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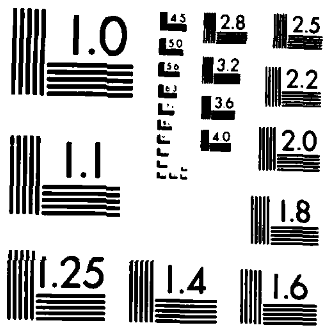

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ON FAILURE MODELING

AD-A151 411

by

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ABSTRACT

A promising approach to failure modeling, in particular to developing failure-time distributions, is discussed. Under this approach, system state or wear and tear is modeled by an appropriately chosen random process, eg, a diffusion process; and the occurrences of fatal shocks are modeled by a Poisson process whose rate function is state dependent. The system is said to fail when either wear and tear accumulates beyond an acceptable or safe level or a fatal shock occurs.

This approach has significant merit. First it provides revealing new insights into most of the famous and frequently used lifetime distributions in reliability theory. Moreover, it suggests intuitively appealing ways for enhancing those standard models. Indeed, this approach provides a means of representing the underlying dynamics inherent in failure processes. Reasonable postulates for the dynamics of failure should lend credence to prediction and estimation of reliability, maintainability, and availability. In other words, accuracy of representation could lead to better, more reliable prediction of failure.

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SECTION 1

INTRODUCTION

Modeling of failure times in reliability theory has generally proceeded by means of ad hoc, albeit clever, data analysis methods. In many cases, however, failure modeling has been done without any apparent attempt to capture the dynamic relationship between the system state and system reliability. (Notable exceptions include Gumbel [10], Kao [12], Birnbaum and Saunders [5], and Bergman [4]; a good discussion is provided in Mann, Schafer, and Singpurwalla [16].) Consequently, it has been very hard to give derived models a natural interpretation. Furthermore, there is no intuitively appealing way to fine tune a given model. The lack of an intuitively appealing and naturally interpretable methodology has motivated us to study reliability models that accurately reflect the dynamic dependency of system failure and decay on the state of the system.

Two relevant stochastic models have aroused interest in the theoretical and applied modeling communities; however, their rich structure and potential for applications to reliability have not been fully exploited. The first of these is the celebrated shot-noise model (cf. Rice [19] and Cox and Isham [7] for background and further references). The shot-noise model supposes that the system is subjected to "shots" or jolts according to a Poisson process. A jolt may consist of an internal component malfunctioning or an external "blow" to the system. Jolts induce stress on the system when they occur. However, if the system survives the jolt it may then recover to some extent. For instance, a sudden and unexpected surge of power in the circuit of a control system may temporarily increase the likelihood of system failure, but the overload itself decays rapidly. For another example, the mortality rate for persons who have suffered a heart attack declines with the elapsed time since the trauma. In this case, the heart actually repairs itself to a degree. The shot-noise model is both easily interpretable and analytically tractable.

The other relevant model involves the inverse Gaussian distribution. Under this approach, system wear and tear is modeled by a Brownian motion that has positive drift. The system fails whenever the wear and tear reaches a certain critical threshold. For example, consider a structural support subject to loadings that vary both in terms of size and in time of application. Each load causes microscopic cracks to form in the material. Eventually those cracks coalesce into a critical break that causes the support to fail. If the successive loadings and their induced cracks are assumed to be stochastically independent, it is then plausible to model this process by a Brownian motion with positive drift. Under this modeling assumption, the time to material failure corresponds to the first passage time of the candidate Brownian motion to the critical level, and this first passage time has an inverse Gaussian distribution (cf. Karlin and Taylor [13, p. 363]). Practical experience has shown that this model provides both good fit and easy interpretation. Moreover, the distribution is extremely tractable from the viewpoint of statistical analysis (cf. Folks and Chhikara [8]).

The rich structure of these models suggests that an appropriate conceptual framework for reliability modeling is the following:

Suppose that a certain component in a physical system begins operating with a given strength or a given "operational age" (eg, extent of wear and tear or stress) that can be measured in physical units. Suppose that, as time goes on, component wear or stress builds up (loss of strength with increasing age), perhaps in a random way. (The concept of wear or stress buildup is dual to that of declining strength.) For instance, consider the thickness of tread on a tire (or brake lining) or the level of fluid in a hydraulic system. The tread wears down with use, and there may be gradual loss of fluid from the hydraulic system. Assume that this wear may be offset, but only in part, by maintenance and repair. Such considerations suggest modeling component strength (or susceptibility to failure) by a stochastic process $X = \{X(t), t \geq 0\}$ with starting state corresponding to the initial level of strength (or initial operational age). This process X should tend to drift downward (decrease) with time as wear builds up; if X is the operational age or wear-and-tear process then it should tend to drift upward. The component may fail when either wear alone has reduced strength below some safe level (as in structural material) or at the epoch of occurence of some hazardous event (eg, an external shock) severe enough to overcome current strength. We denote by τ the time of passage of the X process to the critical level. For the examples cited above, hazardous events might be the tire's abrupt encounter with a sharp portion of road surface or the rupture of tubing in the hydraulic system due to an external blow. (It is clear from these examples that the rate of fatal shocks should be modeled as a decreasing function of component strength or an increasing function of component wear or stress.) We denote by k(x) the Poisson killing rate associated with state x, and by T the time to failure of the component. With the above conventions and modeling assumptions in place, we can express the probability of surviving beyond time t, starting with strength or operational age x, as follows:

$$(1) \quad P^x(T > t) = E^x\left[\exp\left\{-\int_0^t k(X(s))ds\right\} I_{\{\tau > t\}}\right]$$

The next step in the modeling process is the selection of a mathematical model for the strength (wear) process. Two viable candidate classes are diffusion processes and shot-noise processes. We first explore the diffusion model in some detail.

Suppose that component strength evolves in accordance with a diffusion process $X = \{X(t), t \geq 0\}$ having drift parameter $\mu(x)$ and diffusion coefficient $\sigma^2(x)$ in state $x > 0$. (A comprehensive introduction to the subject of diffusion processes is provided in Chapter 15 of Karlin and Taylor [14].) Then $-\mu(x)$ can be interpreted as the rate at which wear builds up in state x. Alternatively, if component wear builds up according to the diffusion process X then $-\mu(x)$ can be interpreted as the rate at which strength declines in state x. If T is the time to failure of the component, we assume that $P(T \leq h \mid X(s) = x) = k(x)h + o(h)$ for each time point s, so that k(x) can be interpreted as the Poisson rate of occurrence of a traumatic shock of magnitude sufficient to overcome or "kill" a component of strength x.

Now let $w(x,t)$ be the probability that a component of strength x survives beyond time t, ie, $w(x,t) = P^x(T > t)$. Indeed, $w(x,t)$ coincides with the right side of equation (1) if τ is the first passage time of the diffusion X to the critical level. It follows from the backward differential equation for the Kac functional of the diffusion process X (cf. [14, pp. 222-224]) that $w(x,t)$ satisfies

$$(2) \quad \frac{\partial w(x,t)}{\partial t} = -k(x)\,w(x,t) + \mu(x)\frac{\partial w(x,t)}{\partial x} + \frac{\sigma^2(x)}{2}\frac{\partial^2 w(x,t)}{\partial x^2}$$

The initial condition for this differential equation is determined by the critical strength or wear threshold Δ. We allow $0 \leqslant \Delta \leqslant +\infty$. If the diffusion represents strength then $w(x,0) = 1$ if $x > \Delta$ and 0 otherwise. If the diffusion represents wear or stress then $w(x,0) = 1$ if $x < \Delta$ and 0 otherwise. Examples of subclasses that lead to mathematically tractable solutions of the backward differential equation (2) for the state-dependent survivor function $w(x,t)$ are presented in Table 1-1. In the table the diffusion process X represents wear or stress buildup. These subclasses overlap one another. Moreover, we certainly do not imply that (2) has a tractable solution for every choice of infinitesimal parameters, killing rate function, and failure threshold in these subclasses. From a failure modeling standpoint there is, rather, a rich and interesting variety of cases for which explicit solutions can be identified. Furthermore, the widespread availability of excellent and efficient computational algorithms to solve (2) approximately, makes it feasible to experiment with various choices of infinitesimal parameters for modeling purposes.

Table 1-1. Model Subclasses and Parameters

Subclass	Infinitesimal Parameters $\mu(x)$, $\sigma^2(x)$	Killing Function $k(x)$	Failure Threshold Δ
Deterministic Wear	$\sigma^2(x) \equiv 0$	arbitrary	arbitrary
Constant Killing Rate	arbitrary	$k(x) \equiv \lambda$	$\Delta < +\infty$
Infinite Level of Wear and Tear	arbitrary	arbitrary	$\Delta = +\infty$

SECTION 2

THE DETERMINISTIC SYSTEM STATE PROCESS

For this model the buildup of wear (or loss of strength) is assumed to be deterministic, and thus $\sigma^2(x) \equiv 0$. Under this assumption the system state process $X = \{X(t), t \geq 0\}$ satisfies the following equation:

$$(3)\quad X(t) = \int_0^t \mu(X(s))ds + x$$

Alternatively, X satisfies the differential equation

$$(4)\quad dX(t) = \mu(X(t))dt$$

with initial condition $X(0) = x$. The probability of system survival through time t is then given by

$$(5)\quad P(T>t) = \exp\left\{-\int_0^t k(X(s))ds\right\} I_{\{\tau>t\}}$$

Equation (5) can be interpreted in terms of standard reliability theory. To simplify the notation suppose for now that $\Delta = +\infty$ and $x = 0$, and write $P(T > t)$ in place of $w(x,t)$. First we need to recall the failure rate function r(t), which is usually defined by

$$(6)\quad r(t) = -(dP(T>t)/dt)/P(T>t)$$

Integrating r over [0,t] and assuming that $P(T=0) = 0$ shows that

$$(7)\quad \int_0^t r(s)ds = -Ln(P(T>t))$$

Consequently

$$(8)\quad \exp\left\{-\int_0^t r(s)ds\right\} = P(T>t)$$

Comparing (5) and (8) we see that

$$(9)\quad r(t) = k(X(t))$$

Therefore the failure rate at time t is equal to the killing rate that corresponds to the "system state" at time t.

Moreover, equations (4) and (9) enable us to reinterpret standard reliability distributions in terms of a killing rate function and a system evolution process. To illustrate this, we will now give new interpretations to some well known distributions in reliability modeling.

Makeham's Life Distribution

(10) $P(T > t) = \exp\{-bt - a(\exp(ct) - 1)\}$

Then

(11) $k(X(t)) = r(t) = d(bt + a(\exp(ct) - 1))/dt = b + ac\exp(ct)$.

By taking the killing rate proportional to the stress, eg,

(12) $k(x) = x$

we find that

(13) $X(t) = b + ac\exp(ct)$

Equation (4) implies that

(14) $\mu(b + ac\exp(ct)) = ac^2 \exp(ct)$

Thus

(15) $\mu(x) = c(x - b)$

Interpret the "stress" function $\mu(x)$ as saying that an individual is born b operational units old and declines at a rate proportional to chronological age. If we set b=0 in (10) we obtain the Gompertz life distribution. Gompertz [9] described the conceptual basis for his distribution as assuming "...the average exhaustion of a man's power to avoid death to be such that at the end of equal infinitely small intervals of time he lost equal portions of his power to oppose destruction which he had at the commencement of these intervals." Gompertz thus assumes that an individual's operational age is proportional to his chronological age. Makeham's improvement to the Gompertz distribution takes into account the fact that a person is born b operational units old.

Weibull Distribution

(16) $P(T > t) = \exp(-at^b)$

Then

(17) $k(X(t)) = r(t) = d(at^b)/dt = abt^{b-1}$

Suppose that $b \neq 1$. Taking $k(x) = x^{b-1}$, gives

(18) $X(t) = t(ab)^{1/(b-1)}$.

Setting $c = (ab)^{1/(b-1)}$ will simplify (18) to

(19) $X(t) = ct$

This implies the wear rate function is constant, ie,

(20) $\mu(x) = c$

If $b = 1$ then $k(X(t)) = a$ for all t. Thus, either the killing rate function is constant or the wear rate is zero or both. Adopting the exponential distribution as a failure model is a strong assumption indeed!

Rayleigh Distribution

(21) $P(T > t) = \exp\{-(at + bt^2/2)\}$

Then

(22) $k(X(t)) = r(t) = a + bt$

Taking

(23) $k(x) = x$

gives

(24) $X(t) = a + bt$

and thus

(25) $\mu(x) = b$ and $x_o = a$

Gumbel Distribution

(26) $P(T > t) = 1 - \exp(-\exp(-(t-a)/b))$

Then

(27) $$\exp\left[-\int_o^t k(X(s))ds\right] = 1 - \exp(-\exp(-(t-a)/b))$$

or

(28) $$\int_o^t k(X(s))ds = -\mathrm{Ln}(1 - \exp(-\exp(-(t-a)/b)))$$

Hence

(29) $$k(X(t)) = \{\exp(-\exp(-(t-a)/b))\exp(-(t-a)/b)/b\}/\{1-\exp(-\exp(-(t-a)/b))\}$$
$$= \mathrm{Ein}(\exp(-(t-a)/b))/b$$

where Ein(x) is the Einstein-Planck function (cf. Abramowitz and Stegun [1]) defined by

(30) $\mathrm{Ein}(x) = x/(\exp(x) - 1)$

Setting

(31) $k(x) = \mathrm{Ein}(x)/b$

we see that

(32) $X(t) = \exp(-(t-a)/b)$

and consequently

(33) $\mu(X(t)) = -\exp(-(t-a)/b)/b$

where $\mu(x) = -x/b$. In contrast to the Makeham, Weibull, and Rayleigh examples, it is appropriate to view this X process for the Gumbel distribution as representing loss of strength.

Repeating the foregoing development with $\Delta < +\infty$ (ie. a finite critical failure threshold) yields truncated versions of these well-known distributions.

Thus, within the framework of our general approach, a "first order model" (taking $\sigma^2(x) = 0$) not only yields some classic failure distributions but provides new and revealing insights into their structure. It is important to note that while the decompositions between state process and killing rate function presented above are somewhat arbitrary, this should not be the case with an actual application. In many reliability modeling efforts there will be a natural candidate for system state, and the corresponding state process will be observable. This observability permits the study of conditional probability of failure given the state process. Indeed, the conditional distribution of failure due to trauma is simply the distribution of the first event in a nonhomogeneous Poisson process whose rate at time t is equal to $k(X(t))$. It is this fact that allows the analyst to decompose the parameter estimation problem into two distinct and largely independent statistical estimation problems; namely, the estimation of the killing rate function and the estimation of the system state process parameters.

Furthermore, this decomposition has significant statistical implications. In most instances the state of the system at *death* will contain all the relevant information about the killing rate function, independent of the *time of death*. Consequently, post-mortem analysis will yield most if not all of the relevant information. The actual time of death becomes a less significant factor.

We will consider these important implications in a future paper.

SECTION 3

NONDETERMINISTIC WEAR (STRENGTH) MODELS (WHY DO WE NEED THEM?)

The deterministic wear model presented above is satisfactory when used in either predicting the behavior of very large ensembles of units (such as assessing the price of warranties for mass produced items) or in predicting the behavior of units whose wear process is nearly constant. In many cases, however, neither of these conditions is satisfied. For example, the setting of safety standards for the replacement of airplane tires is a case in point. The wear of tires is obviously quite random; tire wear depends on many factors such as weather conditions, runway surface, airplane load, etc. The determination of the optimal tire replacement policy requires a reliability model for the tires. Inaccurate reliability analysis will result in either unnecessary tire replacement or in tire failures (with possibly catastrophic consequences). The utility of a "dynamic" reliability model is that it allows the safety engineer to define a replacement policy in terms of tire condition rather than in terms of operational age. By exploiting the information contained in tire condition, it is possible to simultaneously minimize the average number of replacements while decreasing the risk of failure.

One plausible way to obtain reliability models that satisfactorily capture the dynamic connection of component reliability with component condition is to model wear or stress by a diffusion process. A diffusion process can be conceived as a continuous approximation of a first order stochastic difference equation, ie, something of the form

$$(34)\quad X(n+1) - X(n) = \mu(X(n)) + \sigma(X(n))Z_n$$

where $\{Z_n, n \geq 1\}$ is a sequence of independently and identically distributed random variables.

Interpret (34) as saying that during one unit of time (and starting from level x) the wear will increase on average $\mu(x)$ and that the standard deviation from this average increase in wear is $\sigma(x)$. If $\sigma(x)$ is identically zero, then (34) reduces to a first order difference equation whose continuous approximation is given by

$$(35)\quad X'(t) = \mu(X(t))$$

If we assume that the sequence $\{Z_n, n \geq 1\}$ is normally distributed, then (34) is simply a discrete approximation to a diffusion process. It is certainly plausible to model the distribution of the $\{Z_n, n \geq 1\}$ sequence by some other distribution (eg, a gamma); however, the normality assumption seems to produce mathematically more tractable equations. Furthermore, there is an enormous literature on diffusion processes which can be utilized.

Several subclasses of our diffusion approach in failure modeling seem to yield tractable problems. They are discussed in the following paragraphs.

First Passages With Constant Killing

Consider a unit that is replaced when its strength dips below a certain level or when it fails due to a catastrophic shock. Assume that the probability of death due to a catastrophe is independent of the unit's condition. (For example, a car headlight is replaced as soon as its luminescence drops below a certain critical level or it is broken by some road shock.) In this case, the model has constant killing rate function, ie. $k(x) \equiv \lambda$. Thus the probability that the "component is alive and well" at time t is given by

$$(36)\quad P^{x}(T > t) = E^{x}[I_{\{\tau > t\}} \exp(-\lambda t)]$$

where τ is the time of first passage of the diffusion process to the critical level Δ. This expression simplifies to

$$(37)\quad P^{x}(T > t) = \exp(-\lambda t)\, P^{x}(\tau > t)$$

Consequently, it suffices to derive the distribution of the first passage time τ. Calculating the first passage time distribution for an arbitrary diffusion is generally quite difficult. In fact, it corresponds to solving a second order partial differential equation with nonconstant coefficients; in particular, equation (2) with $k(x) \equiv 0$ and the appropriate initial condition.

However, there is a significant class of diffusions that give rise to mathematically tractable first-passage problems.

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Generalized Inverse Gaussian

Any generalized inverse Gaussian distribution with a nonpositive power parameter is a first passage distribution for a certain diffusion process with drift (cf. Barndorff-Nielsen et al [3]). The generalized inverse Gaussian distribution has density

$$(38)\qquad f(t) = \frac{(c/b)^{a/2}}{2K_a(\sqrt{bc})} t^{a-1} \exp\left\{-(bt^{-1} + ct)/2\right\} \quad (t > 0)$$

Here $K_a(\ )$ stands for the modified Bessel function of the third kind (cf. [1]) with index a, and a is called the power parameter. The domain of variation of the parameters (a,b,c) is

$$(39)\qquad \begin{array}{l} a > 0,\ b \geq 0,\ c > 0 \\ a = 0,\ b > 0,\ c > 0 \\ a \leq 0,\ b > 0,\ c \geq 0 \end{array}$$

This rich class of distributions includes the inverse Gaussian ($a = -1/2$), the hyperbola distribution ($a = 0$), and as limit cases, the gamma ($a > 0$ and $b = 0$) and the reciprocal gamma ($a < 0$ and $c = 0$). (For a recent survey of the main results and related references, see Jorgensen [11].)

For a first passage example (cf. [3]) and a striking illustration of the richness of this class, consider the following. Let g be the an arbitrary strictly increasing function on $[0,\infty)$ with continuous second derivative for which $g(0) = 0$. Let $X(t) = g(\sigma B(t) - \mu t)$ where $\sigma > 0$ and $\mu > 0$. Now set

$$(40)\quad \sigma(x) = \sigma g'(g^{-1}(x)),$$

$$(41)\quad \theta(x) = \int_0^x \frac{dy}{\sigma(y)}\ ,$$

and

$$(42)\quad \mu(x) = \sigma(x)\left[-\frac{\mu}{\sigma} + \frac{d\sigma(x)}{2dx}\right]$$

Consider the diffusion on $[0,\infty)$ with infinitesimal parameters $\mu(x)$ and $\sigma^2(x)$ so defined. Let $x_0 > 0$ be the initial position of the process. Then the first hitting time of level 0 has the distribution obtained from (38) with parameter set $(a,b,c) = (-1/2, \theta^2(x_0), \mu^2/\sigma^2)$.

For instance, take $g(x) = x^2$. Then $\sigma^2(x) = 4\sigma^2 x$, $\theta^2(x) = x/\sigma^2$, and $\mu(x) = -2\mu x^{1/2} + \sigma^2$. For another example, take $g(x) = \exp(x) - 1$; this is geometric Brownian Motion centered at the origin. Then $\sigma^2(x) = \sigma^2(x+1)^2$, $\theta^2(x) = [\frac{\ln(x+1)}{\sigma}]^2$, and $\mu(x) = (-\mu + \sigma^2/2)(x+1)$. (The transformation of diffusion processes is discussed in Karlin[14].)

Failure Due to Shock Only

In this class of models, systems can age indefinitely ($\Delta = \infty$); that is, a system is not subject to retirement but must die in the line of duty. As an example, consider home appliances, which

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are literally worked to death at ages well beyond any nominal design lifetime. Another example is automobile usage in third world countries, where cars are driven 500,000 miles or more before retirement.

Mathematically speaking, the probability of survival for this class corresponds to the Kac functional of the state process. More specifically, let X be the state process and let Y(t) be defined as follows:

$$(43) \qquad Y(t) = \int_0^t k(X(s))ds$$

Then the probability of survival beyond time t is given by

$$(44) \qquad w(x,t) = E^x[\exp(-Y(t))]$$

For a remarkable example, consider Brownian motion with quadratic killing.

To motivate this model in a reliability context, imagine that a system ages (wears) on the average μ age units per unit of time, but that the increase in age per unit time deviates from its expected value by σ age units. Furthermore, imagine that the system is subject to externally generated shocks, some of which are potentially fatal. For example, consider the human cardiovascular system and suppose that we are measuring age in terms of some operational criterion (eg, systolic blood pressure). The human heart is constantly subject to shocks (emotional distress, sudden physical exertions, etc). The likelihood that a given shock will cause heart failure depends on the condition of the heart (and the overall health of the person), so it is reasonable to postulate that the occurrence rate of fatal blows increases with age. Moreover, the occurrence rate of fatal blows grows at first very slowly with age, then more rapidly with advancing age. The above considerations would suggest using a convex killing rate function that has a minimum at 0. A plausible candidate function is $k(x) = \lambda x^2$.

Let X be Brownian motion with drift μ and positive variance σ^2 , and initial position x , ie,

(45) $$X(t) = \sigma B(t) + \mu t + x$$

Let the killing function be given by

(46) $$k(x) = \lambda x^2$$

The functional Y(t) is then

(47) $$Y(t) = \lambda \int_0^t [X(s)]^2 ds$$

For this important case, the probability of survival up to time t (starting in state x) is given by

(48) $$w(x,t) = \left[\operatorname{sech}(\sigma t\sqrt{2\lambda})\right]^{1/2} \exp\left\{-\frac{\mu^2 t}{2\sigma^2} + \frac{\mu^2}{2\sigma^3\sqrt{2\lambda}} \tanh(\sigma t\sqrt{2\lambda}) - \frac{x\mu}{\sigma^2} + \frac{x\mu}{\sigma^2}\operatorname{sech}(\sigma t\sqrt{2\lambda}) - \frac{x^2}{\sigma}\sqrt{\frac{\lambda}{2}} \tanh(\sigma t\sqrt{2\lambda})\right\}$$

where

$$\operatorname{sech} y = 2/[\exp(y) + \exp(-y)]$$

and

$$\tanh y = [\exp(y) - \exp(-y)]/[\exp(y) + \exp(-y)]$$

The calculation of this expression for w(x,t) is a nontrivial exercise. To obtain the formula given by the right-side of (48) we first expressed X(t) by means of the Karhunen-Loeve expansion (cf. Ash and Gardner [2]). The calculation then required the use of special function

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theory and the calculus of residues as it applies to the summation of series. Finally, we checked our calculation by verifying that $w(x,t)$ satisfies the backward differential equation (2) with $k(x) = \lambda x^2$, $\mu(x) = \mu$, $\sigma(x) = \sigma$, and initial condition $w(x,0) = 1$. The derivation of (48) and some properties of the distribution are presented in [15].

Shot Noise Model for System Stress

We continue in the vein of the preceding example (failure due to externally induced trauma only) but return to the formalism of equation (1). Suppose that system wear or operational age is modeled by a shot noise process $X = \{X(t), t \geq 0\}$. For example, suppose the system is subjected to shots or jolts according to Poisson process with rate λ. Suppose that if a jolt of magnitude D occurs at epoch S then at time $S + t$ the contribution of the jolt to the system stress is $Dh(t)$, where h is a nonnegative function, vanishing on $(-\infty,0)$, which tends to 0 sufficiently fast as $t \to \infty$ for several integrals of powers of h, over $(0,\infty)$, to be finite. In other words, shot-induced stress is additive and decays with time according to the rate function h (recall the cardiac example mentioned earlier). Thus, if $\{S_n, n \geq 1\}$ are the epochs of shot occurrences and $\{D_n, n \geq 1\}$ the magnitudes of the successive jolts, then the "residual" system stress at time, say $X(t)$, is given by

$$(49) \qquad X(t) = \sum_{n=1}^{\infty} D_n h(t - S_n)$$

An intuitively appealing and customary choice for h is $h(y) = \exp(-ay)$ if $y \geq 0$, and $h(y) = 0$ if $y < 0$, ie, exponential decay. Suppose that system failure is proportional to system stress, say $k(X(t)) = X(t)$ in equation (1), and $X(0) = 0$. Then the time to system failure T is the epoch of the first count in a doubly stochastic Poisson process with rate function $\{X(t), t \geq 0\}$. In particular, if the shot epochs and magnitudes are independent and the shot magnitudes are mutually independent with a common distribution having Laplace transform ϕ, then

$$(50) \qquad P(T>t) = E\left[\exp\left\{-\int_0^t X(s)ds\right\}\right]$$

$$= \exp(-\lambda t)\ \exp\left\{\lambda \int_0^t \phi\left(\int_0^y h(z)dz\right)dy\right\}$$

This distribution has failure rate

$$(51) \qquad \lambda\ [1 - \phi(\int_0^t h(y)dy)]$$

The formulation is intuitively appealing while leading to tractable results. Indeed, system stress $X(t)$ has a limit distribution as $t \to \infty$. In particular, if D has an exponential distribution with parameter μ and $h(y) = \exp(-ay)$ for $y \geq 0$, then the limit distribution of $X(t)$ is gamma with location parameter μ and shape parameter λ/a. This is a remarkable result in that a gamma distribution with arbitrary shape parameter appears as a limit distribution in a physical model (For background and further discussion on the structure of shot-noise distributions, see Bondesson[6].)

Various enhancements are possible, including alternative choices for the attenuation or recovery function h and modeling the pattern of shot occurrences by a nonhomogeneous Poisson process, a renewal process, a semi-Markov process, or a cluster point process. Implications of the shot-noise formulations will be explored in a future paper.

References

[1] Abramowitz, M. and Stegun, I.A., *Handbook of Mathematical Functions*, Dover, New York, 1970.

[2] Ash, R.B. and Gardner, M.F., *Topics in Stochastic Processes*, Academic Press, New York, 1975.

[3] Barndorff-Nielsen, O., Blaesid, P., and Halgreen, C., "First Hitting Time Models for the Generalized Inverse Gaussian Distribution," *Stochastic Processes and their Applications 7*, pp. 49-54, 1978.

[4] Bergman, B., "Optimal Replacement under a General Failure Model," *Advances in Applied Probability 10*, pp. 431-451, 1978.

[5] Birnbaum, Z.W. and Saunders, S.C., "A New Family of Life Distributions," *Journal of Applied Probability, 6*, pp. 319-327, 1969.

[6] Bondesson, L., "On Simulation from Infinitely Divisible Distributions," *Advances in Applied Probability 14*, pp. 855-869, 1982.

[7] Cox, D.R. and Isham, V., *Point Processes*, Chapman and Hall, London, 1980.

[8] Folks J.L. and Chhikara, R.S., "The Inverse Gaussian Distribution and its Statistical Application - A Review," *Journal of the Royal Statistical Society, Series B 40*, pp. 263-289, 1978.

[9] Gompertz, B., "On The Nature of the Function Expressive of the Law of Human Mortality," *Philosophical Transactions, Royal Society of London*, 1825.

[10] Gumbel, E.J., *Statistics of Extremes*, Columbia University Press, New York, 1958.

[11] Jorgensen, B., *Statistical Properties of the Generalized Inverse Gaussian Distribution*, Lecture Notes in Statistics No. 9, Springer-Verlag, Berlin-New York, 1982.

[12] Kao, J.H.K., "Statistical Models in Mechanical Reliability," *Proceedings of the Eleventh National Symposium on Reliability and Quality Control*, pp. 240-247, 1965.

[13] Karlin, S. and Taylor, H.M., *A First Course in Stochastic Processes*, 2nd Ed., Academic Press, New York, 1975.

[14] Karlin, S. and Taylor, H.M., *A Second Course in Stochastic Processes*, Academic Press, New York, 1981.

[15] Wenocur, M.L., "Brownian Motion with Quadratic Killing and Some Implications," 1984. Submitted for Publication.
[16] Mann, N.R., Schafer, R.E. and Singpurwalla, N.D., *Methods for Statistical Analysis of Reliability and Life Data*, Wiley, New York, 1974.
[17] Rice, J., "On Generalized Shot Noise," *Advances in Applied Probability 9*, pp.553-565, 1977.

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