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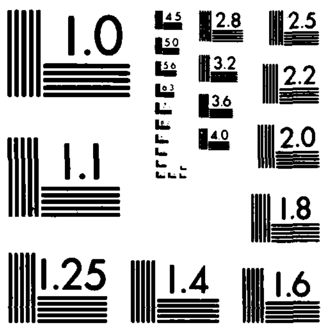

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A SIMULATION STUDY
OF SYLVESTER'S PROBLEM IN THREE DIMENSIONS

BY

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A SIMULATION STUDY
OF SYLVESTER'S PROBLEM IN THREE DIMENSIONS

Kim-Anh Do and Herbert Solomon

Consider five points P_1, P_2, P_3, P_4, P_5 independently and uniformly distributed over a convex region K in three dimensional space. We are interested in the probability that the five points are the vertices of a re-entrant polyhedron, i.e. one of the points falls within the tetrahedron formed by the other four points. This is a three-dimensional generalization of the Sylvester problem which asks for the probability that four points independently and uniformly distributed over a convex region in the plane form a re-entrant quadrilateral, see Solomon (1978).

Hostinsky (1925) has obtained the result for the case when K is a sphere, namely the probability p_S that five random points in a sphere form a re-entrant polyhedron is

$$p_S = \frac{9}{143}.$$

Kingman (1969) also obtained this result by evaluating the probability that n+2 points falling at random in an n-dimensional ball form a convex hull with n+2 rather than n+1 vertices. It is

$$(n+2)\binom{n+1}{\frac{1}{2}(n+1)}^{n+1} \Big/ \left\{\binom{(n+1)^2}{\frac{1}{2}(n+1)^2} 2^n\right\}.$$

For n=3, he shows the probability is

$$1 - p_S = \frac{134}{143}.$$

However, explicit values of p_K for random points in non-spherical convex regions have apparently not yet been successfully calculated. A simulation study is performed here where the probability p_K is approximated when K is a sphere, a cube, a tetrahedron, or an octahedron. Since we know the exact answer for the sphere, an anchor point is provided for simulation in a sphere.

For easy programming purposes, a random point $P = (x, y, z)$ is generated in a cube with vertices

$$(0,0,0),(0,0,1),(0,1,0),(0,1,1),(1,0,0),(1,0,1),(1,1,0),(1,1,1)$$

where x, y, and z are uniformly distributed in $[0,1]$. If K is a sphere, then we are interested only in points belonging to the particular sphere with centre $(0.5, 0.5, 0.5)$ and radius 0.5 . So a point generated inside the unit cube is discarded if

$$(x-0.5)^2 + (y-0.5)^2 + (z-0.5)^2 > 0.25.$$

Since the volume of this sphere is

$$\frac{4}{3}\pi r^3 = \frac{\pi}{6} \approx 0.52$$

and the volume of the unit cube is 1, about 48% of the generated random points will be discarded during simulation.

Another program employing spherical co-ordinates can be produced as follows: generate a point $P = (x, y, z)$ where x, y , and z are independently normally distributed with mean 0 and variance 1. We then perform a projection by letting

$$x' = \frac{x}{||P||}$$
$$y' = \frac{y}{||P||}$$
$$z' = \frac{z}{||P||}$$

where

$$||P|| = \left(x^2 + y^2 + z^2\right)^{\frac{1}{2}}.$$

The point $P' = (x', y', z')$ is then uniformly distributed on the surface of the sphere with centre $(0,0,0)$ and radius 1 . Now generate $u \sim U(0,1)$ and let

$$x'' = \sqrt{u}\, x'$$

$$y'' = \sqrt{u}\, y'$$

$$z'' = \sqrt{u}\, z'$$

then $P'' = (x'', y'', z'')$ is uniformly distributed in the sphere .

Of the two programs described above for the sphere, we employed the former. To generate uniform random numbers, we used the method of combining the linear congruential generator and the Fibonacci generator which produced high resolution, shuffled random numbers on [0,1). It is easy and economical to generate uniform random numbers and so the cost of discarding points in the cube that do not fall in the convex region is negligible.

If the convex region K is a tetrahedron, then we will consider only points belonging to the tetrahedron contained in the unit cube, in particular that body with vertices

$$(0,0,0),(1,0,0),(\frac{1}{2},\frac{\sqrt{3}}{2},0), \text{ and } (\frac{1}{2},\frac{\sqrt{3}}{6},\sqrt{\frac{3}{2}}).$$

This tetrahedron has edge length a = 1 and volume

$$\frac{a^3}{6\sqrt{2}} = \frac{1}{6\sqrt{2}}$$

so about 88% of the generated random points will be discarded during simulation.

Similarly, if K is an octahedron, then the points considered are those belonging to the particular octahedron with vertices

$$(\frac{1}{2\sqrt{2}},\frac{1}{2\sqrt{2}},1),(0,0,\frac{1}{2}),(0,\frac{1}{\sqrt{2}},\frac{1}{2}),(\frac{1}{\sqrt{2}},0,\frac{1}{2}),(\frac{1}{\sqrt{2}},\frac{1}{\sqrt{2}},\frac{1}{2}), \text{ and } (\frac{1}{2\sqrt{2}},\frac{1}{2\sqrt{2}},0).$$

Since this octahedron has edge length a = $\frac{1}{\sqrt{2}}$ and volume

$$\frac{a^3\sqrt{2}}{3} = \frac{1}{6}$$

about 83% of the generated random points will be discarded. However since generating random points is cheap, this method is still quite acceptable.

The probability that the five points $P_1, P_2, P_3, P_4,$ and P_5 form a re-entrant polyhedron is the probability that one of the points falls within the tetrahedron formed by the other four points . There are five possible combinations of a tetrahedron formed by four points from five given points. We have a re-entrant polyhedron if any one of these tetrahedra contains the fifth point.

Suppose we want to check if the point P_5 lies within the tetrahedron formed by (P_1, P_2, P_3, P_4), that is we want to check that P_5 and the fourth point fall on the same side of the plane passing through the other three points . Consider the plane passing through

$$P_1 = (x_1, y_1, z_1)$$

$$P_2 = (x_2, y_2, z_2)$$

$$P_3 = (x_3, y_3, z_3).$$

The equation of this plane is

$$a\,x + b\,y + c\,z = d \qquad (\star)$$

where

$$a = (y_3 - y_1)(z_2 - z_1) - (z_3 - z_1)(y_2 - y_1)$$

$$b = (z_3 - z_1)(x_2 - x_1) - (x_3 - x_1)(z_2 - z_1)$$

$$c = (x_3 - x_1)(y_2 - y_1) - (y_3 - y_1)(x_2 - x_1)$$

$$d = a\,x_1 + b\,y_1 + c\,z_1.$$

By replacing the co-ordinates of P_4 and P_5 in the equation above, we can obtain signed values for P_4 and P_5 . If the signs are the same then we can conclude that P_4 and P_5 fall on the same side of the plane formed by (P_1, P_2, P_3). The process is repeated with the other two planes namely (P_1, P_2, P_4) and (P_2, P_3, P_4).

We perform a simulation for each of the cases when K is a sphere, a cube, a tetrahedron, or an octahedron . Each run is based on 100,000 polyhedra, and the results obtained are displayed below

	# of re-entrant tetrahedra	proportion of re-entrant tetrahedra
Sphere	6274	0.06274
Octahedron	6728	0.06728
Cube	6945	0.06945
Tetrahedron	8604	0.08604

The probability of a re-entrant polyhedron in a sphere follows a binomial distribution with

$$p = \frac{9}{143} \approx 0.0629,$$

so the approximate standard deviation when there are 100,000 trials is

$$\sqrt{\frac{0.0629 \times 0.9371}{100,000}} \approx 0.00077.$$

Since we obtained $p = 0.06274$ which is 0.00016 away from the true value , the simulation result is quite plausible. The standard deviation for the sphere approximated from the simulations is also 0.00077, the same value as that obtained using the exact population proportion p. The standard deviations for the tetrahedron, cube, and octahedron can be estimated by $\sqrt{\frac{\hat{p}(1-\hat{p})}{100,000}}$ from the simulations and are respectively 0.00089, 0.00080, and 0.00080. This suggests 95% confidence intervals of 0.0843 - 0.0878 for the tetrahedron, 0.0679 - 0.0710 for the cube, and 0.0657 - 0.0689 for the octahedron. We also note that the order of the proportions agrees with Sylvester's conjecture that p_K is minimized when K is a sphere and maximized when K is a tetrahedron.

For the two-dimensional problem, the probabilities for a re-entrant quadrilateral go from $\frac{1}{3}$ to $\frac{35}{12\pi^2}$ (=0.2955) as one goes from the triangle to the circle. Thus for two dimensions we have 0.0378 as the difference between the maximum and minimum probabilities; in three dimensions we estimate 0.0233 as the range. This suggests that for higher dimensions, a good estimate for the probability will not differ much among convex regions. Thus the value obtained by Kingman for the ball may serve well for $n \geq 4$ for all convex regions. These values are:

n	p_S
4	0.010710192
5	0.001542677
6	0.000194894
7	0.000022118
8	0.000002293
9	0.000000220

Note that the probability of a re-entrant polyhedron in higher dimensions is exceedingly small.

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This note gives estimates of solutions to the Sylvester problem in three dimensions. These results suggest that the solutions in four and higher dimensions are very close to the solution for the n-dimensional sphere which is known.

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