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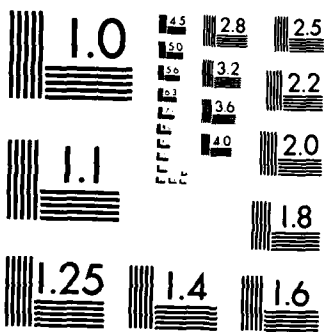
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David Siegmund  
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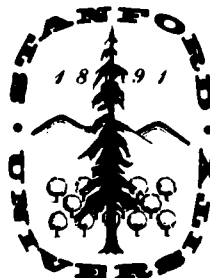
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## **Large Deviations for the Maxima of Some Random Fields**

by

M. L. Hogan \* and D. Siegmund †

Columbia University and Stanford University

Dedicated to Herbert Robbins on the occasion of his 70th birthday

Several statistical problems which involve the distribution of the maximum of Gaussian random fields are described. Specific examples are the pinned Brownian sheet and a Brownian bridge with "reflection," which arises in certain change point problems. In these concrete cases the method of Pickands (1969, *Trans. Amer. Math. Soc.*) is adapted to give large deviation probabilities for the maximum, both for continuous and for discrete indexing sets. A different method is used to give a second order correction for the reflected Brownian bridge and hence for reflected Brownian motion. The numerical accuracy of the approximations is studied via simulation.

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## 1. Introduction and Summary.

A random field is a stochastic process indexed by a more than one dimensional set, typically a subset of the  $n$  dimensional integer lattice or  $n$  dimensional Euclidean space. We shall call such a process an  $n$  dimensional random field. Since the random variables themselves assume real values, no confusion should result from this terminology. For the most part we consider only the case  $n = 2$ , but some of our methods are valid more generally.

We wish to discuss some examples of random fields arising in statistics. The statistical questions give rise to probability questions about the random fields, among which is determining the distribution of the maximum of the field over some subset of its indexing set. We deal with fields which are closely related to simple one dimensional processes such as random walk, Brownian motion, and Brownian bridge. There are many techniques for deriving exact results about the maximum of these one dimensional processes, mainly based on the strong Markov property and the relation of the time at which the process first exceeds a level (first passage time) and the maximum of the process. For higher dimensional random fields there is no natural linear ordering of the indexing set. Consequently there are no first passage times, and the techniques that give exact results for one dimension work only approximately or not at all in more than one.

On the other hand there are methods for approximating the tail of the distribution of the maximum, which are not intrinsically one dimensional. These have been developed mainly in the context of Gaussian random fields, especially by Pickands [13], Bickel and Rosenblatt [4], and Qualls and Watanabe [14]. The techniques can be broken into two parts. The first is to observe that the contribution to the probability of ever crossing a high level comes from a small neighborhood of the subset of the indexing set where the marginal probability of being above the level is greatest. Second, when this subset is not a single point, it can be broken into small pieces which contribute approximately disjointly to the total probability, which consequently can be obtained by adding together the contributions of each small piece. The method does not in general give explicit results, and in fact does so only rarely in the papers quoted above. For the problems in which we are interested, it does give explicit asymptotic approximations for the tail probability of the distribution of

the maximum.

Since our interest in these approximations arises because of their relations to certain statistical problems, these problems provide criteria for judging whether the approximations are adequate or not. Usually the probabilities can be interpreted as the significance levels of statistical tests, so it is important that they be accurate when the true probabilities are in the range .01-.10. We discuss later how accurately the asymptotic expressions approximate the actual distribution.

We begin by describing two statistical problems that lead to random fields: the empirical distribution in more than one dimension and certain change point problems. In each case we consider in some detail two random fields, one of which is common to both problems.

Suppose that  $X_1, X_2, \dots$  are independently and identically distributed with a continuous distribution function  $F$ . Let

$$F_n(x) = n^{-1} \sum_{i=1}^n 1_{(-\infty, x]}(X_i)$$

be the empirical distribution function and

$$\tilde{D}_n(x) = n^{1/2} [F_n(x) - F(x)]$$

the empirical process. The change of variables  $z = F(x)$  converts this to

$$D_n(z) = n^{1/2} [U_n(z) - z],$$

where

$$U_n(z) = n^{-1} \sum_{i=1}^n 1_{(0, z]}(F(X_i)).$$

The random variables  $F(X_i)$  are independently and uniformly distributed on  $(0, 1)$ , so the distribution of  $D_n(z)$  does not depend on  $F$ . It is well known that the limiting distribution of  $D_n(z)$  is that of a Brownian bridge on  $[0, 1]$  (cf. [5], Section 13.6). The Kolmogorov-Smirnov statistic is

$$n^{1/2} \sup_x [F_n(x) - F(x)] = \sup_{0 < z < 1} D_n(z).$$

Thus the distribution of the Kolmogorov-Smirnov statistic does not depend on the underlying distribution function  $F$ , and its limiting distribution is that of the maximum of a Brownian bridge.

Now suppose that we have independent, identically distributed bivariate observations  $(X_i, Y_i)$  with distribution function  $G(x, y)$ . Assume that  $X_i$  and  $Y_i$  have continuous marginal distribution functions  $F_1$  and  $F_2$  respectively. The empirical distribution function of the first  $n$  observations is

$$G_n(x, y) = n^{-1} \sum_{i=1}^n 1_{(-\infty, x] \times (-\infty, y]}(X_i, Y_i).$$

Set

$$\tilde{D}_n(x, y) = n^{1/2}[G_n(x, y) - G(x, y)] \quad -\infty < x, y < \infty.$$

With the change of variables  $z = F_1(x)$ ,  $w = F_2(y)$ ,  $\tilde{D}_n$  transforms to a process on the unit square

$$D_n(z, w) = n^{1/2}[U_n(z, w) - G(F_1^{-1}(z), F_2^{-1}(w))],$$

where

$$U_n(z, w) = n^{-1} \sum_{i=1}^n 1_{(0, z] \times (0, w]}[F_1(X_i), F_2(Y_i)].$$

The Kolmogorov-Smirnov statistic is

$$n^{1/2} \sup_{x, y} [G_n(x, y) - G(x, y)] = \sup_{z, w} D_n(z, w).$$

The random field  $D_n(z, w)$  converges to a limiting Gaussian random field  $W_0(z, w)$  on the unit square, but unlike the one dimensional case the covariance function of the random field depends on the underlying distribution  $G$ . The special case that  $X_i$  and  $Y_i$  are independent is particularly important. Then the covariance function is  $K[(z_1, w_1), (z_2, w_2)] = (z_1 \wedge z_2)(w_1 \wedge w_2)(1 - z_1 \vee z_2)(1 - w_1 \vee w_2)$ . This random field, the pinned Brownian sheet, stands in the same relation to another random field, the Brownian sheet, as the Brownian bridge does to Brownian motion in one dimension. The question suggested by the Kolmogorov-Smirnov statistics is to compute the distribution of the maximum of the pinned Brownian sheet.

In Section 2 we establish



**Theorem 1.** As  $u \rightarrow \infty$

$$P\{\sup_{z,w} W_0(z,w) > u\} \sim 4 \log 2 u^2 \exp(-2u^2). \quad (1.1)$$

Goodman [8] showed that the probability on the left hand side of (1.1) exceeds  $(2u^2 + 1) \exp(-2u^2)$ . An upper bound is given in [6].

Since the distribution of the two dimensional Kolmogorov-Smirnov statistic depends on the underlying distribution  $G$ , Adler and Brown [1] raise the question of finding

$$\sup_G P\{\sup_{z,w} D_n(z,w) > u\},$$

and show that in the asymptotic limit ( $n \rightarrow \infty$ ) the supremum is attained by a two dimensional distribution  $G$  which is uniform on the off-diagonal,  $z+w=1$ . Moreover, the limiting value of this supremum has the following representation in terms of the one dimensional Brownian bridge,  $W_0(t)$ ,  $0 \leq t \leq 1$ :

$$P\{\sup_{0 \leq s < t \leq 1} [W_0(t) - W_0(s)] > u\}. \quad (1.2)$$

A slightly more general quantity than (1.2) arises in a class of change point problems, which we discuss next.

As a second example we consider a class of change point problems, more or less as formulated by Levin and Kline [12]. Let  $X_i$ ,  $i = 1, 2, \dots, m$  be independent, normally distributed random variables with means  $\mu_i$  and variance 1. Consider the problem of testing

$$H_0 : \mu_1 = \mu_2 = \dots = \mu_m \quad (= \mu_0)$$

against

$$H_1 : \exists 1 \leq \rho_1 < \rho_2 < m, \quad \mu_1 = \dots = \mu_{\rho_1} = \mu_0,$$

$$\mu_{\rho_1+1} = \dots = \mu_{\rho_2} = \mu_0 + \delta, \quad \mu_{\rho_2+1} = \dots = \mu_m = \mu_0.$$

The alternative hypothesis  $H_1$  has been called a square-wave or epidemic alternative because an epidemic runs from time  $\rho_1$  through  $\rho_2$  after which the baseline rate  $\mu_0$  is restored. It may be compared to the more common hypothesis of a single change point where, in effect,  $\rho_2 = m$ .

If it is assumed that  $\mu_0$  and  $\delta$  are known, the log likelihood ratio statistic for testing  $H_0$  against  $H_1$  is given by

$$\begin{aligned} Z_1 &= \delta \max_{0 \leq i < j \leq m} [S_j - j\mu_0 - (S_i - i\mu_0) - (j - i)\delta/2] \\ &= \max_{0 \leq i < j \leq m} [\tilde{S}_j - \tilde{S}_i] \\ &= \max_{j \leq m} [\tilde{S}_j - \inf_{i \leq j} \tilde{S}_i], \end{aligned} \quad (1.3)$$

where  $\tilde{S}_i = \delta[S_i - i(\mu_0 + \delta/2)]$ .

When  $\mu_0$  is unknown one possible course, suggested by Levin and Kline [12], is to replace  $\mu_0$  by its estimate under  $H_0$ ,  $S_m/m$ , which leads to the test statistic

$$Z_2 = \delta \max_{0 \leq i < j \leq m} [S_j - j S_m/m - (S_i - S_m/m) - (j - i)\delta/2]. \quad (1.4)$$

Levin and Kline are interested in Bernoulli and Poisson random variables rather than normal. Since  $\mu_0$  is a nuisance parameter, they suggest that the distribution of  $Z_2$  should be calculated conditional on  $S_m$ . The conditional and unconditional distributions of  $Z_2$  are the same in the normal case, but in general this adds another feature to the problem.

Alternatively, the actual likelihood ratio statistic may be computed by maximizing the log likelihood over  $\mu_0$ ,  $\rho_1$ , and  $\rho_2$ . This gives

$$Z_3 = \delta \max_{0 \leq i < j \leq m} [S_j - S_i - (j - i)S_m/m - \frac{1}{2}\delta(j - i)(1 - (j - i)/m)]. \quad (1.5)$$

When  $\delta$  is also not known one might use either  $Z_2$  or  $Z_3$  based on some value  $\delta_0$ , the smallest difference in means which is considered important to detect, or proceed to the full log likelihood ratio statistic by maximizing (1.5) over  $\delta$ , obtaining

$$Z_4 = \max_{0 \leq i < j \leq m} \{[S_j - S_i - (j - i)S_m/m]^+ / [(j - i)(1 - m^{-1}(j - i))]^{1/2}\},$$

where  $x^+ = \max(x, 0)$ .

Each of these statistics is the maximum of a Gaussian random field defined on  $\{(i, j) : 0 \leq i, j \leq m\}$ . It is interesting to compare the third expression for  $Z_1$  given in (1.3) to what would be obtained under the simpler alternative hypothesis of exactly one change point. This is tantamount to setting  $\rho_2 = m$  in  $H_1$ , which leads in the case of known  $\mu_0$  and  $\delta$  to

the log likelihood ratio statistic

$$\begin{aligned} Z_6 &= \max_{i \leq m} [\tilde{S}_m - \tilde{S}_i] \\ &= \tilde{S}_m - \min_{i \leq m} \tilde{S}_i. \end{aligned}$$

This random variable can be shown by means of time reversal to have the same distribution as  $\max(0, \tilde{S}_1, \dots, \tilde{S}_m)$  ([7], p. 198), and consequently the distribution of  $Z_6$  is determined by solving a first passage problem for the random walk  $\tilde{S}_i$ . However, the time reversal technique applied to  $Z_1$  leaves it basically unchanged. In addition, first passage problems for ordinary random walk are analytically tractable because the value of the random walk at the time of first passage is rather well determined. (For Brownian motion it would be known exactly.) In principle the problem of computing the distribution of  $Z_1$  is equivalent to a first passage problem for the "reflecting" barrier process

$$W_j = \tilde{S}_j - \min_{i \leq j} \tilde{S}_i,$$

in the sense that  $P\{Z_1 > u\} = P\{T \leq m\}$ , where  $T = \min\{j : W_j > u\}$ . The fact that the value of  $\tilde{S}_T$  is not known, even approximately, makes this problem substantially more difficult than the corresponding passage problem for  $\tilde{S}_i$ . (The distribution of  $\tilde{S}_T$  has been studied in [18], at least in continuous time, but this does not seem to help us here.) Nevertheless, the observation that (1.3) and (1.4) can be formulated as one dimensional problems is very useful. See Theorem 3 below and its proof in Section 3. There does not seem to be any corresponding transformation of (1.5) to a one dimensional problem.

In order to state the following results it is convenient to let  $W_\xi(t)$  denote one dimensional Brownian motion conditioned to equal  $\xi$  at time  $t = m$ . Then  $Z_c = \max_{0 < t \leq m} [W_\xi(t) - W_\xi(s)]$  is essentially a continuous version of  $Z_2$  defined in (1.4). In Section 3 we prove

**Theorem 2.** Suppose  $u = m\zeta$  and  $\xi = m\xi_0$  for some  $\zeta > 0$  and  $\xi_0 < \zeta$ . Then as  $m \rightarrow \infty$

$$P\left\{\max_{i < j \leq m} [W_\xi(j) - W_\xi(i)] > u\right\} = \nu^2[2(2\zeta - \xi_0)][2(2u - \xi)(u - \xi)/m + o(m)] \exp[-2u(u - \xi)/m], \quad (1.6)$$

where  $\nu(\cdot)$  is defined in (3.2), and  $i, j$  are restricted to be integers.

**Theorem 3.** With the same asymptotic normalization

$$P\{\max_{s < t \leq m} [W_\xi(t) - W_\xi(s)] > u\} = [2(2u - \xi)(u - \xi)/m + 1 + o(1)] \exp[-2u(u - \xi)/m]. \quad (1.7)$$

Theorem 2 was stated and Theorem 3 conjectured in [16]. Related statistics are discussed in [3].

Theorem 2 is a sampled version of Theorem 3. The two approximations differ in the leading term by the factor  $\nu^2[2(2\xi - \xi_0)]$ , which occurs because of the discrete indexing set in Theorem 2. For computational purposes it usually suffices to use the approximation

$$\nu(x) = \exp(-\rho x) + o(x^2) \quad (x \rightarrow 0), \quad (1.8)$$

where  $\rho$  is a numerical constant approximately equal to 0.583 ([17], X.2). Typically the value of  $\nu^2$  is in the range .2 to .5, so failure to account for the discreteness usually overestimates the true probability by a considerable amount. Theorem 3 contains a higher order term in an asymptotic expansion of the tail probability for that process.

If the max in Theorem 3 were taken over all  $s \neq t$ , instead of  $s < t$ , it would be easy to calculate the probability exactly. For example,

$$P\{\max_{s \neq t < m} [W_\xi(t) - W_\xi(s)] > u\} \quad (1.9)$$

$$\sum_{\substack{n=-\infty \\ n \neq 0}}^{\infty} \{m^{-1}[2n(2nu - \xi)(u - \xi) \exp[-2nu(nu - \xi)/m] - (2n + 1) \exp[-2nu(nu + \xi)/m]\}.$$

For the special case  $\xi = 0$  this becomes

$$P\{\max_{s \neq t < m} [W_0(t) - W_0(s)] > u\} = \sum_{n=1}^{\infty} (8n^2u^2/m - 2) \exp(-2n^2u^2/m). \quad (1.10)$$

From considerations of symmetry, it appears that the probability in (1.7) in the case  $\xi = 0$  should be about 1/2 that in (1.10). This heuristic is asymptotically correct at the first order term, but not the second. It may be possible to evaluate the probability in (1.7) exactly, but we have no idea how to do it.

By integrating out  $\xi$  one obtains from Theorems 2 and 3 analogous results for unconditional processes. For example we have,

**Corollary to Theorem 3.** Let  $W(t)$ ,  $0 \leq t < \infty$ , be standard Brownian motion and  $\mu > 0$ . Suppose  $m \rightarrow \infty$  and  $u \rightarrow \infty$  such that  $m\mu u^{-1}$  is some fixed number in  $(1, \infty)$ . Then

$$P\left\{\max_{s \leq t \leq m} [W(t) - W(s) - \mu(t-s)] > u\right\} = [2\mu(m\mu - u) + 3 + o(1)] \exp(-2\mu u).$$

Proofs of Theorems 1, 2, and 3 are given in Sections 2, 3, and 4 respectively. A numerical example illustrating the accuracy of the approximation of Theorem 2 appears in Section 5, which also contains a heuristic attempt to adapt the expansion (1.7) for use with a discrete indexing set.

## 2. Proof of Theorem 1.

In this section we use Pickands' [13] method to prove Theorem 1. The method was also applied to random field problems in [4] and [14]. Our exposition follows closely that given in [11], Chapter 12, in the one dimensional case. Large parts of the proofs carry over almost word for word, but with two novel features. All of the authors above were interested in stationary fields or processes in which the random variables corresponding to each point in the indexing set contribute equally to the maximum. Our processes are nonstationary, and asymptotically the only contribution comes from a neighborhood of that subset of the index set where the marginal probability of being above a high level is a maximum. For the present case of the pinned Brownian sheet,  $W_0(s, t)$  is a zero mean Gaussian variable. Therefore,  $P\{W_0(s, t) > u\}$  is maximized at those values of  $s, t$  which maximize  $E[W_0^2(s, t)] = st(1-st)$ . This set is the section of the hyperbola  $st = 1/2$  lying in the unit square. Technically this means that the major contribution to certain sums comes from a neighborhood of the critical set, resulting in delta-function like approximations. Every argument in [11] must be modified to take this fact into account, but it is straightforward to do so. One example of the necessary changes is given in Lemma 3, but most, along with most detailed proofs, are omitted.

Secondly, the general expressions for the tail of the maximum of Gaussian random fields involve a constant given in terms of a complicated functional of the maximum of a related process, which can be shown to be positive and finite, but otherwise is not obviously

tractable. The related processes which occur in the problems we discuss are simple enough to allow explicit evaluation of all constants. The appearance of the function  $\nu$  in Theorem 2 because of the discrete indexing set is a particularly interesting example. See Lemma 3.4.

The proof of Theorem 1 is given as a series of lemmas, the proofs of which are mostly omitted because they follow closely those of [11], Chapter 12. See also the analogous results of Section 3 where occasionally more detail is given.

We shall use the notation

$$\varphi(x) = (2\pi)^{-1/2} \exp\left(-\frac{1}{2}x^2\right), \quad \Phi(x) = \int_{-\infty}^x \varphi(z) dz.$$

Also, for any random variable  $X$ ,  $P\{X \in dx\}$  denotes the measure corresponding to the distribution function of  $X$ . In particular if  $X$  is absolutely continuous with probability density function  $f$ , then  $P\{X \in dx\} = f(x)dx$ . We write  $X \sim N(\mu, \sigma^2)$  to mean  $P\{X \leq x\} = \Phi[(x - \mu)/\sigma]$ .

For the rest of this section  $u^2q = a$ . Let  $\xi_u^{s,t}(\sigma, \tau) = u[W_0(s - q\sigma, t - q\tau) - u]$ .

**Lemma 2.1.** Suppose  $\sigma, \tau \geq 0$ . Then

$$E(\xi_u^{s,t}(\sigma, \tau) \mid \xi_u^{s,t}(0, 0) = x) = x - s^{-1}a\sigma - t^{-1}a\tau + O(q)$$

and

$$\text{Cov}(\xi_u^{s,t}(\sigma_1, \tau_2), \xi_u^{s,t}(\sigma_2, \tau_2) \mid \xi_u^{s,t}(0, 0) = x) = a[s(\tau_1 \wedge \tau_2) + t(\sigma_1 \wedge \sigma_2)] + O(q),$$

where in both cases  $O(q)$  holds uniformly for  $(\sigma_1, \tau_1, \sigma_2, \tau_2)$  in compact sets and  $s, t$  with  $st$  bounded away from 0.

**Proof.** A straightforward calculation suffices.

**Remarks.** Since the conditional distribution is normal it is determined by its mean and covariance. Furthermore, as  $u \rightarrow \infty$  (hence  $q \rightarrow 0$ ),  $\xi_u^{s,t}(\sigma, \tau)$  converges in distribution to a process having a very simple representation. Let  $X_1(\sigma)$  and  $X_2(\tau)$  be independent standard Wiener processes. It is easily verified by checking covariances that the limiting process can be represented as

$$(at)^{1/2}X_1(\sigma) - a\sigma/s + (as)^{1/2}X_2(\tau) - a\tau/t. \quad (2.1)$$

14. Qualls, C. and Watanabe, H. Asymptotic properties of Gaussian random fields, *Trans. Amer. Math. Soc.* **177** (1973), 155-171.
15. Siegmund, D. Corrected diffusion approximations in certain random walk problems, *Adv. Appl. Probab.* **11** (1979), 701-719.
16. Siegmund, D. Boundary crossing probabilities and statistical applications, submitted to *Ann. Statist.* (1985).
17. Siegmund, D. *Sequential Analysis: Tests and Confidence Intervals*, Springer-Verlag, New York-Heidelberg-Berlin (1985).
18. Taylor, H. A stopped Brownian motion formula, *Ann. Probab.* **3** (1975), 234-246.
19. Woodroffe, M. A renewal theorem for curved boundaries and moments of first passage times, *Ann. Probab.*, **4** (1976), 67-80.
20. Woodroffe, M. *Nonlinear Renewal Theory in Sequential Analysis*, Society for Industrial and Applied Mathematics, Philadelphia (1982).

## References

1. Adler, R. and Brown, L. Tail behavior for suprema of empirical processes, unpublished manuscript (1984).
2. Asmussen, S. Approximations for the probability of ruin in finite time, *Scand. Actuarial J.* (1984), 31-57.
3. Bhattacharya, P. K. and Brockwell, P. J. The minimum of an additive process with applications to signal estimation and storage theory, *Z. Wahrschein. verw. Geb.* **37** (1976), 51-75.
4. Bickel, P. and Rosenblatt, M. Two-dimensional random fields, in *Multivariate Analysis - III*, P. K. Krishnaiah, ed., Academic Press, New York (1973), 3-15.
5. Breiman, L. *Probability*, Addison Wesley, Reading, Massachusetts (1968).
6. Cabaña, E. and Wschebor, M. The two parameter Brownian bridge, *Ann. Probab.* **10** (1982), 289-302.
7. Feller, W. *An Introduction to Probability Theory and Its Applications*, Vol. II, John Wiley and Sons, New York (1972).
8. Goodman, V. Distribution estimates for functionals of the two parameter Wiener process, *Ann. Probab.* **4** (1976), 977-982.
9. Hogan, M. "Problems in boundary crossings for random walks," Stanford University dissertation (1984).
10. Joag-Dev, K., Perlman, M., and Pitt, L. Association of normal random variables and Slepian's inequality, *Ann. Probab.* **11** (1983), 451-455.
11. Leadbetter, M., Lindgren, G., and Rootzen, H. *Extremes and Related Properties of Random Sequences and Processes*, Springer-Verlag, New York-Heidelberg-Berlin (1983).
12. Levin, B. and Kline, J. The cusum test of homogeneity, with an application to spontaneous abortion epidemiology, Columbia University, preprint (1985).
13. Pickands, J. Upcrossing probabilities for stationary Gaussian processes, *Trans. Amer. Math. Soc.* **145** (1969), 51-73.



Table I

Approximations to $P\{\max_{0 \leq i < j < m}(S_j - S_i) > u \mid S_m = \xi\}$					
$u$	$m$	$m^{-1}\xi$	Monte Carlo	Theoretical	
			( $N = 1600$ )	(1.6)	Modified (1.7)
4.0	20	-.4	.020 $\pm$ .004	.024	.028
5.0	20	-.4	.0056 $\pm$ .0019	.0043	.0048
2.0	20	-.5	.344 $\pm$ .012	.298	.348
3.0	30	-.7	.044 $\pm$ .005	.044	.046
3.0	30	-.5	.156 $\pm$ .009	.135	.150
6.0	30	.0	.257 $\pm$ .011	.171	.256
8.0	30	.1	.174 $\pm$ .009	.110	.169
5.0	50	-.4	.053 $\pm$ .006	.049	.055
6.0	75	-.3	.117 $\pm$ .008	.094	.104
5.0	100	-.5	.063 $\pm$ .006	.067	.070
9.0	100	-.15	.117 $\pm$ .008	.098	.113

2. This approximation appears to be moderately good, but consistently too small. It is poor for  $\xi \geq 0$ .

For random walk first passage problems there exists an approximation which uses a completely different normalization than the large deviation normalization considered here, although the resulting approximations are often very similar [2, 9, 15, 17]. These approximations often have the interpretation that they equal the analogous Brownian motion probability corrected for discrete time and (if necessary) for nonnormality of the random walk. Moreover, the correction for discrete time is simply to displace the first passage level  $u$  by the average amount that the discrete time process jumps over the boundary. In the Gaussian case this is just the constant  $\rho$  which appears in (1.8). See [17] for a more detailed discussion of this approximation and a comparison with large deviation approximations.

For random walk problems with reflecting barriers, it is clear that the analogous modification is to displace the first passage boundary by  $2\rho$  (cf. [17], Theorem 10.16). If one were to make this modification to the Brownian motion approximation of Theorem 3, to the extent that (1.8) is an equality, the resulting approximation would be the same as (1.6) to first order. Now, however, there are higher order terms, which might conceivably improve the approximation.

The third entry in each row of Table I gives the approximation of Theorem 3, but with  $u$  replaced by  $u + 1.166$ . This second approximation seems to be slightly better in cases where both approximations are good and substantially better when  $\xi \geq 0$ , where the approximation from Theorem 2 is not particularly good.

It appears to be a very complicated task to find a genuine second order approximation or to justify the one suggested here. The problem becomes even more difficult for other Gaussian fields, e.g. those considered in Theorem 1 and in (1.6), which do not appear to have a convenient one dimensional representation.

$m$ ). To obtain an appropriate bound for the first term let  $\sigma^{(j)} = \sigma^{(j)}(u)$  be defined by

$$W(\sigma^{(j)}) = \inf_{T^{(j-1)} \leq s \leq T^{(j)}} W(s)$$

and observe that

$$\begin{aligned} \{T^{(2)}(u) < m\} \subset \bigcup_{n=1}^m (\{T^{(1)}(u) < n \leq \sigma^{(2)}(u), T^{(2)}(u) < m\} \\ \cup \{ \max_{n-1 \leq s \leq t \leq n} [W(t) - W(s)] \geq \frac{1}{2}u \}) \cup \{T^{(1)}(\frac{3}{2}u) < m\}, \end{aligned}$$

and these events are easily analyzed individually. (Corollary 3 of [10] can be used to show that

$$P_{0,\xi}^{(m)} \{T^{(1)} < n < \sigma^{(2)}, T^{(2)} < m\} \leq (P_{0,\xi}^{(m)} \{T < m\})^2,$$

or a somewhat weaker result may be obtained by "bare hands.") For the rest of the series a crude but more than adequate bound is a consequence of the following inequalities: for any  $k$ ,

$$\sum_{i=2}^{\infty} P_{0,\xi}^{(m)} \{T^{(i)} < \infty\} \leq mk P_{0,\xi}^{(m)} \{T^{(2)} < \infty\} + mk \sum_{i=1}^{\infty} P_{0,\xi}^{(m)} \{T^{(mki)} < m\}$$

and

$$P_{0,\xi}^{(m)} \{T^{(mki)} < m\} \leq \sum_{j=1}^{mk} P_{0,\xi}^{(m)} \{W(t) - W(s) \geq u \text{ for some } (j-1)/ki \leq s < t \leq (j+1)ki\}.$$

## 5. Numerical Examples.

In this section we report the result of a Monte Carlo experiment to indicate the accuracy of the approximations obtained in the preceding sections. As mentioned in the introduction, the statistical origins of these problems, where they arise as significance levels of statistical tests, suggests that we should be particularly interested in cases where the probability is about 0.01 - 0.10. (For the same reason we are more interested in non positive values of the drift parameter,  $\xi$ .)

For selected values of  $u$ ,  $m$ , and  $\xi$  Table 1 gives approximations to the probability in (1.6). The first entry is a Monte Carlo estimate based on a direct frequency count from a 1600 trial experiment. The second entry is the asymptotic approximation given by Theorem

$$\begin{aligned} & \lim_{\epsilon \rightarrow 0} P_{s+\eta_2\epsilon^{1/2}, \xi}^{(m-t-\epsilon)} \{ \tilde{T} < m-t-\epsilon, W(\tilde{T}) = x+u \} / \eta_2 \epsilon^{1/2} \quad (4.3) \\ & = 2(m-t)^{-1} \begin{cases} \sum_{i=1}^{\infty} (2iu+x-\xi) \exp[-2iu(iu+x-\xi)/(m-t)], & x > \xi-u \\ \sum_{i=0}^{\infty} (2iu+\xi-x) \exp[-2iu(iu+\xi-x)/(m-t)], & x < \xi-u. \end{cases} \end{aligned}$$

After a tedious argument to check that these formal substitutions, especially that coming from (4.3), can be justified by the dominated convergence theorem, we obtain the upper bound

$$\begin{aligned} & \lim_{\epsilon \rightarrow 0} \epsilon^{-1} P_{0, \xi}^{(m)} \{ M_{t+\epsilon} \in dx, l_{t+\epsilon} \in (t, t+\epsilon), \tilde{T}_{t+\epsilon} < m, W(\tilde{T}) = u+x \} / \{ 2|x|dx/mt^{3/2}(1-t/m)^{3/2} \} \\ & \leq \varphi \left[ \frac{(x-\xi t/m)}{\{t(1-t/m)\}^{1/2}} \right] \begin{cases} \sum_{i=1}^{\infty} (2iu+x-\xi) \exp[-2iu(iu+x-\xi)/(m-t)], & x > \xi-u \\ \sum_{i=0}^{\infty} (2i+\xi-x) \exp[-2iu(iu+\xi-x)/(m-t)], & x < \xi-u. \end{cases} \quad (4.4) \end{aligned}$$

An exact upper bound for the probability of interest can in principle be obtained by using (4.4) to bound (4.1). A lengthy asymptotic evaluation of the resulting multiple integral yields the right hand side of (1.7).

The lower bound is substantially more complicated, and we briefly outline the argument. Let  $T^{(1)} = T$  and for  $k > 1$  on  $\{T^{(k-1)} < m\}$  let

$$T^{(k)} = \inf \{ t : t > T^{(k-1)}, W(t) - \inf_{T^{(k-1)} \leq s \leq t} W(s) \geq u \}.$$

Then by Fatou's lemma, a lower bound for the right hand side of (4.1) is the upper bound discussed above minus the limit inferior as  $\epsilon \rightarrow 0$  of

$$\begin{aligned} & \sum_k \epsilon^{-1} \int \int P_{0, \xi}^{(m)} \{ T^{(k)} \leq t < T^{(k+1)}, M_{t+\epsilon} \in dx, l_{t+\epsilon} \in (t, t+\epsilon), \tilde{T}_{t+\epsilon} < m, \\ & W(\tilde{T}) = u+x \} dt \leq \sum_k \epsilon^{-1} \int P_{0, \xi}^{(m)} \{ t < T^{(k+1)} \leq t+\epsilon \} dt + \sum_k \epsilon^{-1} \int \int P_{0, \xi}^{(m)} \{ T^{(k)} \leq t, \\ & T^{(k+1)} > t+\epsilon, M_{t+\epsilon} \in dx, l_{t+\epsilon} \in (t, t+\epsilon), \tilde{T}_{t+\epsilon} < m, W(\tilde{T}) = u+x \} dt \\ & \leq 2 \sum_{k=1}^{\infty} P_{0, \xi}^{(m)} \{ T^{(k+1)} < m \}. \end{aligned} \quad (4.5)$$

Intuitively it seems clear that each term in the series (4.5) is exponentially smaller than its predecessor and hence the entire series is exponentially smaller as  $m \rightarrow \infty$  than  $P_{0, \xi}^{(m)} \{ T <$

and

$$P_{\eta_1, \eta_2}^{(\epsilon)} \left\{ \min_{0 \leq s \leq \epsilon} W(s) \in dx \right\} = 2 \exp(-2\eta_1 \eta_2) (\eta_1 + \eta_2) dx / \epsilon^{1/2}.$$

Formally substituting these expressions into the right hand side of (4.2) yields as an upper bound for the left hand side

$$\begin{aligned} & - 4x dx \epsilon \varphi \left[ \left( x - \xi t/m \right) / \left\{ t(1 - t/m) \right\}^{1/2} \right] / t^{3/2} (1 - t/m)^{1/2} \\ & \int_0^\infty \int_0^\infty \eta_1 \eta_2 (\eta_1 + \eta_2) \varphi(\eta_1 + \eta_2) \left[ P_{x + \eta_2 \epsilon^{1/2}, \xi}^{(m-t-\epsilon)} \left\{ \bar{T}_0 < m - t - \epsilon, W(\bar{T}) = x + u \right\} \right. \\ & \quad \left. / \eta_2 \epsilon^{1/2} \right] d\eta_2 d\eta_1 \\ & + o(\epsilon). \end{aligned}$$

A straightforward calculation beginning with (say) Theorem 3.42 of [17] shows that

$W(t) - M_t \geq u$ . In this notation we are interested in

$$P_{0,\xi}^{(m)}\{T < m\}.$$

Also let  $\tau_x = \inf\{t : W(t) = x\}$  and define  $\sigma$  by  $W(\sigma) = M_{T \wedge m}$ . Finally let

$$\tilde{T}_t = \tilde{T}_t(x) = \inf\{s : s > t, W(s) \notin (x, x+u)\}$$

and  $l_t = \sup\{s : s \leq t, W(s) = M(t)\}$ .

**Lemma 4.1.**

$$P_{0,\xi}^{(m)}\{T < m\} = \int_0^m \int_{-\infty}^0 \lim_{\epsilon \rightarrow 0} \epsilon^{-1} P_{0,\xi}^{(m)}\{T > t, M_{t+\epsilon} \in dx, l_{t+\epsilon} \in (t, t+\epsilon), \tilde{T}_{t+\epsilon} < m, W(\tilde{T}) = u+x\} dt. \quad (4.1)$$

**Proof.** We start from the relation

$$P_{0,\xi}^{(m)}\{T < m\} = \int_0^m \int_{-\infty}^0 \lim_{\epsilon \rightarrow 0} \epsilon^{-1} P_{0,\xi}^{(m)}\{T < m, \sigma \in (t, t+\epsilon), W(\sigma) \in dx\} dt.$$

It may be shown that the absolute difference between this integral and the right hand side of (4.1) is majorized by a sum of terms, each of which as a consequence of Fatou's lemma is less than

$$\int_0^m \lim_{\epsilon \rightarrow 0} \epsilon^{-1} P_{0,\xi}^{(m)}\left\{ \sup_{t < s_1 < s_2 < t+\epsilon} |W(s_2) - W(s_1)| > u \right\} dt,$$

which is easily seen to equal 0.

To obtain an upper bound for (4.1), we omit the condition  $T > t$  and rewrite the right hand side by conditioning on  $W(t)$  and  $W(t+\epsilon)$  to obtain

$$\begin{aligned} & P_{0,\xi}^{(m)}\{M_{t+\epsilon} \in dx, l_{t+\epsilon} \in (t, t+\epsilon), \tilde{T}_{t+\epsilon} < m, W(\tilde{T}) = u+x\} \\ &= \int_s^0 \int_s^0 P_{0,\xi}^{(m)}\{\tau_s > t, W(t) \in dy_1\} P_{y_1,\xi}^{(m-\epsilon)}\{W(\epsilon) \in dy_2\} \\ & P_{y_1,y_2}^{(\epsilon)}\{M_\epsilon \in dx\} P_{y_2,\xi}^{(m-t-\epsilon)}\{\tilde{T}_0 < m-t, W(\tilde{T}_0) = x+u\}. \end{aligned} \quad (4.2)$$

In (4.2) each of the first three factors can be evaluated explicitly with the aid of (3.13) of [17]. We do so and then make the change of variables  $y_i = x + \eta_i \epsilon^{1/2}$  to obtain as  $\epsilon \rightarrow 0$

$$\begin{aligned} & P_{0,\xi}^{(m)}\{\tau_s > t, W(t) \in dy_1\} \\ & \sim -2z\eta_1 \epsilon \varphi\left[\frac{(x - \xi t/m)}{t(1-t/m)}\right]^{1/2} d\eta_1 / t^{3/2} (1-t/m)^{1/2}, \end{aligned}$$

$$P_{y_1,\xi}^{(m-t-\epsilon)}\{W(\epsilon) \in dy_2\} \sim \varphi(\eta_2 - \eta_1) d\eta_2,$$

This quantity can be calculated numerically (e.g. [20], Section 2.4) or approximately as suggested in (1.8).

**Lemma 3.4.**

$$\lim_{n \rightarrow \infty} n^{-2} H(\Delta_0, n) = 2(2\zeta - \xi_0)^5 \nu^2 [2(2\zeta - \xi_0)] / (\zeta - \xi_0),$$

where  $\nu$  is defined in (3.2).

**Proof.** It suffices to evaluate the limit on the right hand side of (3.1). By the definition of  $T_x$  and Wald's likelihood ratio identity (cf. [20], Theorem 1.1)

$$\begin{aligned} P_{-\mu_0} \{ \max_{i \leq n} U_i > x \} &= P_{-\mu_0} \{ T_x \leq n \} = E_{\mu_0} \{ \exp(-2\mu_0 U_{T_x}); T_x \leq n \} \\ &= \exp(-2\mu_0 x) E_{\mu_0} \{ \exp(-2\mu_0 R_x); T_x \leq n \}. \end{aligned}$$

Hence it suffices to evaluate the limit as  $n \rightarrow \infty$  of

$$n^{-1} \int_0^\infty E_{\mu_0} \{ \exp(-2\mu_0 R_x); T_x \leq n \} dx. \quad (3.3)$$

We split this integral into three pieces,  $0 \leq x \leq (1 - \epsilon)n\mu_0$ ,  $(1 - \epsilon)n\mu_0 \leq x \leq (1 + \epsilon)n\mu_0$ , and  $(1 + \epsilon)n\mu_0 \leq x < \infty$ . Uniformly for  $x \leq (1 - \epsilon)n\mu_0$ ,  $P_{\mu_0} \{ T_x \leq n \} \geq P_{\mu_0} \{ S_n \geq (1 - \epsilon)n\mu_0 \} \rightarrow 1$ , so by (3.2) the limit inferior of the expression in (3.3) exceeds

$$\liminf_{n \rightarrow \infty} n^{-1} \int_0^{(1 - \epsilon)n\mu_0} E_{\mu_0} \{ \exp(-2\mu_0 R_x) \} dx \geq (1 - \epsilon)\mu_0 \nu(2\mu_0).$$

For an upper bound, we use the obvious inequalities  $E_{\mu_0} \{ \exp(-2\mu_0 R_x); T_x \leq n \} \leq E_{\mu_0} \{ \exp(-2\mu_0 R_x) \}$  or  $P_{\mu_0} \{ T_x \leq n \}$  according as  $x \leq (1 + \epsilon)n\mu_0$  or  $x > (1 + \epsilon)n\mu_0$ . The range  $x \leq (1 + \epsilon)n\mu_0$  is analyzed as above; and for  $x > (1 + \epsilon)n\mu_0$ ,  $P_{\mu_0} \{ T_x \leq n \}$  is bounded by the corresponding probability for a Brownian motion process. Some calculation shows that the lim sup of (3.3) is smaller than  $(1 + \epsilon)\mu_0 \nu(2\mu_0)$ . Since  $\epsilon > 0$  is arbitrary, this completes the proof.

**Proof of Theorem 2.** This follows immediately from Lemmas 3.3 and 3.4.

#### 4. Proof of Theorem 3.

Let  $P_{a,b}^{(t)}$  be the probability measure under which  $W(\cdot)$  is a Brownian bridge starting at  $a$  at time 0 and terminating at  $b$  at time  $t$ . Let  $M_t = \min_{0 \leq s \leq t} W(s)$  and  $T = \inf\{t :$

**Proof.** This is proved just like Lemma 2.3. We indicate the high points. Let

$$B_{k,l} = \left\{ \max_{\substack{kn < i \leq (k+1)n \\ ln < j \leq (l+1)n}} Z_{i,j} > u \right\}.$$

It can be shown that for  $m$  and  $n$  large

$$P\left\{ \max_{0 \leq i,j \leq m} Z_{i,j} > u \right\} \cong \sum_{k,l} P(B_{k,l}).$$

The main contribution to the sum on the right comes from a neighborhood of those  $k$  and  $l$  for which the marginal probability

$$P\{Z_{kn,ln} > u\}$$

is a maximum, i.e. from a neighborhood of  $ln - kn = \Delta_0 m$ , where  $\Delta_0 = \zeta / (2\zeta - \xi_0)$ . Substituting the estimate of  $P(B_{k,l})$  from Lemma 3.2 and analyzing the sum as in Lemma 2.3 gives the stated result.

Before stating Lemma 3.4 we introduce some notation. First note that when  $j_0 - i_0 = m\Delta_0$  the random variables  $U_1$  and  $V_1$  from Lemma 3.1 are both  $N(-(2\zeta - \xi_0), 1)$ , and the exponential appearing in the definition of  $H(\Delta_0, n)$  is

$$\exp[x(\zeta - \Delta_0\xi_0)/\Delta_0(1 - \Delta_0)] = \exp[2(2\zeta - \xi_0)x].$$

Let  $P_\mu$  denote the probability measure which gives the random walks  $U_i$  and  $V_i$  increments having a  $N(\mu, 1)$  distribution. We are particularly interested in the case  $\mu_0 = 2\zeta - \xi_0$ . Using this notation and the identity

$$\int_0^\infty e^{\alpha x} P\{X > x\} dx = \alpha^{-1} E[e^{\alpha X^+} - 1],$$

we obtain

$$\begin{aligned} \lim_{n \rightarrow \infty} n^{-2} H(\Delta_0, n) &= 2(2\zeta - \xi_0)^2 (\zeta - \xi_0)^{-1} \\ &\cdot \lim_{n \rightarrow \infty} \left( n^{-1} \int_0^\infty \exp(2\mu_0 x) P_{-\mu_0} \left\{ \max_{i \leq n} U_i > x \right\} dx \right)^2 \end{aligned} \quad (3.1)$$

Let  $T_z = \inf\{n : U_n > z\}$  and  $R_z = U_{T_z} - z$ . It follows from renewal theory that for  $\mu \geq 0$ , the  $P_\mu$  distributions of  $R_z$  converge weakly as  $z \rightarrow \infty$  (cf. [20], Theorem 2.3). Let

$$\nu(\mu) = \lim_{z \rightarrow \infty} E_{\mu/2}[\exp(-\mu R_z)]. \quad (3.2)$$



Let  $W_\xi(t)$  be a Brownian bridge starting from 0 at  $t = 0$  and equal to  $\xi$  at  $t = m$ . Let  $u = m\zeta$  and  $\xi = m\xi_0$  for some fixed  $\zeta > 0$  and  $\xi_0 < \zeta$ . Let  $Z_{i,j} = W_\xi(j) - W_\xi(i)$ . We are interested in

$$P\left\{\max_{0 \leq i, j \leq m} Z_{i,j} > u\right\}.$$

**Lemma 3.1.** Let  $j_0 - i_0 = m\Delta$  for some  $\Delta > 0$ . Then as  $m \rightarrow \infty$

$$\mathcal{L}(Z_{i_0-i, j_0-j} - u + x, i, j = 0, \dots, n \mid Z_{i_0, j_0} = u - x) \rightarrow \mathcal{L}(U_i + V_j, i, j = 0, 1, \dots, n),$$

where  $U_i$  and  $V_j$  are partial sums of independent, identically distributed random variables with  $U_1 \sim N(-\zeta/\Delta, 1)$  and  $V_1 \sim N(-(\zeta - \xi_0)/(1 - \Delta), 1)$ . The convergence is uniform for  $\Delta$  bounded away from 0 and 1.

**Proof.** Note that

$$Z_{i_0-i, j_0-j} = Z_{i_0, j_0} - Z_{i_0-i, i_0} + Z_{j_0-j, j_0},$$

and conditional on  $Z_{i_0, j_0}$ , the processes  $Z_{i_0-i, j_0}$ ,  $i = 1, \dots, n$  and  $Z_{j_0-j, j_0}$ ,  $j = 1, \dots, n$  are independent of each other, provided  $m$  is sufficiently large. The indicated limiting distributions follow from Lemma 5.1 of [20].

**Lemma 3.2.** Let  $j_0 - i_0 = m\Delta$ . As  $m \rightarrow \infty$

$$uP\left\{\max_{1 \leq i, j \leq n} Z_{i_0-i, j_0-j} > u\right\} / [m\Delta(1 - \Delta)]^{1/2} \varphi\left\{(u - \Delta\xi) / [m\Delta(1 - \Delta)]^{1/2}\right\} \rightarrow 1 + H(\Delta, n),$$

where

$$H(\Delta, n) = \zeta \int_0^\infty P\left\{\max_{i \leq n} U_i + \max_{j \leq n} V_j \geq x\right\} \exp[(\zeta - \Delta\xi_0)x / \Delta(1 - \Delta)] dx / \Delta(1 - \Delta),$$

and the convergence is uniform for  $\Delta$  bounded away from 0 and 1.

**Proof.** This is proved exactly like Lemma 2.2.

**Lemma 3.3.**

$$\lim_{m \rightarrow \infty} P\left\{\max_{0 \leq i, j \leq m} Z_{i,j} > u\right\} \exp[2m\zeta(\zeta - \xi_0)](2\zeta - \xi_0)^4 / m(\zeta - \xi_0)^2 = \lim_{n \rightarrow \infty} n^{-2} H(\Delta_0, n),$$

where  $\Delta_0 = \zeta / (2\zeta - \xi_0)$ .

uniformly in  $t$  bounded away from 0.

**Proof.** This follows from Corollary 3 of [4]. Alternatively it can be proved by the method of Lemma 3.4 below, which yields an explicit evaluation of  $H^*$  in terms of the function  $\nu$  defined in (3.2). The proof is completed by the observation that  $\nu(\mu) \rightarrow 0$  as  $\mu \rightarrow 0$ .

**Lemma 2.5.** Let  $\gamma = a^\beta$  for some  $0 < \beta < 1/2$ . Then as  $u \rightarrow \infty$

$$P\left\{\max_{0 \leq iq, jq \leq 1} W_0(iq, jq) < u - \gamma u^{-1}, \max_{0 \leq s, t \leq 1} W_0(s, t) > u\right\} = o[u^2 \exp(-2u^2)].$$

**Proof.** See Lemma 12.1.5 of [11].

**Lemma 2.6.** With the same notation as above

$$P\{u - \gamma/u \leq \max_{0 \leq iq, jq \leq 1} W_0(iq, jq) < u\} / u^2 \exp(-2u^2) \rightarrow (e^{2\gamma} - 1) \int_{1/2}^1 H^*(t, a) dt / ta^2.$$

**Proof.** This follows exactly as in Lemma 12.2.6 of [11].

**Proof of Theorem 1.** Let  $\epsilon > 0$ . Choose  $\beta < \frac{1}{2}$  and  $a > 0$  so that for all  $\frac{1}{2} \leq t \leq 1$

$$|a^{-2} H^*(t, a) - 32| < \epsilon \quad \text{and} \quad e^{2\gamma} - 1 < \epsilon.$$

Then

$$|a^{-2} \int_{1/2}^1 H^*(t, a) dt / t - 32 \log 2| < 2\epsilon,$$

and by Lemmas 2.5, 2.6,

$$\lim_{u \rightarrow \infty} |P\{\max_{0 \leq iq, jq \leq 1} W_0(iq, jq) > u\} - P\{\sup_{0 \leq s, t \leq 1} W_0(s, t) > u\}| / u^2 \exp(-2u^2) < 5\epsilon,$$

while by Lemma 2.4

$$P\{\max_{0 \leq iq, jq \leq 1} W_0(iq, jq) > u\} \sim (4 \log 2) u^2 \exp(-2u^2).$$

### 3. Proof of Theorem 2.

For the most part the proof given here for Theorem 2 is very similar to that of Theorem 1. The main difference relates to the discrete indexing set. See Lemma 3.4. Lemma 3.1 could be proved by calculation of means and covariances along the lines of Lemma 2.1, but we give a proof which can be generalized to (nonnormal) exponential families of distributions.

Fix  $t = (l+1)nq \geq 1/2$ . According to Lemma 2.2  $\sum_k P(B_{k,l})$  is asymptotically of the form

$$\sum_k P(B_{k,l}) \sim \sum_k \frac{\psi[(k+1)nq, t, u] \{1 + H[(k+1)nq, t, n, a]\}}{u/[(k+1)nqt(1 - (k+1)nqt)]^{1/2}}.$$

The function  $\psi(s, t, u)$  considered as a function of  $s$  has a unique maximum at  $s = 1/2t$ . Let  $k_0$  be such that  $|1/2t - (k_0+1)nq| = \inf_k |1/2t - (k+1)nq|$ . Set  $(k_0+1)nq = 1/2t + \epsilon nq$ , where  $|\epsilon| \leq 1$ , and set  $i = k - k_0 + 1$ . Then

$$\sum_k P(B_{k,l}) \sim \sum_{i=-k_0}^{(nq)^{-1}-k_0} \frac{\psi[inq + 1/2t + \epsilon nq, t, u] \{1 + H[inq + 1/2t + \epsilon nq, t, n, a]\}}{u/[t(inq + 1/2t + \epsilon nq)(1 - t(inq + 1/2t + \epsilon nq))]^{1/2}}.$$

To simplify this expression, note that the sum concentrates in a neighborhood of  $i = 0$ ; and since the function  $H$  as well as the denominator are continuous functions, they can be replaced by their values at  $i = 0$ . Since  $q \rightarrow 0$  as  $u \rightarrow \infty$ , this yields

$$\sum_k P(B_{k,l}) \sim \frac{1 + H(1/2t, t, n, a)}{2u} \sum_{i=-k_0}^{(nq)^{-1}-k_0} \psi(inq + 1/2t + \epsilon nq, t, u). \quad (2.2)$$

The sum in (2.2) is easily approximated by the integral

$$\varphi(2u) \int_{-\infty}^{\infty} \exp(-8z^2) dz / nqtu = \exp(-2u^2) / 4nqtu.$$

Substituting this into (2.2) yields

$$\sum_k P(B_{k,l}) \sim [1 + H(1/2t, t, n, a)] \exp(-2u^2) / 8nqtu^2.$$

The case  $(l+1)nq < 1/2$  can be done similarly and is seen to be of a smaller order of magnitude. Therefore

$$\begin{aligned} \sum_{k,l} P(B_{k,l}) &\sim (8nqu^2)^{-1} \exp(-2u^2) \sum_l \frac{\{1 + H[1/2(1/2 + lnq), 1/2 + lnq, n, a]\}}{1/2 + lnq} \\ &\sim (8n^2q^2u^2)^{-1} \exp(-2u^2) \int_{1/2}^1 \{1 + H(1/2t, t, n, a)\} dt/t \\ &= 8^{-1} u^2 \exp(-2u^2) \int_{1/2}^1 \{1 + H(1/2t, t, n, a)\} dt/t(na)^2. \end{aligned}$$

The proof is completed by letting  $n \rightarrow \infty$  and proceeding as in Lemma 12.2.4 of [11].

**Lemma 2.4.** As  $a \rightarrow 0$

$$a^{-2} H^2(t, a) \rightarrow 32$$

**Lemma 2.2.** For fixed  $n$  and  $a$ , as  $u \rightarrow \infty$

$$P\left\{\max_{0 \leq i, j \leq n} W_0(s-iq, t-jq) > u\right\} / \left\{u^{-1}(st(1-st))^{1/2} \varphi[u/(st(1-st))^{1/2}]\right\} \rightarrow 1 + H(s, t, n, a),$$

where the convergence is uniform for  $st$  bounded away from 0 and 1, and

$$H(s, t, n, a) = \int_0^\infty P\left\{\max_{i \leq n} U_i + \max_{j \leq n} V_j \geq x\right\} \exp[x/st(1-st)] dx / st(1-st),$$

where  $U_i$  and  $V_j$  are partial sums of independent, identically distributed random variables with

$$U_1 \sim N(-a/s, at), \quad V_1 \sim N(-a/t, as).$$

**Proof.** The argument is the same as Lemma 12.2.3 of [11] used in conjunction with the representation (2.1) of the limiting process.

**Lemma 2.3.** There exists a function  $H^*(t, a)$  such that  $\lim_{n \rightarrow \infty} n^{-2} H(1/2t, t, n, a) = H^*(t, a)$  uniformly in  $t$  bounded away from 0. As  $u \rightarrow \infty$

$$P\left\{\max_{0 \leq i, j \leq 1} W_0(iq, jq) > u\right\} / u^2 \exp(-2u^2) \rightarrow 8^{-1} a^{-2} \int_{1/2}^1 H^*(t, a) dt / t.$$

**Proof.** Recall that the major contribution to the indicated probability is expected to come from a neighborhood of  $st = 1/2$ , where  $E\{W_0(s, t)\}^2$  is a maximum.

Let

$$B_{k,l} = \left\{ \begin{array}{l} \max_{kn \leq i \leq (k+1)n} \\ \max_{ln \leq j \leq (l+1)n} \end{array} W_0(iq, jq) > u \right\}.$$

For technical convenience we assume that  $q$  is such that  $(nq)^{-1}$  is an integer. Then

$$\left\{ \max_{0 \leq i, j \leq q^{-1}} W_0(iq, jq) > u \right\} = \bigcup_{k,l} B_{k,l},$$

so the probability of interest is sandwiched between

$$\sum_{k,l} P(B_{k,l}) - \sum_{(k,l) \neq (k',l')} P\{B_{k,l} \cap B_{k',l'}\}$$

and

$$\sum_{k,l} P(B_{k,l}).$$

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Several statistical problems which involve the distribution of the maximum of Gaussian random fields are described. Specific examples are the pinned Brownian sheet and a Brownian bridge with "reflection," which arises in certain change point problems. In these concrete cases the method of Pickands (1969, Trans. Amer. Math. Soc.) is adapted to give large deviation probabilities for the maximum, both for continuous and for discrete indexing sets. A different method is used to give a second order correction for the reflected Brownian bridge and hence for reflected Brownian motion. The numerical accuracy of the approximations is studied via simulation.

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