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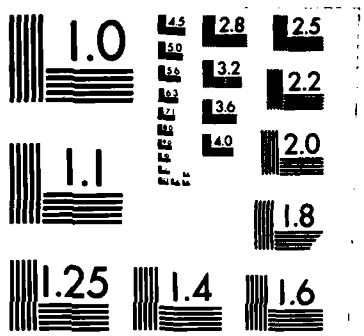

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SMOOTHING SURFACE DATA BY SPLINE FUNCTIONS

FINAL REPORT

PAUL C. STEIN AND WILLIAM L. WHITE

APRIL 4, 1985

U.S. ARMY RESEARCH OFFICE

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Research was performed to develop techniques based on spline functions which can be used to smooth two- and higher-dimensional data.

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STATEMENT OF PROBLEM:

Let $x: a = x_1 < x_2 < --- < x_N = b$

$y: c = y_1 < y_2 < --- < y_N = d$

and $R^2 = [a,b] \times [c,d]$.

Given (probably noisy) data

$f_{ij} = f(x_i, y_j) + \epsilon_{ij}, \quad 1 \leq i, j \leq N$

let $0 < \rho < 1$, and let $w_{ij} > 0$

minimize

$$\rho \sum_{i=1}^{N} \sum_{j=1}^{N} \left[g(x_i, y_j) - f_{ij} \right]^2 w_{ij} + (1-\rho) \int_a^b \int_c^d \sum_{k=0}^{m} \binom{m}{k} \left| \frac{\partial^m g(x,y)}{\partial x^k \partial y^{m-k}} \right|^2 dxdy$$

over appropriate smooth functions g.

SUMMARY OF RESULTS:

The investigation of the problem has included the work of [1,2 , 3, 6 ,7] .

As in the one dimensional case, Chui points out that it is sufficient to minimize the auxiliary expression

$$\int_a^b \int_c^d \sum_{k=0}^{m} \binom{m}{k} \frac{\partial^m g(x,y)}{\partial x^k \partial y^{m-k}} dxdy \qquad (1)$$

over appropriate smooth functions g.

As analyzed in [6,7] a proper abstract setting for a natural 2-dimensional setting of this optimal problem and results is provided by the Beppo Levi space $X = BL^m(R^2)$ of order m over R^2 (m an integer greater than or equal 1). Defined as

$$X = \{ g \in D' : \partial^\alpha g \in L_2 \text{ for } |\alpha| = m \}, \qquad (2)$$

where $\alpha = (\alpha_1, \alpha_2, --- , \alpha_n) \in N^n$ and $|\alpha| = \alpha_1 + \alpha_2 + --- + \alpha_n$,

X is thus simply the vector space of all the (Schwartz) distributions (i.e. continuous linear functionals on the vector space of infinitely differentiable functions with compact support in R^2, provided with the canonical Schwartz topology) for which all the partial derivatives (in the distributional sense) of total order m are square integrable in R^2. X is naturally equipped with the semi-inner product $(\cdot,\cdot)_m$ corresponding to the rotation invariant seminorm

$$|g|_m = \left\{ \int_a^b \int_c^d \sum_{k=0}^{m} \binom{m}{k} \left| \frac{\partial^m g(x,y)}{\partial x^k \, \partial y^{m-k}} \right|^2 dxdy \right\}^{\frac{1}{2}}, \tag{3}$$

the Kernel of $|\cdot|_m$ is known to be simply the vector space $P = P_{m-1}$ of dimension

$$M = \binom{m+1}{2} \tag{4}$$

of all polynomials over R^2 of (total) degree $m-1$. It should be noted that $|g|_m^2$ may be physically interpreted (at least if $m = 2$ and under some simplifying assumptions) as the bending energy of a thin plate of infinite extent, g denoting the deflection normal to the rest position. As they define the equilibrium position of infinite extent that deforms in bending only (under deflections specified at a number of independent points), the solution of the optimal interpolation problem can be appropriately termed surface splines.

Originally introduced for interpolating wing deflections and computing slopes for aeroelastic calculations by [5], this ingenious device proves most interesting to analyze mathematically; in this connection, various deep results have been obtained in [2]. However in [6] a more constructive approach is taken, where a prominent role is played by representation formulas in function and distribution spaces, these complementary results being obtained by resorting to such basic mathematical tools as convolutions and Fourier transforms of distributions. For a more concrete presentation stressing the significant properties of the optimal interpolation process of surface spline interpolation at the algorithmic level we present the approach of Meinguet [7].

Meinguet formulates the optimal problem as follows:
Let there be given:

A finite set $A = (a_i)_{i \in I}$ of distinct points of R^2 containing a P-unisolvent subset, by which we mean a set $B = (a_j)_{j \in J}$ of M points of A, M being defined by (4), such that there exists a unique $p \in P$ satisfying the interpolating conditions

$$p(a_j) = \alpha_j \ , \forall j \in J \tag{5}$$

for any prescribed real scalars $\alpha_j, \forall j \in J$.

A set of real scalars $(\alpha_i)_{i \in I}$, or equivalently provided that $m > \frac{n}{2}$, the linear variety:

$$V = \{g \in X: g(a_i)_i = \alpha_i \ , \forall i \in I\} ; \tag{6}$$

whenever $\alpha_i = f(a_i)$, $\forall i \in I$, where f denotes a function defined (at least) on A, V can be interpreted as the set of X-interpolants of f on A. Thus our problem is to find $h \in V$ such that

$$|h|_m = \inf_{g \in V} |g|_m . \tag{7}$$

By virtue of the P-unisolvence of the subset B of the given set A of interpolation points in R^2, there exists in $P=P_{m-1}$ a unique basis $(p_j)_{j \; J}$ that is dual to the set of shifted dirac measures $(\delta_{(a_j)})_{j \in J}$ (in the sense that $p_i(a_j) = \delta_{ij}$, $i,j \in J$ where δ_{ij} is the Kronecker symbol). For every $g \in X$, $m>1$, the P-interpolant P_g of g on B is given by

$$P_g = \sum_{j \in J} g(a_i) p_j \ ; \tag{8}$$

owing to this definition, the mapping $P: X \rightarrow X$ is a linear projection of X with range $P=P_{m-1}$ and Kernel

$$X_o = \{g \in X : g(a_j) = 0, \forall j \in J\}, \tag{9}$$

so that

$$X = P_{m-1} \oplus X_o , \tag{10}$$

equipped with the seminorm $|\cdot|_{m-1}$, X_o is a Hilbert space.

In view of the direct sum decomposition (10), the restriction of the associated projector $I - P$ to the linear variety $V \subset X$ defined by (6) is clearly an injection. Therefore, finding $h \in V$ such that (7) holds amounts strictly to finding an element w of minimal norm $|\cdot|$ in the image of V under $I - P$, which is the linear variety,

$$W = \{ g \in X_o : g(a_k) = \alpha'_k , \forall k \in K \} , \tag{11}$$

where

$$K \equiv I - J = \{ k \in N : \ 1 \leq k \leq N \equiv \text{card } (K)\} \tag{12}$$

for definiteness, and

$$\alpha'_k = \alpha_k - \sum_{j \in J} \alpha_j p_j(a_k) , \quad k \in K. \tag{13}$$

The unique solution h of the problem is given by

$$h = w + \sum_{j \in J} \alpha_j p_j , \tag{14}$$

depend continuously on the data $(\alpha_i)_{i \in I}$, where

$$w = \sum_{k=1}^{N} \gamma_k K_{a_k} , \tag{15}$$

K_{a_k} denotes the Frechet-Riesz representer of the a shifted Dirac measure δ_{a_k} , γ_k real coefficients satisfying the Cramer system of linear equations

$$\sum_{k=1}^{N} K(a_i, a_k) \gamma_k = \alpha'_k , \quad 1 \leq k \leq N. \tag{16}$$

in fact K_{a_k} involves no functions more complicated than logarithms and is easily coded. The set $\{K_x : \forall x \in R^2\}$ is the so called reproducing kernel of the Hilbert function space X_o; it can be regarded equivalently as the real-valued continuous function $(x,y) \rightarrow K(x,y) = K_x(y)$ on $R^2 x R^2$.

Two dimensional interpolation by radial basis functions. The "thin plate spline" is among the radial basis functions for surface interpolation which were found to be most successful [4]. The drawback in using these basis functions for interpolation is the need to solve large full systems of linear equations which become very ill-conditioned as their order, namely the number of data points, increases. However, the above basis function share the property that

$$\Delta^m w(r) \to 0 \quad \text{as } r \text{ increases}$$

and $$\Delta^m w(r) \to \infty \quad \text{as } r \to 0\ , \quad (r^2 = x^2 + y^2) \tag{17}$$

where Δ^m is the m-iterated Laplacian, $\Delta = \frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial y^2}$.

This property of the basis functions enables us to apply a procedure by which most of the interpolation equations become diagonally dominant.

As pointed out in [3], the major application of the above radial basis functions is for interpolation of scattered data. This presentation considers data given on a square grid, where the discrete analogues of the operators Δ^m are well known. A method for defining the discrete analogues of Δ^m for general domain is being investigated.

Let the data $f_{kl} = f(x_k, y_l)$ be given on the square grid:

$$(x_i, y_j) = ((i-1)h, (j-1)h), \quad 1 \leq i, j \leq N \tag{18}$$

The system of interpolation equations becomes

$$\sum_{i,j=1}^{N} \alpha_{ij} w(r_{ij}^{kl}) = f_{kl}\ , \qquad i \leq k, l \leq N \tag{19}$$

where $r_{ij}^{kl} = \sqrt{(x_i - x_k)^2 + (y_j - y_l)^2}$.

We would like, by row operations, to form a finite difference approximation to the iterated Laplacian of $w(r)$, and thus to generate diagonal dominance in the system (19). The central difference operator δ^2 is replaced here by the 5-point difference approximation to the Laplacian

$$\Delta_h f_{kl} = f_{k+1,l} + f_{k,l+1} + f_{k-1,l} + f_{k,l-1} - 4f_{kl}\ .$$

Operating with Δ_h^m (with respect to the indices k and l) on (19) we obtain

$$\sum_{i,j=1}^{N} \alpha_{ij} \Delta_h^m \, w(r_{ij}^{kl}) = \Delta_h^m \, f_{kl} \ , \ m+1 \leq k,l \leq N-m.$$

By properties of (17) we expect that $\Delta_h^m w(r_{ij}^{kl})$ decreases as r_{ij}^{kl} increases and that the diagonal element becomes large. Furthermore we expect to achieve diagonal dominance in the transformed rows, i.e. ,

$$\left|\Delta_h^m w(r_{ij}^{kl})\right| > \sum_{\substack{i,j=1 \\ (i,j)\neq(k,l)}}^{N} \left|\Delta_h^m w(r_{ij}^{kl})\right| \quad , \quad m=1 \leq k,l \leq N-m.$$

The remaining equations cannot, in general, be made diagonally dominant. Therefore, even if the major part of the system (19) can be transformed into a diagonally dominant system to be solved by iterations, there is a substantial part of the system which must be solved directly by each iteration. A way of overcoming this difficulty, by using special differencing of (19) near the "boundaries" of the system, is presented.

Iterative construction of "thin plate splines", TPS, on a rectangular grid.

We consider TPS with m=2; this surface corresponds to an infinte extent thin plate of minimum bending energy clamped at the data points [3]. For this case the basis functions are fundamental solutions of the biharmonic equation $\Delta^2 w = 0$, i.e.,

$$w(r) = r^2 \log r$$

augmented by the monomials 1,x and y. The interpolation equations for data given on a square grid (18) are

$$\sum_{i,j=1}^{N} \alpha_{ij} w(r_{ij}^{kl}) + a + bx_k + cy_l = f_{kl} \qquad 1 \leq k,l \leq N, \qquad (20)$$

with the constraints

$$\sum_{k,l=1}^{N} \alpha_{kl} = \sum_{k,l=1}^{N} \alpha_{kl} x_k = \sum_{k,l=1}^{N} \alpha_{kl} y_l = 0. \tag{21}$$

Since $\Delta^2 w(r_{ij}) = \delta(r_{ij})$, it is expected that the difference operator Δ_h^2 is appropriate for generating diagonal dominance.

So far the process of differencing the equations is limited to the interior equations with $3 \leq k,l \leq N-2$. The application is extended to the "boundary" equations, i.e., with k or l equal to $1,2,N-1,N$ by defining the difference operator Δ_h^2 on the boundary points of the domain in such a way that $\Delta_h^2 f = 0$ on all grid points if and only if f is a linear grid function of the form $f_{kl} = a + bx_k + cy_l$, $1 \leq k,l \leq N$.

This property guarantees that the polynomial part in (20) vanishes with the application of Δ_h^2 to (20), and can be restored from the solution α of the derived system:

$$\Delta_h^2 \left[f - \sum_{i,j=1}^{N} w(r_{ij}) \alpha_{ij} \right]_{kl} = 0, \quad 1 \leq k,l \leq N. \tag{22}$$

To obtain the right form of Δ_h^2 we use the discrete analogue of the iterated Green's formula:

$$a(f,g) \equiv \int_\Omega (f_{xx} g_{xx} + 2 f_{xy} g_{xy} + f_{yy} g_{yy}) = \int_\Omega g(\Delta^2 f) + \text{boundary terms.}$$

The null space of the functional $a(f,f)$ is the space of linear polynomials. We define a discrete analogue a_h of a, so that the null space of $a_h(f,f)$ is the space of linear functions:

$$a_h(f,g) = \sum_{k=1}^{N} \sum_{l=1}^{N-2} (\delta_x^2 f)_{lk} (\delta_x^2 g)_{lk} + 2 \sum_{l,k=1}^{N-1} (\delta_y \delta_x f)_{lk} (\delta_y \delta_x g)_{lk}$$

$$+ \sum_{k=1}^{N-2} \sum_{l=1}^{N} (\delta_y^2 f)_{lk} (\delta_y^2 g)_{lk} , \tag{23}$$

where $(\delta_x f)_{lk} = f_{l+1,k} - f_{lk}$, $(\delta_y f)_{lk} = f_{l,k+1} - f_{lk}$.

Hence, the desired property of Δ_h^2 is guaranteed by formally defining Δ_h^2 via the summation by parts of (23), namely by the identity

$$a_h(f,g) = \sum_{k,l=1}^{N} g_{kl}(\Delta_h^2 f)_{kl} \ . \tag{24}$$

The following Lemmas are proved in [3] :

Lemma 1. Let the difference operator Δ_h^2 be defined by (23) and Then $\Delta_h^2 f = 0$ if and only if f is a grid function of the form $f_{kl} = a + bx_k + cy_l$, $1 \leq k,l \leq N$.

Lemma 2. The matrix Λ is symmetric, nonnegative definite of rank N^2-3, where Λ is a matrix representation of the difference operator on grid functions, regarded as vectors in R^{N^2}.

As direct conclusions from Lemmas 1 and 2, we have:

Corollary 1. $\sum_{k,l=1}^{N} w_{kl}\cdot(a + bx_k + cy_l) = 0$, for any a,b,c if and only if $w = \Delta_h^2 f$ for some grid function f.

Corollary 2. The α-component of the solution of (20) and (21) is a grid function of the form

$$\alpha_{kl} = (\Delta_h^2 \beta)_{kl} \qquad 1 \leq k,l \leq N,$$

for some grid function β.

The system (22) obtained by the application of Δ_h^2 to (20) can also be written as

$$\sum_{i,j=1}^{N} \Delta_h^2 w(r_{ij}^{\cdot\cdot})_{kl}\alpha_{ij} = (\Delta_h^2 f)_{kl}, \quad 1 \leq k,l \leq N \text{ or} \tag{25}$$

$$\Lambda A \alpha = \Lambda f,$$

where A is the $N^2 x N^2$ matrix with entries $w(r_{ij}^{kl})$, ordered in accordance with the vector form of the grid functions. The system (25)is singular, by Lemma 2. More precisely we have:

Corollary 3. The matrix ΛA of the system (25) is of rank N^2-3.

Due to the symmetry of Λ, the system obtained from (20) by the substitution $\alpha = \Delta_h^2 \beta$,

$$\sum_{i,j=1}^{N} \Delta_h^2 w(r_{\cdot\cdot}^{kl})_{ij}\, \beta_{ij} + a + bx_k + cy_l = f_{kl}, \quad 1 \leq k,l \leq N, \tag{26}$$

corresponds to the representation of the solution in terms of the new basis functions

$$B_{ij}(x,y) = [\Delta_h^2 w(r_{..})]_{ij}, \quad 1 \leq i,j \leq N,$$

which are bell-shaped for $3 \leq i,j \leq N-2$. This system of functions is of dimension N^2-3, by corollary 1.

In particular:

$$\sum_{i,j=1}^{N} B_{ij}(x,y) = \sum_{i,j=1}^{N} x_i B_{ij}(x,y) = \sum_{i,j=1}^{N} y_j B_{ij}(x,y) = 0.$$

The solution of (25) constitutes a three-dimensional subspace of N^2- vectors by corollary 3. In order to obtain that solution of (25) which satisfies (21), we present an iterative scheme for the solution of (25), in which each iterant is of the form $\Delta_h^2 w$, and therefore satisfies (21) in view of corollary 1.

The iterative scheme is:

$$\alpha_{kl}^{(0)} = 0 \quad 1 \leq k,l \leq N \tag{27}$$

for n = 0,1,2,---,

$$e_{kl}^{(n)} = f_{kl} - \sum_{i,j=1}^{N} \alpha_{ij}^{(n)} w(r_{ij}^{kl}), \quad 1 \leq k,l \leq N \tag{28}$$

$$\alpha_{kl}^{(n+1)} = \alpha_{kl}^{(n)} + w\left[\Delta_h^2 e_{..}^{(n)}\right]_{kl}, \quad 1 \leq k,l \leq N. \tag{29}$$

The first iterants in procedure (29) provide smoothing solutions for noisy data $\{f_{kl}\}$.

The development throughout this presentation has been restricted to square grids. However, the application of these ideas to triangular grids of general form is now being investigated. In addition we plan to continue the investigation of the smoothing technique.

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