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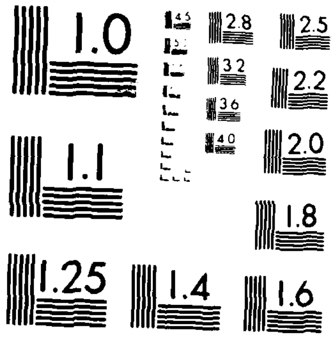

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ON ASYMPTOTIC JOINT DISTRIBUTION OF THE EIGENVALUES OF THE NONCENTRAL MANOVA MATRIX FOR NONNORMAL POPULATIONS

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1. INTRODUCTION

The problem of testing the hypothesis of the equality of the mean vectors of several multivariate populations with a common covariance matrix received considerable attention in the literature. The test procedures are based upon certain functions of the eigenvalues of the multivariate analysis of variance (MANOVA) matrix. In the univariate case, the MANOVA matrix reduces to the ratio of the between group and within group sums of squares. The joint distribution of the eigenvalues of the MANOVA matrix in the noncentral case is useful in studying the power of the tests for the equality of the mean vectors. This distribution is also useful in the problems connected with selection of important discriminant functions in the area of classification. Fisher (1939), Hsu (1939), and Roy (1939) have independently derived the joint distribution of the eigenvalues of the MANOVA matrix in the central case. Hsu (1941) derived the above distribution in the noncentral case when the sample size tends to infinity and the underlying distribution is multivariate normal. In proving the above result, Hsu assumed that the ratios of the sample sizes of the groups to the total sample size tend to constants in the limiting case. In this paper, we extend the result of Hsu to the case when the underlying distribution is not necessarily multivariate normal.

2. PRELIMINARIES

For $t = 1, 2, \ldots, k$, let $\underset{\sim}{X}_t' = (X_{1t}, \ldots, X_{pt})$ be distributed with mean vector $\underset{\sim}{\xi}_t' = (\xi_{1t}, \ldots, \xi_{pt})$ and covariance matrix $\Sigma = (\sigma_{ij})$. Let $\underset{\sim}{X}_{t1}, \ldots, \underset{\sim}{X}_{tm_t}$ be a sample of m_t independent observations on $\underset{\sim}{X}_t$. Also, let

$$\bar{\underset{\sim}{\xi}}. = \frac{1}{N} \sum_{t=1}^{k} m_t \underset{\sim}{\xi}_t, \quad N = m_1 + \cdots + m_k$$

$$\Psi = (\Psi_{ij}) = \frac{1}{N} \sum_{t=1}^{k} m_t (\underset{\sim}{\xi}_t - \bar{\underset{\sim}{\xi}}.)(\underset{\sim}{\xi}_t - \bar{\underset{\sim}{\xi}}.)' . \tag{2.1}$$

Unless stated otherwise, we assume that $m_t = q_t m$. So, $N = qm$ where $q = q_1 + \cdots + q_k$. The eigenvalues of $\Psi\Sigma^{-1}$ are constants independent of m. Let the rank of Ψ be r and the nonzero eigenvalues of $\Psi\Sigma^{-1}$ be denoted by $\lambda_1' \geq \cdots \geq \lambda_r'$. We assume that these eigenvalues have multiplicities, i.e.,

$$\lambda_i' = \lambda_h \quad \text{for } i = a_{h-1} + 1, \ldots, a_h;\ h = 1, 2, \ldots, \nu \tag{2.2}$$

where $a_0 = 0$, $a_h = a_{h-1} + \mu_h$, $a_\nu = r$. Now, let $\phi_1 \geq \cdots \geq \phi_{\ell}$ denote the eigenvalues of the MANOVA matrix $\hat{\Psi}\hat{\Sigma}^{-1}$ where $\ell_1 = \min(p, k - 1)$,

$$\hat{\Psi} = \frac{1}{N} \sum_{t=1}^{k} m_t (\bar{\underset{\sim}{X}}_{t.} - \bar{\underset{\sim}{X}}_{..})(\bar{\underset{\sim}{X}}_{t.} - \bar{\underset{\sim}{X}}_{..})'$$

$$\hat{\Sigma} = \frac{1}{N} \sum_{t=1}^{k} \sum_{\ell=1}^{m_t} (\underset{\sim}{X}_{t\ell} - \bar{\underset{\sim}{X}}_{t.})(\underset{\sim}{X}_{t\ell} - \bar{\underset{\sim}{X}}_{t.})' = (\hat{\sigma}_{ij}). \tag{2.3}$$

$$\bar{\underset{\sim}{X}}_{t.} = \frac{1}{m_t} \sum_{j=1}^{m_t} \underset{\sim}{X}_{tj}, \qquad \bar{\underset{\sim}{X}}_{..} = \frac{1}{N} \sum_{t=1}^{k} \sum_{j=1}^{m_t} \underset{\sim}{X}_{tj}.$$

Here we note that $E(\hat{\Sigma}) = \frac{N - k}{N} \Sigma$ and $E(\hat{\Psi}) = \Psi + \frac{(k - 1)}{N} \Sigma$.

In the sequel, we need the following lemmas:

<u>Lemma 2.1</u> For each t, $t = 1, 2, \ldots, k$, let $\underset{\sim}{y}_{t\ell} : p \times 1$, $\ell = 1, 2, \ldots,$ be a sequence of i.i.d. random vectors with $E\underset{\sim}{y}_{t\ell} = 0$ and with I_p as the covariance matrix. Suppose that the k sequences are independent and that for each t, $E(\underset{\sim}{y}_{t1}' \underset{\sim}{y}_{t1})^2 < \infty$.

If $m_t = q_t m$, $q_t > 0$, $t = 1, 2, \ldots, k$ and $N = m_1 + \cdots + m_k$, then, as m tends to infinity, the k vectors $\sqrt{m_t}\,(\bar{\underset{\sim}{y}}_{t.} - \bar{\underset{\sim}{y}}_{..})$, $t = 1, 2, \ldots, k$ and the $p \times p$ matrix

$$\frac{1}{\sqrt{N}} \sum_{t=1}^{k} \sum_{\ell=1}^{m_t} [(\underset{\sim}{y}_{t\ell} - \bar{\underset{\sim}{y}}_{t.})(\underset{\sim}{y}'_{t\ell} - \bar{\underset{\sim}{y}}'_{t.}) - I_p]$$

converge in distribution to k random vectors $\underset{\sim}{Z}_t = (z_{1t} \ldots z_{pt})'$, $t = 1, 2, \ldots, k$ and a $p \times p$ symmetric matrix $U = (u_{ij})$, which satisfy the following:

1. The joint distribution of $\{z_{jt}$, $j = 1, 2, \ldots, p$, $t = 1, 2, \ldots, k$, u_{ij}, $1 \le i \le j \le p\}$ are $kp + \frac{1}{2}p(p + 1)$-variate normal,
2. The k vectors $\{\underset{\sim}{Z}_1, \ldots, \underset{\sim}{Z}_k\}$ and the matrix U are mutually independent,
3. $E\underset{\sim}{Z}_t = 0$, $EU = 0$ and

$$\mathrm{var} Z_t = (1 - (q_t/q))I_p, \mathrm{cov}(Z_t, Z_s) = -(\sqrt{q_t q_s}/q)I_p, \quad t \neq s$$

$$\mathrm{cov}(u_{ii}, u_{jj}) = \frac{1}{q} \sum_{t=1}^{k} q_t (Ey_{it1}^2 y_{jt1}^2 - 1), \quad 1 \le i, j \le p$$

$$\mathrm{cov}(u_{i_1 i_2}, u_{j_1 j_2}) = \frac{1}{q} \sum_{t=1}^{k} q_t E y_{i_1 t1} y_{i_2 t1} y_{j_1 t1} y_{j_2 t1}$$

$$i_1 \neq i_2 \text{ or } j_1 \neq j_2, \quad 1 \le i_1, i_2, j_1, j_2 \le p.$$

Here

$$\bar{\underset{\sim}{y}}_{t.} = \frac{1}{m_t} \sum_{\ell=1}^{m_t} \underset{\sim}{y}_{t\ell}, \quad \bar{\underset{\sim}{y}}_{..} = \frac{1}{N} \sum_{t=1}^{k} \sum_{\ell=1}^{m_t} \underset{\sim}{y}_{t\ell} = \frac{1}{N} \sum_{t=1}^{k} m_t \bar{\underset{\sim}{y}}_{t.},$$

I_p denotes the $p \times p$ identity matrix and y_{jt1} is the jth element of the p-vector $\underset{\sim}{y}_{t1}$, and $q = q_1 + \cdots + q_t$.

Proof: The proof follows by application of central limit theorem and by direct computations.

Lemma 2.2 Let $\underset{\sim}{X}_n$, $n = 0, 1, 2, \ldots$, be a sequence of random p-vectors with $\underset{\sim}{X}_n \to \underset{\sim}{X}_0$ in distribution. Then we can find a probability space (Ω, F, P) on which we can define a sequence of random vectors $\tilde{\underset{\sim}{X}}_n$, $n = 0, 1, 2, \ldots$, such that

1. $\underset{\sim}{X}_n$ and $\tilde{\underset{\sim}{X}}_n$ are identically distributed

2. $\tilde{\underset{\sim}{X}}_n \to \tilde{\underset{\sim}{X}}_0$ pointwise.

The above lemma was given in Skorokhod (1956).

Lemma 2.3 Let $g_n(x)$ be a sequence of K-degree polynomials with roots $x_1^{(n)},\ldots,x_K^{(n)}$ for each n, and let g(x) be a k-degree polynomial with roots $x_1,\ldots,x_k$, $k \leq K$. If $g_n(x) \to g(x)$ as $n \to \infty$, then after suitable rearrangement of $x_1^{(n)},\ldots,x_K^{(n)}$, we have $x_j^{(n)} \to x_j$, $j = 1, 2,\ldots,k$ and $|x_j^{(n)}| \to \infty$, $j = k + 1,\ldots,K$.

The proof of the above lemma is given in Bai (1984). In the sequel, det A denotes the determinant of A.

3. ASYMPTOTIC JOINT DISTRIBUTION OF THE EIGENVALUES OF THE MANOVA MATRIX

In this section, we derive the asymptotic joint distribution of the eigenvalues of the MANOVA matrix when the population eigenvalues have multiplicities. Let

$$\underset{\sim}{\eta}_t = \Sigma^{-\frac{1}{2}} \underset{\sim}{\xi}_t \text{ and } \underset{\sim}{\eta} = \Sigma^{-\frac{1}{2}} \underset{\sim}{\xi}.$$

Then

$$\frac{1}{N} \sum_{t=1}^{k} m_t (\underset{\sim}{\eta}_t - \underset{\sim}{\eta})(\underset{\sim}{\eta}_t - \underset{\sim}{\eta})' = \Sigma^{-\frac{1}{2}} \| \psi_{ij} \| \Sigma^{-\frac{1}{2}},$$

which has $\lambda_1', \ldots, \lambda_r'$ as its nonzero roots. Consider the matrix

$$A = (\sqrt{q_1/q}\ (\underset{\sim}{\eta}_1 - \underset{\sim}{\eta}) \ \ldots \ \sqrt{q_k/q}\ (\underset{\sim}{\eta}_k - \underset{\sim}{\eta})).$$

It is evident that $AA' = \Sigma^{-\frac{1}{2}} \| \psi_{ij} \| \Sigma^{-\frac{1}{2}}$ and that $A(\sqrt{q_1/q}, \ldots, \sqrt{q_k/q}) = 0$.

Let B_1 be an orthogonal matrix with $(\sqrt{q_1/q}, \ldots, \sqrt{q_k/q})'$ as its last column, and A_1 be the $p \times (k-1)$ matrix constructed by the first $k-1$ columns of AB_1. Noting the last column of AB_1 is zero, we know that

$$A_1 A_1' = \Sigma^{-\frac{1}{2}} \| \psi_{ij} \| \Sigma^{-\frac{1}{2}}. \tag{3.1}$$

Let

$$\underset{\sim}{y}_{t\ell} = \Sigma^{-\frac{1}{2}} (\underset{\sim}{X}_{t\ell} - \underset{\sim}{\xi}_t)$$

$$\bar{\underset{\sim}{y}}_{t.} = \frac{1}{m_t} \sum_{\ell=1}^{m_t} \underset{\sim}{y}_{t\ell} \text{ and } \bar{\underset{\sim}{y}}_{..} = \frac{1}{N} \sum_{t=1}^{k} m_t \bar{\underset{\sim}{y}}_{t.}$$

$$U_m = \frac{1}{N} \sum_{t=1}^{k} \sum_{\ell=1}^{m_t} [(\underset{\sim}{y}_{t\ell} - \bar{\underset{\sim}{y}}_{t.})(\underset{\sim}{y}_{t\ell}' - \bar{\underset{\sim}{y}}_{t.}') - I_p]$$

$$H_m = (\sqrt{m_1}\ (\bar{\underset{\sim}{y}}_{1.} - \bar{\underset{\sim}{y}}_{..}), \ldots, \sqrt{m_k}\ (\bar{\underset{\sim}{y}}_{k.} - \bar{\underset{\sim}{y}}_{..}))$$

According to Lemma 2.1, we have

$$(H_m, U_m) \xrightarrow{\text{in dist.}} (\tilde{H}, \tilde{U})$$

where $\tilde{H} = (\tilde{Z}_1, \ldots, \tilde{Z}_k)$, and $\tilde{U} = (\tilde{u}_{ij})$ are mutually independent, and are distributed respectively as $(Z_1, \ldots, Z_k)$ and U defined in Lemma 2.1.

By simple computation, we find that the eigenvalues of $\hat{S}\hat{\Sigma}^{-1}$ are the same as the solutions

$$\det \left\| \frac{1}{N} H_m H_m' - \frac{1}{\sqrt{N}} H_m A' - \frac{1}{\sqrt{N}} A H_m' + AA' - \phi I - \frac{1}{\sqrt{N}} U_m \right\| = 0 . \tag{3.2}$$

Let G_m be the $p \times (k - 1)$ matrix constructed by the first $k - 1$ columns of $H_m B_1$. Noting that the last column of $H_m B_1$ is zero, we find (3.2) is equivalent to

$$\det \left\| \frac{1}{N} G_m G_m' - \frac{1}{\sqrt{N}} G_m A_1' - \frac{1}{\sqrt{N}} A_1 G_m' + A_1 A_1' - \phi I - \frac{1}{\sqrt{N}} U_m \right\| = 0 . \tag{3.3}$$

Note that the $p \times (k - 1)$ elements of the first $k - 1$ columns of HB_1 are i.i.d. $N(0,1)$'s and the last column of HB_1 is zero. We know that the elements of G_m converge in distribution to $p \times (k - 1)$ i.i.d. $N(0,1)$ variables, which are independent of U. Let H_1 be the $p \times (k - 1)$ matrix constructed by the first $(k - 1)$ columns of HB_1.

Recalling that $A_1 A_1' = \Sigma^{-\frac{1}{2}} \| \psi_{ij} \| \Sigma^{-\frac{1}{2}}$, we find that there exists a $p \times p$ orthogonal matrix B_2 and a $(k - 1) \times (k - 1)$

$$A_1 = B_2 \operatorname{diag}[\sqrt{\lambda_1'}, \ldots, \sqrt{\lambda_r'}, 0 \ldots 0]\, B_3$$

where the $\operatorname{diag}[\sqrt{\lambda_1'} \ldots \sqrt{\lambda_r'}, 0 \ldots 0]$ denotes a $p \times (k - 1)$ matrix whose first r diagonal elements are $\lambda_1', \ldots, \lambda_r'$ respectively, and the remaining elements are all zero.

Let $W_m = B_2' G_m B_3' = \| w_{ij}^{(m)} \|$, $\hat{U}_m = B_2' \tilde{U}_m B_2 = \| \hat{u}_{ij}^{(m)} \|$ and let $v = N^{-\frac{1}{2}}$. Then (3.3) is equivalent to

$$\det \| v^2 W_m' W_m' - v C^{(m)} + D + v \hat{U}_m \| = 0 \tag{3.4}$$

where

$$D = \left\| \begin{matrix} (\lambda_1 - \phi) I_{\mu_1} & & & \\ & \ddots & & \\ & & (\lambda_\nu - \phi) I_{\mu_\nu} & \\ & & & -\phi I_{p-r} \end{matrix} \right\|$$

$$C^{(m)} = \left\| \begin{matrix} C_{11}^{(m)} & \dots & C_{1\nu}^{(m)} & E_1^{(m)\prime} \\ \cdot & \cdot & \cdot & \cdot \\ C_{\nu 1}^{(m)} & \dots & C_{\nu\nu}^{(m)} & E_\nu^{(m)\prime} \\ E_1^{(m)} & \dots & E_\nu^{(m)} & 0 \end{matrix} \right\|$$

and

$$C_{hh}^{(m)} = \left\| \sqrt{\lambda_h}(w_{ij}^{(m)} + w_{ji}^{(m)}) \right\| \quad i, j = a_{h-1} + 1, \dots, a_h \ , \ h = 1, 2, \dots, \nu,$$

$$C_{gh}^{(m)} = \left\| \sqrt{\lambda_g} w_{ij}^{(m)} + \sqrt{\lambda_h}\, w_{ji}^{(m)} \right\| \quad i = a_{h-1} + 1, \dots, a_h, \ j = a_{g-1} + 1, \dots, a_g,$$
$$1 \leq h \neq g \leq \nu$$

and

$$E_h^{(m)} = \left\| \sqrt{\lambda_h}\, w_{ij}^{(m)} \right\| \quad i = r + 1, \dots, p \quad j = a_{h-1} + 1, \dots, a_h, \ j = a_{g-1} + 1, \dots, a_g,$$

As proved earlier, the elements of H_m and of U_m tend in distribution to that of $\tilde{H}$ and $\tilde{U}$. Hence the elements of W_m and of $\hat{U}_m$ will tend in distribution to that of $H = B_2' \tilde{H} B_3' = \| h_{ij} \|$ and of $U = B_2' \tilde{U} B_2 = \| u_{ij} \|$, satisfying that H and U are independent and h_{ij}'s are i.i.d. N(0,1) variables.

According to Lemma 2.2, without loss of generality, we can assume that this convergence is pointwise. From (3.4) it is easily seen that for all $\omega \in \Omega$, when m is large enough, equation (3.4) has p roots, and they are given by $\phi_1 \geq \phi_2 \geq \dots \geq \phi_p$.

Write $U = \| V_{gh} \|$ where $V_{gh} = \| (u_{ij}) \|$ $i = a_{g-1} + 1, \dots, a_g$, $j = a_{h-1} + 1, \dots, a_h$, $1 \leq g, h \leq \nu + 1$, $a_{\nu+1} = p$ and denote by C_{gh}, E_g, the limits of $C_{gh}^{(m)}$, $E_g^{(m)}$, respectively and take the variable transformation $\phi = \lambda_1 + \zeta v$ in (3.4). Multiplying by $\frac{1}{\sqrt{v}}$ the first μ_1 rows and the first μ_1 columns of the determinant on the left hand side of (3.4) and making m tend to infinity, we obtain the following in the limit:

$$\det \left\| \begin{matrix} -C_{11}+U_{11}-I_{\mu_1} & & & & \\ & (\lambda_2-\lambda_1)I_{\mu_2} & & & \\ & & \ddots & & \\ & & & (\lambda_\nu-\lambda_1)I_{\mu_\nu} & \\ & & & & -\lambda_1 I_{\mu_{p-r}} \end{matrix} \right\|$$

Let the p roots of (3.4) after the variable transformation be denoted by $\zeta_{i1}^{(m)} = (\phi_i - \lambda_1)/\sqrt{v}$, $i = 1, 2, \ldots, p$. Then we know that $\zeta_{i1}^{(m)} \to \zeta_i$, $i = 1, 2, \ldots, a_1$, $\zeta_{i1}^{(m)} \to -\infty$, $i = a_1 + 1, \ldots, p$, where ζ_i, $i = 1, 2, \ldots, a_1$ are the roots of

$$\det \| -C_{11} + U_{11} - \zeta I_{\mu_1} \| = 0. \tag{3.5}$$

Similarly, if we denote by $\zeta_{ih}^{(m)} = (\phi_i - \lambda_h)/\sqrt{v}$ the roots of (3.4) after variable transformation $\zeta = (\phi - \lambda_h)/\sqrt{v}$, we can prove that

$$\zeta_{ih} \begin{cases} \longrightarrow +\infty & \text{if} \quad i \leq a_{h-1} \\ \longrightarrow \zeta_i & \text{if} \quad i = a_{h-1} + 1, \ldots, a_h \\ \longrightarrow -\infty & \text{if} \quad i \geq a_h + 1,\ h = 1, 2, \ldots, \nu \end{cases}$$

where ζ_i, $i = a_{h-1} + 1, \ldots, a_h$ are roots of

$$\det \| -C_{hh} + U_{hh} - \zeta I_{\mu_h} \| = 0. \tag{3.6}$$

Finally, if we take the variable transformation $\phi = v^2\zeta$ in (3.4), multiplying by $\frac{1}{v}$ the last p − r rows and columns of the determinantal equation in (3.4), and making $m \to \infty$, we get

$$\det \left\| \begin{matrix} \lambda_1 I_{\mu_1} & \cdots & 0 & E_1' \\ 0 & \cdots & \lambda_\nu I_{\mu_\nu} & E_\nu' \\ E_1 & \cdots & E_\nu & \overline{W} - \zeta I_{p-r} \end{matrix} \right\| = 0 \tag{3.7}$$

where $\overline{W}$ is the $(p - r) \times (p - r)$ right lower submatrix of $B_2'H_1H_1'B_2$.

The equation (3.7) is equivalent to

$$\det \left\| \bar{W} - \frac{1}{\lambda_1} E_1 E_1' - \frac{1}{\lambda_2} E_2 E_2' - \ldots - \frac{1}{\lambda_\nu} E_\nu E_\nu' - \zeta I_{p-r} \right\| = 0. \qquad (3.8)$$

If we write

$$B_2' H_1 B_3' = \begin{Vmatrix} w_{11} & \cdots & w_{1k-1} \\ \cdot & & \cdot \\ \cdot & & \cdot \\ \cdot & & \cdot \\ w_{p1} & \cdots & w_{pk-1} \end{Vmatrix}$$

where the w's are i.i.d. N(0,1) variables, then recalling the definition of $E_1, \ldots, E_\nu$ and $\bar{W}$, we find that

$$\bar{W} - \frac{1}{\lambda_1} E_1 E_1' - \ldots - \frac{1}{\lambda_\nu} E_\nu E_\nu' = \| d_{ij} \| ,$$

where $d_{ij} = \sum_{\ell=r+1}^{k-1} w_{i\ell} w_{j\ell}$ $i, j = r + 1, \ldots, p$. Hence (3.8) is equivalent to

$$\det\left(\| d_{ij} \| - \zeta I_{p-r} \right) = 0. \qquad (3.9)$$

By Lemma 2.3, we see that

$$\zeta_{i\nu+1}^{(m)} = N\phi_i \quad \begin{cases} +\infty & \text{if } i \leq r \\ \zeta_i & \text{if } i = r + 1, \ldots, p \end{cases}$$

where ζ_i, $i = r + 1, \ldots, p$ are the roots of (3.9). Let $\ell_1 = \min(p, k - 1)$. Note that the rank of $\| d_{ij} \|$ is $\ell_1 - r$ with probability one. Thus, there are surely $p - \ell_1$ zero roots of (3.9).

Up to now, we have proved the following theorem.

Theorem 4.1 When m tends to infinity, then $\zeta_1^{(m)} \ldots \zeta_{\ell_1}^{(m)}$ converge in distribution to $\zeta_1 \ldots \zeta_{\ell_1}$, where $\zeta_i^{(m)} = \sqrt{N}\, (\phi_i - \lambda_i')$, $i = 1, 2, \ldots, r$ and $\zeta_i^{(m)} = N\phi_i$, $i = r + 1, \ldots, \ell_1$, and for $h = 1, 2, \ldots, \nu$, $\zeta_{a_{h-1}} + 1, \ldots, \zeta_{a_h}$, in decreasing order, are the roots of

$$\det \| C_{hh} + U_{hh} - \zeta I_{\mu_h} \| = 0. \qquad (3.10)$$

$\zeta_{r+1},\ldots,\zeta_{\ell_1}$ are the positive roots of

$$\det \left\| \, \| d_{ij} \| - \zeta I_{p-r} \right\| = 0 \tag{3.11}$$

$$C_{hh} = \| \sqrt{\lambda_h}\,(w_{ij} + w_{ji}) \| \quad i, j = a_{h-1} + 1,\ldots,a_h \quad h = 1, 2,\ldots,\nu$$

$$d_{ij} = \sum_{\ell=r+1}^{k-1} w_{i\ell} w_{j\ell} \quad i, j = r + 1,\ldots,p.$$

$\{w_{ij}, i = 1, 2,\ldots,p, j = 1, 2,\ldots,k - 1\}$ are i.i.d. $N(0,1)$ variable, and are independent of $\{U_{hh}, h = 1, 2,\ldots,\nu\}$, $U_{hh} = \| u_{ij} \|$, $i, j = a_{h-1} + 1,\ldots,a_h$ satisfying

1) $\{u_{ij}, 1 \le i \le j \le p$ with $u_{ij} = u_{ji}$ are $\frac{1}{2}\, p(p+1)$-variate normal

2) $Eu_{ij} = 0$, $i, j = 1, 2,\ldots,p$

$$Eu_{ii}u_{jj} = \frac{1}{q} \sum_{t=1}^{k} q_t (ER_{it}^2 R_{jt}^2 - 1) \quad 1 \le i, j \le p$$

$$Eu_{i_1 i_2} u_{j_1 j_2} = \frac{1}{q} \sum_{t=1}^{k} q_t E R_{i_1 t} R_{i_2 t} R_{j_1 t} R_{j_2 t}$$

if $i_1 \neq i_2$ or $j_1 \neq j_2$.

Here R_{it} are the elements of

$$\begin{Vmatrix} R_{11} & \cdots & R_{1k} \\ \vdots & & \vdots \\ R_{p1} & \cdots & R_{pk} \end{Vmatrix} = B_2' \Sigma^{-\frac{1}{2}} \left\| (\underset{\sim}{X}_1 - \underset{\sim}{\mu}_1) \;\cdots\; (\underset{\sim}{X}_k - \underset{\sim}{\mu}_k) \right\| .$$

<u>Remark 3.1</u> The admissibility of changing $- C_{hh}$ in (3.6) into C_{hh} in (3.10) can be seen from the symmetry of normality of the entries of C_{hh} and the independence of C_{hh} and U.

<u>Remark 3.2</u> In Theorem 3.1, it is easy to see that $(\zeta_{r+1},\ldots,\zeta_{\ell_1})$ is independent of $(\zeta_1,\ldots,\zeta_r)$. According to Hsu's paper, the density of the distribution of $(\zeta_{r+1},\ldots,\zeta_{\ell_1})$ is given by

$$D_1(\zeta_{r+1},\ldots,\zeta_{\ell_1}) = 2^{-\frac{1}{2}(p-r)(k-1-r)}\,\pi^{\frac{1}{2}(\ell_1-r)}\left\{\prod_{i=1}^{\ell_1-r}\Gamma\left(\frac{\ell_2}{2}-\frac{r}{2}-\frac{i}{2}+\frac{1}{2}\right)\right\}^{-1}$$

$$\times\left\{\prod_{i=r+1}^{\ell_1}\prod_{j=i+1}^{\ell_1}(\zeta_i-\zeta_j)\right\}\left\{\prod_{i=r+1}^{\ell_1}\zeta_i\right\}^{\frac{1}{2}(\ell_2-\ell_1-1)}\exp\left\{-\frac{1}{2}\sum_{i=r+1}^{\ell_1}\zeta_i\right\}$$

$$0 > \zeta_{r+1}\geq\cdots\geq\zeta_{\ell_1}\geq 0, \tag{3.12}$$

where $\ell_2 = \max(p, k-1)$, $\ell_1 = \min(p, k-1)$.

It is evident that we have the following theorems.

<u>Theorem 3.2</u> Under the restrictions of Theorem 3.1 and that $ER_{i_1t}R_{i_2t}R_{j_1t}R_{j_2t} = 0$, for all $t = 1, 2,\ldots,k$ and at least one among $i_1i_2j_1j_2$ is not equal to any one of the rest and that $ER_{it}^2R_{jt}^2 = 1$, for all $t = 1, 2,\ldots,k$ and $i \neq j$, the $\nu + 1$ random vectors $(\zeta_1,\ldots,\zeta_{a_1})$, $(\zeta_{a_1+1},\ldots,\zeta_{a_2})\ldots(\zeta_{a_{\nu-1}+1},\ldots,\zeta_r)$, $(\zeta_{r+1}\ldots\zeta_{\ell_1})$ are mutually independent.

<u>Remark 3.3</u> If for each $t = 1, 2,\ldots,k$, $\underset{\sim}{X}_t$ has an isotropic distribution with mean vector $\underset{\sim}{\mu}_t$ and covariance matrix Σ, then the additional condition of Theorem 3.2 is fulfilled.

<u>Theorem 3.3</u> Under the restrictions of Theorem 3.2 and that $ER_{it}^4 = 3$, for all $t = 1, 2,\ldots,k$, the density of the distribution of $(\zeta_{a_{h-1}+1},\ldots,\zeta_{a_h})/\sqrt{2\lambda_h^2 + 4\lambda_h}$ is given by $D(x_{a_{h-1}+1},\ldots,x_{a_h})$, $h = 1, 2,\ldots,$, where

$$D(x_1\ldots x_\mu) = 2^{-\mu/2}\left(\prod_{i=1}^{\mu}\Gamma\left(\frac{i}{2}\right)\right)^{-1}\left\{\prod_{i=1}^{\mu}\prod_{j=i+1}^{\mu}(x_i-x_j)\right\}\exp\left\{-\frac{1}{2}\sum_{i=1}^{\mu}x_i^2\right\} \tag{3.13}$$

$$(\infty \geq x_1 \geq x_2 \geq\cdots\geq x_\mu > -\infty)$$

When the underlying distribution is multivariate normal, Hsu (1940) proved the expression (3.12) and (3.13). In proving the above result, Hsu used Lemma 1 in his paper. Recently, one of the authors (Liang) pointed out, through a counterexample, that the above lemma is incorrect. However, Bai (1984) had very recently pointed out that the final result of Hsu is correct.

4. APPLICATIONS

In a number of situations, it is of interest to test the hypothesis $H_1 : \underset{\sim}{\xi}_1 = \ldots = \underset{\sim}{\xi}_k$. Let S_B and S_W respectively denote the between group and within group sums of squares and cross products (SP) matrices respectively where $S_B = N\hat{\Psi}$ and $S_W = N\hat{\Sigma}$. The nonzero eigenvalues of $S_B S_W^{-1}$ are given by $\phi_1 \geq \ldots \geq \phi_{\ell_1}$. It is known that

$$\frac{(N - k - p - 1)}{(k - 1)} E\{S_B S_W^{-1}\} = I_p + \frac{\Omega\Sigma^{-1}}{N(k - 1)} . \tag{4.1}$$

The problem of testing for the rank of $\Omega\Sigma^{-1}$ is the same as the problem of testing for the number of significant discriminant functions. A discussion of some procedures for testing for the rank of Ω was given in Krishnaiah (1981). We will discuss as to how the results of Section 3 of this paper can be used in testing for the rank of Ω when the underlying distribution is not multivariate normal but the first four moments of the distribution exist. Unless otherwise stated, we assume in the sequel that $(m_t/m) = q_t$ for $t = 1, 2, \ldots, k$ and $N = mq$ tends to infinity.

We can test (e.g., see Krishnaiah (1981)) for the hypothesis H_{r+1} by using certain functions $\eta(\phi_{r+1}, \ldots, \phi_{\ell_1})$ of the eigenvalues of $\hat{\Psi}\hat{\Sigma}^{-1}$ where H_r denotes the hypothesis that the rank of $\Psi\Sigma^{-1}$ is r. Fisher (1938) suggested using T_1 as a test statistic for testing H_r where $T_1 = \phi_{r+1} + \ldots + \phi_{\ell_1}$. When H_r is true, the joint asymptotic distribution of $N\phi_{r+1}, \ldots, N\phi_{\ell_1}$ is the same as the joint distribution of the eigenvalues of $S_r : (\ell_1 - r) \times (\ell_1 - r)$ where S is distributed as the central Wishart matrix with ℓ_2 degrees of freedom. We can use ϕ_{r+1} also as a test statistic. The asymptotic distribution of $N\phi_{r+1}$, under H_r, is the same as the distribution of the largest eigenvalue of S_r. This distribution has been tabulated (e.g., see Krishnaiah (1981)). Suppose the null hypothesis is not true and the rank of $\Psi\Sigma^{-1}$ is r + s. Then $\lambda'_{r+s+1} = \ldots = \lambda'_{\ell_1} = 0$, and $\lambda'_{r+s} > 0$. Also, if $\lambda'_{r+1}, \ldots, \lambda'_{r+s}$ are distinct, then

$$\frac{\sqrt{N}(\phi_{r+1} - \lambda'_{r+1})}{\sqrt{2\lambda^{2}_{r+1} + 4\lambda'_{r+1}}}, \ldots, \frac{\sqrt{N}(\phi_{r+s} - \lambda'_{r+s})}{\sqrt{2\lambda'^{2}_{r+s} + 4\lambda'_{r+s}}}$$

are distributed independently as normal with mean zero and variance one. So, we can compute the asymptotic power function of the test based upon ϕ_{r+1}. If the eigenvalues $\lambda'_{r+1}, \ldots, \lambda'_{r+t}$ $(t < s)$ are equal, then the joint asymptotic distribution of

$$\frac{\sqrt{N}(\phi_{r+1} - \lambda'_{r+1})}{\sqrt{2\lambda'^{2}_{r+1} + 4\lambda'^{2}_{r+1}}}, \ldots, \frac{\sqrt{N}(\phi_{r+t} - \lambda'_{r+t})}{\sqrt{2\lambda'^{2}_{r+t} + 4\lambda'_{r+t}}}$$

is the same as the joint distribution of the eigenvalues of the Gaussian matrix $A = (a_{ii}) : (r + t) \times (r + t)$ where the elements of A are distributed independently as normal with means zero, $var(a_{ii}) = 2$ and $var(a_{ij}) = 1$ for $i \neq j$. The distribution of the largest eigenvalue of this matrix can be computed by using the method discussed in Krishnaiah and Chang (1971). Percentage points of this distribution are given in Krishnaiah and Schuurmann (1985). So, the power function of the test in this case can be computed.

We will now discuss a sequential method (see Krishnaiah (1981)) method of testing for the rank of Ω. The hypothesis $\Omega = 0$ is accepted or rejected according as

$$\phi_1 \lessgtr c_{\alpha 1}$$

where

$$P[\phi_1 \leq c_{\alpha 1} \mid \Omega = 0] = (1 - \alpha_1).$$

If $\Omega = 0$, we don't proceed further. If $\Omega = 0$ is rejected, we accept or reject H_1 according as

$$\phi_2 \lesseqgtr C_{\alpha 2}$$

where

$$P[\phi_2 \leq C_{\alpha 2} \mid \phi_1 > C_{\alpha 1}; H_1] = (1 - \alpha_2)$$

When H_1 is true, ϕ_1 and ϕ_2 are distributed independent of each other asymptotically. Also, under H_1, $N\phi_2$ is distributed asymptotically as the largest eigenvalue of $S_1 : (\ell_1 - 1) \times (\ell_1 - 1)$ where S_1 is distributed as the central Wishart matrix with ℓ_2 degrees of freedom and $E(S_1) = \ell_2 I$. So, we can compute α_2 for given value of $C_{\alpha 2}$ and vice versa. If H_1 is accepted, we don't proceed further. Otherwise, we accept or reject H_2 according as

$$\phi_3 \lesseqgtr C_{\alpha 3}$$

where

$$P[\phi_3 \leq C_{\alpha 3} \mid \phi_2 \geq C_{\alpha 2}; H_2] = (1 - \alpha_3).$$

When H_2 is true, $N\phi_3$ is distributed independent of ϕ_2 as the largest eigenvalue of $S_2 : (\ell_1 - 2) \times (\ell_1 - 2)$ where S_2 is distributed as the central Wishart matrix with ℓ_2 degrees of freedom and $E(S_2) = \ell_2 I$. So, we can compute $C_{\alpha 3}$ for given value of α_3 and vice versa. This procedure is continued until a decision is made about the rank of Ω.

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