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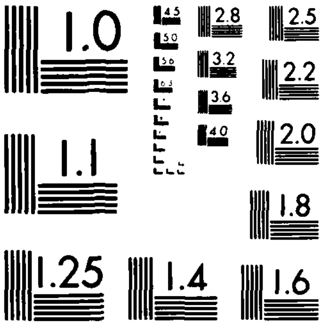
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# INTERNAL VARIABLES IN THE LATTICE FOR CHOLESKY FACTORIZATION, KALMAN FILTERING, AND MODEL IDENTIFICATION

Louis L. Scharf[1]
Cédric Demeure[2]

Department of Electrical Engineering
University of Rhode Island
Kingston, Rhode Island 02881

[1]Electrical and Computer Engineering, University of Colorado, Boulder, CO 80309; formerly with Electrical Department, University of Rhode Island, Kingston, RI 02881.

[2]Electrical Engineering Department, University of Rhode Island, Kingston, RI 02881

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20. ABSTRACT (Continue on reverse side if necessary and identify by block number)

We use the Levinson recursions to derive the LeRoux-Gueguen algorithm for factoring a Toeplitz correlation matrix R, column-by-column. We organize the algorithm into a coupled set of vector recursions and show that it is, indeed, a lattice algorithm. By re-ordering the column variables into rows, we convert the LeRoux-Gueguen algorithm into Friedlander's lattice algorithm for factoring a Toeplitz correlation matrix, row-by-row. This shows the two algorithms to be different ways of looking at a fast

Cholesky factorization that is also called the Shur or Bcrlekamp-Massey algorithm.

We review how fast Cholesky factorizations are usually used to compute reflection coefficients from correlations. Then we show how the factorizations may be run backwards to compute Cholesky factors and correlations from reflection coefficients. This generalizes a result usually attributed to Robinson and Treitel. One of our main purposes is to emphasize that the Levinson recursions for going back and forth between correlations, reflection coefficients, and autoregressive filter parameters may be replaced with a dual set of recursions for going back and forth between correlations, reflection coefficients, and moving average filter parameters.

When the correlation matrix R describes an autoregressive moving average time series, then each column of the Cholesky factor of R consists of a p-dimensional Kalman gain vector in its first p non-zero entries, followed by a homogeneous recursions for subsequent values. This means the fast Cholesky factorization of R may be further speeded up by using these Kalman gains to set initial conditions in a homogeneous recursion. It means also that the Kalman gains obey the same vector recursions as the columns of the Cholesky factor of R. This fast Kalman gain algorithm may be read out of the LeRoux-Gueguen algorithm or out of the Friedlander algorithm.

the Cholesky factor of R. This fast Kalman gain algorithm may be read out of the LeRoux-Gueguen algorithm or out of the Friedlander algorithm.

## 1. Introduction

The Levinson recursions provide an efficient algorithm for factoring the inverse of a Toeplitz correlation matrix R into its upper and lower triangular Cholesky factors. These factors produce a Gram-Schmidt orthogonalization of the underlying time series. The recursions are routinely used to compute reflection coefficients for implementing whitening and predicting filters in lattice form. They are also used to go back and forth between reflection coefficients, order-increasing whiteners, and correlations. In this paper we tell the dual of this story, based on a factorization of the correlation matrix R, itself. One of our main purposes is to show how the Levinson recursion for going back and forth between correlations, reflection coefficients, and order-increasing whiteners may be replaced with a dual set of recursions for going back and forth between correlations, reflection coefficients, and order increasing synthesizers.

We use the Levinson recursions to derive the LeRoux-Gueguen algorithm [1] for factoring a Toeplitz correlation matrix R, column-by-column. We organize the algorithm into a coupled set of vector recursions and show that it is, indeed, a lattice algorithm. By re-ordering the column variables into rows, we convert the LeRoux-Gueguen algorithm into Friedlander's lattice algorithm [2] for factoring a Toeplitz correlation matrix, row-by-row. (Conversely, we could have re-ordered Friedlander's algorithm to obtain the LeRoux-Gueguen algorithm.) This discussion shows the two algorithms to be two different ways of looking at a fast Cholesky factorization that is also called the Shur [3], Berlekamp-Massey [4], or

# Abstract

We use the Levinson recursions to derive the LeRoux-Guegnen algorithm for factoring a Toeplitz correlation matrix R, column-by-column. We organize the algorithm into a coupled set of vector recursions and show that it is, indeed, a lattice algorithm. By re-ordering the column variables into rows, we convert the LeRoux-Guegnen algorithm into Friedlander's lattice algorithm for factoring a Toeplitz correlation matrix, row-by-row. This shows the two algorithms to be different ways of looking at a fast Cholesky factorization that is also called the Shur or Berlekamp-Massey algorithm.

We review how fast Cholesky factorizations are usually used to compute reflection coefficients from correlations. Then we show how the factorizations may be run backwards to compute Cholesky factors and correlations from reflection coefficients. This generalizes a result usually attributed to Robinson and Treitel. One of our main purposes is to emphasize that the Levinson recursions for going back and forth between correlations, reflection coefficients, and autoregressive filter parameters may be replaced with a dual set of recursions for going back and forth between correlations, reflection coefficients, and moving average filter parameters.

When the correlation matrix R describes an autoregressive moving average time series, then each column of the Cholesky factor of R consists of a p-dimensional Kalman gain vector in its first p non-zero entries, followed by a homogeneous recursion for subsequent values. This means the fast Cholesky factorization of R may be further speeded up by using these Kalman gains to set initial conditions in a homogeneous recursion. It means also that the Kalman gains obey the same vector recursions as the columns of

Morf [5] algorithm. When we run the recursion backwards to obtain Cholesky factors and correlations from reflection coefficients we generalize the result of Robinson and Treitel [6] for obtaining correlations from reflection coefficients.

When the correlation matrix R describes an autoregressive moving average (ARMA) time series, then each column of the Cholesky factor for R consists of a p-dimensional Kalman gain vector in its first p non-zero entries, followed by a homogeneous recursion for subsequent values [7]. This means the fast Cholesky factorization of R may be further speeded up by using these Kalman gains, together with ARMA parameters, to set initial conditions and run a homogeneous recursion to generate columns [7]. It means also that the Kalman gains obey the same vector recursions as the columns, themselves [7], [8]. Thus the fast Kalman gains may be read out of the LeRoux-Gueguen algorithm, or out of the Friedlander algorithm, by reading internal variables out of a lattice. The algorithm may be slightly modified to produce the Morf, Sidhu, Kailath algorithm [9] for computing Kalman gains on a fixed length, time-varying lattice [11].

The diagram in Figure 1 summarizes our perspective and our results. The right side of the diagram we take to be well understood: Levinson recursions may be used to factor $R^{-1}$ and to go back and forth between correlations $\{r_n\}$, order increasing whiteners $\{\underline{a}^n\}$, prediction error variances $\{\sigma_n{}^2\}$ and reflection coefficients $\{k_n\}$. The left size of the diagram we take to be less well understood and it is our intention to clarify it and extend known results concerning it. To this end we show how the LeRoux-Gueguen and Friedlander algorithms may be used to factor R and to go back and forth between correlations $\{r_n\}$, order increasing synthesizers $\{\underline{h}^n\}$, prediction error variances $\{\sigma_n{}^2\}$, and reflection coefficients $\{k_n\}$.

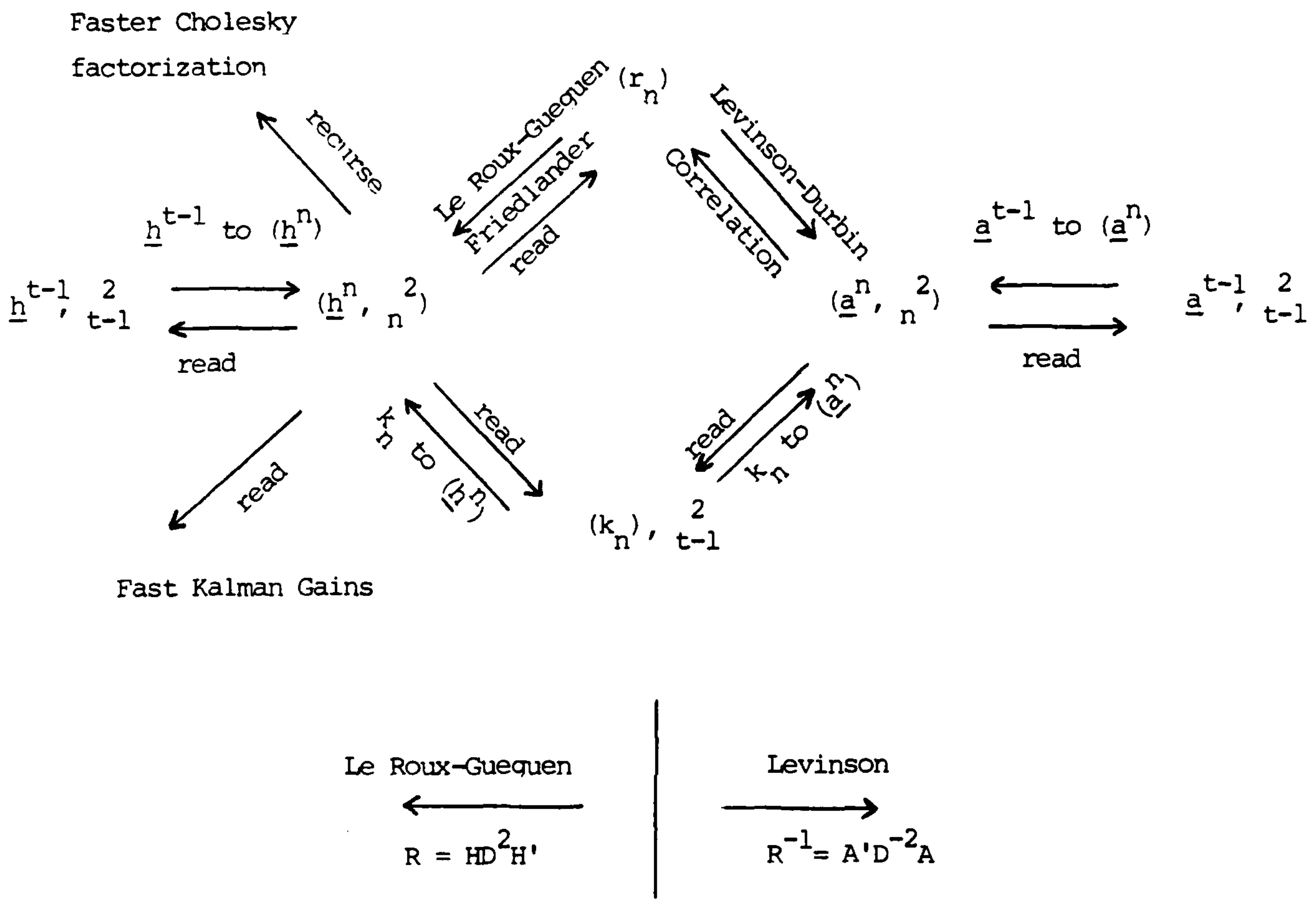

Faster Cholesky factorization
recurse
Le Roux-Gueguen
$(r_n)$
Levinson-Durbin
Friedlander
read
Correlation
$\underline{h}^{t-1}$ to $(\underline{h}^n)$
$\underline{a}^{t-1}$ to $(\underline{a}^n)$
$\underline{h}^{t-1}, {}^2_{t-1}$
$(\underline{h}^n, {}^2_n)$
$(\underline{a}^n, {}^2_n)$
$\underline{a}^{t-1}, {}^2_{t-1}$
read
read
read
$k_n$ to $(\underline{h}^n)$
read
read
$k_n$ to $(\underline{a}^n)$
$(k_n), {}^2_{t-1}$
Fast Kalman Gains
Le Roux-Gueguen
$R = HD^2H'$
Levinson
$R^{-1} = A'D^{-2}A$

Figure 1.

Connections between Algorithms.

These results are used to speed up the Cholesky factorization in the ARMA case and to read the fast Kalman gain algorithm out of the factorization.

## 2. Linear Statistical Models for Stationary Sequences

Let $\underline{x} = (x_o, x_1, \ldots, x_{t-1})'$ denote t real random variables drawn from a wide sense stationary random sequence with mean value sequence zero and correlation sequence $\{r_t\}$. The first two moments of $\underline{x}$ are

$$E\underline{x} = \underline{0}$$

$$E\underline{x}\underline{x}' = R = \{r_{|i-j|}\} = \begin{vmatrix} r_0 & r_1 & \ldots r_{t-1} \\ r_1 & r_0 & r_1 \ldots r_{t-2} \\ \cdot & & \cdot \\ \cdot & & \cdot \\ \cdot & & \cdot \\ r_{t-1} \cdots & r_1 & r_0 \end{vmatrix}$$

Call R a (txt) Toeplitz correlation matrix.

Define the exchange matrix J:

$$J = \begin{vmatrix} & & 1 \\ 0 & 1^{\cdot^{\cdot^{\cdot}}} & \\ 1 & & 0 \end{vmatrix} = J'$$

$$JJ = I$$

This matrix turns column vectors up-side down and row vectors right-side left:

$$J\underline{x} = (x_{t-1}, \ldots, x_o)'$$

$$\underline{x}'J = (x_{t-1}, \ldots, x_o)$$

The vector $\underline{x}$ is drawn from a wide-sense stationary sequence that does not know forward time from reverse time. Therefore $J\underline{x}$ has the same moments as

$\underline{x}$:

$$E J\underline{x} = \underline{0}$$

$$E J\underline{x}(J\underline{x})' = JRJ = R$$

This says R is not only Toeplitz, but also centro- or J-symmetric.

## 2.1 Synthesis

The correlated vector $\underline{x}$ may be synthesized from an uncorrelated vector $D\underline{u}$:

$$\underline{x} = HD\underline{u} \; ; \; E\underline{u}\underline{u}' = I$$

$$R = HD^2H'$$

The diagonal matrix $D^2$ and triangular matrix H are defined as follows:

$$D^2 = \text{diag}\,(\sigma_0^2, \sigma_1^2, \ldots, \sigma_{t-1}^2)$$

$$H = \begin{vmatrix} 1 & & & & \\ h_1^0 & 1 & & & \\ & & & 0 & \\ \vdots & h_0^1 & & & \\ & \vdots & & 1 & \\ h_{t-1}^0 & h_{t-2}^1 & \cdots & h_1^{t-2} & 1 \end{vmatrix} = \begin{vmatrix} \diagdown & & \\ & \ddots & 0 \\ & & \diagdown \\ \underline{h}^0 & \underline{h}^1 \; \cdots & \underline{h}^{t-1} \end{vmatrix}$$

$$\underline{h}^n = (1 \quad h_1^n \; . \; . \; . \; h_{t-n-1}^n)'$$

We call $HD^2H'$ a lower-upper (LU) Cholesky factorization of R and $\underline{x} = HD\underline{u}$ a forward synthesis of $\underline{x}$.

The super and subscripting of elements in H is somewhat arbitrary, but generally, $h_i^j$ tells how the random variable $u_j$ is used to build the random variable $x_{i+j}$.

The vector $J\underline{x}$ may be synthesized as follows:

$$J\underline{x} = JHJJDJJ\underline{u}$$

$$= GCJ\underline{u}$$

$$R = GC^2G'$$

The matrices G and C are defined as follows:

$$C^2 = JDJ = \text{diag}\ (\sigma_{t-1}^2, \ldots, \sigma_1^2, \sigma_0^2)$$

$$G = JHJ = \begin{vmatrix} 1 & h_1^{t-2} & \cdots & h_{t-2}^1 & h_{t-1}^0 \\ & 1 & & & \vdots \\ & & \ddots & h_0^1 & \\ & 0 & & 1 & h_1^0 \\ & & & & 1 \end{vmatrix} = \begin{vmatrix} J\underline{h}^{t-1} & \cdots & J\underline{h}^1 & J\underline{h}^0 \\ & \ddots & & \\ 0 & & \ddots & \\ & & & \ddots \end{vmatrix}$$

We call $GC^2G'$ an upper-lower (UL) Cholesky factorization of R and $J\underline{x} = GCJ\underline{u}$ a backward synthesis of $J\underline{x}$.

## 2.2 Analysis

The random vector $\underline{x}$ may also be analyzed to produce the uncorrelated random variables $D\underline{u}$ in a Gram-Schmidt procedure:

$$A\underline{x} = D\underline{u}$$

$$ARA' = D^2$$

The triangular matrix A' is defined as follows:

$$A' = \begin{vmatrix} 1 & a_1^1 & a_2^2 & \cdots & a_{t-1}^{t-1} \\ & 1 & a_1^2 & & \vdots \\ & & \ddots & & \\ & 0 & & 1 & a_1^{t-1} \\ & & & & 1 \end{vmatrix} = \begin{vmatrix} J\underline{a}^0 & J\underline{a}^1 & \cdots & J\underline{a}^{t-1} \\ & \ddots & & \\ 0 & & \ddots & \\ & & & \end{vmatrix}$$

$$\underline{a}^n = (1 \quad a_1^n \; . \; . \; . \; a_n^n)'$$

We call $ARA' = D^2$ an upper-lower Cholesky factorization and $A\underline{x} = D\underline{u}$ a forward analysis or Gram-Schmidt orthogonalization of The first of these interpretations comes from the following manipulation of $= D^2$:

$$(ARA')^{-1} = D^{-2}$$

$$R^{-1} = A'D^{-2}A$$

The vector $\underline{a}^n$ is the $n^{th}$ order whitener and the vector $(a_1^n, a_2^n, \ldots, a_n^n)$ is the $n^{th}$ order predictor.

The vector $J\underline{x}$ may be analyzed as follows:

$$JAJJ\underline{x} = JDJJ\underline{u}$$

$$BJ\underline{x} = CJ\underline{u}$$

$$BRB' = C^2$$

The matrix B' is defined as follows:

$$B' = JA'J = \begin{vmatrix} 1 & & & & \\ a_1^{t-1} & 1 & & 0 & \\ & & \ddots & 1 & \\ a_{t-1}^{t-1} & \cdots & & a_1^1 & 1 \end{vmatrix} = \begin{vmatrix} \ddots & & 0 & \\ & \ddots & & \\ & & \ddots & \\ \underline{a}^{t-1} & \cdots & \underline{a}^1 & \underline{a}^0 \end{vmatrix}$$

We call $BRB' = C^2$ a lower-upper Cholesky factorization of $R^{-1}$ and $BJ\underline{x} = CJ\underline{u}$ a backward analysis of $J\underline{x}$. The first of these interpretations comes from the following manipulation of $BRB' = C^2$:

$$(BRB')^{-1} = C^{-2}$$

$$R^{-1} = B'C^{-2}B$$

## 2.3 Summary

From these factorizations of R and $R^{-1}$ we have these further connections between analysis and synthesis parameters:

$$H = A^{-1} \qquad A' = H^{-1'}$$

$$G = B^{-1} \qquad B' = G^{-1'}$$

The LU and UL factorizations of R may therefore be rearranged as follows:

$$RA' = HD^2$$

$$RB' = GC^2$$

These results for analysis and synthesis, and for factorization of R and $R^{-1}$, are summarized in Figure 2. It is already obvious, and our fast algorithms will confirm it, that H(G) and B'(A') play dual roles. This leads us to suspect that recursions for columns of B'(A') should lead to similar recursions for columns of H(G).

| | Synthesis | Analysis |
|---|---|---|
| Forward | $\underline{x} = HD\underline{u}$<br>$R = HD^2H'$ (LU)<br>$RA' = HD^2$ | $A\underline{x} = D\underline{u}$<br>$R^{-1} = A'D^{-2}A$ (UL)<br>$R^{-1}H = A'D^{-2}$ |
| Backward | $J\underline{x} = GJ\underline{u}$<br>$R = GC^2G^1$ (UL)<br>$RB' = GC^2$ | $BJ\underline{x} = CJ\underline{u}$<br>$R^{-1} = B'C^{-2}B$ (LU)<br>$R^{-1}G = B'C^{-2}$ |
| Connections | $H = JGJ$ (L)<br>$G = JHJ$ (U)<br>$D = JCJ$<br>$C = JDJ$<br>$H = A^{-1}$<br>$G = B^{-1}$ | $A' = JB'J$ (U)<br>$B' = JA'J$ (L)<br>$D^{-1} = JC^{-1}J$<br>$C^{-1} = JD^{-1}J$<br>$A' = H^{-1'}$<br>$B' = G^{-1'}$ |

Figure 2. Forward and Backward Analysis and Synthesis

## 3. Fast Algorithms

Begin with the correlation matrix R and the following rearrangements of its UL and LU factors:

$RA' = HD^2$

$RB' = GC^2$

### 3.1 Levinson Recursions

Call $J\underline{a}^{n+1}$ the vector of non-zero elements in the (n+1) column of A' and $\underline{a}^{n+1}$ the vector of non-zero elements in the (n+1) column of B', counting from the right-most column of B'. Then, from the factorization above, we have the relations

$$\begin{bmatrix} r_o & r_1 & \cdots & r_{n+1} \\ r_1 & & & \vdots \\ \vdots & & & r_1 \\ r_{n+1} & \cdots & r_1 & r_o \end{bmatrix} \begin{bmatrix} \\ J\underline{a}^{n+1} \\ \\ \\ \end{bmatrix} = \begin{bmatrix} 0 \\ \cdot \\ \cdot \\ \cdot \\ \sigma^2_{n+1} \end{bmatrix}$$

$$\begin{bmatrix} r_o & r_1 & \cdots & r_{n+1} \\ r_1 & & & \vdots \\ \vdots & & & r_1 \\ r_{n+1} & \cdots & r_1 & r_o \end{bmatrix} \begin{bmatrix} \\ \underline{a}^{n+1} \\ \\ \\ \end{bmatrix} = \begin{bmatrix} \sigma^2_{n+1} \\ 0 \\ \cdot \\ \cdot \\ \cdot \\ 0 \end{bmatrix}$$

Note what happens when $J\underline{a}^{n+1}$ and $\underline{a}^{n+1}$ are replaced by the previous columns $J\underline{a}^n$ and $\underline{a}^n$:

$$\begin{bmatrix} r_o & r_1 \cdots & r_{n+1} \\ r_1 & & \\ & & \\ & & \\ r_{n+1} & \cdots r_1 & r_o \end{bmatrix} \begin{bmatrix} 0 \\ \\ J\underline{a}^n \\ \\ \\ \end{bmatrix} = \begin{bmatrix} c^n_{n+1} \\ 0 \\ \cdot \\ \cdot \\ \cdot \\ 0 \\ (\sigma^2_n) \end{bmatrix}$$

$$c^n_{n+1} = \sum_{i=0}^{n} a^n_i \, r_{n+1-i}$$

$$\begin{bmatrix} r_o & r_1 \cdots & r_{n+1} \\ r_1 & & \\ \cdot & & \\ \cdot & & \\ \cdot & & \\ r_{n+1} & \cdots r_1 & r_o \end{bmatrix} \begin{bmatrix} \\ \\ \underline{a}^n \\ \\ \\ 0 \end{bmatrix} = \begin{bmatrix} \sigma^2_n \\ 0 \\ \cdot \\ \cdot \\ \cdot \\ 0 \\ c^n_{n+1} \end{bmatrix}$$

Choose $J\underline{a}^{n+1}$ and $\underline{a}^{n+1}$ to be the following coupled recursions in order to satisfy the factorizations:

$$\begin{bmatrix} \\ J\underline{a}^{n+1} \end{bmatrix} = \begin{bmatrix} 0 \\ J\underline{a}^n \end{bmatrix} + k_{n+1} \begin{bmatrix} \underline{a}^n \\ 0 \end{bmatrix}$$

$$\begin{bmatrix} \underline{a}^{n+1} \\ \\ \end{bmatrix} = \begin{bmatrix} \underline{a}^n \\ 0 \end{bmatrix} + k_{n+1} \begin{bmatrix} 0 \\ J\underline{a}^n \end{bmatrix}$$

$$\sigma^2_n \; k_{n+1} = - \, c^n_{n+1}$$

$$\sigma^2_{n+1} = (1- k^2_{n+1})\sigma^2_n$$

These are the celebrated Levinson recursions for sequentially building the whiteners $\underline{a}^n$, or equivalently the columns of the A' and B' in the Cholesky factorization of $R^{-1}$.

Vectorized Form #1: These equations may be put into a concise matrix form by defining the following matrices:

$$\Delta = \begin{vmatrix} 1 & & & & \\ & 1 & & & 0 \\ & & \cdot & & \\ & 0 & & \cdot & \\ & & & \cdot & \\ & & & & 1 \\ 0 & \cdot & \cdot & \cdot & 0 \end{vmatrix} \qquad \Omega = \begin{vmatrix} 0 & \cdot & \cdot & \cdot & 0 \\ 1 & & & & \\ & 1 & & & \\ & & \cdot & & 0 \\ & 0 & & \cdot & \\ & & & & \cdot \\ & & & & 1 \end{vmatrix}$$

$$J \; \Delta \; J = \Omega \qquad J \; \Omega \; J = \Delta$$

Now the Levinson recursions are

$$\underline{a}^{n+1} = \Delta \; \underline{a}^n + k_{n+1} \; \Omega \; J\underline{a}^n$$

$$J\underline{a}^{n+1} = \Omega \; J\underline{a}^n + k_{n+1} \; \Delta \; \underline{a}^n$$

A cross matrix representation for these recursions is

$$\underline{a}^{n+1} = K_{n+1} \; \Delta \; \underline{a}^n$$

$$J\underline{a}^{n+1} = JK_{n+1} \; JJ \; \Delta \; JJ \; \underline{a}^n = K_{n+1} \; \Omega \; J\underline{a}^n$$

where the cross matrix $K_{n+1} = JK_{n+1}J$ is defined as follows:

$$K_{n+1} = \begin{vmatrix} 1 & & 0 & & k_{n+1} \\ & \cdot & & \cdot & \\ & & \cdot \;\; \cdot & & \\ 0 & & \cdot & & 0 \\ & k_{n+1} & & 1 & \\ k_{n+1} & & 0 & & 1 \end{vmatrix}$$

When the diagonals cross, the center term is $(1+k_{n+1})$.

This representation may be used over and over again to obtain the iterated cross matrix recursion

$$\underline{a}^{n+1} = K_{n+1} \Delta K_n \Delta \ . \ . \ . \ K_2 \Delta K_1 \Delta \underline{1}$$

where $\underline{1}$ is the 1x1 matrix equal to unity. The representation may also be inverted to give a handy formula for running the Levinson recursions backwards:

$$\Delta \underline{a}^n = K_{n+1}^{-1} \underline{a}^{n+1}$$

The inverse is

$$K_{n+1}^{-1} = (1 - k_{n+1}^2)^{-1} \begin{vmatrix} 1 & & 0 & & -k_{n+1} \\ & 1 & & \cdot & \\ 0 & & \cdot & & 0 \\ & \cdot & & & \\ & & & \cdot & \\ -k_{n+1} & & 0 & & 1 \end{vmatrix}$$

Of course $k_{n+1} = a_{n+1}^{n+1}$. In these recursions, $\underline{a}^n$ and $J\underline{a}^n$ are vectors of increasing dimension.

<u>Vectorized Form #2:</u> To obtain vectorized equations in which dimensions are constant, let $\underline{A}^n$ denote a t-vector consisting of $J\underline{a}^n$, followed by zeros, and let $\underline{B}^n$ denote a t-vector consisting of $\underline{a}^n$, followed by zeros:

$$\underline{A}^n = \begin{bmatrix} J\underline{a}^n \\ \underline{0} \end{bmatrix} \qquad \underline{B}^n = \begin{bmatrix} \underline{a}^n \\ \underline{0} \end{bmatrix}$$

The vector $\underline{A}^n$ is the $n^{th}$ column of A', but $B^n$ is <u>not</u> the $n^{th}$ column of B'. $\underline{A}^n \neq J\underline{B}^n$

The Levinson recursions produce the following recursions for $\underline{A}^n$ and $\underline{B}^n$:

$$\underline{B}^{n+1} = \underline{B}^n + k_{n+1} T \underline{A}^n$$

$$\underline{A}^{n+1} = T \underline{A}^n + k_{n+1} \underline{B}^n$$

The matrix T in these equations is a delay matrix:

$$T = \begin{bmatrix} 0 & \cdot & \cdot & \cdot & & & 0 \\ 1 & 0 & \cdot & \cdot & \cdot & & 0 \\ 0 & 1 & & & & & \\ \cdot & & & & & & \\ \cdot & & & & & & \\ \cdot & & & & & & \\ 0 & & & & & 1 & 0 \end{bmatrix}$$

This form of the Levinson recursions will simplify our discussion of fast algorithms as lattice algorithms.

<u>Remarks:</u> 1. Given the reflection coefficients $\{k_n\}_1^{t-1}$, the iterated cross-matrix recursion may be used to generate all of the order increasing whiteners $\underline{a}^n$. This is equivalent to factoring $R^{-1}$.

2. Given $\sigma_0^2 = r_0$ and the reflection coefficients $\{k_n\}_1^{t-1}$, prediction errors $\sigma_n^2$ may be generated recursively.

3. Given the highest order whitener $\underline{a}^{t+1}$, with $k_{t+1} = a_{t+1}^{t+1}$, the cross-matrix recursion may be used to generate all of the order-decreasing whiteners $\underline{a}^n$, and all of the order-decreasing reflection coefficients $k_n = a_n^n$. In this way lattice parameters are built from filter coefficients. The stability of a filter built from $\underline{a}^{t+1}$ is tested by checking for $|k_n| < 1$ for $n=1,2,\ldots,t-1$. This is the Shur-Cohn test.

4. The first row of the equation $RA' = HD^2$ gives the following recursion for computing correlations from whitening filters:

$$r_n = \sigma_0^2 \, \delta_n - \sum_{i=1}^{n} a_i^n \, r_{n-i}$$

These remarks, coupled with the Levinson recursions themselves, complete the discussion of the right side of Figure 1.

## 3.2 LeRoux-Gueguen Algorithm

Return again to the following re-arrangement of the UL factorization of R:

$$RA' = HD^2$$

Write out the $n^{th}$ column of this equation:

$$\begin{bmatrix} r_0 & r_1 \cdot \cdot \cdot & r_n \\ r_1 & r_2 \cdot \cdot \cdot & r_{n+1} \\ \cdot & & \cdot \\ \cdot & & \cdot \\ \cdot & & \cdot \\ r_{t-1} & \cdot \cdot \cdot & r_{t-1-n} \end{bmatrix} \begin{bmatrix} J\underline{a}^n \end{bmatrix} = \sigma_n^2 \begin{bmatrix} \underline{0} \\ \\ \\ \underline{h}^n \end{bmatrix}$$

Ignore the zero terms in the right-hand column and write

$$\begin{bmatrix} r_n & r_{n-1} & \cdot \cdot \cdot & r_0 \\ r_{n+1} & \cdot & R_n & r_1 \\ \cdot & \cdot & & \cdot \\ \cdot & & & \cdot \\ \cdot & & & \cdot \\ r_{t-1} & & \cdot \cdot \cdot & r_{t-1-n} \end{bmatrix} \begin{bmatrix} J\underline{a}^n \end{bmatrix} = \sigma_n^2 \begin{bmatrix} \underline{h}^n \end{bmatrix}$$

or

$$\sigma_n^2 \underline{h}^n = R_n J \underline{a}^n$$

A related vector, which will be needed in our development, is $R_n \underline{a}^n$:

$$\begin{bmatrix} r_n & r_{n-1} \cdots & r_0 \\ r_{n+1} & R_n & r_1 \\ \vdots & & \vdots \\ r_{t-1} & \cdots & r_{t-1-n} \end{bmatrix} \begin{bmatrix} \underline{a}^n \end{bmatrix} = \left[\begin{array}{cccc} r_n & \cdots & r_0 & x \\ \hline r_{n+1} & & r_1 & r_0 \\ \vdots & R_{n+1} & \vdots & \vdots \\ r_{t-1} \cdots & & r_{t-1-n} & r_{t-n-2} \end{array}\right] \begin{bmatrix} \underline{a}^n \\ 0 \end{bmatrix} =$$

$$\sigma_n^2 \begin{bmatrix} g^n \end{bmatrix} = \sigma_n^2 \begin{bmatrix} 0 \\ \hat{g}^n \end{bmatrix}$$

The first term of $g^n$, denoted $g^n{}_0$, is zero. Note the definition of $R_{n+1}$ and summarize these results as follows:

$$\sigma_n^2 \underline{h}^n = R_n \, J\underline{a}^n$$

$$\sigma_n^2 g^n = R_n \, \underline{a}^n$$

$$g^n = \begin{bmatrix} 0 \\ \hat{g}^n \end{bmatrix} \quad ; \quad \sigma_n^2 \, \hat{g}_n = R_{n+1} \begin{bmatrix} \underline{a}^n \\ 0 \end{bmatrix}$$

Vectorized Form #1: Our objective is to derive recursions for $\underline{h}^n$, $g^n$, and $\sigma_n^2$. In the Appendix, the Levinson recursions for $J\underline{a}^n$ are used to derive the following results:

$$\sigma_{n+1}^2 \; \underline{h}^{n+1} = \sigma_n^2 \; \Delta' \; \underline{h}^n + k_{n+1} \, \sigma_n^2 \; \Omega' \; g^n$$

$$\sigma_{n+1}^2 \; g^{n+1} = \sigma_n^2 \, \Omega' \; g^n + k_{n+1} \, \sigma_n^2 \; \Delta' \; \underline{h}^n$$

$$\sigma_{n+1}^2 = \sigma_n^2 (1 - k_{n+1}^2)$$

$k_{n+1} = -2^{nd}$ element of $g^n$

$$\sigma_0^2 \, \underline{h}^0 = (r_0, \ \ldots, \ r_{t-1})'; \; \sigma_0^2 \, g^0 = (0, r_1, \ldots, r_{t-1})'$$

$$\sigma_0^2 = r_0$$

Because $g^n \neq J\underline{h}^n$, there is no handy cross-matrix representation.

Vectorized Form #2: Now define $\underline{H}^n$ to be a t-vector consisting of $\sigma_n^2\,\underline{h}^n$, preceded by zeros, and $\underline{G}^n$ to be a t-vector consisting of $\sigma_n^2\,g^n$, preceded by zeros:

$$\underline{H}^n = \sigma_n^2 \begin{bmatrix} \underline{0} \\ \underline{h}^n \end{bmatrix} \qquad \underline{G}^n = \sigma_n^2 \begin{bmatrix} \underline{0} \\ g^n \end{bmatrix}$$

The vector $\underline{H}^n$ is the $n^{th}$ column of H, but $\underline{G}^n$ is not the $n^{th}$ column of G. $\underline{H}^n \neq J\underline{G}^n$.

The LeRoux-Gueguen recursions may now be written

$$\underline{G}^{n+1} = \underline{G}^n + k_{n+1}\,T\underline{H}^n$$

$$\underline{H}^{n+1} = T\underline{H}^n + k_{n+1}\,\underline{G}^n$$

where T is still the delay matrix.

## 3.3 Summary

In order to emphasize the striking similarity in the Levinson-Durbin and LeRoux-Gueguen algorithms, we say that they both obey the recursion

$$\underline{U}^{n+1} = U^n + k_{n+1}\,T\underline{V}^n$$

$$\underline{V}^{n+1} = T\underline{V}^n + k_{n+1}\,\underline{U}^n$$

These equations are reversed (more on this in the next section) as follows:

$$\underline{U}^n = \underline{U}^{n+1} - k_{n+1}\,T\underline{V}^n$$

$$\underline{V}^{n+1} = T\underline{V}^n + k_{n+1}\,\underline{U}^n$$

The iteration for increasing order filters in the Levinson recursions is obtained by setting initial conditions as follows:

$\underline{V}_0 = \underline{U}_0 = (1,0,\ldots,0)'$

The LeRoux-Gueguen algorithm is obtained by setting

$\underline{U}_0 = (r_0,r_1,\ldots,r_{t-1})'$; $\underline{V}_0 = (0,r_1,\ldots,r_{t-1})'$

In the LeRoux-Gueguen recursions, the variance and reflection coefficients are obtained as follows:

$\sigma_n{}^2$ = (n+1)st element of $\underline{V}^n$

$k_n$ = -(n+2)nd element of $\underline{U}^n/\sigma_n{}^2$

These recursions are discussed as vector recursions for a vector processing machine in [10].

## 4. Lattice Implementations

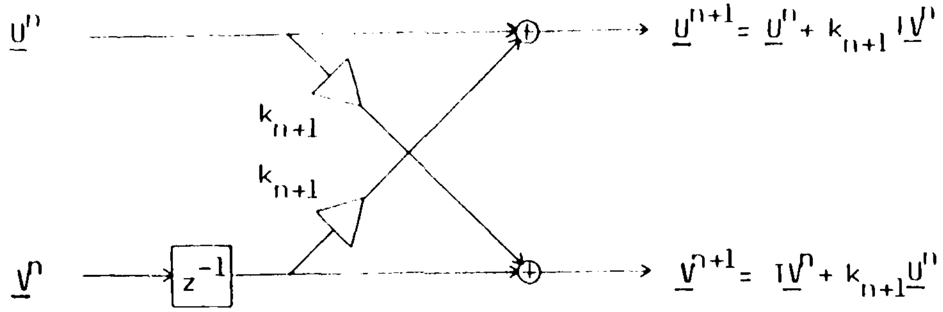
Figure 3 contains four different lattice cells: (a) and (b) are unnormalized and normalized forward lattice cells; (c) and (d) are unnormalized and normalized inverse lattice cells. The equations implemented by each of these cells are illustrated on the figure. The $z^{-1}$ operator is interpreted as the delay matrix T when the cells map vectors $\underline{U}^n$ and $\underline{V}^n$ into vectors $\underline{U}^{n+1}$ and $\underline{V}^{n+1}$. When the vectors $\underline{U}^n$ and $\underline{V}^n$ fill up sequentially in time, the lattice cells see scalar sequences and the $z^{-1}$ operator works just like a scalar delay. When these cells are concatenated, then the lattice structures of Figure 4 are constructed.

Denote the internal variables in the lattice as follows:

$$\underline{U}^n = \begin{bmatrix} u_0^n \\ u_1^n \\ \cdot \\ \cdot \\ \cdot \\ u_{t-1}^n \end{bmatrix} \qquad \underline{V}^n = \begin{bmatrix} v_0^n \\ v_1^n \\ \cdot \\ \cdot \\ \cdot \\ v_{t-1}^n \end{bmatrix}$$

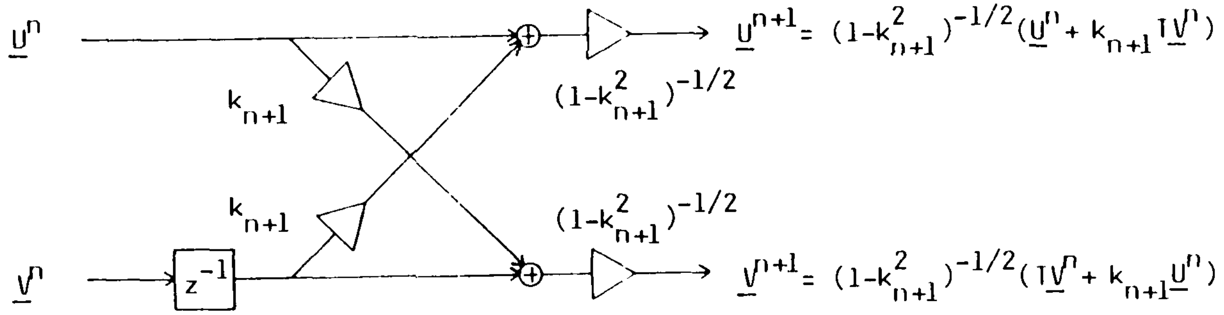
In the forward (reverse) lattice, $u_i^n$ is the internal variable at the output (input) of cell n at time i; $v_i^n$ is the input to cell n+1 at time i in both lattices.

Think of the forward lattice as a linear transformation that maps the input sequence $\underline{U}^0$ into the space-time sequence



$\underline{U}^n$
$\underline{V}^n$
$z^{-1}$
$k_{n+1}$
$k_{n+1}$
$\underline{U}^{n+1} = \underline{U}^n + k_{n+1} I \underline{V}^n$
$\underline{V}^{n+1} = I \underline{V}^n + k_{n+1} \underline{U}^n$

(a)



$\underline{U}^n$
$\underline{V}^n$
$z^{-1}$
$k_{n+1}$
$k_{n+1}$
$(1-k_{n+1}^2)^{-1/2}$
$(1-k_{n+1}^2)^{-1/2}$
$\underline{U}^{n+1} = (1-k_{n+1}^2)^{-1/2}(\underline{U}^n + k_{n+1} I \underline{V}^n)$
$\underline{V}^{n+1} = (1-k_{n+1}^2)^{-1/2}(I \underline{V}^n + k_{n+1} \underline{U}^n)$

(b)

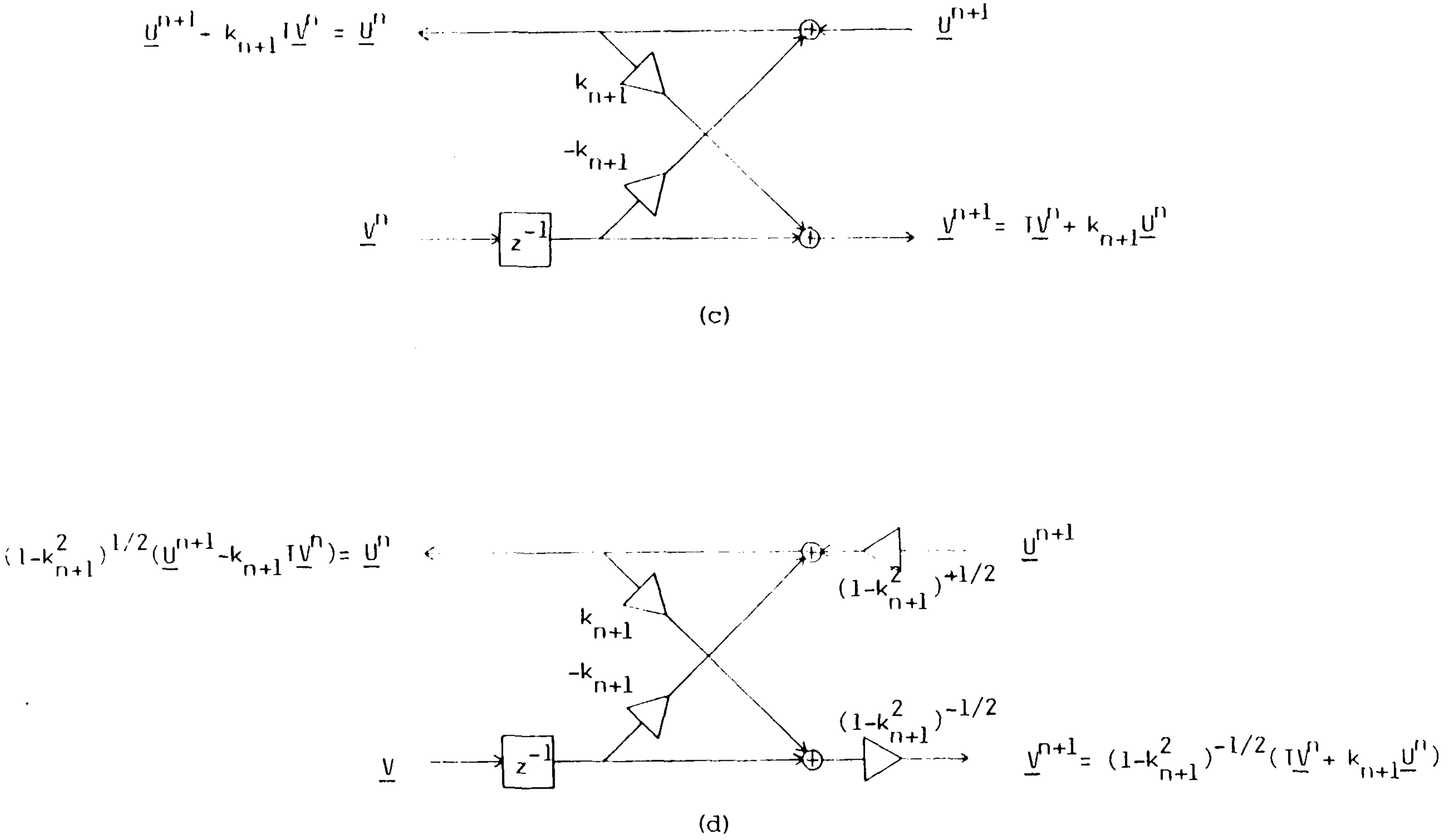

$\underline{U}^{n+1} - k_{n+1} I \underline{V}^{n} = \underline{U}^{n}$
$\underline{U}^{n+1}$
$k_{n+1}$
$-k_{n+1}$
$\underline{V}^{n}$
$z^{-1}$
$\underline{V}^{n+1} = I\underline{V}^{n} + k_{n+1}\underline{U}^{n}$
(c)
$(1-k_{n+1}^{2})^{1/2}(\underline{U}^{n+1} - k_{n+1} I \underline{V}^{n}) = \underline{U}^{n}$
$\underline{U}^{n+1}$
$(1-k_{n+1}^{2})^{+1/2}$
$k_{n+1}$
$-k_{n+1}$
$(1-k_{n+1}^{2})^{-1/2}$
$\underline{V}$
$z^{-1}$
$\underline{V}^{n+1} = (1-k_{n+1}^{2})^{-1/2}(I\underline{V}^{n} + k_{n+1}\underline{U}^{n})$
(d)

Figure 3.

Lattice Cells : (a) forward, (b) Variance-normalized forward
(c) inverse, (d) Variance-normalized inverse .

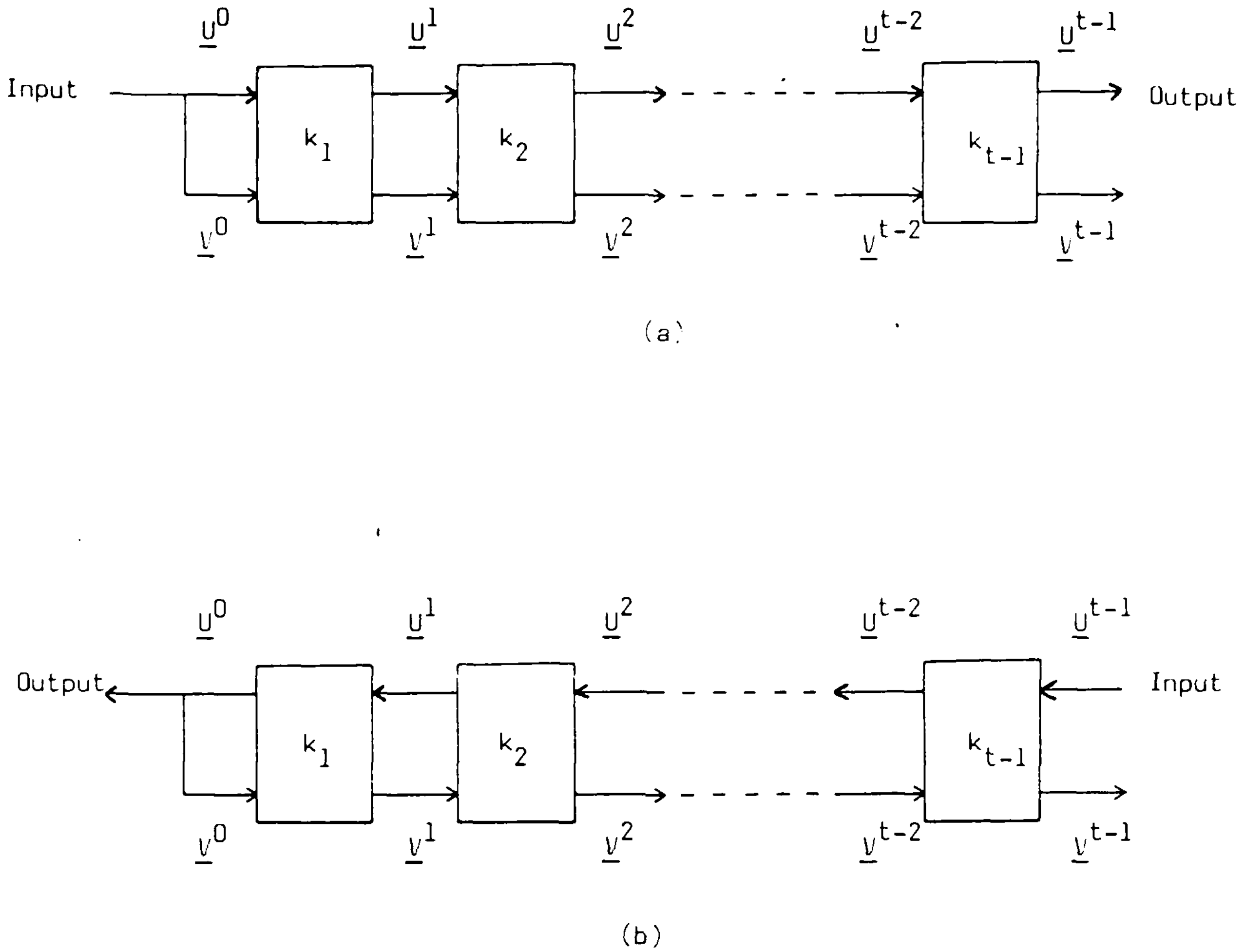


Figure 4.

Lattices : (a) Forward, (b) Inverse.

$$\underline{W} = \begin{bmatrix} u_0^0 \\ u_1^1 \\ \cdot \\ \cdot \\ \cdot \\ u_{t-1}^{t-1} \end{bmatrix}$$

while generating the interval variables $\underline{U}^n$ and $\underline{V}^n$. The space-time sequence $\underline{W}$ contains upper lattice variable at cell n and time n for n=0,1,...,t-1. Think of the inverse lattice as a linear transformation that maps $\underline{W}$ into $\underline{u}^0$ while generating the internal variables $\underline{U}^n$ and $\underline{V}^n$.

## 4.1 Forward Lattice

The forward lattice may be used to (i) convert correlations $\{r_n\}_0^{t-1}$ into reflection coefficients $\{k_n\}_0^{t-1}$, variances $\{\sigma_n^2\}_0^{t-1}$, and "impulse responses" $\{\underline{h}^n\}_0^{t-1}$ in the factorization $R^2$ =HDH', (ii) convert reflection coefficients $\{k_n\}_1^{t-1}$ into variances $\{\sigma_n^2\}_0^{t-1}$ and whiteners $\{\underline{a}^n\}$ in the factorization $R^{-1} = AD^{-2}A'$, and (iii) whiten correlated data $\{x_n\}_0^{t-1}$ to obtain uncorrelated random variables in the analysis representation $D\underline{u} = A\underline{x}$. The Levinson procedure is really a composite procedure consisting of (i) and (ii) above.

(i) Correlations to reflection coefficients, variances, and impulse responses. When $\underline{U}^0$ and $\underline{V}^0$ are set with the initial conditions required in the LeRoux-Gueguen algorithm then at the upper and lower branches of the forward lattice one observes

$$\underline{U}^n = \underline{G}^n = \begin{bmatrix} \underline{0} \\ \sigma_n^2 \underline{g}^n \end{bmatrix}$$

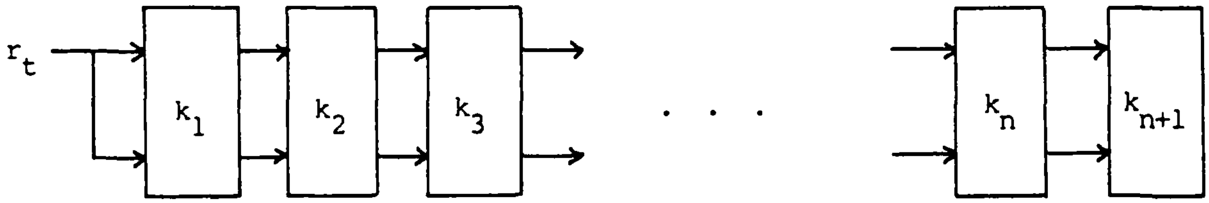
$$\underline{V}^n = \underline{H}^n = \begin{bmatrix} \underline{0} \\ \sigma_n^2 \underline{h}^n \end{bmatrix}$$

Note that the non-zero entries in $\underline{V}^n$ that contain information about $\underline{h}^n$ appear as <u>trailing</u> entries with delay n. This means the n<sup>th</sup> cell need not be switched into the lattice until time n. At this time the output $g_1^{n-1}$ in $\underline{U}^{n-1} = \underline{G}^{n-1}$ is available to compute $k_n = -g_1^{n-1}$ for use in cell n. In summary, the correlation sequence $(r_0, r_1, \ldots, r_{t-1})$ may be fed sequentially into a forward lattice in which cells are computed and switched in sequentially to produce reflection coefficients, variances, and "impulse responses $\sigma_n^2 \underline{h}^n$. The elements of $\underline{h}^n$ in the LeRoux-Gueguen algorithm appear sequentially in time for all values of n less than or equal to the current value of time. Thus the lattice procedure produces the Cholesky factor H, row-by-row. This was Friedlander's insight. The procedure is illustrated in Figure 5. Note that initial conditions are set to zero.

To complete the discussion, we note that one can stand at cell n and observe the evolution of $\sigma_n^2 \underline{h}^n$ sequentially in time. This is a column-by-column procedure.

(ii) <u>Reflection coefficients to variances and whiteners</u>. When $\underline{U}^0 = \underline{V}^0 = (1,0,\ldots,0)'$, then at the upper and lower branches of the forward lattice, one observes

$$\begin{array}{cccccccc}
 & \sigma_0^2 g_0^0 & x & x & x & \cdots & & x \\
 & -\sigma_0^2 k_1 & \sigma_1^2 g_0^1=0 & x & x & \cdots & & x \\
\text{t} & \sigma_0^2 g_2^0 & -\sigma_1^2 k_2 & \sigma_2^2 g_0^2=0 & x & \cdots & & x \\
\text{i} & . & . & . & 0 & \cdots & & x \\
\text{m} & & & & & & & \\
\text{e}\downarrow & \vdots & \vdots & \vdots & \vdots & & & \vdots \\
 & \sigma_0^2 q_n^0 & . & . & . & \cdots & & -\sigma_n^2 k_{n+1}
\end{array}$$



$$\begin{array}{cccccccc}
 & h_0^0=\sigma_0^2 & x & x & x & \cdots & & x \\
\text{t} & h_1^0 & h_0^1=\sigma_1^2 & x & x & \cdots & & . \\
\text{i} & h_2^0 & h_1^1 & h_0^2=\sigma_2^2 & x & \cdots & & . \\
\text{m} & . & . & . & . & \cdots & & . \\
\text{e}\downarrow & \vdots & \vdots & \vdots & \vdots & & & \vdots \\
\rightarrow & h_n^0 & h_{n-1}^1 & h_{n-2}^2 & . & \cdots & & h_0^n=\sigma_n^2 \\
 & h_{n+1}^0 & . & . & . & & & h_1^n \\
 & . & . & . & . & \cdots & & . \\
 & & & & & & & \uparrow
\end{array}$$

This row is
read out at time n.

This column is
observed for sequential
evolution of $\underline{h}^n$.

Figure 5.

Reading Cholesky Factors out of the lattice.

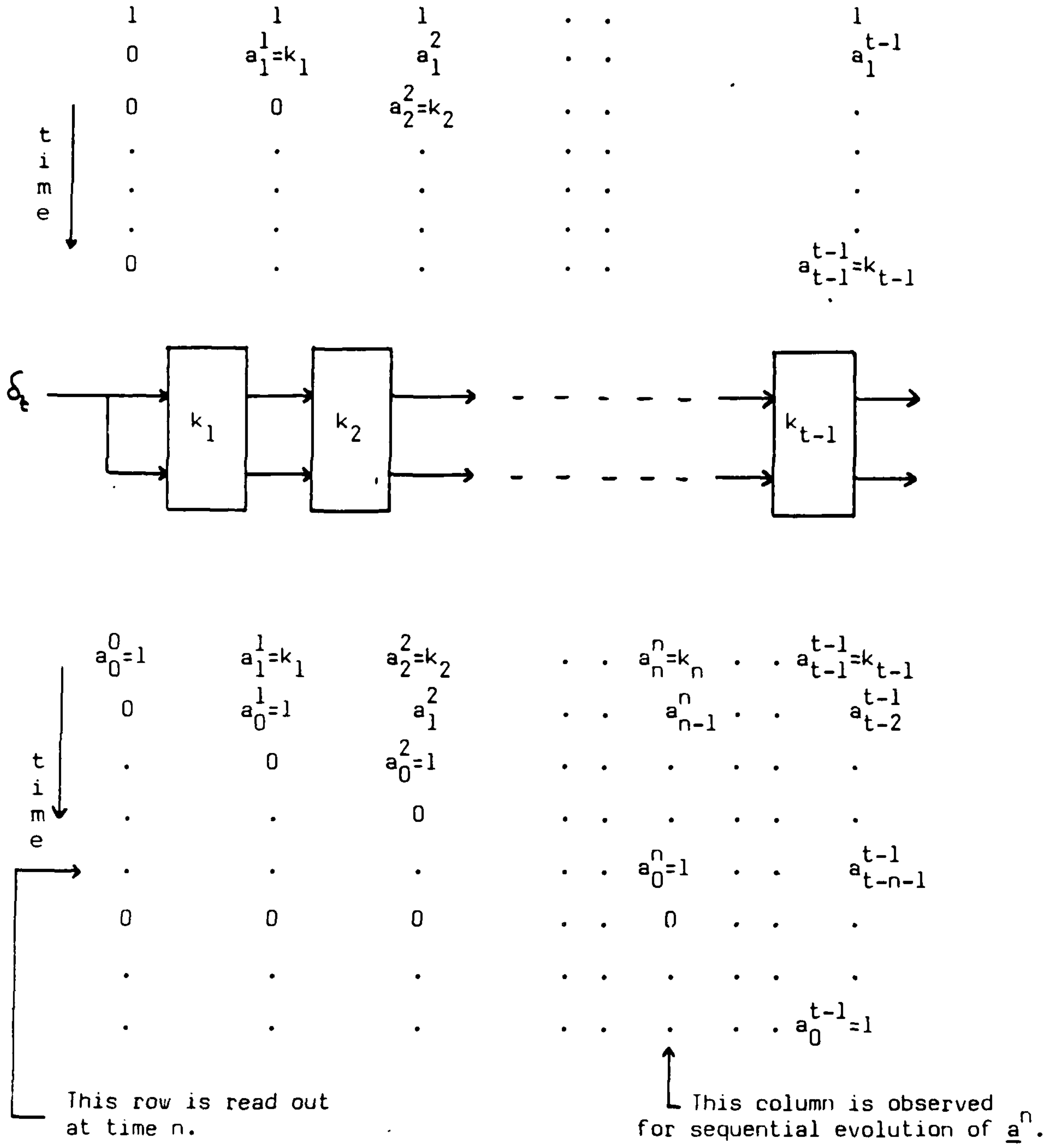
$$\underline{U}^n = \underline{B}^n = \begin{bmatrix} \underline{a}^n \\ \underline{0} \end{bmatrix}$$

$$\underline{V}^n = \underline{A}^n = \begin{bmatrix} J\underline{a}^n \\ \underline{0} \end{bmatrix}$$

If the lattice cells are wired together, as in Figure 4, then the entries in $\underline{U}^0 = (1,0,\ldots,0)'$ may be sent in sequentially and the entries in $\underline{U}^n$ and $\underline{V}^n$ may be read out sequentially. This follows from the causal dependence of $\underline{U}^{n+1}$ and $\underline{V}^{n+1}$ on $\underline{U}^n$ and $\underline{V}^n$. Thus, given the reflection coefficients $\{k_n\}$ required to build the lattice, an impulse may be sent in to generate order increasing whiteners $J\underline{a}^n$ which may be read out as <u>leading</u> entries of $\underline{V}^n = \underline{A}^n$. They may be read out sequentially to produce the factorization $R^{-1} = AD^{-2}A'$, row-by-row. But, as the whiteners $\underline{a}^n$ show up immediately as leading entries of $\underline{A}^n$ in each lattice cell, the lattice cannot be sequentially wired as in Figure 5. It must be completely wired from all of the reflection coefficients $\{k_n\}_1^{t-1}$. This is illustrated in Figure 6.

An alternative is to generate cells sequentially, and then after each new cell is computed and switched into the lattice, compute the impulse response of the new lattice. Each step of the procedure produces $\underline{A}^n = (J\underline{a}^n, \underline{0})'$. This composite procedure is, in fact, a Levinson-Durbin procedure.

<u>(iii) Correlated data to uncorrelated data</u>. If the forward lattice is excited with correlated inputs $(x_0,x_1,\ldots,x_{t-1})$, then the variables observed in $\underline{U}^n$ are



$$\begin{array}{ccccc}
1 & 1 & 1 & \cdots & 1 \\
0 & a_1^1=k_1 & a_1^2 & \cdots & a_1^{t-1} \\
0 & 0 & a_2^2=k_2 & \cdots & \cdot \\
\cdot & \cdot & \cdot & \cdots & \cdot \\
\cdot & \cdot & \cdot & \cdots & \cdot \\
\cdot & \cdot & \cdot & \cdots & \cdot \\
0 & \cdot & \cdot & \cdots & a_{t-1}^{t-1}=k_{t-1}
\end{array}$$

$$\begin{array}{ccccccc}
a_0^0=1 & a_1^1=k_1 & a_2^2=k_2 & \cdots & a_n^n=k_n & \cdots & a_{t-1}^{t-1}=k_{t-1} \\
0 & a_0^1=1 & a_1^2 & \cdots & a_{n-1}^n & \cdots & a_{t-2}^{t-1} \\
\cdot & 0 & a_0^2=1 & \cdots & \cdot & \cdots & \cdot \\
\cdot & \cdot & 0 & \cdots & \cdot & \cdots & \cdot \\
\cdot & \cdot & \cdot & \cdots & a_0^n=1 & \cdots & a_{t-n-1}^{t-1} \\
0 & 0 & 0 & \cdots & 0 & \cdots & \cdot \\
\cdot & \cdot & \cdot & \cdots & \cdot & \cdots & \cdot \\
\cdot & \cdot & \cdot & \cdots & \cdot & \cdots & a_0^{t-1}=1
\end{array}$$

time
$\delta_t$
$k_1$
$k_2$
$k_{t-1}$
time
This row is read out at time n.
This column is observed for sequential evolution of $\underline{a}^n$.

Fiqure 6.

Reflection Coefficients to Whiteners.

$$\begin{bmatrix} u_0^n \\ u_1^n \\ \cdot \\ \cdot \\ \cdot \\ u_n^n \\ \cdot \\ \cdot \\ u_{t-1}^n \end{bmatrix} = \begin{bmatrix} x_0 & & & & 0 & & \\ x_1 & & x_0 & & & & \\ \cdot & & & \cdot & & & \\ \cdot & & & & \cdot & & \\ \cdot & & & & & \cdot & \\ x_n & \cdots & & & & x_1 & x_0 \\ \cdot & & & & & & \cdot \\ \cdot & & & & & & \cdot \\ x_{t-1} & \cdots & & & & & x_{t-1-n} \end{bmatrix} \begin{bmatrix} 1 \\ a_1^n \\ \cdot \\ \cdot \\ \cdot \\ a_n^n \end{bmatrix}$$

$$= \begin{bmatrix} 1 & & & & 0 & & & & \\ a_1^n & 1 & & & & & & & \\ \vdots & & \cdot & & & & & & \\ \vdots & & & \cdot & & & & & \\ a_n^n & \cdots & a_1^n & & 1 & & & & \\ & & a_n^n & \cdots & a_1^n & 1 & & & \\ & & & & & & \cdot & & \\ & & & & & & & \cdot & \\ 0 & & & & 0 & & a_n^n \cdots & a_1^n & 1 \end{bmatrix} \begin{bmatrix} x_0 \\ x_1 \\ \cdot \\ \cdot \\ \cdot \\ \\ x_{t-1} \end{bmatrix}$$

The variable $u_n^n$ is the $n^{th}$ entry in the vector $D\underline{u}$:

$$D\underline{u} = A\underline{x}$$

If the normalized forward lattice is excited with $(x_0,\ldots,x_{t-1})$, then the variable $u_n^n$ is the $n^{th}$ entry in the vector $\underline{u}$, itself. So, by reading out time variables $u_0^0, u_1^1, \ldots, u_{t-1}^{t-1}$ from the appropriate cells, at the appropriate times (in fact the time equals the cell number), we read out a white sequence of uncorrelated random variables with normalized variances. The same story holds for the unnormalized lattice, except now the variable $u_n^n$ has variance $\sigma_n^2$. This Gram-Schmidt orthogonalization is illustrated in Figure 7.

## 4.2 Inverse Lattice

The inverse lattice may be used to (i) convert reflection coefficients $\{k_n\}_0^{t-1}$ and variances $\{\sigma_n^2\}_0^{t-1}$, into correlations $\{r_n\}_0^{t-1}$ and "impulse responses" $\{\underline{h}^n\}_0^{t-1}$, (ii) convert the highest order whitener $\underline{a}^{t-1}$ into reflection coefficients $\{k_n\}_0^{t-1}$, and lower order whiteners, and (iii) color uncorrelated random variables $D\underline{u}$ to obtain correlated random variables in the representation $\underline{x} = HD\underline{u}$.

(i) Reflection coefficients and variances to correlations and impulse responses. From Figure 5 we see that initial conditions may be set to zero and the correlation sequence may be sent into the lattice, to generate the space-time vector

$$\underline{W} = \begin{bmatrix} \sigma_0^2 \\ 0 \\ \cdot \\ \cdot \\ \cdot \\ 0 \end{bmatrix}$$

This means initial conditions may be set in the inverse lattice in exactly the same way they were set in the forward lattice, with $\sigma_0{}^2$ at the input to the first cell and zeros elsewhere, to produce an impulse response for

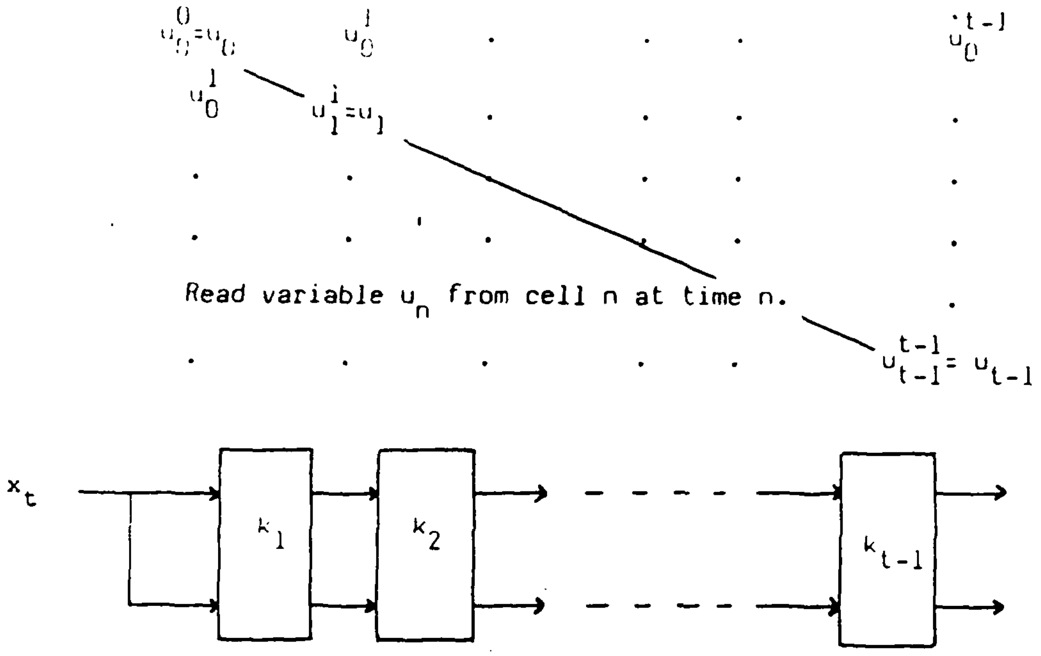


Figure 7.

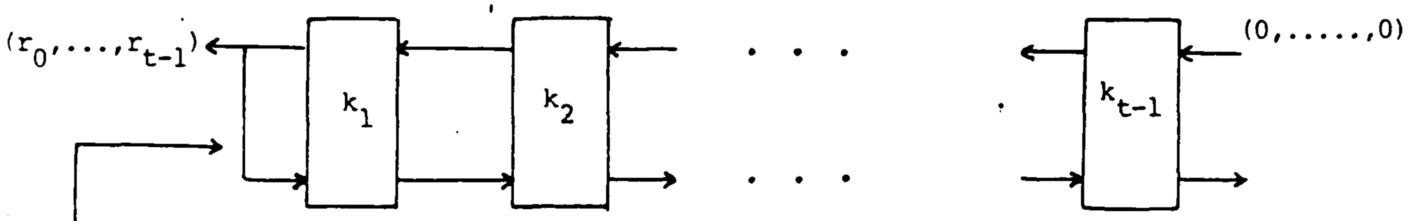
Whitening, Analysis or Gram-Schmidt orthogonalization of correlated data.

the inverse lattice which generates the internal variables $\underline{H}^n = (\underline{0}, \underline{h}^n)$ and the output sequence $\{r_n\}_0^{t-1}$. This generalizes the Robinson-Treitel result, to include the generation of Cholesky factors as internal lattice variables. It is illustrated in Figure 8.

(ii) Highest order whitener to reflection coefficients and lower order whiteners. From Figure 6 we see that initial conditions may be set to zero, and a 1 placed at the input to the first cell, to produce the order-increasing whiteners $\underline{a}^n$ (or $J\underline{a}^n$) as internal variables in the lattice. The ouput of the (t-1)st cell contains $\underline{B}^{t-1} = \underline{a}^{t-1}$ on the upper branch. This means the initial conditions may be set to zero, and the sequence $(1, a_1^{t-1}, \ldots, a_{t-1}^{t-1})$ sent into the inverse lattice to produce all of the lower order whiteners as internal variables. The lattice must be fully connected to achieve this. See Figure 9.

(iii) Uncorrelated data to correlated data. Recall Figure 7: with initial conditions set to zero and the input equal to $(x_0, x_1, \ldots, x_{t-1})$, the uncorrelated random variables $u_n^n$ are observed at cell n at time n. Thus, an inverse lattice may be set with zero initial conditions and excited with uncorrelated random variables $u_0^0, u_1^1, \ldots, u_n^n, \ldots, u_{t-1}^{t-1}$, each entered at the appropriate time and place (cell n, time n), to produce the correlated output $(x_0, x_1, \ldots, x_{t-1})$. This is illustrated in Figure 10.

$$\begin{array}{cccccc}
\sigma_0^2 g_0^0 = r_0 & 0 & 0 & \cdots & 0 & 0 \\
\sigma_0^2 g_1^0 = r_1 & \sigma_1^2 g_0^1 = 0 & 0 & \cdots & \cdot & \cdot \\
\sigma_0^2 g_2^0 = r_2 & \sigma_1^2 g_1^1 & 0 & \cdots & \cdot & \cdot \\
\vdots & \vdots & \vdots & \cdots & \vdots & \vdots \\
 & & & & 0 & \cdot \\
\sigma_0^2 g_{t-1}^0 & \cdot & \cdot & \cdots & \sigma_{t-2}^2 g_1^{t-2} & 0
\end{array}$$



$$\begin{array}{cccccc}
h_0^0 & 0 & \cdot & \cdots & 0 & 0 \\
h_1^0 & h_0^1 & 0 & \cdots & \cdot & \cdot \\
\vdots & \vdots & \vdots & \cdots & \vdots & \vdots \\
h_{t-2}^0 & \cdot & \cdot & \cdots & h_0^{t-2} & 0 \\
h_{t-1}^0 & h_{t-2}^1 & \cdot & \cdots & h_1^{t-2} & h_0^{t-1}
\end{array}$$

This cell delay is initialized at $h_0^0$.

Figure 8.

Generating Cholesky Factors and Correlations from Reflection Coefficients.

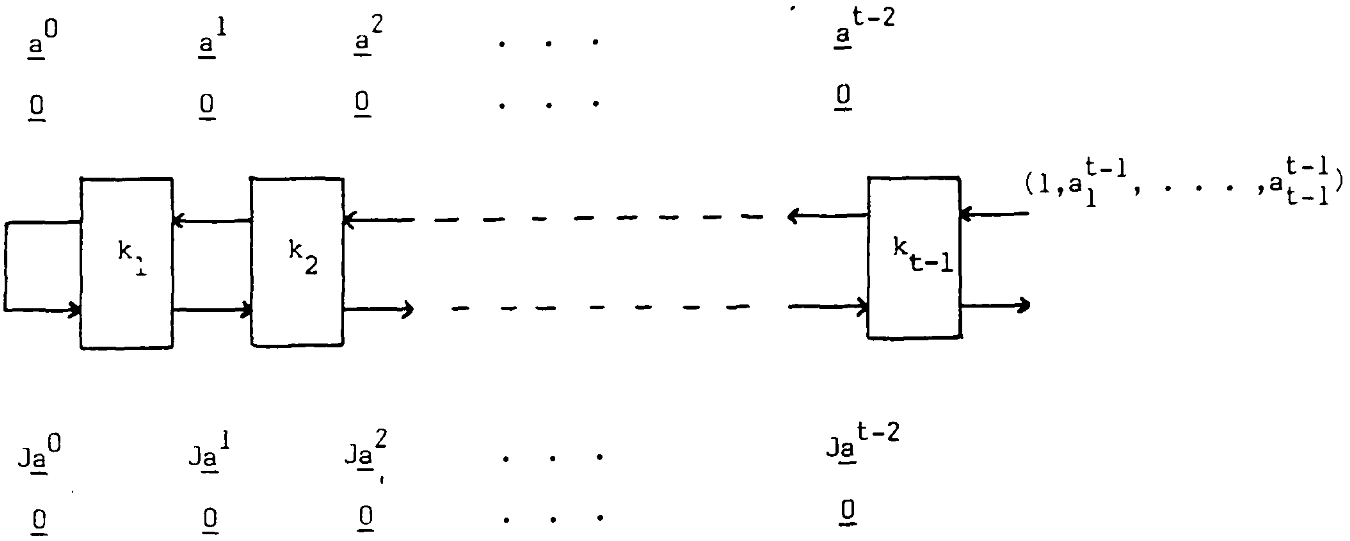


Figure 9.

Generating low Order Whiteners from a High Order One.

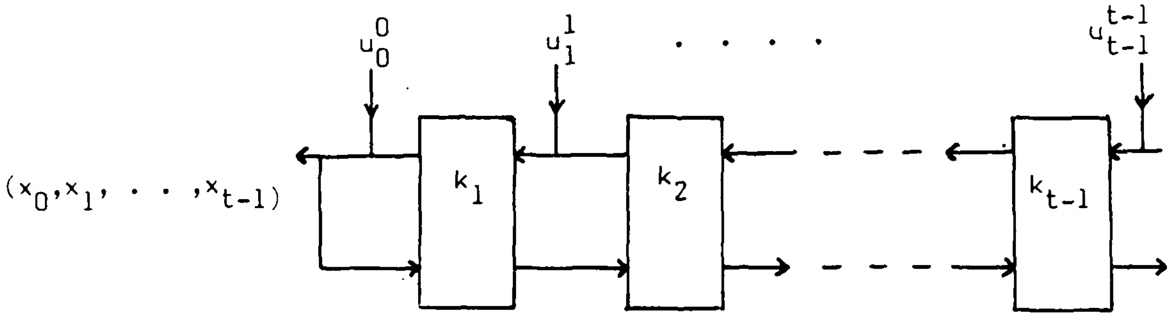


Figure 10.

Coloring or synthesis of correlated data.

## 5. Autoregressive Moving Average Sequences

So far we have dealt exclusively with a wide-sense stationary time series $\{x_n\}$ whose correlation sequence $\{r_n\}$ is even and non-negative definite. A finite record of the series $\{x_n\}_0^{t-1}$ has a correlation matrix R which is symmetric, Toeplitz, and non-negative definite. We now specialize our results to the case where the correlation sequence obeys the coupled autoregressive moving average (ARMA) recursions

$$\sum_{n=0}^{p} a_n r_{t-n} = \sum_{n=0}^{p} b_n h_{t+n} \quad ; a_0 = 1$$

$$\sum_{n=0}^{p} a_n h_{t-n} = b_t \qquad ; b_0 = b_0$$

Such a correlation sequence is said to be ARMA (p,p).

Whenever these recursions hold for the correlation sequence, the underlying time series itself is said to be ARMA (p,p). It obeys the autoregressive moving average recursion

$$\sum_{n=0}^{p} a_n x_{t-n} = \sum_{a=0}^{p} b_n u_{t-n}$$

$\{u_t\}$: sequence of uncorrelation random variables with unit variance.

The operator or transfer function representation of $\{x_n\}$ is

$A(z)\{x_t\} = B(z)\ \{u_t\}$

$$A(z) = \sum_{n=0}^{p} a_n z^{-n}; \; B(z) = \sum_{n=0}^{p} b_n z^{-n}$$

This may also be written as an infinite moving average,

$$\{x_t\} = H(z) \{u_t\} ,$$

where the transfer function H(z) obeys the recursion

$$A(z)H(z) = B(z)$$

$$\sum_{n=0}^{p} a_n h_{t-n} = b_t$$

The spectrum $R(z) = H(z)H(z^{-1})$ obeys the recursion

$$A(z)R(z) = B(z)H(z^{-1})$$

$$\sum_{n=0}^{p} a_n r_{t-n} = \sum_{n=0}^{p} b_n h_{t+n}$$

## 5.1 Stationary Wold Representation

The sequence $\{x_n\}$ may be decomposed as follows:

$$x_t = \hat{x}_{t/t-1} + h_0 u_t$$

$$\hat{x}_{t/t-1} = (H(z) - h_0)u_t$$

$$= \{B(z)/A(z) - h_0\}u_t$$

This may be rewritten as

$$A(z) \hat{x}_{t/t-1} = (B(z) - A(z)h_0) \{u_t\}$$

$$x_t = \hat{x}_{t/t-1} + h_0 u_t$$

Define the polynomials

$$z^{-1}Q(z) = B(z) - h_0 A(z)$$

$$z^{-1} P(z) = 1 - A(z)$$

The previous equations can then be re-cast as follows:

$$\hat{x}_{t+1/t} = P(z) \hat{x}_{t/t-1} + Q(z)u_t$$

$$x_t = \hat{x}_{t/t-1} + h_0 u_t$$

A block diagram is illustrated in Figure 11. The variable $\hat{x}_{t/t-1}$ is the minimum mean-squared error prediction of $x_t$ based on the infinite past $\{\ldots u_1, u_0, \ldots, u_{t-1}\}$.

For this structure to be used as a synthesizer of a stationary time series, the inputs $u_t$ must be initiated infinitely far in the past. Alternatively, the initial conditions in the structure must be set appropriately. This brings us to the Markovian representation.

## 5.2 Markovian Representation

The Markovian state space representation corresponding to the stationary Wold representation is

$$\underline{x}_{t+1} = F\, \underline{x}_t + \underline{h}\, u_t$$

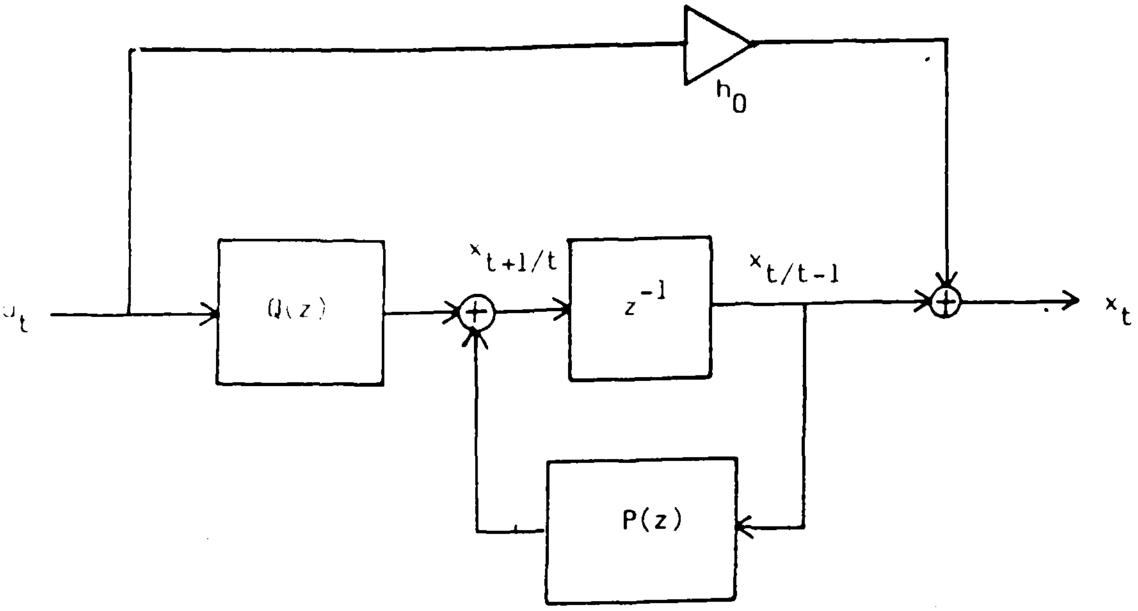
$$x_t = \underline{\delta}'\, \underline{x}_t + h_0 u_t$$

$u_t$: sequence of uncorrelated, unit variance random variables

In this state space model, the vectors and matrices are defined as follows:

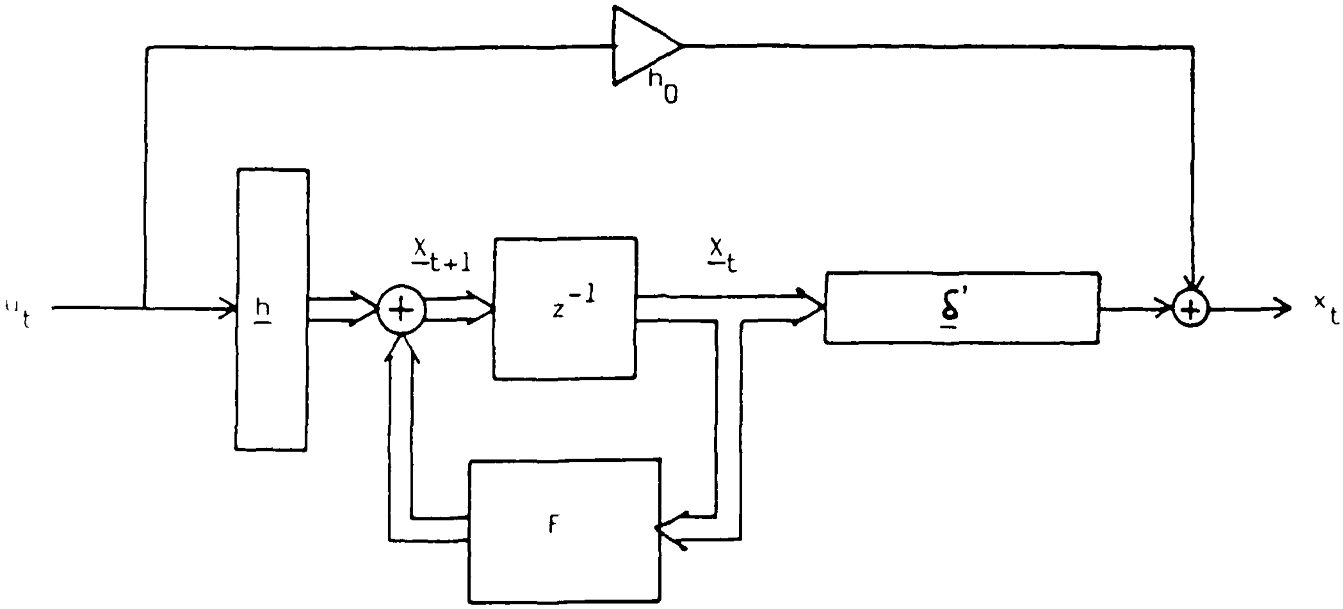
$$\underline{x}_t = \begin{bmatrix} \hat{x}_{t/t-1} \\ \hat{x}_{t+1/t-1} \\ \cdot \\ \cdot \\ \cdot \\ \hat{x}_{t+p+1/t-1} \end{bmatrix} \qquad \delta = \begin{bmatrix} 1 \\ 0 \\ \cdot \\ \cdot \\ \cdot \\ 0 \end{bmatrix}$$

$$F = \begin{bmatrix} 0 & 1 & 0 \ldots & 0 \\ 0 & 0 & 1 \ldots & \\ \cdot & & \ddots & \\ \cdot & & & \\ \cdot & & & 1 \\ -a_p & & \ldots & -a_1 \end{bmatrix} \qquad \underline{h} = \begin{bmatrix} h_1 \\ h_2 \\ \cdot \\ \cdot \\ \cdot \\ h_p \end{bmatrix}$$



Fiqure 11.

Stationary Wold Representation.



Fiqure 12.

Markovian State-Space Representation.

$$\begin{bmatrix} 1 & & & \\ a_1 & 1 & 0 & \\ \cdot & & & \\ \cdot & & & \\ \cdot & & & \\ a_p & a_{p-1} & \cdots & 1 \end{bmatrix} \begin{bmatrix} h_0 \\ h_1 \\ \cdot \\ \cdot \\ \cdot \\ h_p \end{bmatrix} = \begin{bmatrix} b_0 \\ b_1 \\ \cdot \\ \cdot \\ \cdot \\ b_p \end{bmatrix}$$

The block diagram is illustrated in Figure 12.

The impulse response and correlation sequence for the Markovian representation has to match that of the stationary Wold representation. The impulse response is

$$h_t = \left| \begin{array}{ll} 0, & t < 0 \\ h_0, & t = 0 \\ \underline{\delta}' F^{t-1} \underline{h}, & t > 0 \end{array} \right.$$

The correlation sequence is

$$r_t = \underline{\delta}'\ F^t\ Q_0\ \underline{\delta} + h_0\ h_t$$

$$(r_o = \underline{\delta}'\ Q_0\ \underline{\delta} + h_0{}^2)$$

$$Q_0 = FQ_0F' + \underline{h}\ \underline{h}'$$

The matrix $Q_0$ is the zero-lag state covariance:

$$Q_0 = E\underline{x}_0\underline{x}_0'$$

In order for this structure to synthesize a stationary sequence, the initial condition $\underline{x}_0$ must be drawn from a distribution with mean zero and covariance $Q_0$.

## 5.3 Innovations Representation

An innovations representation allows us to replace the constant vector

$\underline{h}$ with a time-varying Kalman gain $\underline{k}_t$, replace the random initial conditions with zero initial conditions, and replace the unit variance input sequence $\{u_n\}$ with a sequence $\{\sigma_n u_n\}$ whose variance is time varying. The representation is

$$\underline{x}_{t+1} = F\underline{x}_t + \underline{k}_t \, \sigma_t \, u_t$$

$$x_t = \underline{\delta}' \, \underline{x}_t + h_0 \, \sigma_t \, u_t$$

$u_t$: sequence of uncorrelated random variables with unit variance

This representation is illustrated in Figure 13.

For the innovations representation to produce a stationary sequence, we require the correlation of $\{x_t\}$ to be $\{r_n\}$. That is, we require

$$r_n^t = E \, x_t \, x_{t+n} = r_n$$

The expression for $r_n^t$ is

$$r_n^t = \underline{\delta}' \, F^n \, Q_t \, \underline{\delta} \; + h_0 \, \underline{\delta}' \, F^{n-1} \, \underline{k}_t \, \sigma_t^2$$

where $Q_t$ is the zero-lag state covariance at time t, for this to equal $r_t$ we require

$$\underline{k}_t \, \sigma_t = -F(Q_t - Q_0) + \underline{h}$$

$$\sigma_t^2 = \underline{\delta}'(Q_0 - Q_t)\underline{\delta} + h_0^2 = r_0 - \underline{\delta}' \, Q_t \, \underline{\delta}$$

This is the usual Ricatti solution given for the Kalman gain $\underline{k}_t$ and innovations variance $\sigma_t^2$ . But there is another way to go.

In the innovations representation, the output may be written as a time-varying convolution of a time-varying impulse response with the input $\{u_n\}$:

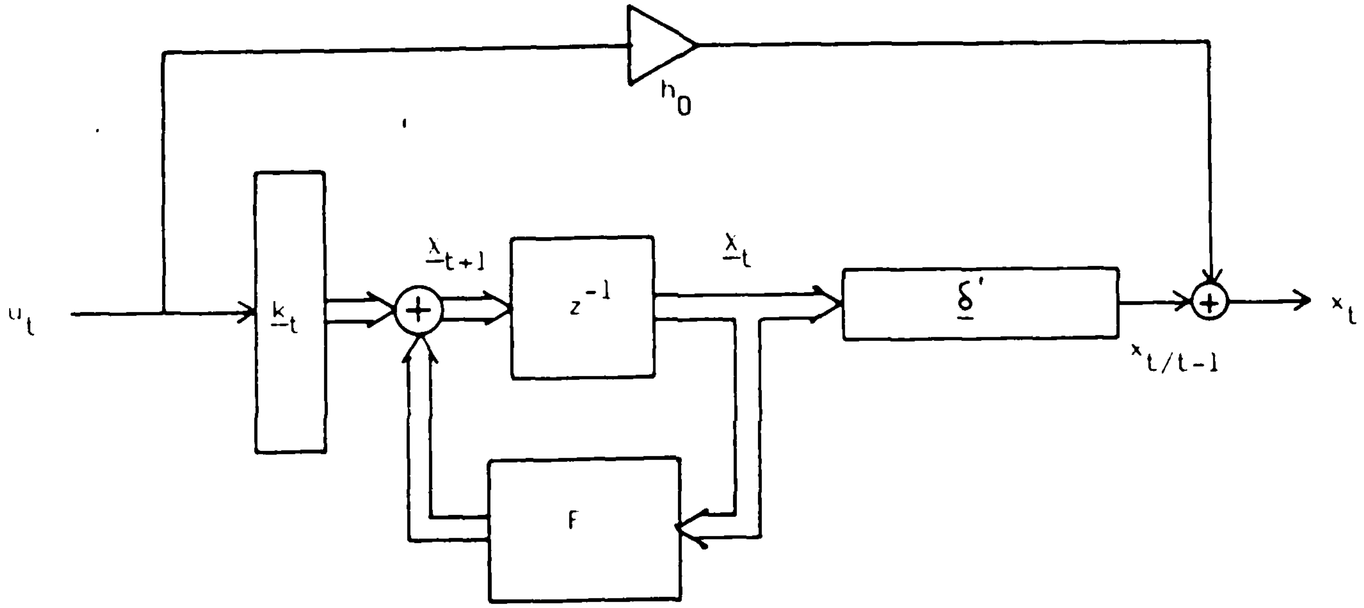


Figure 13.

Innovations Representation.

$$x_t = \sum_{n=0}^{t} h^n_{t-n} \quad \sigma_n u_n$$

The impulse response is

$$h^n_{t-n} = \left| \begin{array}{l} 0 , \quad t < n \\ h^0_0 = 1 , \; t = n \\ \underline{\delta}' \, F^{t-n-1} \, \underline{k}_n , \quad t > n \end{array} \right.$$

This means the vector of outputs $\underline{x} = (x_0, x_1, \ldots, x_{t-1})'$ may be written

$$\begin{bmatrix} x_0 \\ x_1 \\ \vdots \\ x_{t-1} \end{bmatrix} = \begin{bmatrix} h^0_0 & & & \\ h^0_1 & h^1_0 & & 0 \\ \vdots & h^1_1 & & \\ \vdots & \vdots & & \\ h^0_{t-1} & h^1_{t-2} & \cdots & h^{t-1}_0 \end{bmatrix} \begin{bmatrix} \sigma_0 & & & \\ & \sigma_1 & & 0 \\ & & \ddots & \\ & 0 & & \\ & & & \sigma_{t-1} \end{bmatrix} \begin{bmatrix} u_0 \\ u_1 \\ \vdots \\ u_{t-1} \end{bmatrix}$$

or as follows:

$$\begin{bmatrix} \\ \underline{x} \\ \\ \end{bmatrix} = \begin{bmatrix} & & & 0 \\ & & H & \\ \underline{h}^0 & \underline{h}^1 & \cdots & \underline{h}^{t-1} \end{bmatrix} \begin{bmatrix} \\ D \\ \\ \end{bmatrix} \begin{bmatrix} \\ \underline{u} \\ \\ \end{bmatrix}$$

So, in fact, the time varying impulse response in the innovations representation just fills in the Cholesky factorization

$$R = HD^2H'$$

corresponding to the synthesis

$$\underline{x} = HD\underline{u}$$

Why is this important? Remember we have a fast algorithm for obtaining the columns (or rows) of H. And, from the expression that relates the time varying impulse response to the Kalman gains we have the result

$$\underline{k}_n = \begin{bmatrix} h_1^n \\ h_2^n \\ \cdot \\ \cdot \\ \cdot \\ h_p^n \end{bmatrix}$$

This means we can run a fast Cholesky algorithm (either by columns or rows) and read out the Kalman gains $\underline{k}_n$ as the entries $h_1^n$ through $h_p^n$. This is a fast Kalman gain algorithm. An attractive implementation uses the lattice excited with a correlation sequence $\{r_n\}_0^{t-1}$. The gains are read out as internal lattice variables. This is illustrated in Figure 14. Of course the gains $\underline{k}_t$ inherit the same recursions as $\underline{h}^n$ and $\underline{g}^n$ in Chapter 4.

As the final topper to this story, note that the companion matrix F satisfies its own characteristic equation:

$$\sum_{t=n}^{n+p} a_{t-n} F^{n+p-t} = 0$$

Thus the time varying impulse response $h_{t-n}^n$ obeys the following recursion:

$$\underline{\delta}' \sum_{t=n}^{n+p} a_{t-n} F^{n+p-t} \underline{k}_n = \sum_{t=n}^{n+p} a_{t-n}\, h_{n+p-t+1}^n = \sum_{t=n+1}^{n+p+1} a_{p+1-t+n}\, h_{t-n}^n = 0$$

This means each column $\underline{h}^n$ may be generated by computing the variance $\sigma_n^2$, the

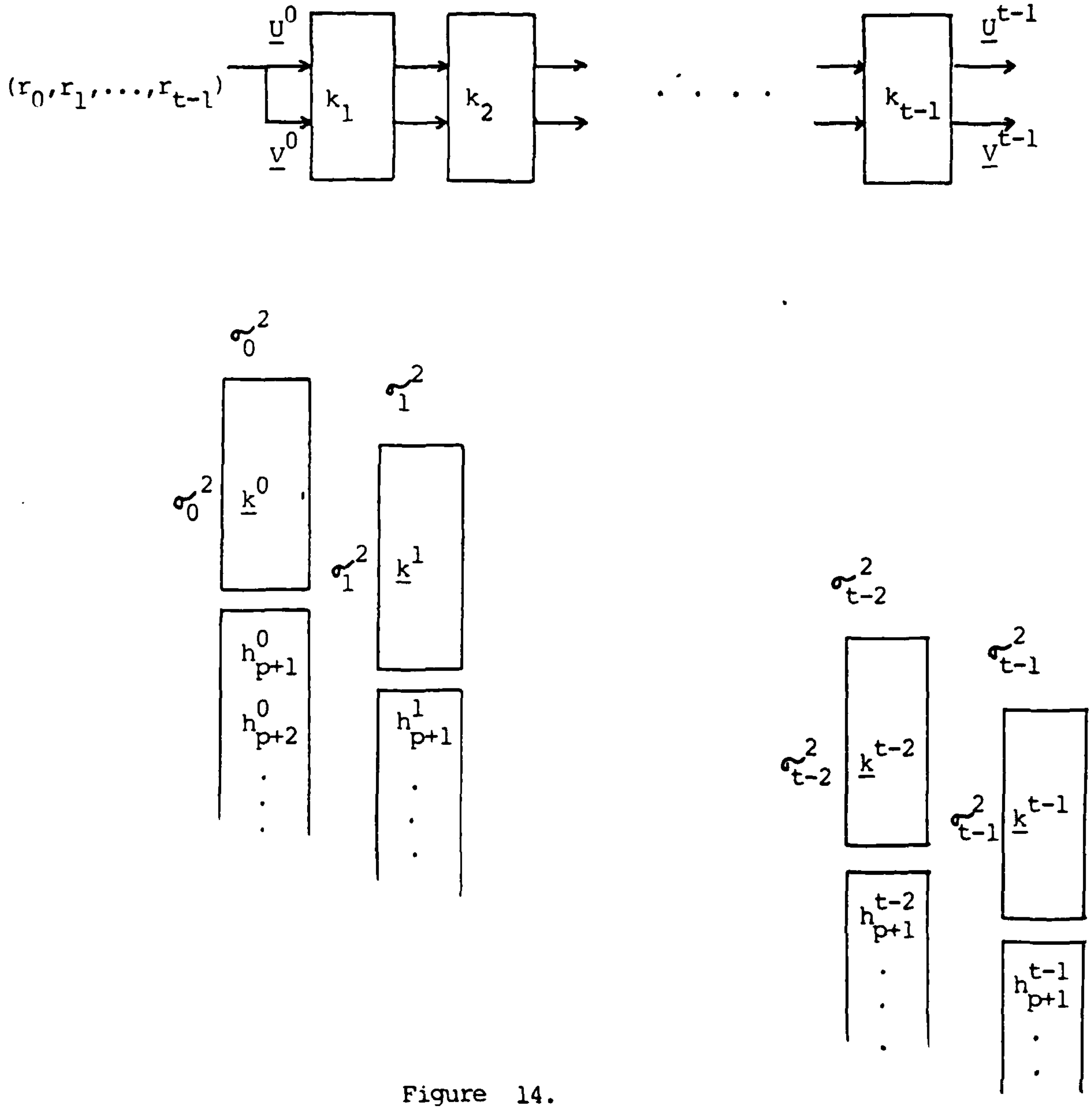


Figure 14.

Using the lattice to compute Kalman Gains.

Kalman gain $\underline{k}_n$, and then using the gain to initialize the recursion above to

fill out the columns. The same recursions can be derived for $\underline{g}_{p+1}^{n}$ [7]. This leads to the Morf, Sidhu, Kailath recursions for $\underline{k}_n$ [9].

## 6. Conclusions

Our conclusions are much like our introductory comments. Forward and inverse lattices may be used with a variety of excitations and initial conditions to produce internal lattice variables which are Cholesky factors of correlation matrices and their inverses. When the correlation matrix is ARMA, then these internal variables may be used to generate Kalman gains and to identify autoregressive parameters. The highest order Kalman gain may be identified with a stationary impulse response to obtain an estimate of the moving average coefficients.

In references [10] the vector recursions of Section 3.0 are presented as algorithms that may be implemented on a vector processing machine. In [11] a fixed length, time varying lattice is derived for implementing the Morf, Sidhu, Kailath recursions.

## 8. Appendix

Begin with the UL factorization of R:

$$RA' = HD^2$$

Write out the $n^{th}$ column of this equation:

$$\begin{bmatrix} r_0 & r_1 \cdot \cdot \cdot & r_n \\ r_1 & r_2 \cdot \cdot \cdot & r_{n+1} \\ \cdot & & \cdot \\ \cdot & & \cdot \\ \cdot & & \cdot \\ r_{t-1} & \cdot \cdot \cdot & r_{t-1-n} \end{bmatrix} \begin{bmatrix} J\underline{a}^n \end{bmatrix} = \sigma_n^2 \begin{bmatrix} \underline{0} \\ \\ \\ \underline{h}^n \end{bmatrix}$$

Ignore the zero terms in the right-hand column and write

$$\begin{bmatrix} r_n & r_{n-1} & \cdot \cdot \cdot & r_0 \\ r_{n+1} & \vdots & R_n & r_1 \\ & & & \vdots \\ r_{t-1} & & \cdot \cdot \cdot & r_{t-1-n} \end{bmatrix} \begin{bmatrix} J\underline{a}^n \end{bmatrix} = \sigma_n^2 \begin{bmatrix} \underline{h}^n \\ \\ \\ \end{bmatrix}$$

or

$$\sigma_n^2 \underline{h}^n = R_n J \underline{a}^n$$

A related vector, is $R_n \underline{a}^n$:

$$\begin{bmatrix} r_n & r_{n-1} \cdot \cdot \cdot & r_0 \\ r_{n+1} & R_n & r_1 \\ \vdots & & \vdots \\ r_{t-1} & \cdot \cdot \cdot & r_{t-1-n} \end{bmatrix} \begin{bmatrix} \underline{a}^n \end{bmatrix} = \left[\begin{array}{ccc} r_n \cdot \cdot \cdot & r_0 & x \\ \hline r_{n+1} & r_1 & r_0 \\ \vdots \;\; R_{n+1} & \vdots & \vdots \\ r_{t-1} \cdot \cdot \cdot & r_{t-1-n} & r_{t-n-2} \end{array}\right] \begin{bmatrix} \underline{a}^n \\ 0 \end{bmatrix} =$$

$$\sigma_n^2 \begin{bmatrix} g^n \\ \\ \end{bmatrix} = \sigma_n^2 \begin{bmatrix} 0 \\ \hat{g}^n \\ \\ \end{bmatrix}$$

The first term of $g^n$, denoted $g^n{}_0$, is zero. Note the definition of $R_{n+1}$ and summarize these results as follows:

$$\sigma_n^2 \underline{h}^n = R_n \, J\underline{a}^n$$

$$\sigma_n^2 \underline{g}^n = R_n \, \underline{a}^n$$

$$\underline{g}^n = \begin{bmatrix} 0 \\ \hat{\underline{g}}^n \end{bmatrix} \; ; \quad \sigma_n^2 \, \hat{g}_n = R_{n+1} \begin{bmatrix} \underline{a}^n \\ 0 \end{bmatrix}$$

Our objective is to derive recursions for $\underline{h}^n$, $\underline{g}^n$, and $\sigma_n^2$. To this end, add one more row to the equation governing $\sigma_{n+1}^2 \, \underline{h}^{n+1}$:

$$\sigma_{n+1}^2 \left[\begin{array}{c} \\ \underline{h}^{n+1} \\ \\ \hline 1_t^{n+1} \end{array}\right] = \left[\begin{array}{ccc} r_{n+1} & r_n & r_0 \\ & R_{n+1} & \\ \hline r_t & r_{t-1} & r_{t-n-1} \end{array}\right] J\underline{a}^{n+1}$$

$$\sigma_{n+1}^2 \, 1_t^{n+1} = r_t \, a_{n+1}^{n+1} + r_{t-1} \, a_n^{n+1} + \ldots + r_{t-n-1} \, a_0^{n+1}$$

Use the Levinson-Durbin recursion for $J\underline{a}^{n+1}$:

$$\sigma_{n+1}^2 \left[\begin{array}{c} \\ \underline{h}^{n+1} \\ \\ \\ \hline 1_t^{n+1} \end{array}\right] = \left[\begin{array}{ccc} r_{n+1} & r_n & r_0 \\ & R_{n+1} & \\ \\ r_{t-1} & & r_{t-n-2} \\ \hline r_t & & r_{t-n-1} \end{array}\right] \left( \begin{bmatrix} 0 \\ J\underline{a}^n \end{bmatrix} + k_{n+1} \begin{bmatrix} \underline{a}^n \\ 0 \end{bmatrix} \right)$$

$$= \sigma_n^2 \begin{bmatrix} \underline{h}^n \\ \\ \\ \end{bmatrix} + k_{n+1}\sigma_n^2 \begin{bmatrix} \hat{g}^n \\ \\ c_t^n/\sigma_n^2 \end{bmatrix}$$

From this recursion we note the following recursion for $\sigma_n^2$:

$$\sigma_{n+1}^2 = \sigma_n^2\ (1-k_{n+1}\ \hat{g}_0^n) = \sigma_n^2(1-k_{n+1}\ g_1^n)$$

To obtain a recursion for $g^n$, consider

$$\sigma_{n+1}^2 \begin{bmatrix} g^{n+1} \\ \\ \end{bmatrix} = R_{n+1}\ \underline{a}^{n+1}$$

Add one more row and use the Levinson-Durbin recursions to obtain

$$\sigma_{n+1}^2 \left[\begin{array}{c} g^{n+1} \\ \\ \\ \hline m_t^{n+1} \end{array}\right] = \left[\begin{array}{ccc} r_{n+1} & & r_0 \\ & R_{n+1} & \\ r_{t-1} & & r_{t-n-2} \\ \hline r_t & & r_{t-n-1} \end{array}\right] \left( \begin{bmatrix} \underline{a}^n \\ \\ 0 \end{bmatrix} + k_{n+1} \begin{bmatrix} 0 \\ \\ J\underline{a}^n \end{bmatrix} \right)$$

$$= \sigma_n^2 \begin{bmatrix} \hat{g}^n \\ \\ m_t^n \end{bmatrix} + k_{n+1}\ \sigma_n^2 \begin{bmatrix} \\ \underline{h}^n \\ \\ \end{bmatrix}$$

$$\sigma_n^2\ m_t^n = r_t + a_1^n\ r_{t-1} + \ .\ .\ .\ a_n^n\ r_{t-n}$$

From this recursion we note

$$k_{n+1} = -\hat{g}_0^n = -g_1^n$$

Substitute into the recursion for $\sigma_n^2$ to obtain

$$\sigma_{n+1}^2 = \sigma_n^2 \; (1-k_{n+1}^2)$$

We may summarize the recursions as follows:

$$\sigma_{n+1}^2 \begin{bmatrix} 0 \\ \underline{h}^{n+1} \\ 1_t^{n+1} \end{bmatrix} = \sigma_n^2 \begin{bmatrix} 0 \\ \underline{h}^n \\ \phantom{0} \end{bmatrix} + k_{n+1}\sigma_n{}^2 \begin{bmatrix} \phantom{0} \\ g^n \\ c_t^n/\sigma_n^2 \end{bmatrix}$$

$$\sigma_{n+1}^2 \begin{bmatrix} 0 \\ g^{n+1} \\ m_t^{n+1} \end{bmatrix} = \sigma_n^2 \begin{bmatrix} \phantom{0} \\ g^n \\ m_t^n \end{bmatrix} + k_{n+1}\sigma_n{}^2 \begin{bmatrix} 0 \\ \underline{h}^n \\ \phantom{0} \end{bmatrix}$$

If we keep only the relevant terms, and use our previous definitions of $\Delta$ and $\Omega$ we may write

$$\sigma_{n+1}^2 \;\; \underline{h}^{n+1} = \sigma_n^2 \;\; \Delta' \; \underline{h}^n + k_{n+1} \; \sigma_n^2 \;\; \Omega' \; g^n$$

$$\sigma_{n+1}^2 \;\; g^{n+1} = \sigma_n^2 \; \Omega' \; g^n + k_{n+1} \; \sigma_n^2 \;\; \Delta' \; \underline{h}^n$$

These recursions are initialized as follows:

$$\sigma_0^2\, \underline{h}^0 = (r_0,\ldots,r_{t-1})' \; ; \; \sigma_0^2\, \underline{g}^0 = (0,r_1,\ldots,r_{t-1})'$$

$$\sigma_0^2 = r_0$$

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