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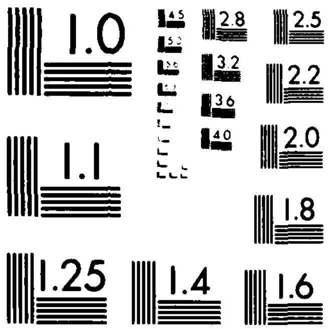

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A NEW APPROACH TO "QUEER" DIFFERENTIAL EQUATIONS

Peter Laurence and E. W. Stredulinsky

Mathematics Research Center
University of Wisconsin—Madison
610 Walnut Street
Madison, Wisconsin 53705

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ABSTRACT

We study the variational problem

$$J(\psi) = \int_\Omega |\nabla\psi|^2 + \int_0^{|\Omega|} (\psi^{*\prime}(v))^2 dv$$

$$\psi = 0 \text{ on } \partial\Omega, \ \psi^*(0) = 0, \ \psi^*(|\Omega|) = 1$$

where ψ^* is the increasing rearrangement of ψ. An approximate problem is introduced which involves a variational problem with n free boundaries ($n \to \infty$). Various estimates are established. In particular when Ω is convex it is shown that the solution to the approximate problem is superharmonic and has bounded gradient.

AMS (MOS) Subject Classification: 35J20

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*Courant Institute of Mathematical Sciences, New York University.

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SIGNIFICANCE AND EXPLANATION

Queer differential equations first arose in the work of Harold Grad on controlled thermonuclear fusion. They relate in particular to models for the slow adiabatic evolution and resistive diffusion of a plasma. They are queer in that ~~they~~ share aspects of partial, ordinary and functional differential equations. In ~~the present~~ work the authors give a new way of thinking of these equations by relating them to free boundary problems. This is the first of a series of papers intending to demonstrate that solutions of such a queer differential equation can be thought of as limits of solutions of free boundary problems with n-free boundaries. A long term hope is that this work will complement and further refine existing numerical schemes for finding such solutions.

The responsibility for the wording and views expressed in this descriptive summary lies with MRC, and not with the authors of this report.

A NEW APPROACH TO "QUEER" DIFFERENTIAL EQUATIONS

Peter Laurence* and E. W. Stredulinsky

Introduction.

In this work we study the so called queer differential equations. We introduce a new approach which gives insight into the types of estimates one can expect and which provides a setting in which it is hoped they can be rigorously established. In this approach we introduce an approximate problem involving many free boundaries. Exploiting the associated free boundary conditions and properties of harmonic functions we show that solutions are superharmonic and satisfy gradient bounds. It appears likely that these estimates will carry over to the original problem.

Queer differential equations (Q.D.E.s) arise in nuclear fusion research. They were introduced by Harold Grad as a way of more accurately modeling adiabatic compression [8] and resistive diffusion [5] of a plasma.

Historically "plasma equations" have been studied by many authors. These studies have essentially involved nonlinear elliptic equations and the associated free boundary problems. Other models which lead to Q.D.E.'s have been considered by J. Mossino, R. Temam [10,11], G. Vigfusson [15,16]. The origin of all the plasma equations is the Grad-Shafranov equation which in 2-dimensions (the case of physical interest) is

$$\Delta\psi = -p'(\psi) - \left\{\frac{f^2(\psi)}{2}\right\}'. \tag{1}$$

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*Courant Institute of Mathematical Sciences, New York University.

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Here $p(\psi)$, $f(\psi)$ are profile functions related respectively to the pressure and the total poloidal current [6]. Traditionally p, f have been prescribed.

The work of Harold Grad demonstrates that the quantities which are prescribed by the dynamics of the plasma are not $p(\psi)$, $f(\psi)$ but rather $\mu(\psi)$ and $\nu(\psi)$ given respectively by (see [5])

(2)
$$\mu(\psi) = \frac{p}{(\psi^{*\prime})^{\gamma}}$$

$$\nu(\psi) = \frac{f}{\psi^{*\prime}}$$

where $\gamma = 2$ in the two dimensional case. Here ψ^* is the increasing rearrangement of a function. To recall its definition we first introduce

(3)
$$V_\psi(t) = |\{x : \psi(x) < t\}|$$

(where $|E|$ is the Lebesgue measure of E). Then $\psi^*(v)$ is the inverse function of V_ψ. More exactly

(4)
$$\psi^*(v) = \inf\{s : V_\psi(s) > v\} .$$

Eliminating p, f from (1) in favor of μ and ν leads to

(5)
$$\Delta\psi = -\mu'(\psi)(\psi^{*\prime})^{\gamma} - \gamma\mu(\psi)(\psi^{*\prime})^{\gamma-2}\psi^{*\prime\prime}$$
$$- \tfrac{1}{2}(\nu^2(\psi))'(\psi^{*\prime})^2 - \nu^2(\psi)\psi^{*\prime\prime} .$$

Here $\psi^{*\prime}$ and $\psi^{*\prime\prime}$ denote the compositions

$$\psi^{*\prime}(V_\psi(\psi(x))) \quad \text{and} \quad \psi^{*\prime\prime}(V_\psi(\psi(x)))$$

respectively. From (5) it is clear that in a typical case second derivatives of ψ^* appear. For this reason we propose to study the model equation

(6)
$$\Delta\psi(x) = -\psi^{*\prime\prime}(V_\psi(\psi(x)))$$

obtained in 2-dimensions (where $\gamma = 2$) from the choice

$$\mu(\psi) = \tfrac{1}{2} , \quad \nu(\psi) = 0 .$$

An equation of this form was first considered by R. Temam in [13] where a variational formulation was given and the existence of a minimizer was established. There the author considered a domain $\Omega \subset \mathbb{R}^n$ and imposed the condition $\psi = 0$ on $\partial\Omega$. However physically [6] what one would like to also prescribe is the difference in ψ (the flux) between its absolute maximum and minimum (here the minimum is zero). This is equivalent to specifying $\psi^*(0)$ and $\psi^*(|\Omega|)$. We will take

$$\psi^*(0) = 0, \ \psi^*(|\Omega|) = 1 \ . \tag{7}$$

This is no loss of generality as it is easily checked that (6) is invariant under the transformation

$$\psi \to \lambda \psi + c$$

for constants λ, c.

In section 1 of this paper we will begin as in [13] by formulating a variational principle in Ω (an open set in $\mathbb{R}^n$) and establishing the existence of a minimizer in a function class which incorporates (7). We then derive certain estimates for the minimizer. Obstacles towards obtaining further estimates are the apriori unknown structure of the level sets and of the set of critical points. Thus, as an attempt to isolate regularity questions from these we introduce in section 2 a variational problem within a class of functions with convex level sets. In section 3 we approximate this problem by an n-shell free boundary problem which makes clearer the separate influences of the Laplacian term and the Q.D.E. term. Similar problems with one free boundary were first studied by Acker [1] and Caffarelli and Spruck [3]. In a future paper we intend to show that a solution of (6) is obtained as a limit of solutions to the approximate problem as $n \to \infty$.

In the present paper in section 4 we show that the solution to the approximate problem is superharmonic and satisfies a gradient bound. Superharmonicity is preserved under taking a weak limit of such solutions. There is strong evidence that such a limit is a solution of the Q.D.E. (6) and that the gradient bounds are preserved although all details have not been worked out as of this writing. Recent work of Caffarelli and Friedman [2] may make possible the extension of this approach to the nonconvex case.

We would like to thank H. Grad for suggesting this problem and for his warm encouragement and helpful conversations throughout this work. In addition we would like to thank L. Caffarelli, A. Friedman, J. Spruck and W. Ziemer for useful conversations.

1. A Variational Problem and Estimates for its Minimizers

In this section we will introduce a variational formulation for the problem (6). This is similar to the variational problem considered in [13], however in [13] boundary conditions were not imposed on ψ^*. We begin with a brief discussion of the formal relationship between the Q.D.E. (6) and the variational problem (8,9). We then establish the existence of minimizers and go on to prove various estimates for stationary solutions ψ of the variational problem. For instance it is shown that ψ^* is Lipshitz.

Let Ω be an open bounded set in $\mathbb{R}^n$ and define

$$J(\psi) = \int_\Omega |\nabla\psi|^2 dx + \int_0^{|\Omega|} (\psi^{*\prime})^2 dv \,. \tag{8}$$

We seek to minimize $J(\psi)$ over ψ in

$$W = \{\psi \in \overset{\circ}{W}{}^{1,2}(\Omega) : \psi^* \in W^{1,2}(0,|\Omega|),\ \psi^*(0) = 0,\ \psi^*(|\Omega|) = 1\} \,. \tag{9}$$

By $W^{1,2}(E)$ we mean the Sobolev space of square integral functions on E with square integral first derivatives and by $\overset{\circ}{W}{}^{1,2}(E)$ we mean the closure of $C_0^\infty(E)$ in $W^{1,2}(E)$.

Clearly we have imposed the boundary condition

$$\psi = 0 \quad \text{on} \quad \partial\Omega.$$

In addition the constraints $\psi^*(0) = 0$, $\psi^*(|\Omega|) = 1$ are equivalent to specifying the inf and sup of ψ to be zero and one respectively. The condition $\psi^*(0) = 0$ is actually superfluous, see Remark 1.1.

We will now give a brief derivation of the Euler equation for (8,9). The calculations are somewhat formal but illustrative.

Suppose ψ is a minimizer or simply a stationary point and $\psi \in C^2(\bar{\Omega})$. Let

$$p_t = \{x \in \Omega : \psi(x) = t\}.$$

Since $\psi \in C^2(\bar{\Omega})$ we have that $\Omega\backslash(p_0 \cup p_1)$ is open and consequently

$$\sup_K \psi < 1, \qquad \inf_K \psi > 0$$

for any compact set $K \subset \Omega\backslash(p_0 \cup p_1)$. Given $\eta \in C_0^\infty(\Omega\backslash(p_0 \cup p_1))$ we then have $\psi_\varepsilon = \psi + \varepsilon\,\eta \in W$ for small ε and we can carry out the following variation.

$$(10) \qquad 0 = \frac{d}{d\varepsilon} J(\psi_\varepsilon)\Big|_{\varepsilon=0} = 2 \int_\Omega \nabla\psi \cdot \nabla\eta \, dx$$

$$+ 2 \int_0^{|\Omega|} \psi^{*\prime} < \eta >' dv$$

where

$$< \eta > (v) = \int_{s(v)} \frac{\eta}{|\nabla\psi|} dH^{n-1} \Big/ \int_{s(v)} \frac{1}{|\nabla\psi|} dH^{n-1}$$

with

$$s(v) = \{x : V_\psi(\psi(x)) = v\}$$

and H^{n-1} being $n - 1$ dimensional surface measure (Hausdorff measure). The second term on the right hand side of (10) is obtained by using the following formula along with the assumption that differentiation with respect to ε, v can be interchanged,

$$\frac{d}{d\varepsilon} (\psi + \varepsilon\eta)^*(v)\Big|_{\varepsilon=0} = < \eta > (v) .$$

This formula can be rigorously derived under the assumption that $|\nabla\psi| \neq 0$ in $\bar{\Omega}$, [11]. An integration by parts now yields,

$$\int_\Omega \eta\Delta\psi \, dx + \int_0^{|\Omega|} \psi^{*\prime\prime} < \eta > dv = 0$$

which can be expressed after use of the coarea formula (12) as

$$\int_\Omega \eta\Delta\psi \, dx + \int_\Omega \psi^{*\prime\prime}(V_\psi(\psi(x)))\eta(x) \, dx = 0 .$$

Therefore

$$\Delta\psi = -\psi^{*\prime\prime} \quad \text{in} \quad \Omega \backslash (P_0 \cup P_1) .$$

It is possible to perform variations involving η's whose support does not lie in $P_0 \cup P_1$ (see remark 1.4). These however bring into play certain boundary terms due to the integration by parts in the second term of (10). The interpretation of such terms requires an understanding of the set of critical points of the minimizer ψ and its level sets.

We now present an existence result and various estimates for stationary points of the variational problem. For reasons of clarity we defer the proofs until later.

Theorem 1.1. The infimum of $J(\psi)$ in W is attained.

Remark 1.1. There is a weak maximum principle for the variational problem in the following sense. It is not necessary to impose the condition $\psi^*(0) = 0$. If this assumption is dropped from the definition of W, theorem 1.1 remains true and it can be demonstrated that $\psi^*(0) = 0$ still holds for any minimizer ψ. See the end of the proof of theorem 1.1 for details.

Remark 1.2. It is not known whether the minimizer is unique in general. It is unique however in the radial case, see remark 1.5.

We would now like to introduce a regularity result for ψ^*. To do this we introduce a function

$$W(t) = \int_{\{\psi<t\}} |\nabla\psi|^2 dx \ .$$

Clearly W is an increasing function. Let dW be the induced Lebesque-Stieltjes measure. Note that if we interpret ψ properly (see (12)) it follows from the coarea formula that dW is absolutely continuous and

$$\frac{dW}{dt}(t) = \int_{\{\psi=t\}} |\nabla\psi| dH^{n-1} \text{ a.e. } .$$

Theorem 1.2. If ψ is a stationary point for the variational problem (8,9) then

$$\frac{dW}{dt} + \psi^{*\prime}(V_\psi) = J(\psi) \text{ a.e.}$$

and therefore ψ^* is Lipshitz.

Remark 1.3. The methods of theorem 1.2 apply to the problem of minimizing $J(\psi)$ over $\psi \in W$ having a given collection of level sets $\{\psi = t\} = H(t)$. This is reminiscent of a numerical scheme introduced by Harold Grad [7], see remark 1.6.

In order to present several further estimates we need a general version of the coarea formula. In this respect we discuss averages of Sobolev functions. For $f \in W^{1,2}(\Omega)$ let

$$f_r(x) = \frac{1}{|B_r(x)|} \int_{B_r(x)} f(y)dy$$

where $B_r(x) = \{y : |x - y| < r\}$. Also let $\lambda(x)$, $\mu(x)$ be the approximate lower and upper limits respectively of f at x (see [4], 2.9.12) and $\bar{\lambda}(x) = \liminf_{r\to 0} f_r(x)$, $\bar{\mu}(x) = \limsup_{r\to 0} f_r(x)$. It is easily shown that $f_r \to f$ in $W^{1,2}_{loc}(\Omega)$. From this it is seen that $f_r \to f$ pointwise quasi-everywhere, that is everywhere except possibly on a set of zero Newtonian capacity. Let $\bar{f}(x) = \lim_{r\to 0} f_r(x)$ where it exists. Clearly $\bar{f} = f$ in $W^{1,2}(\Omega)$. Throughout the rest of this section the particular representative (from a $W^{1,2}$ equivalence class) which we use will be the above limit of averages (which is defined quasi-everywhere). Using the fact that $f_r \to f$ in $W^{1,2}_{loc}(\Omega)$ it can be shown in addition that $\lambda(x) = \bar{\lambda}(x) = \mu(x) = \bar{\mu}(x)$ quasi-everywhere. Using this, part 14 of theorem 4.5.9 [7] and the fact that the Newtonian capacity of a set being zero implies that its H^{n-1} measure is zero we get the following result,

$$\int_\Omega g|\nabla f| = \int_{-\infty}^{\infty} \int_{\{f=t\}} g \, dH^{n-1} dt \tag{12}$$

for measurable functions f,g such that $g|\nabla f|$ is integrable and $f \in W^{1,2}$ (with the particular representative of f being chosen as above).

Using (12) we can now establish the following lemma.

__Lemma 1.1.__ Let $dV_u(t)$ be the Lebesgue Stieltjes measure induced by $V_u(t)$ for $u \in W^{1,2}(\Omega)$ then

i) $$dV_u(t) = (u^{*\prime}(V_u(t)))^{-1}dt + \sum_i |\{u = t_i\}| d\delta_{t_i}(t)$$

where $d\delta_{t_i}(t)$ is a unit mass concentrated at t_i and $\{t_i\}$ is the collection of points t for which $|\{u = t\}| > 0$.

ii) $$dV_u(t) = \int_{\{u=t\}} |\nabla u|^{-1} dH^{n-1} dt + dS(t)$$

where $S(t) = |\{u < t\} \cap \{|\nabla u| = 0\}|$

Proof of Remark 1.1 (weak maximum principle). Remove the condition $\psi^*(0) = 0$ from the definition of W. The proof of theorem 1.1 still holds. Assume ψ is a minimizer. Let

$$\psi_+ = \operatorname{Max}\{\psi, 0\} .$$

It is easily checked that $(\psi_+)^* = (\psi^*)_+$. Also $\psi_+ \in \overset{\circ}{W}{}^{1,2}(\Omega)$ with $\nabla\psi_+ = \chi_{\{\psi>0\}}\nabla\psi$ a.e. and $\psi_+^* \in W^{1,2}(0,|\Omega|)$ with $\psi_+^{*\prime} = \chi_{\{\psi^*>0\}}\psi^{*\prime}$ a.e. . Here $\chi_E = 1$ on E and $\chi_E = 0$ on the complement of E. Clearly $J(\psi_+) < J(\psi)$ unless $\psi > 0$ a.e.

Remark 1.4. This weak maximum principle in conjunction with a rescaling leads to the fact that W may be replaced by

$$\overline{W} = \{\psi \in \overset{\circ}{W}{}^{1,2}(\Omega) : \psi^* \in W^{1,2}(0,|\Omega|),\ \psi^*(|\Omega|) > \tag{13}$$

without altering the infimum of J.

Remark 1.5. In the case when Ω is a ball the variational problem has only one radial stationary point. This is in fact the unique minimizer. To see this let $\Omega = \{x : |x| < R\}$ and assume $\psi(x) = f(|x|)$ is a stationary point. The problem is easily reduced to considering an appropriate O.D.E., for which there is a unique solution satisfying

$$f'(r) = -J(\psi)(n\omega_n r^{n-1} + (n\omega_n r^{n-1})^{-1})^{-1}$$

where ω_n is the measure of the unit ball in $\mathbb{R}^n$. By using symmetrization (which leaves ψ^* fixed) we see that any minimizer must be radial and therefore unique.

Proof of Theorem 1.2. Let $\psi_\varepsilon = f_\varepsilon \circ \psi$ where $f_\varepsilon(x) = x + \varepsilon h(x)$, $h \in C_0^\infty(0,1)$ and ε is small enough that $\varepsilon h' < 1$ on $[0,1]$. We have

$$\nabla\psi_\varepsilon = f_\varepsilon'(\psi)\nabla\psi .$$

Also f is strictly increasing with C^∞ inverse so

$$V_{f\circ\psi}(f(t)) = V_\psi(t) .$$

Consequently

$$(f\circ\psi)^*(v) = f(\psi^*(v))$$

and

$$(f\circ\psi)^{*\prime}(v) = f'(\psi^*(v))\psi^{*\prime}(v) .$$

From this it is seen that

$$J(\psi_\varepsilon) = \int_\Omega (f'_\varepsilon(\psi))^2 |\nabla\psi|^2 dx + \int_0^{|\Omega|} (f'_\varepsilon(\psi^*(v))^2 (\psi^{*\prime}(v))^2 dv \ .$$

It is easily seen that the following formula holds for continuous g.

$$\int_0^{|\Omega|} g(t)dW(t) = \int_\Omega g(\psi)|\nabla\psi|^2 dx \ .$$

Therefore

$$\int_0^{|\Omega|} h'(t)dW(t) = \int_\Omega h'(\psi)|\nabla\psi|^2 dx \ .$$

Furthermore if $r(v)$ is increasing and absolutely continuous and g is Borel measurable then (see [12])

$$\int_a^b g(r(v))r'(v)dt = \int_{r(a)}^{r(b)} g(t)dt \ .$$

To apply this formula we introduce the following device. Let

$$\bar{V}_\psi(t) = \frac{V_\psi(t^+) + V_\psi(t^-)}{2}$$

where $V_\psi(t^+)$, $V_\psi(t^-)$ represent the limit of V_ψ from the right and left respectively. We then have

$$\psi^{*\prime}(v) = \psi^{*\prime}(\bar{V}_\psi(\psi^*(v))) \quad \text{a.e.}$$

since off of a countable set either both sides are zero or

$$v = \bar{V}_\psi(\psi^*(v)) \ .$$

Consequently letting $r(v) = \psi^*(v)$, $g(t) = h'(t)\psi^{*\prime}(\bar{v}_\psi(t))$ in this formula we get

$$\int_0^{|\Omega|} h'(\psi^*(v))\psi^{*\prime}(v)^2 dv = \int_0^1 h'(t)\ \psi^{*\prime}(\overline{V}_\psi(t))dt$$

$$= \int_0^1 h'(t)\psi^{*\prime}(V_\psi(t))dt\ .$$

Therefore

$$0 = \int_0^1 h'(t)(dW(t) + \psi^{*\prime}(V_\psi(t))dt)$$

and

(14) $$dW(t) + \psi^{*\prime}(V_\psi(t))dt = \overline{c}\quad \text{a.e.}\ .$$

Clearly ψ^* is Lipshitz since $dW(t) \geq 0$. Integrating (14) over (0,1) and reversing the above arguments leads to

$$\overline{c} = J(\psi).$$

Remark 1.6. The methods used above are closely related to a more geometrical way of looking at the variational problem. This approach which we outline below is made use of in numerical schemes devised by Harold Grad. The relationship between it and the variational problem (8,9) will remain formal until it can be proved that there are no intervals on which $\psi^{*\prime} = 0$ when ψ is a minimizer to (8,9).

Let

$$\overline{V}(x) = V_\psi(\psi(x))\ .$$

Clearly $\overline{V}$ is determined solely by the level sets of ψ and thus is a purely geometrical quantity. Let

$$K(v) = \int_{\{\overline{V}(x)=v\}} |\nabla v|\, dH^{n-1}\ .$$

It is possible to show that $J(\psi)$ is minimized by ψ satisfying

$$\psi^{*\prime}(v) = \frac{1}{1 + K(v)}$$

with $J(\psi)$ being

$$(15) \qquad \left\{ \int_0^{|\Omega|} \frac{1}{1 + K(v)}\, dv \right\}^{-1} .$$

The problem of minimizing (8) can thus be viewed as finding a collection of level sets for which (15) is minimized.

Proof of Lemma 1.1. By [14] $u^* \in W^{1,2}(0,a)$ for any $a < |\Omega|$. We establish the result on $(0,a)$ from which the general result follows.

Remove the "flat places" in the graph of u. That is let $\{(a_n,b_n)\}$ be the collection of intervals on which u^* is constant and define

$$\bar{u}^*(\bar{v}) = u^*(v)$$

for $\bar{v} = v - \sum_{b_n < v} (b_n - a_n)$ and $v \notin \bigcup_n (a_n,b_n)$. $\bar{u}^*$ is absolutely continuous and strictly increasing so it has absolutely continuous inverse g with

$$g'(t) = (\bar{u}^{*\prime}(g(t)))^{-1} \quad \text{a.e.} \; .$$

Also

$$\bar{u}^{*\prime}(\bar{v}) = u^{*\prime}(v)$$

for almost all $v \in (0,a)\setminus(\bigcup_n (a_n,b_n))$, in fact precisely for those v which are Lebesgue points for the set $(0,a)\setminus(\bigcup_n (a_n,b_n))$. It is easy to reconstruct $V_u(t)$ as the sum of g and an increasing step function with jumps corresponding to the "flat places" on the graph of u and so i) follows.

We can conclude ii) from an application of the coarea formula. To see this let

$$g = \chi_{E(t)}|\nabla u|^{-1}, \; E(t) = \{|\nabla u| > 0,\; u < t\}$$

so that

$$|E(t)| = \int_\Omega g|\nabla u|dx$$

$$= \int_{\min u}^{t} \int_{\{u=s\}} \frac{\chi_{\{|\nabla u|>0\}}}{|\nabla u|}\, dH^{n-1} ds \; .$$

But

$$0 = \int_\Omega \chi_{\{|\nabla u|=0\}} |\nabla u| dx = \int_{-\infty}^{\infty} \int_{\{u=t\}} \chi_{\{|\nabla u|=0\}} dH^{n-1} dt$$

so

(16) $$H^{n-1}(\{u = t,\ |\nabla u| = 0\}) = 0 \quad \text{a.e.}$$

therefore

$$V_u(t) = \int_{\min u}^{t} \int_{\{u=s\}} |\nabla u|^{-1} dH^{n-1}\, ds + |\{|\nabla u| = 0,\ u < t\}|$$

and ii) follows.

Since by ii) $ds(t)$ contains the singular part of $dV_u(t)$, that is (by i))

$$\sum_i |\{u = t_i\}| d\delta_{t_i}(t) \ ,$$

it is clear that iii) must hold.

<u>Proof of Proposition 1.1</u>. Let $s(t) = \{\psi = t\}$. For almost all t we have

$$\int_{s(t)} |\nabla\psi| < \infty, \ H^{n-1}(\{\psi = t,\ |\nabla\psi| = 0\}) = 0$$

(see (16)) and lemma 1 iii) holding with $u = \psi$. For such a t choose ε such that

$$\int_{s(t)} |\nabla\psi| dH^{n-1} = \varepsilon H^{n-1}(s(t))$$

and let

$$E = \{x \varepsilon s(t) : |\nabla\psi| \le 2\varepsilon\} .$$

From this we see that

$$2\varepsilon H^{n-1}(s(t) \backslash E) \le \varepsilon H^{n-1}(s(t))$$

so

$$H^{n-1}(s(t)) \le 2H^{n-1}(E)$$

but

$$\begin{aligned}\psi^{*\prime}(V_\psi(t)) &\le \Big(\int_{s(t)} |\nabla\psi|^{-1}\Big)^{-1} \le 2\varepsilon(H^{n-1}(E))^{-1} \\ &\le 4\varepsilon(H^{n-1}(s(t)))^{-1} \\ &= 4(H^{n-1}(s(t)))^{-2} \int_{s(t)} |\nabla\psi| dH^{n-1} .\end{aligned}$$

Combining this with theorem 1.2 gives i).

We now wish to use the following "isoperimetric inequality",

$$(|\Omega| - V_\psi(t))^{\frac{n-1}{n}} \leq n^{-1}\omega_n^{-1/n} H^{n-1}(\{\psi = t\}) \quad \text{a.e.} \tag{17}$$

where ω_n is the volume of the unit ball in $\mathbb{R}^n$. For smooth ψ it is a direct consequence of the standard isoperimetric inequality and the fact that almost all level sets of ψ are smooth (Sard's theorem, implicit function theorem). For $\psi \in \overset{\circ}{W}{}^{1,2}(\Omega)$ one can combine the Sobolev inequality

$$\left\{ \int_\Omega u^{n/(n-1)} \right\}^{\frac{n-1}{n}} \leq n^{-1}\omega_n^{-1/n} \int_\Omega |\nabla u|$$

with the coarea formula (12) to prove (17).

By (16) we have

$$H^{n-1}(\{\psi = t, |\nabla\psi| = 0\}) = 0 \quad \text{a.e.}$$

so

$$(H^{n-1}(\{\psi = t\}))^2 \leq \int_{\{\psi=t\}} |\nabla\psi|^{-1} dH^{n-1} \int_{\{\psi=t\}} |\nabla\psi| dH^{n-1} \quad \text{a.e.}$$

$$= \int_{\{\psi=t\}} |\nabla\psi|^{-1} dH^{n-1} (J(\psi) - \psi^{*\prime}(V_\psi(t)))$$

by theorem 1.2. Now combining this with lemma 1.1 iii) and (17) we get

$$\psi^*(V_{\psi(t)}) \leq J(\psi)\left\{1 + n^2\omega_n^2 (|\Omega| - V_\psi(t))^{\frac{2(n-1)}{n}}\right\}^{-1}$$

almost everywhere. We can easily recover ii) since $\psi^{*\prime}$ is zero on (a_i,b_i), the intervals where ψ^* is constant, and if F has measure zero then

$$|V_\psi((0,1)\backslash F)| = |(0,|\Omega|)| - \sum_i (b_i - a_i)$$

(recall the proof of lemma 1.1 i)). From remark 1.5 it can be seen that ii) is sharp if Ω is a ball.

2. The Variational Problem for Functions with Convex Level Sets

For a convex domain Ω we expect minimizers to the variational problem (8,9) to have nested convex level sets. This motivates us at the present stage to introduce such a condition as a constraint on the variational problem. The hope is that solutions of the constrained problem will correspond to solutions of the original problem. This is justified to some extent by the results to be presented in [9]. For simplicity we restrict our attention to $\Omega \subset \mathbb{R}^2$. This is the case of physical interest.

We will first present the technical setting for the constrained problem and then prove existence and continuity of solutions.

Let us begin by giving a weak definition for the class of functions with convex level sets. We have chosen this particular definition since it accomodates the "almost everywhere defined" nature of functions in $W^{1,p}(\Omega)$ and more importantly allows us to avoid the technicalities associated with "flat places" in the graph of a function, that is places where level sets have positive measure.

For an open convex set Ω let

(18) $C_p = \{u \in W^{1,p}(\Omega)$ *: There is a dense set* $\{t_n\}$, $t_n \in (\inf u, \sup u)$ and there are convex sets C_n such that $u > t_n$ a.e. in C_n and $u < t_n$ a.e. in $\Omega \backslash C_n\}$.

In this context it is easy to see what form "flat places" take. To see this consider a function $u \in C_p$. Let $\{t_n\}$, $\{C_n\}$ be as in (18). Given $t \in (\inf u, \sup u)$ choose subsequences $\{t_{1,n}\}$, $\{t_{2,n}\}$ of $\{t_n\}$ such that $t_{1,n} \uparrow t$ and $t_{2,n} \downarrow t$ and let the associated convex sets be $C_{1,n}$, $C_{2,n}$. It is easy to check that $\{u = t\}$ differs from

$$\Gamma(t) = (\bigcap_n C_{1,n}) \backslash (\bigcup_n C_{2,n}) \tag{19}$$

by a set of measure zero. If $|\{u=t\}| > 0$ then $\Gamma(t)$ is a region bounded by the convex curves

$$\partial(\bigcap_n C_{1,n}), \quad \partial(\bigcup_n C_{2,n}) .$$

If $|\{u=t\}| = 0$ then $\Gamma(t)$ is a convex curve. In the case where $t = \inf u$ or $t = \sup u$ one can use $C_{2,n}$, $C_{1,n}$ respectively to show that $\{u=t\}$ differs from

$\Omega\backslash(\bigcup_n C_{2,n})$ and $\bigcap_n C_{1,n}$ respectively by a set of measure zero. In the first case a "flat place" is bounded by $\partial\Omega$ and $\partial(\bigcup_n C_{2,n})$ and in the second by $\partial(\bigcap_n C_{1,n})$.

We can now state the constrained problem as that of minimizing $J(\psi)$ over $W \cap C_2$, that is

$$\mathrm{Inf}\{J(\psi) : \psi \in W \cap C_2\} \ . \tag{20}$$

Theorem 2.1. The infimum in (20) is attained in $W \cap C_2$.

Remark. Theorem 2.1 is an immediate consequence of the proof of theorem 1.1 and the following result on the weak closedness of C_2.

Proposition 2.1. Given $\psi_n \in C_p$ such that $\psi_n \to \psi$ weakly in $W^{1,p}(\Omega)$, $p > 1$ and $t \in (\inf \psi, \sup \psi)$ then there exists a convex set C such that $\psi > t$ a.e. in C and $\psi < t$ a.e. in $\Omega\backslash C$.

Corollary 2.1. C_p is weakly closed in $W^{1,p}(\Omega)$ for $p > 1$.

In proving proposition 2.1 we need the following lemma. It is actually true that $u^* \in W^{1,p}_{loc}(0,|\Omega|)$ (see [13]) but only the weaker result is needed.

Lemma 2.1. If $u \in W^{1,p}(\Omega)$, $p > 1$ then u^* is continuous on $(0,|\Omega|)$.

In addition we will prove the following regularity result.

Theorem 2.2. If $u \in C_2 \cap L^\infty(\Omega)$ then u is continuous.

Remark. Since $\Omega \subset \mathbb{R}^2$, $W^{1,p}(\Omega)$ consists of continuous functions if $p > 2$.

Proof of Lemma 2.1. Assume u^* is discontinuous at $v \in (0,|\Omega|)$. There then exists t_1, t_2 such that $\inf u < t_1 < t_2 < \sup u$ and $|\{t_1 < u < t_2\}| = 0$. Let

$$f(t) = \begin{cases} 1 & t > t_2 \\ \dfrac{t - t_1}{t_2 - t_1} & t_1 < t < t_2 \\ 0 & t < t_1 \end{cases}$$

so $f(u) \in W^{1,p}(\Omega)$ and $\nabla f(u) = f'(u)\nabla u$ a.e. . Since $|\{t_1 < u < t_2\}| = 0$ we have $\nabla f(u) = 0$ a.e. and also $f(u) = f(u) = \chi_{\{u>t_2\}}$ a.e. (χ_F equals 1 on F and zero on $\Omega\backslash F$). Since $0 < v < |\Omega|$ we have $|\{u > t_2\}| > 0$ and $|\{u < t_2\}| > 0$. Let

$E = \{u < t_2\}$. Using the Sobolev inequality (for $|E| > 0$)

$$\int_\Omega |\phi - \frac{1}{|E|}\int_E \phi|^p \le C(\Omega,|E|)\int_\Omega |\nabla\phi|^p$$

with $\phi = f(u)$ we conclude that

$$|\{u > t_2\}| = 0$$

which contradicts $|\{u > t_2\}| > 0$ and we are done.

<u>Proof of Proposition 2.1</u>. Assume $\psi_n \to \psi$ weakly in $W^{1,p}(\Omega)$ and $\psi_n \in \mathcal{C}_p$, $p > 1$. Choose a subsequence (still represented by ψ_n) for which $\psi_n \to \psi$ both in $L^2(\Omega)$ and pointwise a.e. and $\psi_n^* \to \psi^*$ pointwise a.e.. Given $t \in (\inf u, \sup u)$ we can conclude by lemma 2.1 and the intermediate value theorem that there exists a v such that $\psi^*(v) = t$. Since ψ_n^*, ψ^* are increasing there is a $\bar{v} < |\Omega|$ such that $\psi_n^*(\bar{v}) \to \psi^*(\bar{v}) > t$. Choose t_n, C_n from the definition of $\psi_n \in \mathcal{C}_p$ such that $t_n \to t$ and choose v_n such that v_n is the largest v satisfying $\psi_n^*(v) = t_n$. From the monotonicity of ψ_n^* we have $v_n < \bar{v}$ for large n so

$$|C_n| \ge |\Omega| - v_n \ge |\Omega| - \bar{v} > 0 .$$

Since $C_n \subset \Omega$, Ω is bounded and C_n is convex there exist balls $B_n \subseteq C_n$ with $|B_n| \ge \lambda > 0$, λ independently of n. By choosing a subsequence and redefining the balls B_n we can assume in addition that they have a common center. Parameterizing ∂C_n using an angle θ we can apply Arzela-Ascoli to conclude that a subsequence of $\{\partial C_n\}$ converges uniformly to ∂C for some convex set C. The desired conclusion now follows easily from the properties of C_n, t_n (see (18)) and the fact that $\psi_n \to \psi$ pointwise a.e.. The corollary follows immediately.

<u>Proof of Theorem 2.2</u>. Assume that $\psi \in \mathcal{C}_2$ and that we have real numbers t_1, t_2 and convex sets C_1, C_2 such that $\inf \psi < t_1 < t_2 < \sup \psi$, $\psi \ge t_i$ a.e. on C_i and $\psi < t_i$ a.e. in $\Omega\backslash C_i$. The following calculation will show that ∂C_1, ∂C_2 do not intersect.

Let $d = \mathrm{dist}\,\{\partial C_1, \partial C_2\}$. Choose $x_0 \in \partial C_1$ such that $d = \mathrm{dist}\,\{x_0, \partial C_2\}$. Let $C(r)$ be the circle concentric to x_0 of radius r. Clearly $C(r)$ intersects

$\{\psi \geq t_2\}$ and $\{\psi \leq t_2\}$ on a set of positive one-dimensional measure for almost every $r > d$. Also for almost all r, ψ restricted to $C(r)$ is an absolutely continuous function with (1-dimensional) derivative given by $T \cdot \nabla\psi$ where T is the unit tangent to $C(r)$. Because of this we have for almost all $r > d$ that

$$t_2 - t_1 = \int_{A(r)} T \cdot \nabla\psi dH^1$$

for some arc $A(r) \subset C(r) \cap \Omega$. Therefore

$$\frac{(t_2 - t_1)^2}{r} \leq c \int_{C(r)\cap\Omega} |\nabla\psi|^2 dH^1$$

and integrating with respect to r from d to 1 we have

(21) $$(t_2 - t_1)^2 \log d^{-1} \leq c \int_{\Omega} |\nabla\psi|^2 .$$

Equivalently we have

(22) $$\exp(-(t_2 - t_1)^{-2} \int_{\Omega} |\nabla\psi|^2) \leq d .$$

We will now redefine ψ on a set of measure zero so that its level sets $\{\psi = t\}$ coincide with $\Gamma(t)$ (see(19)). (21) then becomes an estimate of the modulus of continuity of ψ and we are done.

Let t_n, C_n be as in the definition of $\psi \in C_2$. Let

$$E_n = \{x \in C_n : \psi < t\}, \; F_n = \{x \in \Omega \backslash C_n : \psi > t\} .$$

Clearly if $G = \bigcup_n (E_n \cup F_n)$ then $|G| = 0$. Let

$$\bar{\psi}(x) = \begin{cases} \sup \{t_n : x \in C_n\} & \text{for } x \in G \\ \psi(x) & \text{otherwise} . \end{cases}$$

It is easily checked that redefining ψ a.e. in this way forces $\{\bar{\psi} = t\} = \Gamma(t)$ as required.

3. An Approximating Free Boundary Problem

We now introduce an approximate problem which we will demonstrate to be a free boundary problem. This is done for several reasons. One, it regularizes the problem thus allowing estimates to be made. Two, it is a first step in achieving one of the goals of section 2, that is to produce a solution to the original variational problem which has convex level sets. A third advantage to this approach is that it allows us to seperate the influence of the two terms in $J(\psi)$.

In the next section we will establish free boundary conditions and present applications.

The approximation of $J(\psi)$ is achieved by replacing $\psi^{*'}$ by a difference quotient. This yields the following functional. Let

$$J_{\bar{n}}(\psi) = \int_\Omega |\nabla\psi|^2 + \sum_{i=1}^{\bar{n}} \frac{\bar{n}^{-2}}{|A_i|} \tag{23}$$

where $A_i = \{t_{i-1} < \psi < t_i\}$, $t_i = i/\bar{n}$, $i = 1,\dots,\bar{n}$. Clearly $|A_i| = V_\psi(t_i) - V_\psi(t_{i-1})$.

We seek to minimize $J_{\bar{n}}(\psi)$ over

$$F_0 = \{\psi \in \overset{\circ}{W}{}^{1,2}(\Omega) \cap C_2 : \psi^*(0) = 0,\ \psi^*(|\Omega|) = 1\} . \tag{24}$$

It turns out that $J_{\bar{n}}$ does not achieve its minimum on F_0. What one would expect to be the minimizer will be seen to satisfy $\psi^*(|\Omega|) = 1 - \frac{1}{\bar{n}}$. For this reason we introduce the following two function spaces

$$F_1 = \{\psi \in \overset{\circ}{W}{}^{1,2}(\Omega) \cap C_2 : \psi^*(0) = 0,\ 1 - 1/\bar{n} < \psi^*(|\Omega|) < 1\}$$

(25)

$$F_2 = \{\psi \in \overset{\circ}{W}{}^{1,2}(\Omega) \cap C_2 : \psi^*(0) = 0\} .$$

An approximate version of the constraint $\psi^*(|\Omega|) = 1$ is embedded in the functional itself. This is demonstrated by the following proposition. Note that as in section 1 the constraint $\psi^*(0) = 0$ is superfluous.

Proposition 3.1. $$J = \inf_{\psi \in F_i} J_{\bar{n}}(\psi)$$

is independent of i, $i = 0, 1, 2$.

This will be proved in conjunction with the following existence result.

Theorem 3.1. Let J be as in Proposition 3.1. Then $J_{\bar{n}}$ is minimized on F_1, that is there exists $\psi_0 \in F_1$ such that $J = J_{\bar{n}}(\psi_0)$. Also if $J_{\bar{n}}$ is minimized by some $\psi \in F_2 (F_2 \supset F_1)$, that is $J = J_{\bar{n}}(\psi)$ then we can redefine ψ on a set of measure zero so that ψ satisfies the following:

i) ψ is continuous

ii) There are nested convex curves γ_i, $i = 1,\ldots,\bar{n} - 1$ such that $\gamma_i = \{\psi = i/\bar{n}\}$ for $i = 1,\ldots,\bar{n} - 2$ and

$$\gamma_{\bar{n}-1} = \partial\{\psi = 1 - \frac{1}{\bar{n}}\}$$

where $\{\psi = 1 - 1/\bar{n}\}$ is a convex set. Consequently

iii) ψ is harmonic in $\Omega \setminus \{\bigcup_{i=1}^{\bar{n}-1} \gamma_i\}$.

Proof of Theorem 3.1. We first introduce a simple way of altering a function $\psi \in F_2$ with $J_{\bar{n}}(\psi) < \infty$ in such a way that $J_{\bar{n}}$ is reduced. To carry this out first note that $J_{\bar{n}}(\psi) < \infty$ implies that $|A_n| \neq 0$ so that $\sup \psi > 1 - 1/\bar{n}$. Now recall the remarks made subsequent to the definition of C_p. For each $i = 1,\ldots,\bar{n} - 1$ let $C^{(i)}_{j,n}$, $j = 1,2$ be the convex sets associated there with $t = i/\bar{n}$ (if $\sup \psi = 1 - 1/\bar{n}$ then we only have $C^{(i)}_{1,n}$). Let $\bar{\psi}$ be the continuous function which is $i/\bar{n}$ on $\gamma_i = \partial(\bigcap_n C^{(i)}_{1,n})$, $i = 1,\ldots,\bar{n} - 1$, and is harmonic elsewhere. Since V_ψ is left continuous we have $V_{\bar{\psi}}(i/\bar{n}) = V_\psi(i/\bar{n})$ and therefore $J_{\bar{n}}(\bar{\psi}) \le J_{\bar{n}}(\psi)$. By [3] we see that $\bar{\psi} \in C_2$ so $\bar{\psi} \in F_2$ as well. A consequence of the above is that if ψ is a minimizer then $\psi = \bar{\psi}$ a.e. . This establishes the second part of Theorem 3.1.

We now apply the above methods to proving existence. It is easy to see that there exists a function ψ_0 such that $J_{\bar{n}}(\psi_0) < \infty$ so choose a minimizing sequence $\{\psi_n\}$. As above replace ψ_n by $\bar{\psi}_n$. Choose a subsequence such that $\bar{\psi}_n \to \psi$ weakly in $W^{1,2}(\Omega)$.

As in the proof of proposition 2.1 we can choose a subsequence so that the curves $\gamma_i^{(n)}$, associated as above with $\bar{\psi}_n$, converge uniformly to curves γ_i with $\psi < i/\bar{n}$ a.e. outside γ_i and $\psi > i/\bar{n}$ a.e. inside γ_i. Let $\bar{\psi}$ be the continuous function which is $i/\bar{n}$ on γ_i and is harmonic elsewhere. It is clear from the uniform convergence of the $\gamma_i^{(n)}$ that $|A_i(\bar{\psi}_n)| \to |A_i(\bar{\psi})|$. Also

$$\int |\nabla\bar{\psi}|^2 \le \int |\nabla\psi|^2 \le \underline{\lim} \int |\nabla\psi_n|^2$$

so $\bar{\psi}$ minimizes $J_{\bar{n}}$. Clearly $\bar{\psi} \in \overset{\circ}{W}{}^{1,2}(\Omega)$, $\psi^*(0) = 0$ and $\bar{\psi}^*(|\Omega|) = 1 - 1/\bar{n}$ so $\bar{\psi} \in F_1$.

<u>Proof of Proposition 3.1</u>. Clearly

$$\inf_{F_0} J_{\bar{n}} \ge \inf_{F_1} J_{\bar{n}} \ge \inf_{F_2} J_{\bar{n}}$$

since $F_0 \subset F_1 \subset F_2$. From theorem 3.1 we know that there exists a minimizer of $J_{\bar{n}}$ in F_1 and that if $\psi \in F_2$ minimizes $J_{\bar{n}}$ over F_2 then $\psi \in F_1$. Therefore

$$\inf_{F_1} J_{\bar{n}} = \inf_{F_2} J_{\bar{n}} .$$

To finish we will show that

$$\inf_{F_0} J_{\bar{n}} \le \inf_{F_1} J_{\bar{n}}$$

by altering the minimizer ψ whose existence was proved in theorem 3.1.

Let C be the interior of the convex set $\{\psi = 1-1/\bar{n}\}$. Recall $|C| = |A_n| > 0$. Since the Newtonian capacity of a point is zero it is easy to construct functions $f_n \in C_0^\infty(C)$ with $\sup f_n = 1/\bar{n}$ such that

$$\int |\nabla f_n|^2 = o(n^{-1})$$

and $|\{f = 1/\bar{n}\}| = 0$. Now consider $\psi_n = \psi + f_n$. Clearly $\psi_n \in F_0$ and

$$J_{\bar{n}}(\psi_n) = J_{\bar{n}}(\psi) + o(1/n)$$

so that

$$\lim_{n\to\infty} J_{\bar{n}}(\psi_n) = J_{\bar{n}}(\psi)$$

and we are done.

4. Free Boundary Conditions and Applications

Having established existence for the approximate problem (23,24) we now go on to establish the jump conditions across the free boundaries γ_i (see theorem 3.1). These will be applied to show that minimizers ψ of $J_{\bar{n}}$ are superharmonic and satisfy L^∞ gradient bounds.

The derivation of the free boundary conditions which we give here is formal. Our first purpose is to shed light on their consequences so we have left the justification of these formulas to another paper [9].

We will now formally apply the Hadamard variational principle. The result will be a formula for how the normal derivative of ψ jumps across each free boundary γ_i.

Given i we restrict our variations to $\Omega_i = A_i \cup A_{i+1}$. Let T_ε be a one parameter family of smooth diffeomorphisms of Ω_i onto Ω_i which coincide with the identity map on some fixed neighborhood of $\partial\Omega_i$. Let us denote

$$\frac{\partial T_\varepsilon}{\partial \varepsilon}\bigg|_{\varepsilon=0} = \underset{\sim}{f} .$$

Using the trial function $\psi o T_\varepsilon$ we compute the variation of $J_{\bar{n}}$.

$$\frac{d}{d\varepsilon}\,(J_{\bar{n}}(\psi o T_\varepsilon))\bigg|_{\varepsilon=0} = \int_{\Omega_i} |\nabla\psi|^2 \nabla \cdot \underset{\sim}{f} + \nabla\psi^T \cdot D\underset{\sim}{f} \cdot \nabla u$$

$$+ \int_{\gamma_i} \bar{n}^{-2} (|A_i|^{-2} - |A_{i+1}|^{-2}) \underset{\sim}{f} \cdot \underset{\sim}{n}$$

where $\underset{\sim}{n}$ is the unit normal to γ_i directed into A_i. The first term on the right hand side may as usual be turned into a divergence and then a surface integral,

$$\frac{d}{d\varepsilon} J_{\bar{n}}(\psi o T_\varepsilon)\Big|_{\varepsilon=0} = \int_{\gamma_i} \left[(\frac{\partial\psi}{\partial n_e})^2 - (\frac{\partial\psi}{\partial n_i})^2\right] \underset{\sim}{f} \cdot \underset{\sim}{n}$$

$$+ \int_{\gamma_i} \bar{n}^{-2} (|A_i|^{-2} - |A_{i+1})^{-2}) \underset{\sim}{f} \cdot \underset{\sim}{n} ,$$

$\frac{\partial\psi}{\partial n_e}$, $\frac{\partial\psi}{\partial n_i}$, being the exterior and interior normals respectively. As this holds for a "sufficiently large" class of f we have the *free boundary condition across* γ_i being

$$(26) \qquad (\frac{\partial\psi}{\partial n_e})^2 - (\frac{\partial\psi}{\partial n_i})^2 = \bar{n}^{-2}(\frac{1}{|A_{i+1}|^2} - \frac{1}{|A_i|^2}) \qquad i = 1,\ldots,n-1 .$$

One of the main reasons this calculation is formal is that it must be justified that variations preserving the convex function class $W \cap C_2$ provide a sufficiently large collection of functions $f \cdot \underset{\sim}{n}$. on γ_i to conclude (26). This and other considerations such as smoothness of γ_i will be dealt with in [9].

We now present the applications of (26).

<u>Theorem 4.1</u>. If ψ is a minimizer of $J_{\bar{n}}$ on F_1 then ψ is superharmonic.

<u>Theorem 4.2</u>. If ψ is a minimizer of $J_{\bar{n}}$ on F_1 then

$$|\nabla\psi| \le \frac{\bar{n}^{-1}}{|A_{\bar{n}}|} .$$

<u>Remark</u>. Recall that $|\psi^{*\prime}| \le J(\psi)$ for solutions of the variational problem (8,9) and that $\bar{n}^{-1}/|A_{\bar{n}}|$ is the difference quotient corresponding to $\psi^{*\prime}(|\Omega|)$. From this and other considerations we expect that theorem 4.2 will yield an estimate independent of $\bar{n}$.

As a precursor to theorems 4.1, 4.2 we present several results on harmonic functions. Let A be an open region bounded by two nested nonintersecting convex curves. Let Γ_1 be the exterior curve and Γ_2 the interior curve.

<u>Proposition 4.1</u>. If h is subharmonic in A with $h = c_i$ on Γ_i, $i = 1,2, c_1 < c_2$ then $|\nabla h|$ is nondecreasing along curves of steepest ascent.

Remark. A consequence of this result is that h restricted to a curve of steepest descent (parameterized with respect to arclength) is a convex function. This provides the intuition that the volume function $V_h(t) = |\{h<t\}|$ should be convex. This is made rigorous by the following result.

Proposition 4.2. If h is as in proposition 4.1 then

$$V_h''(t) \leq 0 \quad \text{on} \quad (c_1,c_2) .$$

From this we conclude that

$$|\{c_1 < h < \frac{c_1+c_2}{2}\}| \geq |\{\frac{c_1+c_2}{2} < h < c_2\}| .$$

Proof of Proposition 4.1. First note that all level curves of h are convex curves [3]. Given $p_0 \in A$ we can assume without loss of generality that $p_0 = (0,0)$ and that $(x_0,y_0) = (0,1)$ is the outward normal to the level curve of h passing through p_0. Also we can assume that $y = g_t(x)$ gives a local parametrization for level sets $\{h = t\}$. That is

$$h(x,g_t(x)) = t$$

for x,t small. Note that $g_0'(0) = 0$, $g_0''(0) \leq 0$. From the chain rule we get

$$h_{xx} + h_{xy}g_t' + h_{yy}(g_t')^2 + h_y g_t'' = 0 .$$

Using $\Delta h \geq 0$ and $g_0'(0) = 0$ we have

$$-h_{yy} + h_y g_0'' \geq 0 \quad \text{at} \quad p_0 .$$

Also

$$\frac{\nabla h}{|\nabla h|} \cdot \nabla(|\nabla h|^2) = 2|\nabla h|^{-1}(h_x^2 h_{xx} + 2h_x h_y h_{xy} + h_y^2 h_{yy})$$

and

$$h_x(p_0) = 0, \quad -h_y = |\nabla h|$$

so

$$\frac{\nabla h}{|\nabla h|} \cdot \nabla(|\nabla h|^2) = -2h_y h_{yy} \geq -2h_y^2 g_0''$$

at p_0. Since level curves are convex and $(0,1)$ is an outward normal we see that $g_0''(0) \leq 0$. Therefore $|\nabla h|$ decreases as we go outward along curves of steepest descent, that is in the direction h is decreasing. Phrased differently we say that $|\nabla h|$ increases along curves of steepest ascent.

Proof of Proposition 4.2. Recall that ([15], pg. 28) using the coarea formula and the divergence theorem we get

$$V_h'(t) = \int_{\{h=t\}} |\nabla h|^{-1} dH^1$$

and

$$V_h''(t) = \int_{\{h=t\}} \nabla \cdot \left(\frac{\nabla h}{|\nabla h|^2}\right) \frac{dH^1}{|\nabla h|} .$$

Let $k(p)$ be the curvature of a level set of h at a point p so

$$V_h''(t) = \int_{\{h=t\}} \frac{\Delta h}{|\nabla h|^3} - 2 \int_{\{h=t\}} \frac{\nabla h \cdot \nabla(|\nabla h|)}{|\nabla h|^4}$$

$$= - \int_{\{h=t\}} \frac{\Delta h}{|\nabla h|^3} - 2 \int_{\{h=t\}} \frac{k}{|\nabla h|^2} .$$

But $\Delta h > 0$ and $k > 0$ so we have

$$V_h''(t) < 0 .$$

Clearly

$$V\left(\frac{c_1+c_2}{2}\right) > \frac{V(c_1) + V(c_2)}{2} .$$

But

$$|\{c_1 < h < (c_1 + c_2)/2\}| = V_h((c_1 + c_2)/2) - V_h(c_1)$$

and

$$|\{(c_1 + c_2)/2 < h < c_2\}| = V_h(c_2) - V_h((c_1 + c_2)/2)$$

so $|\{c_1 < h < (c_1 + c_2)/2\}| > |\{(c_1 + c_2)/2 < h < c_2\}|$ as required.

Proof of Theorem 4.1. Assume ψ is a minimizer. To prove that ψ is superharmonic we will show that

$$\int_\Omega \nabla\psi \cdot \nabla \eta \geq 0$$

for all $\eta \in C_0^\infty(\Omega)$. For simplicity we prove this only for $\eta \in C_0^\infty(\Omega_i)$, $i = 1,\ldots,\bar{n} - 1$, $\Omega_i = A_i \cup A_{i+1}$. We use the divergence theorem to evaluate the above integral. Since the free boundary condition is only weakly attained the following calculation is rigorous only with the addition of a limiting argument. The technical details of such an argument are left to [9]. Formally the divergence theorem says that

$$\int_{\Omega_i} \nabla\psi \cdot \nabla\eta = \int_{\gamma_i} \left(\left|\frac{\partial\psi}{\partial n_e}\right| - \left|\frac{\partial\psi}{\partial n_i}\right|\right) \eta dH^1 .$$

From the free boundary condition we see that the difference of th normal derivatives is positive if $|A_i| > |A_{i+1}|$ and negative if $|A_i| < |A_{i+1}|$. In the first case ψ would be superharmonic in Ω_i and in the second subharmonic. Assume $|A_i(\psi)| < |A_{i+1}(\psi)|$ so that ψ is subharmonic in Ω_i. Let h be the harmonic function on Ω_i which agrees with ψ on γ_{i-1} and γ_{i+1}. Clearly $\psi \leq h$ in Ω_i. From proposition 4.2 we have $|A_i(h)| > |A_{i+1}(h)|$ but since $\psi \leq h$ the level set $\{\psi = i/\bar{n}\}$ must be surrounded by $\{h = i/\bar{n}\}$ in which case $|A_i(\psi)| > |A_{i+1}(\psi)|$. But this contradicts our assumption so we conclude that $|A_i(\psi)| > |A_{i+1}(\psi)|$ and so ψ is superharmonic.

Proof of Theorem 4.2. To establish the gradient bound we use proposition 4.1 and the free boundary condition. Given $x \in A_i$ for some i, consider the curve of steepest ascent which leads from x to ∂A_n. By proposition 4.1, $|\nabla\psi|^2$ is nondecreasing along this curve except across the free boundaries where the decrease is given by the free boundary condition. Thus we have

$$|\nabla\psi(x)|^2 < \sum_{k=i}^{\bar{n}-1} \bar{n}^{-1}(|A_{k+1}|^{-2} - |A_k|^{-2})$$

$$< \bar{n}^{-2}/|A_{\bar{n}}|^{-2}$$

as required.

Remark. The results of this paper except theorem 2.2 have been generalized to n-dimensions. This will be dealt with to some extent in future papers. In addition some of the results generalize to the variational problems associated with functionals of the type

$$\int_\Omega |\nabla\psi|^2 + (\gamma - 1)^{-1} \int_0^{|\Omega|} \mu(\psi)(\psi^{*\prime})^\gamma, \quad \gamma > 1$$

where μ is smooth and satisfies $\mu > \lambda > 0$. This more general class is used in [8] to model adiabatic compression of a plasma.

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19. KEY WORDS (Continue on reverse side if necessary and identify by block number)

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20. ABSTRACT (Continue on reverse side if necessary and identify by block number)

We study the variational problem

$$J(\psi) = \int_\Omega |\nabla\psi|^2 + \int_0^{|\Omega|} (\psi^{*\prime}(v))^2 dv$$

$$\psi = 0 \text{ on } \partial\Omega, \ \psi^*(0) = 0, \ \psi^*(|\Omega|) = 1$$

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20. ABSTRACT - cont'd.

where ψ^* is the increasing rearrangement of ψ. An approximate problem is introduced which involves a variational problem with n free boundaries ($n \to \infty$). Various estimates are established. In particular when Ω is convex it is shown that the solution to the approximate problem is superharmonic and has bounded gradient.

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