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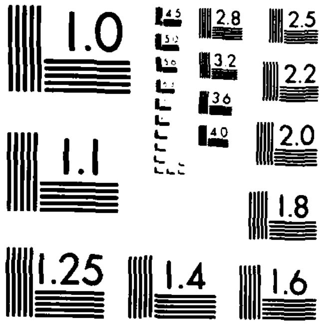

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A NEW METHOD OF APPROXIMATING NORMAL AND t-TAIL AREAS

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A NEW METHOD OF APPROXIMATING NORMAL AND t-TAIL AREAS

Andrew P. Soms*

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ABSTRACT

The bounds of Boyd (1959) and Soms (1980, 1984) for the tail areas of the normal and t-distributions are used to obtain a new method of approximating the tail areas. The absolute and relative errors are given.

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Key Words: absolute error, normal percentile, normal tail area, relative error, t-percentile, t-tail area

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SIGNIFICANCE AND EXPLANATION

A reasonably accurate and simple method of evaluating normal and t-tail areas and percentiles is described. This approach controls both absolute and relative errors, unlike the usual methods which only control the absolute error. The control of both errors is of importance in calculating Bonferroni percentiles and Bonferroni descriptive probability levels, which are useful in simultaneous inference. For example, a nonspecialist in statistics, unaware of the limitations of standard routines, might report a Bonferroni descriptive probability level that had a relative error of 100% or more.

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A NEW METHOD OF APPROXIMATING NORMAL AND t-TAIL AREAS

Andrew P. Soms*

1. THE METHOD

We begin by introducing notation and stating the main results of Boyd (1959) and Soms (1980, 1984). Boyd (1959) showed that if

$$\phi(x) = (2\pi)^{-1/2}\exp(-x^2/2), \quad \bar{F}(x) = \int_x^\infty \phi(t)dt ,$$

and $R_x = \bar{F}(x)/\phi(x)$, $x > 0$, then

$$p(x,\gamma_{min}) < R_x < p(x,\gamma_{max}) ,$$

where $p(x,\gamma) = (\gamma + 1)/[(x^2 + (2/\pi)(\gamma + 1)^2)^{1/2} + \gamma x]$, $\gamma_{max} = 2/(\pi - 2)$, $\gamma_{min} = \pi - 1$, and the bounds are the best possible in the class $\{p(x,\gamma), \gamma > -1\}$. This is also discussed in Johnson and Kotz (1970, Ch. 33). Soms (1980), 1984) extended the above results and showed that if for arbitrary real $k > 0$ and $x > 0$,

$$f_k(t) = c_k(1 + t^2/k)^{-(k+1)/2}, \quad c_k = \frac{\Gamma((k + 1)/2)}{\Gamma(k/2)(\pi k)^{1/2}} ,$$

$$\bar{F}_k(x) = 1 - F_k(x) = \int_x^\infty f_k(t)dt ,$$

$$R_k(x) = \bar{F}_k(x)/[(1 + x^2/k)f_k(x)] ,$$

for $k > 2$, $\gamma_{max} = 4c_k^2/(1 - 4c_k^2)$ and $\gamma_{min} = \frac{k}{2(k + 2)c_k^2} - 1$,

and for $k < 2$, γ_{min} and γ_{max} are interchanged, and

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$$p(x,\gamma) = \frac{1+\gamma}{(x^2 + 4c_k^2(1+\gamma)^2)^{1/2} + \gamma x} ,$$

then

$$p(x,\gamma_{min}) < R_k(x) < p(x,\gamma_{max}) ,$$

or equivalently,

$$\left(1 + \frac{x^2}{k}\right) f_k(x) P(x,\gamma_{min}) < \bar{F}_k(x) < \left(1 + \frac{x^2}{k}\right) f_k(x) p(x,\gamma_{max}) ,$$

and the bounds again are best in the same sense as for the normal. It was also shown there that if $k = 2$, $\gamma_{max} = \gamma_{min} = \gamma_2$ and $R_k(x) = p(x,\gamma_2)$.

The numerical properties of these bounds are discussed in the above references. The important fact to be noted here is that the bounds control both absolute and relative error. Using the bounds as a starting point we now develop a simple method of evaluating normal and t-tail areas that controls both absolute and relative error, as opposed to the usual methods, which generally only control absolute error.

We consider estimates of the tail area of the form

$$\left(\frac{a+bx}{c+dx}\right) p(x,\gamma_{min}) \phi(x) + \left(1 - \frac{a+bx}{c+dx}\right) p(x,\gamma_{max}) \phi(x) \qquad (1.1)$$

for the tail area of the normal and

$$\left(\frac{a+bx}{c+dx}\right) p(x,\gamma_{min}) g_k(x) + \left(1 - \frac{a+bx}{c+dx}\right) p(x,\gamma_{max}) g_k(x) \qquad (1.2)$$

for the tail area of the t, where $g_k(x) = (1 + x^2/k) f_k(x)$. We want the estimates to lie between the upper and lower bounds for the tail area and be strictly decreasing functions of x and therefore impose the added restrictions that

$$bc > ad$$

and

$$0 < \frac{a + bx}{c + dx} < 1 ,$$

all $x > 0$. Since $f(0) = a/c$, we may, without loss of generality, assume that $c = 1$ and so our weight functions f are of the type

$$f(x) = \frac{a + bx}{1 + dx} , \tag{1.3}$$

where $0 < a < 1$, $d > 0$, $b > ad$ and $b/d < 1$. We then seek that particular choice of f which minimizes the absolute error. A direct computer search led to

$$f(x) = \frac{.71x}{1 + .71x} \tag{1.4}$$

for the normal and

$$f(x) = \frac{b_k x}{1 + b_k x} , \tag{1.5}$$

$$b_k = .71 + 1.8/k - .19/k^2 , \tag{1.6}$$

for the t, where, as noted before, k is the degrees of freedom. (1.6) was obtained by finding the optimal constants for $k = 25, 10, 5, 3, 1.5, 1, .5$ and fitting a constrained (constant equal to .71) regression line. However, in the interests of simplicity, for $k < 2$, we did not interchange γ_{min} and γ_{max} and so (1.5) and (1.6) are understood to apply for all k with γ_{min} and γ_{max} defined as for $k > 2$. Numerical evidence indicates that, at least for $k = 1$, the above optimal estimate is still a decreasing function of x.

The maximum absolute and relative errors of the optimal estimates are remarkably constant over the range $1 < k < \infty$ and hence we only give the normal figures. For (1.4), the maximum absolute error is $.66 \times 10^{-4}$ and the maximum relative error is $.97 \times 10^{-3}$. We emphasize once more, that, unlike the usual methods, which generally control only absolute error and give misleading results for large x or small tail areas, the above controls both absolute and relative error and hence can be used to calculate ordinary and Bonferroni descriptive levels and ordinary and Bonferroni percentiles.

As a check, we calculated the standard textbook table of the normal, given, e.g., in Brown and Hollander (1977) and found at most a difference of 1 in the fourth decimal place. We also compared the small normal percentiles given in Abramowitz and Stegun (1965, p. 977) to the ones obtained from (1.4) and after rounding both to three decimal places found that there was at most a difference of 1 in the third decimal place. Similar results apply to the t.

2. CONCLUDING REMARKS

We have given a method of calculating normal and t-tail areas and percentiles which controls both absolute and relative errors and is simple to implement even on small computers. Four short FORTRAN programs that do this are given in the Appendix. The programs are completely self-contained, using a short FORTRAN subroutine to calculate c_k based on Stirling's formula for large k and an approximation to the gamma function given in Abramowitz and Stegun (1965, p. 257) for small k.

APPENDIX

The listings of the four programs referred to in the paper are given below. If $k < 2$ and α is small, the t-percentile program may be slow.

```
      SUBROUTINE NORT(X,TP)
C NORMAL TAIL AREA FROM X TO INFINITY, X NON-NEGATIVE
      DATA PIE/3.1415927/
      PHI(X)=1./SQRT(2*PIE)*EXP(-X**2/2.)
      P(X,G)  =(1.+G)/((X**2+2./PIE*(1.+G)**2)**.5+G*X)
      PAMIN(X)=.71*X/(1.+.71*X)
      TPL(X,GL,GU)=(PAMIN(X)*P(X,GL)+(1.-PAMIN(X))*P(X,GU))*PHI(X)
      GMIN=PIE-1.
      GMAX=2./(PIE-2.)
      TP=TPL(X,GMIN,GMAX)
      RETURN
      END

      SUBROUTINE NORTI(ALPHA,XALP)
C UPPER NORMAL PERCENTILE XALP CORRESPONDING TO TAIL
C PROBABILITY ALPHA, ALPHA BETWEEN 0 AND .5
      DATA EPS/.000005/
      DATA PIE/3.1415927/
      PHI(X)=1./SQRT(2*PIE)*EXP(-X**2/2.)
      P(X,G)  =(1.+G)/((X**2+2./PIE*(1.+G)**2)**.5+G*X)
      PAMIN(X)=.71*X/*1.+.71*X)
      TPL(X,GL,GU)=(PAMIN(X)*P(X,GL)+(1.-PAMIN(X))*P(X,GU))*PHI(X)
      GMIN=PIE-1.
      GMAX=2./(PIE-2.)
      X=1.
102   TP=TPL(X,GMIN,GMAX)
      IF (TP.LT. ALPHA) GO TO 103
      X=X+1
      GO TO 102
103   BL=X-1.
      BU=X
104   X=(BL+BU)/2.
      TP=TPL(X,GMIN,GMAX)
      IF (TP.LT. ALPHA) BU=X
      IF (TP.GE. ALPHA) BL=X
      IF (ABS(BU-BL).LE. EPS) GO TO 105
      GO TO 104
105   XALP=(BL+BU)/2.
      RETURN
      END
```

```
      SUBROUTINE TT(DF,X,TP)
C T TAIL AREA FROM X TO INFINITY, X NON-NEGATIVE
      P(X,G,D)=(1.+G)/((X**2+4.*CK(D)**2*(1.+G)**2)**.5+G*X)
      FKX(X,D)=CK(D)*(1.+X**2/D)**((-D+1.)/2.)
      GMAX(X)=4.*CK(X)**2/(1.-4.*CK(X)**2)
      GMIN(X)=X/(2.*(X+2.)*CK(X)**2)-1.
      TCON(DF)=.71+1.8/DF+1.9/DF**2
      PAMIN(DF,X)=TCON(DF)*X/(1.+TCON(DF)*X)
      TPL(DF,X,GL,GU)=(PAMIN(DF,X)*P(X,GL,DF)+(1.- PAMIN(DF,X))*
     1P(X,GU,DF))*FKX(X,DF)
      TP=TPL(DF,X,GMIN(DF),GMAX(DF))
      RETURN
      END

      SUBROUTINE TTI(DF,ALPHA,XALP)
C UPPER T PERCENTILE XALP CORRESPONDING TO TAIL PROBABILITY ALPHA,
C ALPHA BETWEEN 0 AND .5
      P(X,G,D)=(1.+G)/((X**2+4.*CK(D)**2*(1.+G)**2)**.5+G*X)
      FKX(X,D)=CK(D)*(1.+X**2/D)**((-D+1.)/2.)
      GMAX(X)=4.*CK(X)**2/(1.-4.*CK(X)**2)
      GMIN(X)=X/(2.*(X+2.)*CK(X)**2)-1.
      TCON(DF)=.71+1.8/DF+1.9/DF**2
      PAMIN(DF,X)=TCON(DF)*X/(1.+TCON(DF)*X)
      TPL(DF,X,GL,GU)=(PAMIN(DF,X)*P(X,GL,DF)+(1.- PAMIN(DF,X))*
     1P(X,GU,DF))*FKX(X,DF)
      DATA EPS/.000005/
      X=1.
102   TP=TPL(DF,X,GMIN(DF),GMAX(DF))
      IF (TP.LT. ALPHA) GO TO 103
      X=X+1.
      GO TO 102
103   BL=X-1.
      BU=X
104   X=(BL+BU)/2.
      TP=TPL(DF,X,GMIN(DF),GMAX(DF))
      IF (TP.LT. ALPHA) BU=X
      IF (TP.GE. ALPHA) BL=X
      IF(ABS(BU-BL).LE. EPS) GO TO 105
      GO TO 104
105   XALP=(BL+BU)/2.
      RETURN
      END

      FUNCTION CK(X)
      ALNG(X)=(X-.5)*ALOG(X)-X+1/(12.*X)-1/(360.*X**3)
      GAMMA(T)=1./T-.57719165+.98820589*T-.89705694*T**2+.91820686*
     1T**3-.75670408*T**4+.48219939*T**5-.19352782*T**6+.03586834*
     2T**7
      DATA PIE/3.1415927/
      Z1=(X+1)/2.
      Z2=X/2.
```

```
      IF (Z2 .GT. 10.) GO TO 11
      TEMP1=1.
201   IF (Z1 .LE. 1.) GO TO 202
      TEMP1=TEMP1*(Z1-1.)
      Z1=Z1-1.
      GO TO 201
202   TEMP1=TEMP1*GAMMA(Z1)
      TEMP2=1.
203   IF (Z2 .LE. 1.) GO TO 204
      TEMP2=TEMP2*(Z2-1.)
      Z2=Z2-1.
      GO TO 203
204   TEMP2=TEMP2*GAMMA(Z2)
      CK=TEMP1/TEMP2/SQRT(PIE*X)
      RETURN
 11   CK=EXP(ALNG((X+1.)/2.)-ALNG(X/2.)-.5*ALOG(PIE*X))
      RETURN
      END
```

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The bounds of Boyd (1959) and Soms (1980, 1984) for the tail areas of the normal and t-distributions are used to obtain a new method of approximating the tail areas. The absolute and relative errors are given.

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