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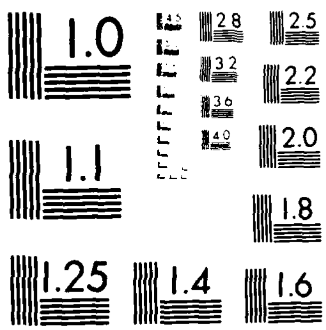

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The Eigenstructure Assignment of Deadbeat Control Systems *

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Abstract

All the available freedom in selecting the closed-loop Jordan block structure associated with deadbeat controllers is described and the parameters associated with this freedom are characterized. It is shown that in general one has freedom in selecting the Jordan block structure as well as the eigenvectors of deadbeat controllers. Although in general the feedback matrix is a nonlinear function of the eigenvectors that are assigned it is shown that for one important Jordan block structure the deadbeat controller feedback matrix is a linear function of the parameters of the system. The feedback matrix of minimum norm is then calculated for this special case.

1. Introduction

In this paper we will examine the problem of deadbeat control. This problem involves the return to the origin of an arbitrary initial state x_o of the linear discrete time system

$$x_{n+1} = Ax_n + B u_n \tag{1}$$

in as few steps as possible. It has been shown in [1,2] that the solution is achieved with linear, time-invariant, state feedback and the resulting closed-loop matrix is nilpotent. It was suggested in [1] that one possible structure of the closed-loop system has m Jordan blocks of dimensions $\{\mu_1, \ldots, \mu_m\}$, the controllability indices, and all subsequent work in this area has taken this to be an inviolable fact. It is shown here that when the controllability indexes are not all identical there is considerably more freedom in the selection of the closed-loop Jordan block structure for the deadbeat control problem, beyond merely selecting the closed-loop eigenvectors. The results of [5,6] are applied to the analysis of the relationship between the feedback matrix that produces deadbeat control and the possible closed-loop eigenvectors. In general there is a nonlinear relationship between the feedback matrix and the parameters associated with the assignable eigenvectors. However, it is shown that when the dimensions of the Jordan blocks are selected to be the controllability indexes, the feedback matrix is a linear function of the parameters describing the freedom in selecting the closed-loop eigenvectors. Some applications are discussed and an example is presented to illustrate the results.

2. Background and Notation

The notation will follow that of [3] and [4]. Specifically, for the linear map M, we denote the image of the subspace spanned by the columns of M as Im(M), the dimension of Im(M) by dim(Im(M)), the nullspace by ker(M) and the Frobenius norm of M as

$$\|M\|_F = \left(\sum_{i,j} m^2_{i,j} \right)^{1/2}$$

The space of polynomials with coefficients in the field R^m is denoted by $P^m[\lambda]$ and the set of integers $(1,2,\ldots, k)$ by $\underline{k}$

The discrete time system is modelled by (1) with $A \in R^{m,m}$, $B \in R^{n,n}$ and the pair (A,B) is assumed controllable. The controllability indexes will be assumed to be ordered so that

$$\mu_1 \geq \mu_2 \geq \quad \ldots \geq \mu_m$$

The associated free generators for ker $[A- \lambda I,B]$ given by $z_i(\lambda) \in P^{n+m}[\lambda]$. are of degree μ_i

where

$$z_i(\lambda) = \begin{bmatrix} s_i(\lambda) \\ t_i(\lambda) \end{bmatrix}$$

$s_i(\lambda) \in P^n[\lambda]$, $t_i(\lambda) \in P^m[\lambda]$

and $\deg[s_i(\lambda)] = \mu_i - 1$

All results will be given in terms of this set (arbitrary) of $[z_i, i\ \underline{m}]$. If

$$[A-\lambda I, B]\, z_i(\lambda) = 0$$

then it follows from the results of [1,2] that one solution to the deadbeat control problem satisfies

$$FV=W \tag{2}$$

where

$$\left.\begin{aligned} &V=[V_1,\dots,V_m],\ W=[W_1,\dots,W_m] \\ &V_i=[v_{i,1},\dots,v_{i,\mu_i}],\ W_i=[w_{i,1},\dots,w_{i,\mu_i}] \\ &v_{i,j}=\frac{d^{(j-1)}s_i(\lambda)}{d\lambda^{j-1}}\bigg|_{\lambda=0} \quad w_{i,j}=\frac{d^{j-1}t_i(\lambda)}{d\lambda^{j-1}}\bigg|_{\lambda=0} \end{aligned}\right\} \tag{3}$$

The closed-loop system (A+BF) then satisfies

$$(A+BF)\,v_{i,j}=v_{i,j-1}, \quad v_{i,0}=0$$

One can "link" these eigenvector chains together to form longer chains. For example, if $w_{i,1}$, were replaced in (4) by $w_{i,1}+w_{j,\mu_j+1}$ then (A+BF) satisfies

$$(A+BF)\,v_{i,1}=v_{j,\mu_j}$$

i.e. the eigenvector chains of length μ_i and μ_j have been "linked" to form a chain of length $\mu_i+\mu_j$. This corresponds to the construction of the polynomial

$$\bar{z}(\lambda)=z_j(\lambda)+\lambda^{\mu_j}z_i(\lambda)$$

to generate a controllability subspace of dimension $\mu_i+\mu_j$ [7,lemma 2] The key observation to note is that eigenvector chains can be "linked" by adding linear combinations of the columns of the w_{i,μ_i+1} to the columns of W in (2). This idea will be developed further.

3. Discussion

3.1 Eigenstructure Constraints of Deadbeat Control

The key observations to understanding deadbeat control are that the closed-loop system matrix must be nilpotent and the longest closed-loop eigenvector chains must be of length at most . The first observation is well known [1,2]. However, in [1,2] and most subsequent work on deadbeat control, it has been assumed that the eigenvector chains must have the lengths

as determined by the controllability indexes, the degrees of the $z_i(\lambda)$.
Now it is clear that the number of steps required to achieve a deadbeat response is determined by the length of the longest eigenvector chain (or chains) and none other. Thus there is in fact no reason to impose any special structural constraints on the eigenvector chains other than the longest chain or chains be of length μ_1. From the discussions in [1,4] it follows that this length constraint represents a min-max relationship; the smallest possible length of the longest eigenvector chain of a deadbeat controlled system is given by μ_1.

The other constraint that one need be concerned about is that the closed-loop set of generalized eigenvectors be linearly independent. This merely involves using a linearly independent combination. of the columns of V in (2,3). In terms of the set $\{s_i(\lambda), i \in \underline{m}\}$, a linearly independent combination of the coefficients must be used to determine the feedback matrix F. This mathematical constraint is fairly simple to comply with.

In summary, a deadbeat controller must comply with two major constraints. The feedback matrix must of course make the closed-loop system nilpotent but it must also

(1) assign generalized eigenvector chains of length at most

(2) assign a set of linearly independent generalized eigenvectors

Any choice of eigenstructure that complies with these two requirements is in fact acceptable for deadbeat control. The set of all deadbeat controllers can then be characterized by examining all possible eigenstructures and the class of all feedback matrices that assign them, using the results of [5,6,7]. One can thus select the lengths of the eigenvector chains as well as the eigenvectors comprising the chains. We also note that the structural information about the polynomial set $\{z_i, i \in \underline{m}\}$ in [4,10] is useful in understanding the available freedon in selecting deadbeat controllers.

3.2 Parameterization of Deadbeat Controllers

The selection of a deadbeat controller involves both the choice of eigenstructure or eigenvector chain lengths as well as a choice of the generalized eigenvectors themselves. Let us first examine the freedon associated with the selection of the eigenvectors for the simplest case where the lengths of the chains are the same as the controllability indexes. In this case, it was shown in [8] that the set of all controllers that assigns chains of these canonical lengths can almost always be described through

$$S = nm - \sum_{i=1}^{m} (2i-1)\mu_i$$

parameters. An alternate proof of this result is included in the statement of the following results:

Proposition 1

Given [$z_i(\lambda)$, $i \in \underline{m}$] a set of free generators of Ker $[A-\lambda I, B]$ then

1) any other set of free generators [$\bar{z}_i(\lambda)$, $i \in \underline{m}$] can be uniquely written as

$$\bar{z}_i(\lambda) = \sum_{j:\mu_j \le \mu_i} \alpha_{ij}(\lambda) z_j(\lambda)$$

where

$$\alpha_{ij} \in P[\lambda]$$

$$\deg(\alpha_{ij}) \le \mu_i - \mu_j$$

2) There are S free parameters associated with the coefficients of the $\alpha_{ij}(\lambda)$ that parameterize the choice for F in [2]

3) The coefficients of each of the $\bar{s}_i(\lambda)$ can be assigned to the closed-loop system as an eigenvector chain with eigenvalue 0. The entire set of eigenvector chains results in an eigenstructure that produces deadbeat response.

Proof:

1) This result is stated for completeness and is found in [4, Prop. 1].

2) This result is based on the observation that if the length of the eigenvector chains is given by the controllability indexes, then the feedback matrix that assigns the coefficients of the $\bar{s}_i(\lambda)$ as eigenvector chains is invariant for all α if $\bar{z}_i$ is changed to

$$\bar{z}_i \rightarrow \bar{z}_i + \alpha \lambda^k z_j$$

where

$$k = \deg(\bar{z}_i) - \deg(z_j)$$

This result is shown in the appendix. Thus, the total number of coefficients of the $\alpha_{ij}(\lambda)$ that will affect the feedback matrix is given by

$$\sum_{i=1}^{m} \sum_{j=i}^{m} (\mu_i - \mu_j) = \sum_{i=1}^{m} (m+1-i)\mu_i - i\mu_i$$

$$= mn - \sum_{i=1}^{m} (2i-1)\mu_i$$

$$= S$$

3) The first part of this result follows from [7, Prop. 1]. Since the chains have lengths given by the controllability indexes, the closed-loop system has a valid deadbeat control eigenstructure. □

Note that the freedom in selecting the coefficients of the α_{ij} is directly related to the freedom in selecting the generalized eigenvectors once the chain lengths have been specified.

One is of course not restricted to eigenvector chains of lengths given by the μ_i. Consider a general polynomial

$$\hat{z}_i(\lambda) = \sum_{i=1}^{m} \alpha_{ij}(\lambda) z_j(\lambda) \quad , \quad \alpha_{ij} \in P[\lambda]$$

where none of the coefficients of $\hat{s}_i(\lambda)$ are zero. It was shown that the space spanned by the coefficients of $\hat{s}_i$ is in fact a controllability subspace and one can select the dimension of this space or the degree of according to the results of [4, Thm 1]. These coefficients can also be assigned as a closed-loop eigenvector chain which thus spans a controllability subspace. But, in the selection of deadbeat controllers, one need not assign only eigenvector chains that span controllability subspaces. In general, for deadbeat control, one can assign eigenvector chains of virtually any length from 1 to μ_i provided the two constraints described earlier are met. This is summarized by the following result:

Proposition 2:

Given z_i and z_j, $\mu_1 \geq \mu_i \geq \mu_j$ the lengths of the eigenvector chains that can be formed to be compatible with deadbeat control is given by

$$\{(q,p), (q+1, p-1), \ldots, (\mu_j, \mu_i)\}$$

where

$$p = \min(\mu_i + \mu_j, 1)$$
$$q = \mu_i + \mu_j - p$$

Furthermore, all the chains except those of lengths (μ_i, μ_j) can be formed in two distinct configurations.

Proof:

Let

$$\hat{z}_1 = z_i + \alpha \lambda^k z_j \qquad \mu_i - \mu_j \leq k \leq p - \mu_j$$

Then, from the previous discussion the coefficients of $\hat{s}_1$ form an eigenvector chain of length $k + \mu_j$. To be compatible with constraint 2 one must have a total of $\mu_i + \mu_j$ generalized eigenvectors generated from the coefficients of s_i and s_j. Thus a complementary chain of length $\mu_i - k$ must be assigned to the closed loop system using the first $\mu_i - k$ coefficients of $s_j(\lambda)$. The feedback matrix that asssigns these two chains satisfies

$$F[V_i, V_j] = [W_i, W_j + \frac{1}{\alpha} w_{i,\mu_i+1} e^T(\mu_i - k + 1)]$$

where $e(\ell)$ is a vector of appropriate length with zeros everywhere except the ℓth element which is 1.

The two distinct configurations result from the formation of either $\hat{z}_1$, or

$$\hat{z}_2 = z_j + \alpha \lambda^k z_i$$

These two configurations result from linking the chains in different orders. The maximum chain length is a result of constraint 1. □

The two configurations in Prop. 2 are distinct in that each configuration has a parameter not referred to in Prop.1. These parameters are associated with the selection of the chain lengths.

One can form eigenvector chains in a more general way than indicated in Prop. 2. Given d, the desired length of an eigenvector chain, one can form

$$\hat{z}_i = \sum_{j: \mu_j \le d} \alpha_{ij}(\lambda) z_j(\lambda) \qquad \alpha_{ij} \in P[\lambda] \qquad (4)$$

and assign the related generalized eigenvectors. This can be done provided that none of the coefficients of $\hat{S}_i$ are 0 which is assured if for some subset U of the controllability indexes one has [4]

$$\max\{\mu_i : \mu_i \in U\} \le d \le \sum_{\mu_i \in U} \mu_i$$

and constraint 2 is met for the entire set of assigned generalized eigenvectors. This latter requirement might necessitate assigning one or more eigenvector chains of length less than μ_m. Therefore, the allowable lengths of eigenvector chains compatible with deadbeat control are given by the following result:

Proposition 3:

Given the controllability indexes $\{\mu_i, i \in \underline{m}\}$

the allowable lengths of eigenvector chains compatible with deadbeat control are:

$$\begin{array}{l} 1, 2, \ldots, \mu_m \\ \mu_{m-1}, \mu_{m-1}+1, \ldots, \min(\mu_1, \mu_m + \mu_{m-1}) \\ \mu_{m-2}, \ldots \qquad\qquad , \min(\mu_1, \mu_m + \mu_{m-1} + \mu_{m-2}) \\ \text{etc.} \end{array}$$

Proof:

Follows directly from the previous discussion and [4, Thm, 1] or [3, Thm, 5.1]. □

Note especially that even the number of eigenvector chains can be adjusted within a range. Because there are at most m polynomials that span ker [$A-\lambda I$, B] the maximum number of chains is m. One can of course always construct one eigenvector chain of length n provided the system is controllable but for deadbeat control the smallest number of chains possible is given by k+1 where

$$n = k\mu_1 + j \quad , \quad j < m$$

and k and j are integers

The total number of free parameters is a function of the number and dimension of the Jordan blocks or eigenvector chains. A naive calculaton can be performed given the chain lengths $\{d_i , i \in \underline{k}\}$ to show that the number of free parameters is

$$N = \sum_{i \in \underline{k}} \left[\left(\sum_{j : \mu_j \le d_i} (d_i - \mu_j + 1) \right) - 1 \right]$$

The term ($d_i - \mu_j + 1$) represents the total number of coefficients of α_j while the -1 takes into account the reduncancy associated with multiplying each polynomial by a nonzero scale factor. This scale factor clearly has no affect on the calculation of the feedback matrix. A discussion of the number of redundant parameters will be deferred to a later date.

3.3 Minimum Norm Deadbeat Control

Consider now the problem of minimizing the Frobenius norm of the deadbeat controller. The problem is of course dependent on the Jordan block structure that is selected. The feedback matrix that produces deadbeat control can be determined from the following:

Proposition 4:

Let V, W and be defined as in section (2) and let J be defined as

$$J = \begin{bmatrix} J_1 & 0 & \cdots & 0 \\ 0 & J_2 & \ddots & \vdots \\ \vdots & \ddots & \ddots & 0 \\ 0 & \cdots & 0 & J_m \end{bmatrix}$$

with each J_i having 1's on the first super diagonal and zeros everywhere else and also

$$\dim(J_1) \geq \dim(J_2) \geq \cdots \geq \dim(J_m)$$

If the dimensions of the Jordan blocks correspond to a set of dimensions consistent with a deadbeat control eigenstructure then the feedback matrix that realizes the eigenstructure satisfies

$$\left.\begin{aligned} &FVT = WT + \bar{W}TJ \\ &\bar{W} = [\bar{W}_1, \ldots, \bar{W}_m] \\ &\bar{W}_i = [0, 0, \ldots, 0, w_{i,\mu_i+1}] \leftarrow \mu_i \times \mu_i \end{aligned}\right\} 5$$

or

$$F = WV^{-1} + \bar{W}TJT^{-1}V^{-1}$$

where T is a nonsingular matrix whose entries are determined from the coefficients of the α_{ij} in (4).

Proof:

Straightforward but tedious algebra. It is important to note that T is not an arbitrary matrix but has a specific structure. This can be seen when the polynomial relationships are translated to the matrix form of (5). Since T relates $\{\bar{z}_i\}$ to $\{z_i\}$ it must be invertible to ensure that the generalized eigenvectors are linearly independent. □

The feedback matrix is in general a complex function of the coefficients of the α_{ij} . However, when the dimensions of the Jordan blocks are chosen to be the controllability indexes the relationship simplifies and the minimum norm solution can be found explicitly as shown by the following.

Theorem 1

Assume that the dimensions of the Jordan blocks in (5) are given by the controllability indexes. Then the feedback matrix is a linear function of the parameters describing the freedom and can be written as

$$F V = W + \bar{W} T$$

where T is a matrix of parameters. The feedback matrix of minimum Frobenius norm is achieved for

$$t = -[u_1^* \otimes \bar{W} , \ldots , u_n^* \otimes \bar{W}]^+ f_o$$

where:

t= vector formed from the columns of T

fo= vector formed from the columns of F

u*= conjugate transpose of the ith row of V^{-1}

Proof:

The first part can be shown in a recursive manner by noting that for μ_m , one always has

$$F V_{\mu_m} = W_{\mu_m} \qquad (6)$$

For $\mu_j > \mu_m$ if

$$\bar{V}_j = V_j + V_{\mu_m} T_1$$

then

$$F (V_j + V_{\mu_m} T_1) = W_j + W_{\mu_m} T_1 + \bar{W}_j T_1 J_j$$

and by using (6) one has $F \bar{V}_j = W_j + \bar{W}_j T_1 J_j$

where $\bar{W}_j$ and $\bar{J}_j$ are the appropriate blocks from $\bar{W}$ and J in (5). A similar approach can be used to show the more general case.

The second part follows from the results of [7,Prop.3] $\square$

4. Example

The system matrices from [11] were

$$A=\begin{bmatrix} 1 & 1 & 0 & 1 & 0 \\ 0 & 0 & 1 & 0 & 0 \\ 0 & -1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 \\ 0 & 1 & 0 & 0 & 1 \end{bmatrix} \qquad B=\begin{bmatrix} 0 & 1 & 0 \\ 0 & 0 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 1 \\ 0 & 0 & 1 \end{bmatrix}$$

The controllability indexes were found to be [3,1,1] and so the only possible chain lengths compatible with deadbeat control are [3,1,1] or [3,2]. The matrix

$$F=\begin{bmatrix} 0 & 0 & -1 & 1 & -1 \\ -1 & -1 & -1 & 0 & -1 \\ 0 & -1/3 & 0 & -2/3 & -1/3 \end{bmatrix}$$

$$\|F\|_F^2 = 6\ 2/3$$

is the minimum norm feedback matrix that assigns the eigenvector chain lengths [3,1,1] But, the feedback matrix

$$F=\begin{bmatrix} 0 & 0 & -1 & 1 & -1 \\ -1 & -1.5 & 0 & 0 & .5 \\ 0 & -.25 & -.25 & -.75 & -.25 \end{bmatrix}$$

$\|F\|_F^2 = 5\ 1/4$

assigns an eigenstructure of chain lengths [3,2] and has smaller norm.

5. Conclusions

The restrictions on the eigenstructure of systems with deadbeat response were described. These observations were then used to describe all the allowable freedom one has in selecting the eigenstructure of such a system. The freedom in selecting the eigenvectors was than described in terms of the allowable eigenstructures. Finally, it was shown that the feedback matrix that assigns the eigenstructure that has Jordan blocks of dimensions given by the controllability indexes is a linear function of the available parameters. An explicit analytic expression was then derived for the feedback matrix of minimum Frobenius norm that assigns this canonical eigenstructure.

References

[1] V. Kucera, "The Structure and properties of time optimal discrete linear control," IEEE Trans. Automat. Contr, Vol. AC-16 pp. 375-77, 1971.

[2] B. Leden, 'Multivariable deadbeat control,' Automatica, Vol 13, pp 185-188, 1977.

[3] W.M. Wonham, Linear Multivarable Control: A Geometric Approach, New York, Springer-Verlag, 1979

[4] M.E. Warren and A.E. Eckberg, Jr., 'On the dimensions of controllability subpaces: A characterization via polynomial matrices and Kronecker invariants, SIAM J. Contr., Vol 13, pp. 434-455, Feb. 1975

[5] B.C. Moore, 'On the flexibility offered by state feedback in multivariable systems beyond closed loop eigenvalue assignment,' IEEE Trans. Automat. Contr., Vol. AC-21, pp 689-692, ot. 1976

[6] G. Klein, and B.C. Moore,' Eigenvalue-generalized eigenvector assignment with state feedback', IEEE Trans. Automat. Contr., Vol. AC-22 pp 140-141, Feb. 1972

[7] G. Klein, 'On the relationship between controllability indexes, eigenvector assignment and deadbeat control', IEEE Trans. Automat. Contr., Vol. AC-29, pp. 77-79, Jan. 1984

[8] M. Schlegel, 'Parameterization of the class of deadbeat controllers,' IEEE Trans. Automat. Contr., Vol AC-27, pp 727-729, June 1982

[9] O.A. Sebakhy, 'A note on parameterization of the class of deadbeat controllers,' IEEE Trans. Automat. Contr., Vol. AC-28, pp. 1124-1127, Dec. 1983

[10] V.M. Popov, 'Invariant Description of Linear, Time Invariant Controllable Systems', SIAM J. Contr., Vol 10, pp32-44, 1972.

[11] O.A. Sebakhy and T.M. Abdel-Moneim, 'State Regulation in Linear Discrete-Time Systems in Minimum Time', IEEE Trans. Aut. Contr., Vol. AC-24, pp84-87, Feb. 1979.

Appendix

Lemma

Given $[z_i, i \in \underline{m}]$ and $[\bar{z}_i, i \in \underline{m} : \deg(\bar{z}_i) = \mu_i]$

with

$$\bar{z}_i = \sum_{j : \mu_j \le \mu_i} \alpha_{ij}(\lambda) z_j \qquad \text{(A1)}$$

then the feedback matrix that assigns the coefficients of $[\bar{s}_i, i \in \underline{m}]$ as closed-loop eigenvector chains remains invariant if

$$\bar{z}_i \rightarrow \bar{z}_i + \alpha \lambda^k z_j$$

for

$$\deg(z_j) \le \deg(\bar{z}_i)$$
$$k = \deg(\bar{z}_i) - \deg(z_j)$$

Furthermore, the feedback matrix that assigns the $\bar{s}_i$ satisfies

$$FV = W + \bar{W}$$

where $\bar{W}$ is a matrix whose columns are linear combinations of w_{i,μ_i}

Proof:

Consider the sequence of feedback matrices that assign the coefficients of the polynomioals in (A1) as the new polynomials $\bar{z}_i$ are introduced to replace the z_i. First, consider the set $[\bar{z}_1, \ldots, \bar{z}_k, z_{k+1}, \ldots, z_m]$ where

$$\deg(\bar{z}_i) = \deg(z_i) = \mu_i \qquad i \in \underline{k}$$

The feedback matrix associated with this set satisfies

$$FV_i = W_i \qquad i \in [k+1, \ldots, m] \qquad (A2)$$

$$F\bar{V}_i = \bar{W}_i \qquad i \in \underline{k} \qquad (A3)$$

where the $\bar{V}_i$ and $\bar{W}_i$ are the coefficients of the $\bar{s}_i$ and $\bar{t}_i$. Let us assume that $\bar{z}_i$ in (A1) can be written as

$$\bar{z}_i = (*) + \alpha \lambda^k z_j$$

where (*) represents all the other terms not involving $\lambda^k z_j$ and $k \leq \mu_i - \mu_j$. The coefficient matrix $\bar{V}_i$ can then be written as

$$\bar{V}_i = V^* + \alpha [0, \ldots, 0, V_j, 0, \ldots, 0] \qquad (A4)$$

where the zeros represent the shift produced by λ^k. Now a similar relationship holds for $\bar{W}_i$ with one important exception. If $k < \mu_i - \mu_j$ then $\bar{W}_i$ involves w_{j,μ_j} and if $k = \mu_i - \mu_j$ then $\bar{W}_i$ does not involve w_{j,μ_j}. One can now simplify (A3) by using the appropriate expression form (A2) and (A4) to show that

$$\begin{aligned} F\bar{V}_i &= F[V^* + \alpha[0, \ldots, 0, V_j, 0, \ldots 0]] \\ &= FV^* \\ &= F[W^* + \alpha[0, \ldots, 0, w_{j,\mu_j}, 0, \ldots, 0]] \end{aligned} \qquad (A5)$$

The term involving w_{j,μ_j} is present only if $k = \mu_i - \mu_j$ Therefore, if k satisfies

$$k = \deg(\bar{z}_i) - \deg(z_j)$$

then the feedback matrix is invariant for any value of α. It is important to emphasize that this resulting expression (A5) does not involve either V_j or W_j. This approach can of course be repeated to eliminate all the references to

$$[V_i, W_i, i = k+1, \ldots, m]$$

in (A2). Note however that there can still be terms involving the $[w_{j,\mu_j}, j = k+1, \ldots m]$ in the equations that define F. Thus the V* and W* only involve linear combinations of $[V_i, W_i, i \in \underline{k}]$ in (A5). Now the set $[\bar{V}_i, i \in \underline{k}]$ must incorporate a linearly independent combination of the $[V_i, i \in \underline{k}]$ if the entire set of eigenvectors is to span the whole space.

This means that the equations in (A5) can be written as

$$F[V_1, \ldots, V_k]\,\Gamma \otimes I = [W_1, \ldots, W_k]\,\Gamma \otimes I + X \tag{A6}$$

where X is a matrix involving the $[w_{j,\mu_j}, j = k+1, \ldots, m]$, $\otimes$ indicates the Kroenicker matrix product and

$$\bar{V}_i = \sum_{j=1}^{k} \gamma_{ij} V_j$$

$$\bar{W}_i = \sum \gamma_{ij} W_j + i\text{th block of } X$$

$$\Gamma = [\gamma_{ij}]$$

Since the $[V_j, j \in \underline{k}]$ and $[\bar{V}_j, j \in \underline{k}]$ must both be linearly independent sets, the matrix Γ must be invertible and so (A6) can be rewritten as

$$F[V_1, \ldots, V_k] = [W_1, \ldots, W_k] + X(\Gamma^{-1} \otimes I) \tag{A7}$$

Now the process just described can be repeated on the set

$$[\bar{z}_{k+1}, \ldots, \bar{z}_\ell, z_{\ell+1}, \ldots, z_m]$$

where

$$\deg(\bar{z}_i) = \mu_\ell < \mu_1 \qquad i = k+1, \ldots, \ell$$

Since none of the $\bar{z}_i$ involves the polynomials of degree μ_1, the feedback matrix that assigns these cofficients as eigenvector chains can also satisfy (A7). This process can be repeated for all the distinct μ_i. Finally we note that the equations relating to the polynomials of degree μ_m satisfy

$$F V_j = W_j$$

since none of these can involve any w_{i,μ_i}.

As a final point, we emphasize that the assumption that the $\bar{z}_i$ have degrees given by the μ_i is crucial. If a polynomial

$$\bar{z} = z_i + \lambda^{\mu_i} z_j \tag{A8}$$

is defined then the terms involving V_j in polynomials of degree less than $\mu_i + \mu_j$ are no longer "redundant" and cannot be eliminated by the previously described process, even if a polynomial of degree less than $\mu_i + \mu_j$ is of the form

$$\bar{Z} = (*) + \alpha \lambda^k Z_j$$

with

$$k = deg(Z) - deg(Z_j)$$

This is due to the fact that the inclusion of (A8) eliminates the equation

$$F V_j = W_j$$

from (A2) and is no longer required to define F □

Premultiplying (19) by $N_1 + 1$ block matrices of dimension $(nh \times nh)$, successively in order will result in (10).

$$\left[\begin{array}{ccc|ccc} I_n & & & L_0 & & \\ & I_n & O & L_1 & L_0 & O \\ O & & & & & \\ & & I_n & L_{N_1-1} & & L_0 \\ \hline & O & & & L_{h-N_1,M_1}(N_1) & \end{array}\right] \left[\begin{array}{c} Y_0 \\ Y_1 \\ \vdots \\ Y_{N_1-1} \\ \hline X_0 \\ X_1 \\ \vdots \\ X_{M_1-1} \end{array}\right] = \left[\begin{array}{c} P_0 \\ P_1 \\ \vdots \\ P_{h-1} \end{array}\right] \tag{20}$$

The block matrix equation (20) is composed of (15) and (18). Therefore, a solution of (15) and (18) is also a solution of (20).

From condition (5) it follows that if the degree of some rows of $B_1(s)$ are less than $M_1 - 1$, then some rows in Z_{M_1} (16) must vanish identically. This implies that the corresponding columns of $L_{h-N_1,M_1}(N_1)$ may be omitted resulting in $\bar{L}$ and $\bar{Z}$. Furthermore, denoting the matrix composed of the independent rows of $\bar{L}$ by $\tilde{L}$ and the matrix composed of the same rows of P by $\tilde{P}$, (15) can be rewritten as

$$\tilde{\bar{L}}_{h-N_1,M_1}(N_1) Z_{M_1} = \tilde{P}_{h-N_1}(N_1). \tag{21}$$

From the uniqueness of $\{X_1(s), Y_1(s)\}$ it follows that the matrix $\tilde{L}_{h-N_1,M_1}(N_1)$ must be square and nonsingular.

In order to get the unique solution $\{X_1(s), Y_1(s)\}$, we increase, at each stage, the degree of $Y_1(s)$ by one, starting with $\deg Y = \deg A - 1$, and thus examine the existence of a solution (using consistency rank conditions) and continue this process until condition (17), with $\deg Y_1 = N_1 - 1$, is satisfied. The solution $\{X_1(s), Y_1(s)\}$ is then obtained by solving (21) for Z_{M_1}, i.e., for $X_1(s)$, and finally $Y_1(s)$ is given by (18).

IV. Example

$$A(s) = \begin{bmatrix} 1+s+s^2 & s \\ s^2 & 2 \end{bmatrix}; \quad B(s) = \begin{bmatrix} 1+s & s^2 \\ s & 1+s \end{bmatrix}$$

$$C(s) = \begin{bmatrix} 1+2s+s^2 & s \\ s & 1+s^2 \end{bmatrix}$$

From (9) and (16), we get

$$L_0 = \begin{bmatrix} 1 & 0 \\ 0 & 2 \end{bmatrix}; \quad L_1 = \begin{bmatrix} 0 & 1 \\ -1 & -2 \end{bmatrix}; \quad L_2 = \begin{bmatrix} 1 & 3 \\ 2 & 1 \end{bmatrix}; \quad L_3 = \begin{bmatrix} 0 & 5 \\ -3 & 2 \end{bmatrix}$$

$$L_{2,2}(2) = \begin{bmatrix} 1 & 3 & 0 & 1 \\ 2 & 1 & 1 & -2 \\ 0 & 5 & 1 & -3 \\ 3 & 2 & 2 & 1 \end{bmatrix}; \quad \bar{L}_{2,2}(2) = \begin{bmatrix} 2 & 13 & 5 & 9 \\ 1 & 10 & -4 & 7 \\ 1 & 1 & 0 & 1 \\ 2 & 17 & -7 & 12 \end{bmatrix}.$$

From (11), we get

$$P_0 = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}; \quad P_1 = \begin{bmatrix} 1 & 1 \\ 1 & -1 \end{bmatrix}; \quad P_2 = \begin{bmatrix} 0 & -2 \\ -1 & 1 \end{bmatrix}; \quad P_3 = \begin{bmatrix} 0 & 3 \\ 1 & 1 \end{bmatrix}.$$

The unique solution obtained by solving (16) for $X_1(s)$ and (23) for $Y_1(s)$ is given by

$$X_1(s) = \begin{bmatrix} -4 & 3 \\ -3-5s & 3+4s \end{bmatrix}; \quad Y_1(s) = \begin{bmatrix} 5+4s & -3-2s \\ 6 & -5 \end{bmatrix}$$

since

$$\operatorname{rank} \begin{bmatrix} X_1(s) \\ Y_1(s) \end{bmatrix} = 2$$

for all s, therefore $X_1(s)$ and $Y_1(s)$ are right coprime.

V. Discussion and Conclusions

An algorithm for solving a matrix polynomial equation has been presented. This algorithm, besides being intuitively simple, has the important advantage of requiring operations on constant matrices rather than polynomial matrices.

It should be noted that this algorithm can be applied, as well, to the solution of

$$A_1(s)X_1(s) + A_2(s)X_2(s) + B(s)Y(s) = C(s)$$

by rewriting it in the form

$$\left[A_1(s) A_2(s)\right] \begin{bmatrix} X_1(s) \\ X_2(s) \end{bmatrix} + B(s)Y(s) = C(s)$$

where $C(s)$ is not necessarily a square matrix, or to the solution of

$$B(s)Y(s) = C(s).$$

References

[1] H. H. Rosenbrock, *State Space and Multivariable Theory*. London: Nelson, 1970.
[2] W. A. Wolovich, *Linear Multivariable Systems*. New York: Springer-Verlag, 1974.
[3] H. H. Rosenbrock and G. E. Hayton, "The general problem of pole assignment," *Int. J. Contr.*, vol. 27, pp. 837–852, 1978.
[4] J. A. Feinstein, "Polynomials and polynomial equations in control theory," Ph.D. dissertation, School of Engineering, Tel-Aviv University, Tel-Aviv, Israel, 1979.
[5] C. A. Desoer, Ruey-Wen Liu, J. Murray, and R. Seaks, "Feedback system design: The fractional representation approach to analysis and synthesis," *IEEE Trans. Automat. Contr.*, vol. AC-25, pp. 399–412, 1980.
[6] D. C. Youla, J. J. Bongiorno, and H. A. Jabr, "Modern Wiener-Hopf design of optimal controllers—Part II: The multivariable case," *IEEE Trans. Automat. Contr.*, vol. AC-21, pp. 319–338, 1976.
[7] V. Kucera, "Stochastic multivariable control: A polynomial equation approach," *IEEE Trans. Automat. Contr.*, vol. AC-25, pp. 913–919, 1980.
[8] T. Kailath, *Linear Systems*. Englewood Cliffs, NJ: Prentice-Hall, 1980.
[9] Y. Shamash, "Construction of the inverse of linear time-invariant multivariable systems," *Int. J. Syst. Sci.*, vol. 6, pp. 733–740, 1976.
[10] C. C. Macduffee, *The Theory of Matrices*. New York: Chelsea, 1956.

On the Relationship Between Controllability Indexes, Eigenvector Assignment, and Deadbeat Control

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***Abstract*—The subspaces, to which closed-loop generalized eigenvectors are restricted, are described in terms of the controllability indexes of the pair (A, B) and the polynomials of minimal degree that span $\ker[A - \lambda I, B]$. This characterization of the eigenspaces is then used to calculate the deadbeat controller of minimum Frobenius norm.**

Introduction

The freedom afforded by state feedback beyond pole placement was described in [3], [12] as that of assigning generalized eigenvectors from specific subspaces. This characterization has been used [5], [15], [16] to design state feedback controllers with desirable properties. In this note, the available freedom in selecting eigenvector chains is examined and clarified to facilitate the design of such controllers. An algebraic relationship between the subspaces from which successive elements of eigenvector chains must be selected is developed in terms of the controllability

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indexes and the set of polynomials of minimal degree that span ker[$A - \lambda I, B$]. This interrelationship is then used to express the freedom in selecting the eigenvector chains. These results are then used to provide an analytic expression for the deadbeat controller of minimum Frobenius norm that will return an arbitrary initial state to the origin in the least number of steps.

Notation

The notation will follow that of [1] and [7]. Specifically, for the linear map M, we denote the image of the subspace spanned by the columns of M as Im(M), the dimension of Im(M) by dim(Im(M)), the nullspace by Ker(M), the Moore–Penrose inverse of M as M^+, and the Frobenius norm of M as

$$\|M\|_F = \left(\sum_{i,j} m_{ij}^2\right)^{1/2}.$$

The space of polynomials with coefficients in the field R^n is denoted by $P^n[\cdot]$ and the set of integers $\{1,2,\cdots,k\}$ by k. Finally, [expression] will denote the logical value of "expression."

Discussion

Let the controllability indexes be ordered as

$$r_1 \geq r_2 \geq \cdots \geq r_m$$

with the associated free generators for ker[$A - \sigma I, B$] given by $z_i(\sigma) \in P^{n+m}[\sigma]$ of degree r_i where

$$z_i(\sigma) = \begin{bmatrix} x_i(\sigma) \\ y_i(\sigma) \end{bmatrix}, \quad x_i(\sigma) \in P^n[\sigma], \quad y_i(\sigma) \in P^m[\sigma]$$

and $r_{m+1} = n$. Also let

$$z_i^{(k)} = \frac{d^k z_i}{d\sigma^k}, \quad (z_i^{(0)} = z_i)$$

$$Z(\sigma) = [z_1(\sigma), \cdots, z_m(\sigma)]$$

and define $X(\sigma)$ and $Y(\sigma)$ in a similar way. The subspaces from which the eigenvector chains can be selected [3] are characterized by the following chain of interrelationship.

Proposition 1: For ($k = 1, 2, \cdots$), the general solution to

$$[A - \lambda I, B] X = S^{(k-1)}(\lambda) \tag{1}$$

is

$$X(\lambda) = Z^{(k)} + Z\alpha(\lambda), \quad \alpha(\lambda) \in P^{n+m}[\lambda].$$

Proof: With $k = 1$, it is a matter of substitution to show that a solution of (1) is $X(\lambda) = Z^{(1)}$ and by induction on k, one solution of (1) can be shown to be $X(\lambda) = Z^{(k)}$. The most general solution [2] is then the sum of any particular solution plus a linear combination of the columns of $Z(\lambda)$, the matrix whose columns span ker[$A - \lambda I, B$].

The subspaces to which the generalized eigenvectors are restricted [3], [12] and the freedom in selecting the eigenvector chains associated with the z_i is described by the following.

Proposition 2: Let (A, B) be controllable, $\lambda \in \mathbf{C}$ be a specified closed-loop eigenvalue, and v^i the generalized eigenvector of grade i associated with λ. Then

i) rank[$X(\lambda)$] = m

ii) for $k = 1, 2, 3, \cdots$, rank[$S^{(k)}(\lambda)$] = $\Sigma_{i=1}^m [r_i - 1 \geq k]$

iii) for $k = 0, 1, \cdots$, rank[$Z^{(k)}(\lambda)$] = $\Sigma_{i=1}^m [r_i \geq k]$

iv) for $v^1 = S^{(0)}(\lambda) h$ the feedback matrix that assigns it satisfies

$$F v^1 = Y^{(0)}(h + q)$$

where $q \in$ ker[$S^{(0)}(\lambda)$], $r \geq 1$

v) for v^i as in iv),

$$v^{i+1} = S^{(i)} h + S [illegible] q [illegible]$$

and is assigned by F [illegible] h.

Proof: Result i) (see [1], [4]) is stated for completeness. [illegible] ii) and iii) follow directly from Proposition 1, and [illegible] from Proposition 1 and the results of [3], [12].

We note that the addition of arbitrary elements from [illegible] does *not* change the freedom in selecting eigenvector [illegible] solution to

$$F[X(\lambda)c_1, (X'(\lambda) + X(\lambda))c [illegible]] = [Y(\lambda)c_1, (Y^{(1)}(\lambda) + [illegible])]$$

is the same as the solution to

$$F[X(\lambda)(c_1 [illegible]), X'(\lambda)c [illegible]] = [Y(\lambda)(c_1 + c) [illegible]]$$

The closed-loop eigenvectors assigned in both cases [illegible] $X^{(1)}(\lambda)c$]. The freedom demonstrated in the example [illegible] fact the freedom indicated in Proposition 2 iv), v). Eigenvectors of [illegible] length greater than r_i (provided $r_i < n$) and eigenvectors not [illegible] $S^{(i)}(\lambda)$ can be assigned by taking linear combinations of the [illegible] $P(\lambda)$.

Given the $z_i(\sigma)$ (calculated as in [14], for example) the [illegible] selecting eigenvector chains can be used to minimize the Frobenius [illegible] of the feedback matrix that assigns eigenvector chains of length [illegible] m. The minimum norm deadbeat controller that returns an [illegible] initial state to the origin in at most r_m steps can be calculated [illegible] following results.

Proposition 3: Let R_i be the matrix whose columns span

$$\ker[S^{(i)}(0)] \cap \ker[S^{(i-1)}(0)] \cap \cdots \cap \ker[S(0)]$$

and let V and W be matrices of proper generalized eigenvector [illegible] assignment vectors for the zero eigenvalue, respectively. [illegible] feedback matrices that assigns the columns of V as [illegible] eigenvectors satisfies

$$FV = W + [R_1 c_1, R_2 c_2, \cdots, R_n c_n]$$

where the c_i are arbitrary vectors of appropriate dimension. [illegible] the minimum norm feedback matrix that satisfies (2) [illegible]

$$\begin{bmatrix} c_1 \\ \vdots \\ c_n \end{bmatrix} = [u_1^* \otimes R_1, \cdots, u_n^* \otimes R_n]^+ f$$

where u_i^* is the conjugate transpose of the ith row of [illegible] column sequenced vector structure of the elements of [illegible] refers to the Kronecker product.

Proof: The freedom in selecting the assignment vectors [illegible] by Proposition 2 iv) is used to derive (2). This equation [illegible] of *all* deadbeat controllers that return an arbitrary initial state to the origin in r_m steps with chain of length [illegible]. The solution is then given by

$$F(c_1, c_2, \cdots, c_n) = WV^{-1} + [R_1 c_1 \cdots] [illegible]$$

The minimum norm solution is then found by rewriting [illegible] (3) as

$$[u_1^* \otimes R_1, \cdots, u_n^* \otimes R_n]\begin{bmatrix} c_1 \\ \vdots \\ c_n \end{bmatrix} = [illegible]$$

and applying the results of [2, p. 120].

When all the r_i are identical (the generic case when n is an integer multiple of m), the eigenstructure is fixed since ker[$X(0)$] [illegible] $r_i = r_m$.

Example: The system matrices for the deadbeat control example [illegible] are

$$A = \begin{bmatrix} 1 & 1 & 0 & 1 & 0 \\ 0 & 0 & 1 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 \\ 0 & 1 & 0 & 0 & 1 \end{bmatrix} \quad B = \begin{bmatrix} 0 & 1 & 0 \\ 0 & 0 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 1 \\ 0 & 0 & 1 \end{bmatrix}$$

[illegible] of the $z_i(\lambda)$ is found to be

$$z_1(\lambda) = (1,0,0,0,0,0,[illegible]1,0)^* + \lambda(0,[illegible],0,0,0,0,1,0)^*$$

$$z_2(\lambda) = (0,0,0,1,1,0,[illegible]1,[illegible]1)^* + \lambda(0,0,0,0,0,[illegible],0,1)^*$$

$$z_3(\lambda) = (2,[illegible]1,0,0,1,1,[illegible]1,0)^* + \lambda(1,1,[illegible]1,0,0,1,0,0)^* + \lambda^2(0,0,1,0,0,[illegible]1,1,0)^* + \lambda^3(0,0,0,0,0,1,0,0)^*$$

[illegible] eigenvectors are chosen with chain lengths of 1, 1, and 3, then the [illegible] matrix and assignment matrix are

$$[illegible] = \begin{bmatrix} 1 & 2 & 0 & 1 & 0 \\ [illegible] & 1 & 0 & 1 & 0 \\ [illegible] & 0 & 0 & 1 & 1 \\ 0 & [illegible] & 1 & 0 & 0 \\ 0 & 1 & 1 & 0 & 0 \end{bmatrix} \qquad W = \begin{bmatrix} 0 & 1 & 0 & 1 & 1 \\ 1 & 1 & 1 & 0 & 1 \\ 0 & 0 & 1 & 0 & 0 \end{bmatrix}$$

The results of Proposition 2 indicate that the elements $W_{[illegible]}, W_{[illegible]}, W_{[illegible]}$, [illegible] are freely selectable. The minimum norm feedback matrix as given in Proposition 3 for eigenvector chains of length 1, 1, and 3 is then found to be

$$F = \begin{bmatrix} 0 & 0 & 1 & 1 & 1 \\ 1 & 1 & 0 & 1 & 0 \\ 0 & 1/3 & 0 & 2/3 & 1/3 \end{bmatrix} \qquad \|F\|_F = 6.2[illegible]3$$

The minimum norm feedback matrix for eigenvector chains of length 2 and 3 is

$$F = \begin{bmatrix} 0 & 0 & 1 & 1 & 1 \\ 1 & 0.5 & 0 & 0 & 0.5 \\ 0 & 0.25 & 0.25 & 0.75 & 0.25 \end{bmatrix} \qquad \|F\|_F = 5.1[illegible]4$$

Conclusions

The freedom in assigning eigenvector chains and the relationship between the subspaces from which they are selected was characterized in terms of the free generators of $\ker[A - \lambda I, B]$ and the controllability indices of (A, B). These results were then used to characterize the [illegible] controller of minimum norm.

References

[1] [illegible] Warren and A. E. Eckberg, Jr., "On the dimensions of controllability subspaces: A characterization via polynomial matrices and Kronecker invariants," *SIAM* [illegible], vol. 13, pp. 434–445, [illegible] 1975.
[2] [illegible] and [illegible] Ben-Israel, *Generalized Inverses: Theory and Applications*. New York: Wiley, 1974.
[3] [illegible] Moore, "On the flexibility offered by state feedback in multivariable systems beyond closed loop eigenvalue assignment," *IEEE Trans. Automat. Contr.*, vol. AC-[illegible], pp. [illegible], Oct. 1976.
[4] [illegible] Brockett, *State Space and Multivariable Theory*. London: Nelson-Wiley, [illegible].
[5] [illegible] and [illegible] Abdel-Moneim, "State regulation in linear discrete [illegible] in minimum time," *IEEE Trans. Automat. Contr.*, vol. AC-24, pp. 84–8[illegible], [illegible].
[6] [illegible] H. Golub and J. H. Wilkinson, "Ill-conditioned eigensystems and the computation of the Jordan canonical form," *SIAM Rev.*, vol. 18, pp. [illegible], 1976.
[7] [illegible] M. Wonham, *Linear Multivariable Control: A Geometric Approach*. New York: Springer-Verlag, 1979.
[8] [illegible] R. Gantmacher, *The Theory of Matrices*, vol. I, II. New York: Chelsea, 1960.
[9] [illegible] and [illegible] Laub, "The singular value decomposition: Its computation [illegible]," *IEEE Trans. Automat. Contr.*, vol. AC-25, pp. 164–17[illegible], Apr. 1980.
[10] [illegible] "The [illegible] structure and properties of [illegible] control," *IEEE Trans. Automat. Contr.*, vol. AC-2[illegible], pp. [illegible], [illegible] 19[illegible].
[11] [illegible] "[illegible] control," [illegible], pp. [illegible], [illegible].
[12] [illegible] C. Kl[illegible] and [illegible] C. Moore, "[illegible]," *IEEE Trans. Automat. Contr.*, vol. AC-[illegible], pp. [illegible], Feb. [illegible].
[13] G. A. [illegible] and [illegible] K[illegible], "Exact [illegible] state feedback [illegible]," in *Proc. IEEE Conf. Dec.*, pp. [illegible], 197[illegible].
[14] [illegible] W. W[illegible], *Linear Multivariable Systems*. New York: [illegible].
[15] [illegible] Kl[illegible] and B. C. Moore, "[illegible] response [illegible]," in *Proc. [illegible] IEEE CDC*, [illegible].
[16] [illegible] Kl[illegible], "[illegible]," [illegible].

Decoupling by Restricted Static-State Feedback: The General Case

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Abstract—In this paper we tackle the general block decoupling problem, for linear constant dynamical systems (C, A, B) with m inputs and p outputs, with restricted static-state feedback, in other words with control laws of type $u = GFx + Gv$. We give a necessary and sufficient condition of existence for such laws which generalizes the one previously given in [2] for the simple case $k = p = m$, where k denotes the number of blocks to be decoupled.

I. Introduction

During the last two decades considerable interest has been shown in the decoupling problem of linear constant dynamical systems with static-state feedback. However, the related theory is still not complete: we only know necessary and sufficient conditions of existence for some particular cases [1]. In fact, we can conclude the existence of solutions like $u = Fx + Gv$ only in the case when the input transformation G has to be an isomorphism. If the latter constraint on G is dropped, when this type of solution does not exist we cannot in general conclude that the decoupling problem does not admit any solution.

Here, we shall consider a special class of control laws which fulfill $\mathrm{Im}\,F \subset \mathrm{Im}\,G$, G being not necessarily an isomorphism. The main advantage of this type of control law is that it leads to a necessary and sufficient condition of existence (Theorem 3.1). Nevertheless, it remains true that their existence is not necessary for the initial problem. In any case, it has to be noted that, for the latter, Theorem 3.1 provides a [illegible] sufficient condition of existence than the previous ones, related to G isomorphism, in the sense that G is now only constrained to be a monomorphism.

Initially, the idea of restricted static-state feedback was [illegible] Kamiyama and Furuta [2]. They have given a result dealing with the simplest case of decoupling, i.e., $k = p = m$, where k denotes the number of blocks to be decoupled. They have used a classical matrix approach which does not make possible an easy extension, if any, of their results to more general situations of decoupling. Here, we solve the problem in the most general case. The approach is purely geometric. Its main advantage is to provide a very compact result (Theorem 3.1).

II. Notation and Problem Setting

We consider the linear constant dynamical system (C, A, B) described by

$$\begin{cases} \dot{x} = Ax + Bu \\ y = Cx \end{cases}$$

where $x \in \mathscr{X} = \mathbf{R}^n$, $u \in \mathscr{U} = \mathbf{R}^m$ and $y \in \mathscr{Y} = \mathbf{R}^p$, where [illegible]

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Robust Noninteracting Controller Design*

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Introduction

The design of a linear state feedback noninteracting (decoupling) controller for the system

$$\dot{x} = Ax + Bu \qquad (1a)$$
$$y_i = C_i x \qquad i \in \underline{k} \ (\underline{k} = 1, 2, \ldots, k) \quad (1b)$$

requires finding the matrices F and G_i, $i \in \underline{k}$, so that for the closed loop system

$$\dot{x} = (A+BF)x + \sum_{i=1}^{k} B\, G_i\, u_i$$

input u_i only affects output y_i. This problem was solved for the case when there are an equal number of inputs and outputs (dim (B) = k) in [1,2], while the more general problem for a system with more inputs than outputs (dim (B) > k) was solved in a theoretic sense in [3,4]. Concerning the practical application of decoupling controllers, it was shown in [5] that a solution to the decoupling problem is in general structurally unstable. Thus, arbitrarily small perturbations to the system in (1) will cause loop interactions in (2) resulting in output y_i to be affected by input u_j when $i \neq j$.

This note addresses this problem by attempting to use the available design freedom to reduce the magnitude of these loop interactions. When the number of inputs equals the number of outputs, the only available design freedom is in selecting some of the closed-loop poles. When there are more inputs than outputs, it was shown in [6] that there is a limited freedom in selecting the closed-loop eigenstructure, i.e. both closed-loop poles and their eigenvectors, of the decoupled system.

Notation and Preliminaries

For the system (1) under consideration, it is assumed that the pair (A,B) is controllable. It will also be assumed that the basic definitions and properties of (A,B) i.s. and (A,B) c.s. are known. The notation will follow that used in [4]:

capital letters	-	denotes matrices
script letters	-	denotes subspaces
d (X)	-	dimension of the subspace
k	-	1, 2, . . . , k
(A,B) i.s.	-	(A,B) invariant subspace
(A,B) c.s.	-	(A,B) controllability subspace
sup C(A,B,K)	-	the largest (A,B) c.s. in K
ker (C)	-	kernel of matrix C

We will also assume that for the given system, the decoupling problem is generically solvable [4, pg. 178] and satisfies the appropriate constraints. It will also be assumed, unless otherwise stated, that rank (CB) = min (m,p), thus ensuring that the system will have a full set of transmission zeros (henceforth referred to as (t.z.'s). We will also define the following quantities

$$V_i^* = \sup I(A,B,K)$$
$$K_i = \bigcap_{j \neq i} \ker (C_j)$$
$$R_i^* = \sup C(A,B,K_i)$$

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$\hat{C}_i$ ... [illegible]

A family of (A,B) i.s. V_i, $i \in \hat{k}$ ... [illegible] compatible if there exists ... [illegible]

$$\ldots \quad V_i \ldots$$

As ... [illegible] ... this ... [illegible] that (A,B) i.s. ... [illegible] ... [illegible] each ... [illegible] the closed ... [illegible]

Discussion

The approach used ... [illegible] controllers in ... (A,B) i.s. ... [illegible] are spanned by closed-loop eigenvectors ... [illegible] tion of ... [illegible] decoupling ... [illegible] thus be posed in terms of selecting ... [illegible] eigenstructure, especially ... [illegible] actions. The following ... [illegible] proof, will be used ... [illegible] design.

Prop. 1 [7]

Let ... $\ker$ ... [illegible]

Then the set of eigenvectors in ker ... [illegible] with the closed-loop poles ... [illegible] feedback matrix that assigns them ... [illegible]

$$F\, V\, k = Wk$$

for some nonzero vector k. □

This result characterizes the freedom in ... [illegible] closed-loop modes that are in the kernels of specific output maps i.e. modes that will not ... [illegible] outputs associated with C_i. The interaction ... [illegible] these modes is discussed in the following ... [illegible]

Prop. 2 [6]

A family of (A,B) i.s., V_i, $i \in \hat{k}$ ... [illegible]

$$V_i \cap \Big(\sum_{j\neq i} V_j\Big) = \sum_{j \neq i} \ldots \cap \ldots \quad i \in \hat{k}$$

and

$$V_i \cap V_j \in I(A,B) \quad i \neq j \qquad \square$$

This result characterizes the compatibility of closed-loop modes that produce a decoupled ... [illegible] This result indicates that any overlap ... [illegible] noninteracting subsets (i.e.(A,B) i.s. ... [illegible] c.s.) must be spanned by modes common ... [illegible] subsets.

This approach to decoupling leads to ... [illegible] interpretation of when and why decoupling is ... [illegible] ble. Let the integer n_i be defined by

$$n_i = \min \left\{ k \mid \hat{C}_i A^{k-1} B \neq 0,\ k = 1, 2, \ldots \right\}$$

Prop. 3

Let (C,A,B) be a square invertible system ... [illegible]

$$V_i^* = \sup (A,B, \ker C_i)$$
$$= \bigcap_{j=0} \ker \hat{C}_i A^{j-1}$$

then the system can be decoupled ... [illegible] iff rank $[B \cap V_1^*, \ldots, B \cap V_m^*]$ = ... [illegible]

Proof:

The result follows from [8] and [9]. (Space limitations prohibit a detailed explanation)

Prop. 3 shows that a system can be decoupled ... [illegible] there are a sufficient number of appropriate ... [illegible] to control each block of the decoupled system. This result also shows that if a system (with m ... [illegible]) ... decoupled, it can be decoupled by state ... feedback. (A similar result is being developed ... [illegible] more general case m>p).

Dynamic compensators allow a greater freedom in the selection of the closed loop eigenstructure. In the generic case for [illegible], the eigenvector and assignment vector must satisfy

$$\begin{bmatrix} A-\lambda I & B \\ \hat{C}_i & 0 \end{bmatrix} \begin{bmatrix} v \\ w \end{bmatrix} = 0$$

and since $\dim^{\perp}(B) = 1+\dim(\hat{C}_i)$ it is clear that there is no freedom in selecting the eigenvectors. The addition of a single dynamic compensator results in

$$\ker \left[\begin{array}{cc|cc} A-\lambda I & 0 & B & 0 \\ 0 & -\lambda & 0 & 1 \\ \hline C_i & 0 & 0 & 0 \end{array}\right] = \operatorname{Im} \begin{bmatrix} V & 0 \\ 0 & 1 \\ W & 0 \\ 0 & \lambda \end{bmatrix}$$

and the closed loop eigenvector can be written as

$$V(\alpha) = \begin{bmatrix} V \\ 0 \end{bmatrix} + \alpha \begin{bmatrix} 0 \\ 1 \end{bmatrix}, \quad \alpha \in R$$

Any nonzero value of α causes the dynamics to be incorporated into the system. This freedom in choosing the α's for each eigenvector affords the selection of a less skewed set of closed-loop eigenvectors. The dynamic compensation can also be used to filter noisy control signals from imperfect sensors. Algorithms to select a "good" decoupling eigenstructure have been developed from the results of [illegible]. There is an additional freedom available in selecting the G_i's as a result of the dynamic compensation. This freedom determines the level of interaction of the augmented dynamics with the input/output [illegible] of the system. It appears that this [illegible] freedom can be used to reduce the steady-state decoupling sensitivity to perturbations.

Example

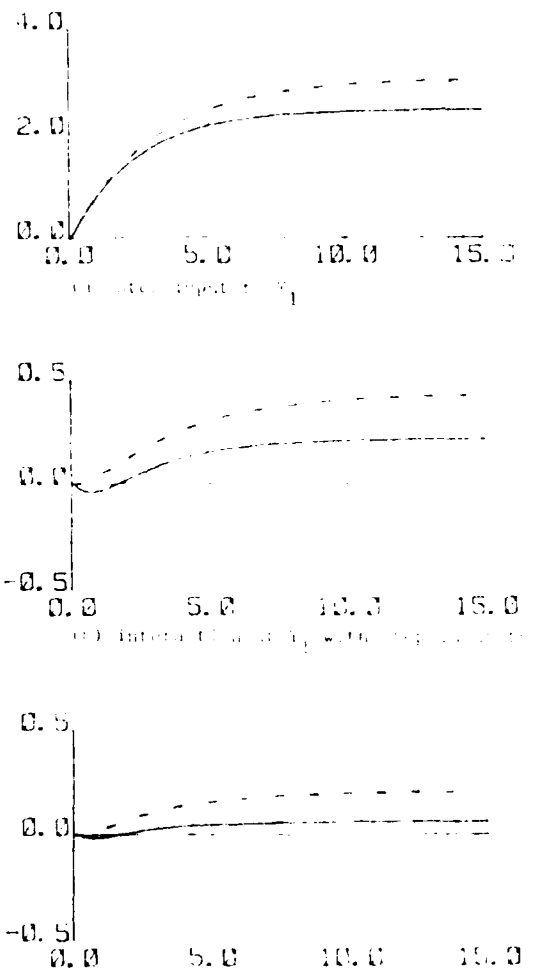
The heuristics outlined above were used to decouple the longitudinal control systems of an aircraft model [11]. This system has no transmission [illegible] but can be decoupled using static feedback. To increase the design freedom, an extra mode was [illegible] to the dynamics of Y_1 and Y_2. The results of [illegible] perturbation experiments are presented in the figures. The original decoupling controller [illegible] in [11] was found to be extremely robust but some improvements are clearly evident.

Conclusions

The eigenstructure constraints necessary to achieve decoupling were examined. Heuristic guidelines were outlined to make use of dynamic compensators in the design of insensitive decoupling controllers.

References

1. E.G. Gilbert, 'The decoupling of multivariable systems by state feedback,' SIAM J. of Cont. 7(1), 1969, pp. 50-53.
2. E.G. Gilbert, J.R. Pivnichny, 'A computer program for the synthesis of decoupled multivariable feedback systems,' IEEE Trans. Aut. Cont., 14(6), 1969, pp. 652-59.
3. [illegible] Morse, W.M. Wonham, 'Decoupling and pole assignment by dynamic compensation,' SIAM J. of Cont., 8(3), 1970, pp. 317-37.
4. W.M. Wonham, Linear Multivariable Control: A Geometric Approach, Springer-Verlag, New York, 1974.
5. J.S.C. Yuan, 'Structural instability of a class of decoupling solutions,' IEEE Trans. Aut. Cont., 22(5), 1977, pp. 843-46.
6. G. Klein, 'An eigenstructure approach to noninteracting control synthesis,' C.D.C., 1981, pp. 688-89.
7. B.C. Moore, 'On the flexibility offered by state feedback in multivariable systems,' IEEE Trans. Aut. Cont., 21(5), 1976, pp. 689-92.
8. [illegible], 'On the structure at infinity of [illegible] are decoupled systems,' IEEE Trans. Aut. Cont., 27(4), 1982, pp. 97[illegible]-74.
9. P.L. Falb, W.A. Wolovich, 'Decoupling in the design and synthesis of multivariable control systems,' IEEE Trans. Aut. Cont., 12(6), 1967, pp. 651-59.
10. B.C. Moore, G. Klein, 'Eigenstructure selection in the linear regulator problem: combining modal and optimal control,' C.D.C., 1976, pp.
11. H.A. Hamer, K.G. Johnson, 'Effect of errors on decoupled control systems,' NASA Tech. Paper 1184, July 1978.



(a) [illegible] step input [illegible] Y_1

(b) interaction of Y_2 with [illegible]

(c) interaction of Y_1 with [illegible]

Fig. 1 The element $B_{[illegible]}$ [illegible]

Only Y_1 produced [illegible] interactions.
dashed line - unaugmented [illegible]
solid line - augmented [illegible]

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