\, g\delta z \tag{4.13}$$

while the work done by the particle originally located at Q_2 is

$$W_2 = \frac{\rho_0 + \delta\rho_0}{2}\left\{\frac{(V + \delta\ V)^2}{(r + \delta r)} + \frac{[(r + \delta r)(V + \delta\ V)]^2}{r^3}\right\}\delta\ r - (\rho_0 + \delta\ \rho_0)g\ \delta z. \tag{4.14}$$

Condition (4.12) can be reached following the argument that the stability of the system requires the leading terms of the total work done by the interchange of the two particles to be non-negative, i.e.,

$$W_1 + W_2 = \frac{1}{r^3}\ \delta[\rho_0\, r^2 V^2]\ \delta\ r + \delta\ (\rho_0\, g)\ \delta z \geqslant 0. \tag{4.15}$$

The necessary conditions in Eqs. (4.8) can then be recovered if we apply the normalization technique in the Appendix to the condition in Eq. (4.12).

If the variation in the axial direction is omitted, Eq. (4.12) reduces to the well-known Rayleigh-Synge criterion in Eq. (3.17), which is a requirement for centrifugal stability. It can also be shown,

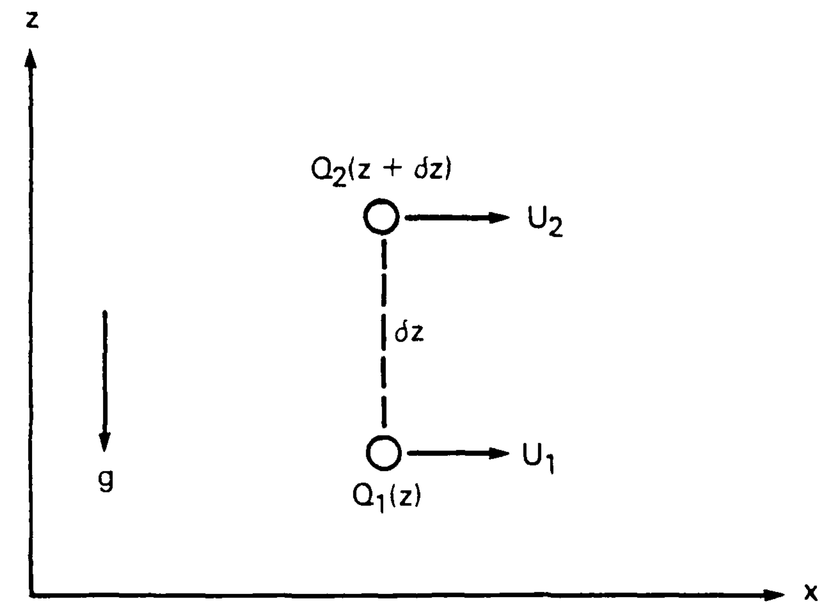


Figure 4 — Coordinate of the fluid elements in the gravitational force field

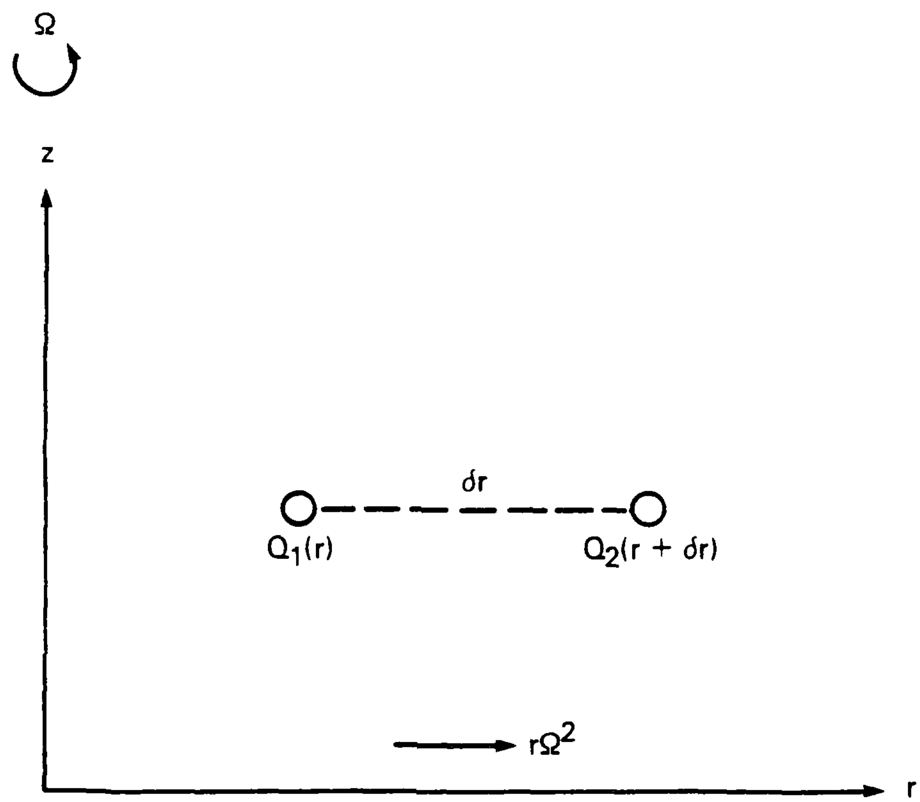


Figure 5 — Coordinate of the fluid elements in the centrifugal force field

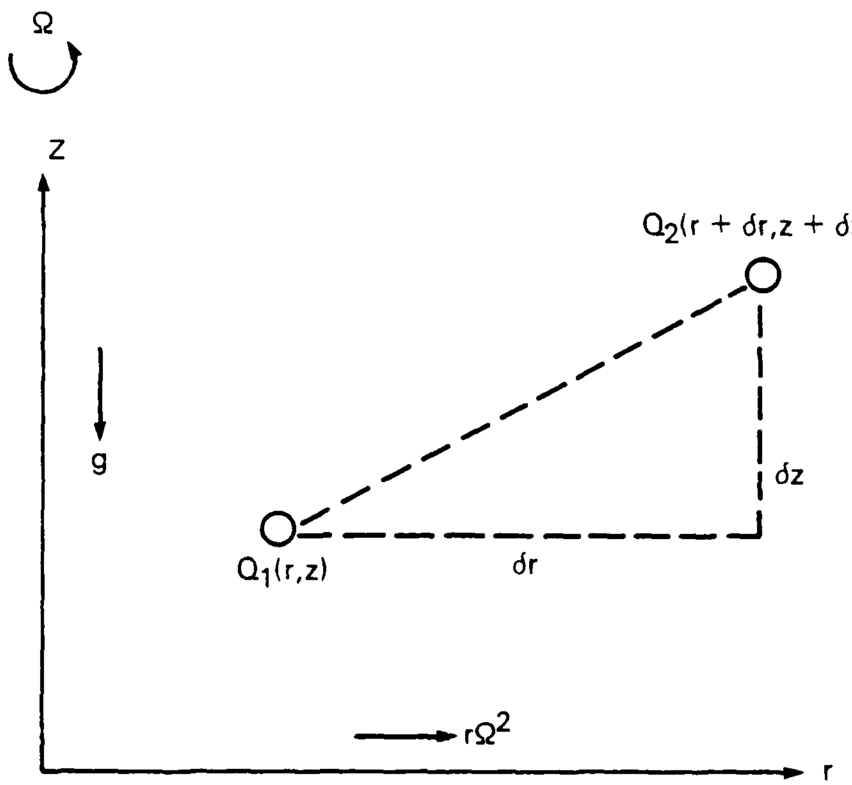
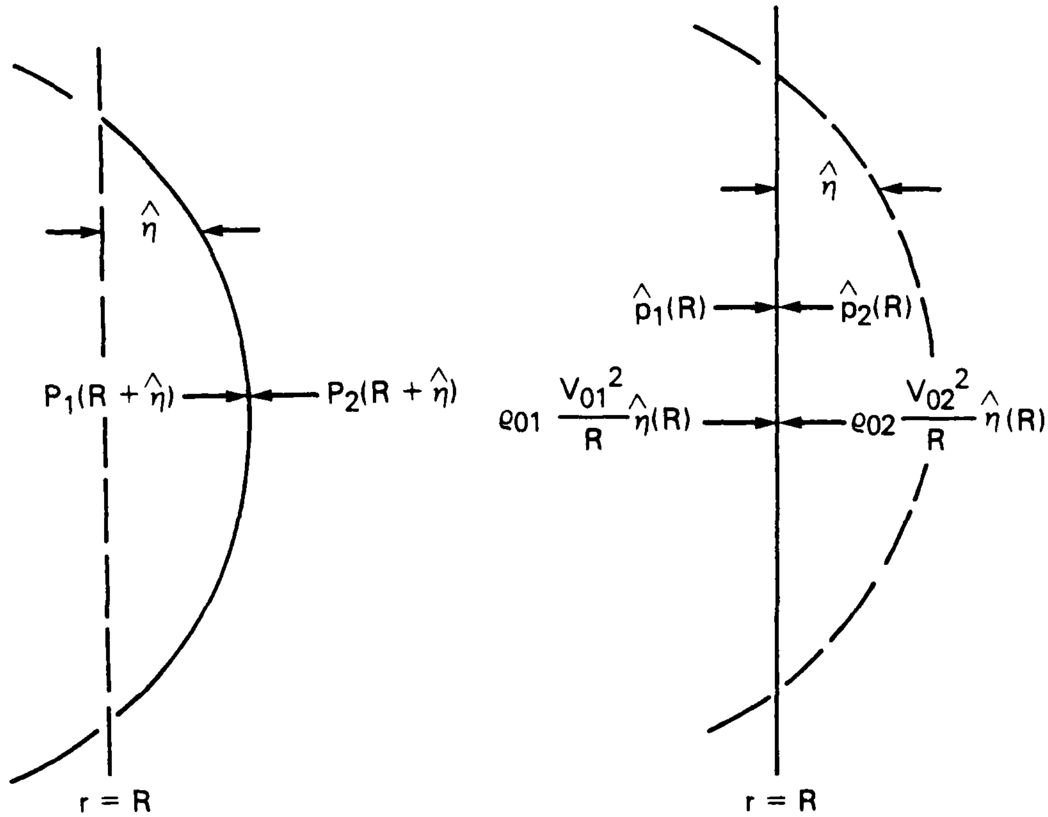


Figure 6 — Coordinate of the fluid elements in the centrifugal and gravitational force fields



(a) (b)

Figure 7 — Dissolution of the total force into the force components at the interface

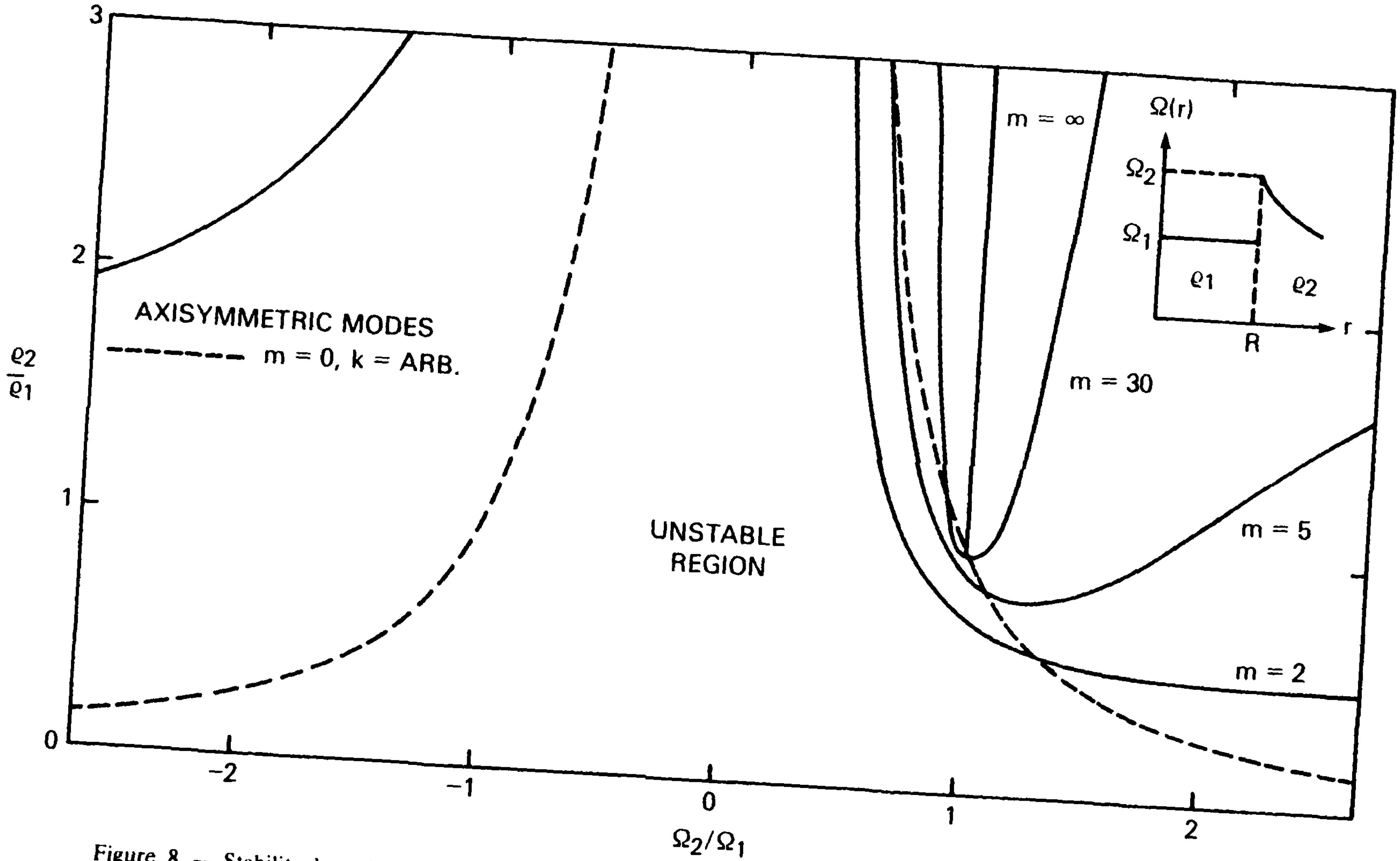


Figure 8 — Stability boundaries of axisymmetric and azimuthal disturbances for different azimuthal wave numbers

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