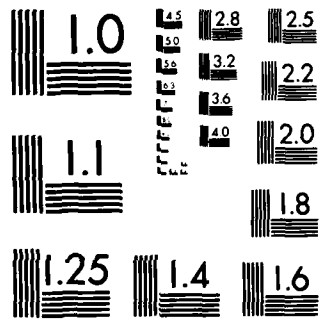END

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ON INTERACTING POPULATIONS THAT
DISPERSE TO AVOID CROWDING:
PRESERVATION OF SEGREGATION

M. Bertsch, M. E. Gurtin,
D. Hilhorst, and L. A. Peletier

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ON INTERACTING POPULATIONS THAT DISPERSE TO AVOID CROWDING:
PRESERVATION OF SEGREGATION

M. Bertsch[*], M. E. Gurtin[**], D. Hilhorst[***], and L. A. Peletier[*]

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ABSTRACT

This paper discusses the dispersal of two interacting biological species.
The dispersal - a response to population pressure alone - is modelled by the
degenerate parabolic system

$$u_t = [u(u+v)_x]_x$$

$$v_t = k[v(u+v)_x]_x$$

in conjunction with an initial prescription of the individual densities u
and v together with standard zero-flux boundary conditions. We demonstrate
here the following interesting feature of this model: segregated initial data
give rise to solutions which are segregated for all time.

AMS (MOS) Subject Classifications: 65P05, 92A15

Key Words: Degenerate parabolic equations, free-boundary problems, dispersal
            of biological populations.

Work Unit Numbers 1 and 2 - Applied Analysis and Physical Mathematics.

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SIGNIFICANCE AND EXPLANATION

This document
~~We~~ considers, here a mathematical model for interacting biological species
that disperse as a response to population pressure. *The authors* ~~We~~ demonstrate here an
interesting feature of the model: species which are initially segregated
remain segregated for all time.

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ON INTERACTING POPULATIONS THAT DISPERSE TO AVOID CROWDING: PRESERVATION OF SEGREGATION

M. Bertsch[*], M. E. Gurtin[**], D. Hilhorst[***], and L. A. Peletier[*]

1. Introduction

Consider two interacting biological species with populations sufficiently dense that a continuum theory is applicable, and assume that the species are undergoing dispersal on a time scale sufficiently small that births and deaths are negligible. Granted these assumptions, conservation of population requires that

$$u_t = -\text{div}(uq),$$

$$(1.1)$$

$$v_t = -\text{div}(vw),$$

where $u(x,t)$ and $v(x,t)$ are the spatial densities of the species, while the vector fields $q(x,t)$ and $w(x,t)$ are the corresponding dispersal velocities.

We restrict our attention to situations in which dispersal is a response to population pressure and express this mathematically by requiring that the dispersal of each of the species be driven by the gradient $\nabla(u+v)$ of the total population,[1] $u+v$. We therefore assume that

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[1]For a single species this type of constitutive assumption was introduced by Gurney and Nisbet [1], Gurtin and MacCamy [2].

$$q = -k_1 \nabla(u+v),$$

$$w = -k_2 \nabla(u+v), \tag{1.2}$$

with k_1, k_2 strictly-positive[1] constants, and this leads to the system[2]

$$u_t = k_1 \text{div}[u \nabla(u+v)],$$

$$v_t = k_2 \text{div}[v \nabla(u+v)]. \tag{1.3}$$

For convenience, we limit our attention to one space-dimension and we choose the time-scale so that $k_1 = 1$. Then writing $k = k_2$ we have the system

$$u_t = [u(u+v)_x]_x,$$

$$v_t = k[v(u+v)_x]_x. \tag{1.4}$$

We shall suppose that the two species live in a finite habitat

$$\Omega = (-L, L), \qquad L > 0;$$

that individuals are unable to cross the boundary of Ω,

$$u(u+v)_x = v(u+v)_x = 0 \quad \text{for} \quad x = \pm L, \quad t > 0; \tag{1.5}$$

and that the two populations are prescribed initially,

---

[1] The system (1.3) with $k_2 = 0$ was studied by Bertsch and Hilhorst [3] and by Bertsch, Gurtin, Hilhorst, and Peletier [4].

[2] Gurtin and Pipkin [5]. See also Busenberg and Travis [6]. An alternative theory was developed by Shigesada, Kawasaki, and Teramoto [7]. This theory is discussed in [4] and [5].

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$$u(x, 0) = u_0(x), \quad v(x, 0) = v_0(x) \quad \text{for} \quad x \in \Omega. \tag{1.6}$$

In this paper we shall study the problem (1.4)-(1.6) for _initial data which are segregated_ in the sense that, for some $a \in \Omega$,

$$u_0(x) \equiv 0 \quad \text{for} \quad x > a, \quad v_0(x) \equiv 0 \quad \text{for} \quad x < a. \tag{1.7}$$

As our main result we establish the existence of a solution in which _the two species are segregated for all time_. This result is quite surprising[1] as it is independent of the initial distributions[2] of the species and of the ratio k of their dispersivities.

---

[1]Actually, Gurtin and Pipkin [5] gave a particular solution to (1.2) - corresponding to initial Dirac distributions - in which the two species are segregated for all time. Being a specific solution, it is not clear from this result whether "preservation of segregation" is a generic property of the equations (1.4).

[2]Granted they are segregated.

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2. The problem. Results.

We shall use the notation

$$\mathbb{R}^+ = (0,\infty), \quad Q = \Omega \times \mathbb{R}^+, \quad Q_T = \Omega \times (0,T),$$

and, for any function $f: Q \to \mathbb{R}$,

$$Q^+(f) = \text{interior}\{(x,t) \in Q: \ f(x,t) > 0\}.$$

Our problem consists in finding functions $u(x,t)$ and $v(x,t)$ on $\bar{Q}$ such that

$$(I) \quad \begin{cases} u_t = [u(u+v)_x]_x \\[2mm] v_t = k[v(u+v)_x]_x \end{cases} \quad \text{in} \quad Q, \qquad (2.1)$$

$$u(u+v)_x = v(u+v)_x = 0 \quad \text{on} \quad \partial\Omega \times \mathbb{R}^+, \qquad (2.2)$$

$$u(x,0) = u_0(x), \ v(x,0) = v_0(x) \quad \text{in} \quad \Omega. \qquad (2.3)$$

We shall assume throughout that:

(A1) $k > 0$, $u_0,v_0 \geq 0$, $u_0,v_0 \in C(\bar{\Omega})$;

(A2) the initial data are segregated, so that (1.7) holds for some $a \in \Omega$;

(A3)[1] each of the sets $\{x: \ u_0(x) > 0\}$ and $\{x: \ v_0(x) > 0\}$ is connected.

The purpose of this paper is to establish - for such segregated initial data - solutions of (I) which are segregated for all time.

---
[1] We make this assumption for convenience only.

Proceeding formally, let (u,v) be a segregated solution. Then the sets $Q^+(u)$ and $Q^+(v)$ are disjoint; hence (assuming $u,v \geq 0$)

$$u \equiv 0 \quad \text{in} \quad Q^+(v), \quad v \equiv 0 \quad \text{in} \quad Q^+(u),$$

and, by (2.1),

$$u_t = (uu_x)_x \quad \text{in} \quad Q^+(u), \quad v_t = k(vv_x)_x \quad \text{in} \quad Q^+(v). \qquad (2.4)$$

Thus where positive u and v obey porous-media equations. As is well known,[1] solutions of the porous-media equation may not be smooth, and for that reason it is advantageous to work with a weak formulation of Problem (I). This is reinforced by the observation that (I) is a free-boundary problem and conditions at the free boundary are generally *indigenous to a weak formulation*, not required as separate restrictions. (The free boundary is the set

$$\mathfrak{F} = \{ \partial Q^+(u) \cup \partial Q^+(v) \} \cap Q \qquad (2.5)$$

which separates the region with $u(x,t) > 0$ from that with $u(x,t) = 0$ and separates the region with $v(x,t) > 0$ from that with $v(x,t) = 0$.)

With this in mind, assume for the moment that (u,v) is a smooth solution of (I). If we multiply (2.1) by an <u>arbitrary</u> smooth function $\psi(x,t)$, integrate over Q_T, and use (2.2)

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[1] Cf., e.g., the survey article of Peletier [8].

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and (2.3), we arrive at the relations

$$\int_\Omega \{u(T)\psi(T) - u_0\psi(0)\} = \int_{Q_T} \{u\psi_t - u(u+v)_x\psi_x\},$$

$$\int_\Omega \{v(T)\psi(T) - v_0\psi(0)\} = \int_{Q_T} \{v\psi_t - kv(u+v)_x\psi_x\},$$
(2.6)

where we have used the notation $u(t) = u(\cdot,t)$, etc. We shall use (2.6) as the basis of our definition of a weak solution.

Definition. A (weak) solution of Problem (I) is a pair (u,v) with the following properties:

(i) $u,v \in L^\infty(Q_T)$ for $T > 0$; $u(t),v(t) \in L^\infty(\Omega)$ for $t \geq 0$; $(u+v)^2 \in L^2(0,T; H^1(\Omega))$ for $T > 0$;

(ii) $u(t),v(t) \geq 0$ almost everywhere in Ω for $t > 0$;

(iii) u and v satisfy (2.6) for all $\psi \in C^1(\bar{Q})$ and $T > 0$.

If, in addition, there is a continuous function $\xi: [0,\infty) \to \Omega$ such that, given any $t > 0$,

$$v(x,t) = 0 \quad \text{for} \quad -L < x < \xi(t),$$

$$u(x,t) = 0 \quad \text{for} \quad \xi(t) < x < L,$$
(2.7)

then (u,v) is segregated. We will refer to ξ as a separation curve.

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<u>Remarks:</u>

1. The terms $u(u+v)_x$ and $v(u+v)_x$ in (2.6) are defined as follows:

$$u(u+v)_x = \begin{cases} \frac{1}{2} \frac{u}{u+v} \left[(u+v)^2\right]_x & \text{if } u > 0 \\ \\ 0 & \text{if } u = 0 \end{cases}$$

and similarly for $v(u+v)_x$. Then, since $u/(u+v) \leq 1$, while $\left[(u+v)^2\right]_x \in L^2(Q_T)$, we have

$$u(u+v)_x \in L^2(Q_T) \quad \text{for } T > 0.$$

2. The integral identities (2.6) imply that, as $t \to 0$,

$$u(t) \to u_0, \quad v(t) \to v_0 \quad \text{weakly in } L^2(\Omega);$$

i.e., that

$$\int_\Omega [u(t)-u_0]\psi \to 0, \quad \int_\Omega [v(t)-v_0]\psi \to 0 \tag{2.8}$$

for all $\psi \in L^2(\Omega)$. To verify (2.8) we simply apply (2.6) with $\psi \in C^1(\overline{\Omega})$ (independent of time). This yields (2.8) for $\psi \in C^1(\overline{\Omega})$ and hence - using a standard argument - for $\psi \in L^2(\Omega)$.

3. The choice $\psi \equiv 1$ in (2.6) leads to the global conservation laws

$$\int_\Omega u(t) = \int_\Omega u_0, \quad \int_\Omega v(t) = \int_\Omega v_0 \tag{2.9}$$

for $t > 0$.

We close this section by stating our main results; the corresponding proofs will be given in Section 3.

Theorem 1. Problem (I) <u>has</u> <u>exactly</u> <u>one</u> <u>segregated</u> <u>solution</u>.

Remark. It is important to emphasize that we have established uniqueness only within the class of segregated solutions. Thus we have not ruled out the possibility - for segregated initial data - of solutions which mix. <u>We</u> <u>conjecture</u> <u>that</u> <u>this</u> <u>cannot</u> <u>happen</u>.

Theorem 2. <u>Let</u> (u,v) <u>be</u> <u>the</u> <u>segregated</u> <u>solution</u> <u>of</u> Problem (I). <u>Then</u>:

(i) $u + v \in C(\overline{Q})$;

(ii) $u_t = (uu_x)_x$ <u>classically</u>[1] <u>in</u> $Q^+(u)$;

(iii) $v_t = k(vv_x)_x$ <u>classically</u> <u>in</u> $Q^+(v)$.

---

[1]That is, u is C^∞ on $Q^+(u)$ and there satisfies $u_t = (uu_x)_x$ pointwise.

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Our next theorem is concerned with the free boundary $\mathfrak{J}$
(cf. (2.5)). In view of (2.4), the portion of $\mathfrak{J}$ along which
the two species are <u>not</u> in contact should have properties
similar to those of the free boundary for the porous-media problem:

$$
(PM) \begin{cases}
\rho_t = (\rho \rho_x)_x & \text{in } Q, \\
\rho \rho_x = 0 & \text{on } \partial\Omega \times \mathbb{R}^+, \\
\rho(x,0) = \rho_0(x) & \text{in } \Omega.
\end{cases}
$$

As is known,[1] when the initial data have the form

$$\rho_0(x) > 0 \quad \text{in } (a_1, a_2), \qquad \rho_0(x) = 0 \quad \text{otherwise,}$$

$-L < a_1 < a_2 < L$, the free boundary $Q \cap \partial Q^+(\rho)$ consists of
two continuous, time-parametrizable curves, one emanating from
a_1, one from a_2. If $b(t)$, $0 \leq t < T_b$, designates the curve
from a_1 (resp., a_2), then:

(F_1) $b(t) = b(0)$ on $(0, \tau_b)$ for some $\tau_b \in [0, T_b]$;

(F_2) $b(t)$ is C^1 and strictly decreasing (resp., strictly
increasing) on (τ_b, T_b);

(F_3) $b(T_b^-) \in \partial\Omega$.

This discussion should motivate the following definition in which
"FB" is shorthand for "free boundary".

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[1] Cf., e.g., the review article by Peletier [8]. See also
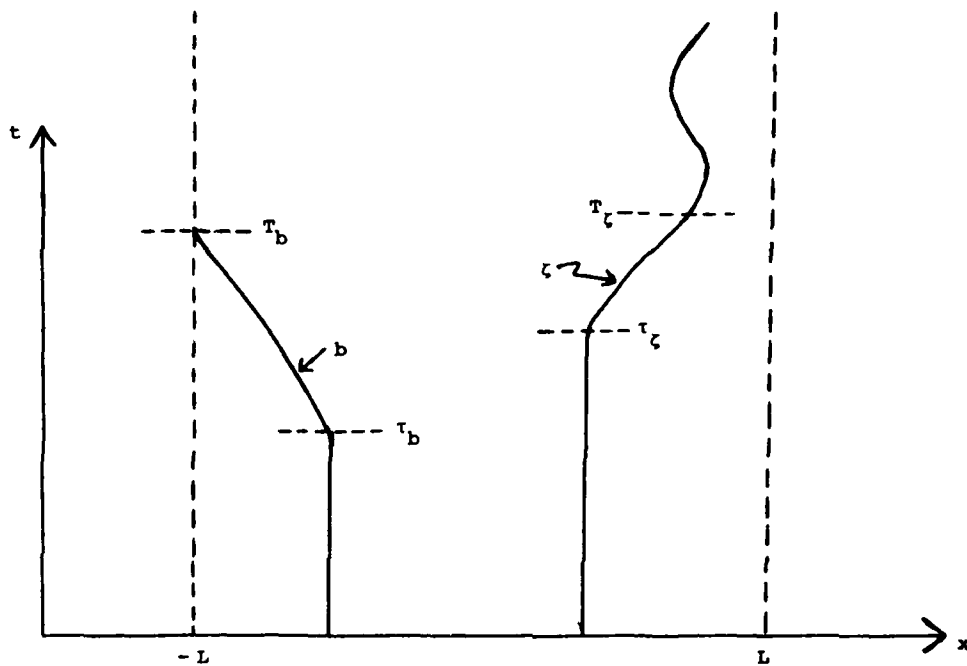Aronson and Peletier [9].

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Figure 1. b is a left free-boundary curve extending
to ∂Ω. ζ is an internal free-boundary
curve that is right up to time T_ζ.

<u>Definition</u>. An F^R <u>curve</u> is a continuous function

$$b: \quad [0,t_b) \to \Omega$$

$(t_b$ may be $\infty)$. Moreover:

(i) b is <u>internal</u> if $t_b = \infty$;

(ii) b is <u>left</u> (resp., <u>right</u>) up to time $T_b \in [0,t_b]$
if $(F_1),(F_2)$ hold;

(iii) b <u>extends</u> <u>to</u> $\partial\Omega$ if b is right or left up to time
t_b with $t_b < \infty$, and $b(t_b^-) \in \partial\Omega$.

Let $b: [0,t_b) \to \Omega$ be an FB curve and let $q: Q \to \mathbb{R}$. Then
<u>FB</u> <u>conditions</u> <u>with</u> <u>velocity</u> q <u>hold</u> <u>from</u> <u>the</u> <u>left</u> (resp., <u>right</u>)
<u>on</u> b if given any $t \in (0,t_b)$ at which b is C^1,

$$b'(t) = q(b(t)^+, t) \quad (\text{resp., } b'(t) = q(b(t)^-, t)).$$

Theorem 3. <u>Let</u> (u,v) <u>be</u> <u>the</u> <u>segregated</u> <u>solution</u> <u>of</u> <u>Problem</u>
(I). <u>Then</u> <u>there</u> <u>exist</u> FB <u>curves</u> $b_u, b_v, \zeta_u, \zeta_v$ <u>with</u> <u>the</u>
<u>following</u> <u>properties</u>:

(i)[1] $b_u < \zeta_u \leq \zeta_v < b_v$ <u>with</u> b_u <u>and</u> ζ_u <u>forming</u> <u>the</u>
<u>boundary</u> <u>of</u> $Q^+(u)$ <u>in</u> Q, b_v <u>and</u> ζ_v <u>forming</u> <u>the</u> <u>boundary</u>
<u>of</u> $Q^+(v)$ <u>in</u> Q;

(ii) b_u <u>and</u> b_v <u>extend</u> <u>to</u> $\partial\Omega$, <u>with</u> b_u <u>left</u> <u>and</u> b_v
<u>right</u>;

---

[1]Here each inequality is assumed to hold at those times at which
the underlying functions are defined.

(iii) ζ_u and ζ_v are internal, and there is a time $T \in [0,\infty)$ such that ζ_u is right up to time T, ζ_v is left up to time T, and

$$\zeta_u(t) = \zeta_v(t) =: \zeta(t) \quad \text{on} \quad [T,\infty)$$

with $\zeta \in C^1(T,\infty)$; in addition,

$$(u+v)(\zeta(t),t) > 0 \quad \text{for} \quad t > T; \tag{2.10}$$

(iv) FB conditions with velocity $-u_x$ hold from the right on b_u, from the left on ζ_u;

(v) FB conditions with velocity $-kv_x$ hold from the right on ζ_v, from the left on b_v.

Remarks.

1. The curve ζ marks that portion of the free boundary on which the two species are in contact. By (2.10), the functions u and v suffer jump discontinuities across ζ; more precisely for $t > T$,

$$u(\zeta(t)^-,t) > 0, \quad u(\zeta(t)^+,t) = 0,$$

$$v(\zeta(t)^-,t) = 0, \quad v(\zeta(t)^+,t) > 0.$$

Further, (iii)-(v) of Theorem 3 in conjunction with the continuity of $u+v$ imply that, for $t > T$,

$$u(\zeta(t)^-, t) = v(\zeta(t)^+, t),$$

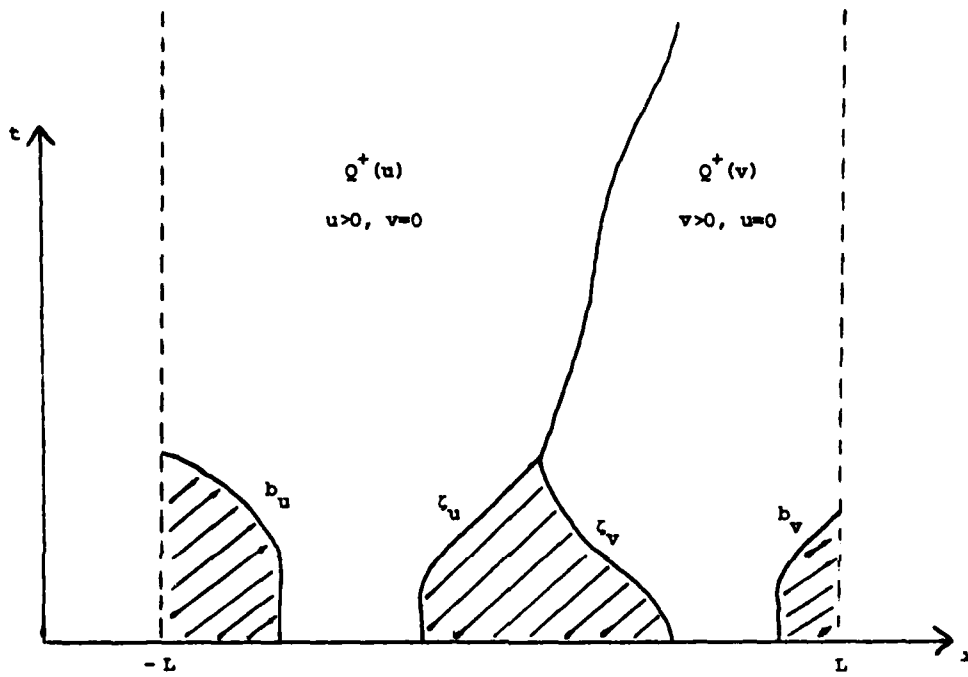$$u_x(\zeta(t)^-, t) = kv_x(\zeta(t)^+, t) = -\zeta'(t). \tag{2.11}$$

-12-

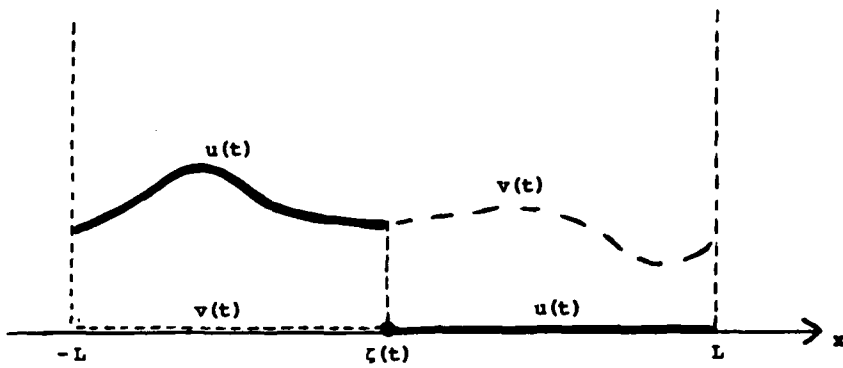Figure 2. The free boundaries. The shaded areas correspond to $u = v = 0$.

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Figure 3. The functions $u(\cdot,t)$ and $v(\cdot,t)$
at a fixed time $t > T$.

2. The results (iv) and (v) assert that each of the "fronts" $b_u(t)$, $\zeta_u(t)$, $\zeta_v(t)$, $b_v(t)$ propagates with the velocity of individuals situated on it; condition (2.11) is the requirement that at the <u>contact</u> <u>front</u> $\zeta(t)$ the two species move together.

Our final result concerns the asymptotic behavior of segregated solutions. Proceeding formally, suppose that $u_\infty(x)$, $v_\infty(x)$ is an equilibrium solution of Problem (I). Then (2.1) and the boundary conditions (2.2) yield

$$u_\infty(u_\infty + v_\infty)' = v_\infty(u_\infty + v_\infty)' = 0 \quad \text{in} \quad \Omega;$$

hence

$$[(u_\infty + v_\infty)^2]' = 0 \quad \text{in} \quad \Omega$$

and

$$u_\infty + v_\infty = \text{constant}.$$

If u_∞ and v_∞ are segregated with habitats in $[-L, x_\infty]$ and $[x_\infty, L]$, respectively, then there exists a constant p such that

$$
u_\infty(x) = \begin{cases} p & \text{if } x \in (-L, x_\infty) \\ \frac{1}{2} p & \text{if } x = x_\infty \\ 0 & \text{if } x \in (x_\infty, L) \end{cases}
$$

$$
v_\infty(x) = \begin{cases} 0 & \text{if } x \in (-L, x_\infty) \\ \frac{1}{2} p & \text{if } x = x_\infty \\ p & \text{if } x \in (x_\infty, L). \end{cases} \tag{2.12}
$$

Moreover, if the equilibrium solution (u_∞, v_∞) is reached from the initial data (u_0, v_0), the conservation law (2.9) implies that

$$
p = \frac{U + V}{2L}, \qquad x_\infty = \frac{U - pL}{p}, \tag{2.13}
$$

where

$$
U = \int_\Omega u_0, \qquad V = \int_\Omega v_0. \tag{2.14}
$$

That these formal calculations are indeed correct is a consequence of the following theorem.

Theorem 4. Let (u,v) be the segregated solution of (I). Then, as $t \to \infty$,

$$
\zeta(t) \to x_\infty, \quad u(t) \to u_\infty, \quad v(t) \to v_\infty,
$$

the latter two limits being pointwise in $\Omega \setminus \{x_\infty\}$. Here $\zeta(t)$ is the contact front (cf. Theorem 3), while x_∞, u_∞, and v_∞ are defined in (2.12), (2.13).

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3. Reformulation of the problem.

Let (u,v) be a sufficiently smooth segregated solution of Problem (I) and define

$$z(x,t) = -U + \int_{-L}^{x} [u(y,t) + v(y,t)]dy; \qquad (3.1)$$

$z(x,t)$ represents the total population, at time t, in the interval $[-L,x]$. In view of the conservation laws (2.9),

$$z(-L,t) = -U, \quad z(\zeta(t), t) = 0, \quad z(L,t) = V, \qquad (3.2)$$

where U and V are defined in (2.14), so that separation curves $\zeta(t)$ (cf. (2.7)) are level curves $z(\zeta(t), t) = 0$; in fact,

$$z(x,t) \leq 0 \quad \text{for} \quad x < \zeta(t),$$
$$z(x,t) \geq 0 \quad \text{for} \quad x > \zeta(t). \qquad (3.3)$$

Further, if we differentiate (3.1) with respect to t and use (2.1), (2.2), and (2.7), we find that

$$z_t = \begin{cases} z_x z_{xx} & \text{for} \quad x < \zeta(t) \\ k z_x z_{xx} & \text{for} \quad x > \zeta(t). \end{cases} \qquad (3.4)$$

Thus defining $c: \mathbb{R} \to \mathbb{R}$ by

$$c(s) = \begin{cases} s, & s \leq 0 \\ s/k, & s > 0, \end{cases} \qquad (3.5)$$

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we may use (3.3) to reduce (3.4) to the single equation

$$c(z)_t = z_x z_{xx}$$

on all of Q. Therefore, if we write

$$z_0(x) = -U + \int_{-L}^{x} (u_0 + v_0), \tag{3.6}$$

we are led to the following problem for $z(x,t)$:

$$(\text{II}) \begin{cases} c(z)_t = z_x z_{xx} & \text{in } Q, \\ z(-L,t) = -U, \quad z(L,t) = V, \quad t > 0, \\ z(x,0) = z_0(x) & \text{in } \Omega. \end{cases}$$

We assume, for the remainder of the section, that c and z_0 are defined by (3.5) and (3.6), and that (A1)-(A3) are satisfied.

Problem (II) under hypotheses more general than ours, has been analyzed in [10]. We shall simply state, without proof, a version of the results of [10] appropriate for our use. With this in mind, we first define what we mean by a solution; in that definition, and in what follows, $z_\infty(x)$ designates the unique equilibrium solution of (II):

$$z_\infty(x) = \frac{(U+V)(x+L)}{2L} - U.$$

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Definition. A (weak) solution of Problem (II) is a function $z \in C([0,\infty); W^{1,\infty}(\Omega))$ with the following properties:

(i) $z(\cdot,t) - z_\infty \in H_0^1(\Omega)$ for $t > 0$;

(ii) $z_t \in L^2(Q_T)$ for $T > 0$;

(iii) for all $\psi \in C^1(\overline{Q})$ with $\psi = 0$ on $\partial\Omega \times (0,\infty)$ and all $T > 0$,

$$\int_\Omega \{c(z(T))\psi(T) - c(z_0)\psi(0)\} = \int_{Q_T} \{c(z)\psi_t - \tfrac{1}{2}(z_x)^2\psi_x\}. \qquad (3.7)$$

Theorem 5 ([10]). Problem (II) has exactly one solution z. Moreover:

(i) $z_x \in C(\overline{Q})$ with $z_x \geq 0$;

(ii) $c(z)_t = z_x z_{xx}$ classically in $Q^+(z_x)$;

(iii) $Q^+(z_x)$ is the union of the sets

$$Q_1 := \{(x,t) \in Q: -U < z(x,t) < 0\},$$

$$Q_2 := \{(x,t) \in Q: 0 < z(x,t) < V\},$$

and there exist free-boundary curves $b_1, b_2, \zeta_1, \zeta_2$ such that (i) - (v) of Theorem 3 hold with $Q^+(u), Q^+(v), b_u, b_v, \zeta_u, \zeta_v$ replaced by $Q_1, Q_2, b_1, b_2, \zeta_1, \zeta_2$, respectively, with u+v in (iii) replaced by z_x, and with u_x and v_x in (iv) and (v) replaced by z_{xx};

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(iv) $z(t) \to z_\infty$ in $C^1(\overline{\Omega})$ as $t \to \infty$;

(v)   given any $x_0 \in \Omega$ with $z_0'(x_0) > 0$, there exists a continuous function $\zeta: [0,\infty) \to \Omega$ such that

$$\{(x,t) \in \overline{Q}: \ z(x,t) = z_0(x_0)\} = \{(x,t) \in \overline{Q}: \ x = \zeta(t)\};$$

moreover, $\zeta \in C^1(0,\infty)$ and, for $t > 0$,

$$\zeta'(t) = -\kappa z_{xx}(\zeta(t), \ t),$$

where $\kappa = 1$ or k according as $z_0(x_0) < 0$ or $z_0(x_0) > 0$.

The next result asserts the equivalence of Problems I and (II) and, when combined with Theorem 5,

yields the validity of Theorems 1-4.

Theorem 6.   Problems (I) and (II) are equivalent:

(i)   Let z be a solution of Problem (II) and define u and v on $\overline{Q}$ by

$$u(x,t) = z_x(x,t), \quad v(x,t) = 0 \quad \underline{if} \quad z(x,t) < 0,$$

$$u(x,t) = 0, \quad v(x,t) = z_x(x,t) \quad \underline{if} \quad z(x,t) > 0, \qquad (3.8)$$

$$u(x,t) = v(x,t) = \tfrac{1}{2} z_x(x,t) \quad \underline{if} \quad z(x,t) = 0.$$

Then (u,v) is a segregated solution of Problem (I).

(ii)   Conversely, let (u,v) be a segregated solution of Problem (I) and define z on $\overline{Q}$ by (3.1). Then z solves Problem (II).

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Proof.

(i) Let z be a solution of $(\Box)$ and define (u,v) through (3.8). By Theorem 5(i), the only nontrivial step in showing that (u,v) solves (I) is proving that u and v satisfy the integral identities (2.6). We shall only verify the first of (2.6); the verification of the second is completely analogous.

For convenience, we write $b = b_1$, $\zeta = \zeta_1$ for the FB curves established in Theorem 5, and we extend $b(t)$ continuously to $[0,\infty)$ by defining $b(t) = -L$ for $t \geq t_b$. By Theorem 5(iii) and (3.8),

$$\text{supp } u(t) = [b(t), \zeta(t)].$$

Choose $\epsilon > 0$ sufficiently small and let $b_\epsilon(t)$ and $\zeta_\epsilon(t)$, respectively, be the level curves $z = -U + \epsilon$ and $z = -\epsilon$ (cf. Theorem 5(v)). Then, by Theorem 5(ii) and (3.8), $u_t = (uu_x)_x$ classically and $v \equiv 0$ must both be satisfied in a neighborhood of any (x,t) such that $b_\epsilon(t) \leq x \leq \zeta_\epsilon(t)$ and $t > 0$. Further, Theorem 5(v) yields

$$b_\epsilon'(t) = -u_x(b_\epsilon(t),\, t), \quad \zeta_\epsilon'(t) = -u_x(\zeta_\epsilon(t),\, t)$$

for $t > 0$. Thus, choosing $\delta > 0$, the identity

$$\int_\delta^t f'(\tau)\, d\tau = f(t) - f(\delta)$$

applied to

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$$f(\tau) = \int_{b_\varepsilon(\tau)}^{\zeta_\varepsilon(\tau)} u(\tau)\psi(\tau)$$

yields, when $\psi \in C^1(\overline{Q})$,

$$\int_{b_\varepsilon(t)}^{\zeta_\varepsilon(t)} u(t)\psi(t) - \int_{b_\varepsilon(\delta)}^{\zeta_\varepsilon(\delta)} u(\delta)\psi(\delta) = \int_\delta^t \left\{ \int_{b_\varepsilon(\tau)}^{\zeta_\varepsilon(\tau)} (u\psi_\tau - u(u+v)_x\psi_x)\,dx \right\}d\tau. \tag{3.9}$$

Next, since $b_\varepsilon(t) \downarrow b(t)$ and $\zeta_\varepsilon(t) \uparrow \zeta(t)$ as $\varepsilon \downarrow 0$ for each $t \in [0,\infty)$, it follows from Lebesgue's dominated convergence theorem that (3.9) holds with b_ε and ζ_ε replaced by b and ζ. Also, since $z_x \in C(\overline{Q})$, it follows that $z_x(\delta) \to z_0'$ in $C(\overline{\Omega})$ as $\delta \downarrow 0$ and

$$\int_{b(\delta)}^{\zeta(\delta)} u(\delta)\psi(\delta) \to \int_{b(0)}^{\zeta(0)} u_0\psi(0) \quad \text{as} \quad \delta \downarrow 0.$$

Thus a second application of Lebesgue's theorem yields

$$\int_{b(t)}^{\zeta(t)} u(t)\psi(t) - \int_{b(0)}^{\zeta(0)} u_0\psi(0) = \int_0^t \left\{ \int_{b(\tau)}^{\zeta(\tau)} (u\psi_\tau - u(u+v)_x\psi_x)\,dx \right\}d\tau,$$

and, since $u(t) \equiv 0$ on $\Omega \backslash (b(t), \zeta(t))$, the first of (2.6) follows.

(ii) Let (u,v) solve (I). We are to prove that z - defined by (3.1) - solves ($\overline{\text{I}}$). Choose $T > 0$. Then $u,v \in L^\infty(Q_T)$ and hence, by (2.9) and the definition of z_∞, $z(\cdot,t) - z_\infty \in H_0^1(\Omega)$. Note also that, since $z_x = u + v$ and $(u+v)^2 \in L^2(0,T;H^1(\Omega))$, it follows that

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$$z_x^2 \in L^2(0,T;H^1(\Omega)), \qquad\qquad (3.10)$$

where $z_x^2 = (z_x)^2$.

Our next step will be to establish the integral identity (3.7). Thus choose $\chi \in C^1(\bar{Q})$ with $\chi = 0$ on $\partial\Omega \times (0,\infty)$ and take

$$\psi(x,t) = \int_{-L}^{x} \chi(y,t)\,dy$$

in (2.6); in view of (2.7), (3.1), and (3.6), the result is

$$\int_{-L}^{\xi(T)} z_x(T)\psi(T) - \int_{-L}^{\xi(0)} z_0'\psi(0) = \int_0^T \int_{-L}^{\xi(t)} \{z_x\psi_t - \tfrac{1}{2}(z_x^2)_x\chi\}\,dx\,dt,$$

$$\int_{\xi(T)}^{L} z_x(T)\psi(T) - \int_{\xi(0)}^{L} z_0'\psi(0) = \int_0^T \int_{\xi(t)}^{L} \{z_x\psi_t - \tfrac{k}{2}(z_x^2)_x\chi\}\,dx\,dt. \qquad (3.11)$$

Since $\psi_x = \chi$, $\psi(-L,t) = 0$, and $\chi(\pm L,t) = 0$, while $z(x,t)$ satisfies (3.2) and (3.3), if we integrate (3.11) by parts and then add the first of the resulting relations to k^{-1} times the second, we arrive at (3.7) (with ψ replaced by χ).

We have only to show that $z_t \in L^2(Q_T)$. But this follows from (3.10) and the fact that, by (3.7), $c(z)_t = \tfrac{1}{2}(z_x^2)_x$ in the sense of distributions on Q_T. This completes the proof of Theorem 6.

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4. <u>Remarks</u>. <u>Open problems</u>. <u>Conjectures</u>.

1. <u>Problem (I) with nonsegregated initial data is open</u>. Here
the problem does not reduce to a free boundary problem for a <u>single</u>
scalar field z, as one must solve the <u>system</u> (2.1) in regions of
interaction (cf. Remark 2).

2. The system (2.1) with $k = 1$ is far simpler to analyze.
There the total density $\rho = u + v$ satisfies (PM) with initial
data $\rho_0 = u_0 + v_0$, and once ρ is known (2.1) are linear
hyperbolic equations for u and v:

$$u_t = (u\rho_x)_x, \quad v_t = (v\rho_x)_x$$

(cf. [5]). Using this reduction one can prove uniqueness within
the class of <u>all</u> solutions (as opposed to all segregated solutions), and
one can show that solutions which begin mixed remain mixed for all
time, including $t = \infty$. (Details will appear elsewhere.)

3. Assume, in place of (1.2), that the dispersal of each
of the species is driven by a weighted sum of the densities; i.e.,
that (in one space-dimension),

$$q = -(k_{11}u + k_{12}v)_x$$
$$w = -(k_{21}u + k_{22}v)_x \qquad (4.1)$$

with all

$$k_{ij} > 0. \qquad (4.2)$$

This constitutive assumption, when combined with the conservation law (1.1) and corresponding zero-flux boundary conditions, leads to the problem

$$
(\text{III}) \begin{cases} u_t = [u(k_{11}u + k_{12}v)_x]_x \\ v_t = [v(k_{21}u + k_{22}v)_x]_x \end{cases} \quad \text{in} \quad Q, \qquad (4.3) \\ u(k_{11}u + k_{12}v)_x = v(k_{21}u + k_{22}v)_x = 0 \quad \text{on} \quad \partial\Omega \times \mathbb{R}^+, \\ u(x,0) = u_0(x), \quad v(x,0) = v_0(x) \quad \text{in} \quad \Omega.
$$

This formulation is greatly simplified if we define new independent variables

$$
\alpha(x,t) = k_{11}u(x,t), \quad \beta(x,t) = k_{12}v(x,t)
$$

and new constants

$$
k = \frac{k_{22}}{k_{12}}, \qquad \mu = \frac{k_{12}k_{21}}{k_{11}k_{22}},
$$

for then (III) reduces to

$$
(\text{IV}) \begin{cases} \alpha_t = [\alpha(\alpha+\beta)_x]_x \\ \beta_t = k[\beta(\mu\alpha + \beta)_x]_x \end{cases} \quad \text{in} \quad Q, \qquad (4.4) \\ \alpha(\alpha + \beta)_x = \beta(\mu\alpha + \beta)_x = 0 \quad \text{on} \quad \partial\Omega \times \mathbb{R}^+, \\ \alpha(x,0) = \alpha_0(x), \quad \beta(x,0) = \beta_0(x) \quad \text{in} \quad \Omega,
$$

with

$$
\alpha_0 = k_{11}u_0, \qquad \beta_0 = k_{12}v_0.
$$

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4. Consider Problem (III), or equivalently (IV). The terms on the right side of (4.3) involving second derivatives are

$$
\begin{pmatrix} uk_{11} & uk_{12} \\ vk_{21} & vk_{22} \end{pmatrix} \begin{pmatrix} u_{xx} \\ v_{xx} \end{pmatrix}. \tag{4.5}
$$

Writing $A = A(u,v)$ for the coefficient matrix in (4.5) and confining our attention to $u > 0$, $v > 0$, we conclude from (4.2) that there are exactly three possibilities for the eigenvalues $\lambda_1 \leq \lambda_2$ of A, namely:

(i) $\lambda_1 > 0$, $\lambda_2 > 0$;   (ii) $\lambda_1 = 0$, $\lambda_2 > 0$;   (iii) $\lambda_1 < 0$, $\lambda_2 > 0$.

Moreover, writing K for the matrix

$$
K = (k_{ij}),
$$

it is not difficult to verify that

(i) $\lambda_1 > 0$, $\lambda_2 > 0$ $\Longleftrightarrow$ $\det K > 0$,

(ii) $\lambda_1 = 0$, $\lambda_2 > 0$ $\Longleftrightarrow$ $\det K = 0$,

(iii) $\lambda_1 < 0$, $\lambda_2 > 0$ $\Longleftrightarrow$ $\det K < 0$.

We consider the three cases separately.

Case (i)   (det $K > 0$). Here the system (4.3) is _degenerate parabolic_, as it is parabolic when $u > 0$ and $v > 0$, but not when $uv = 0$. Because of this property, we expect that initially-segregated solutions will eventually mix. We also expect them to mix for another reason. Indeed, assume to the contrary that Problem (III)

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has a segregated solution (u,v). For such a solution we would
expect the two populations to spread until they meet, and then
to remain in contact along a contact front $\zeta(t)$ (cf. Theorem 3
and Remark 2 following it). From (4.2) one might expect that
both $k_{11}u + k_{12}v$ and $k_{21}u + k_{22}v$ would be continuous across
ζ, and hence both zero along ζ, a condition which cannot generally
be satisfied (cf. Remark 1 following Theorem 3). We are therefore
led to the following conjecture: for $\det K > 0$ there are no
segregated solutions of Problem (III). In this regard it would be
interesting to look at (III) with[1]

$$K = \begin{pmatrix} 1+\varepsilon & 1 \\ 1 & 1 \end{pmatrix},$$

$\varepsilon > 0$; in particular, the limit $\varepsilon \to 0$.

Finally, within the context of the biological model, the
off-diagonal elements of K drive the segregation of the species,
while the diagonal elements, by themselves, result in the usual
diffusive behavior. Since $\det K > 0$ yields $k_{11}k_{22} > k_{12}k_{21}$,
it would seem reasonable that in this case the two species
ultimately mix.

Case (ii) ($\det K = 0$). Here $\mu = 1$ and Problem (IV) is identical
to Problem (I). Thus all of our results generalize trivially to
populations whose interaction is described by (4.1) with K
singular.

---

[1]This choice of K arose in discussions with R. Rostamian.

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For the case det K = 0 we would like to call the system (4.3)

__degenerate__ __parabolic-hyperbolic__.   Indeed, if we set $w = \alpha + \beta$,

then, assuming $k \geq 1$, (4.4) can be written as

$$w_t = [\{w + (k-1)\beta\}w_x]_x,$$

$$\beta_t = k(w_x\beta)_x$$

i.e., as a system composed of a degenerate-parabolic equation and

a hyperbolic equation. The presence of this last equation makes

the discontinuity of u and v at the contact front less

surprising.

In this case one can speak of "passive segregation": if

the species start segregated, they may remain segregated, as we

have seen in the previous sections, and if they start mixed,

then, when $k = 1$, they remain mixed for all $t \geq 0$ (see

Remark 2 of this section).

Case (iii)   (det K < 0). The system (4.3) is now not parabolic,

and Problem (II) is probably not well posed. Since the off-diagonal

terms in K dominate in this case, one might expect a tendency

towards segregation, even in a mixed population.

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5. The system (2.1) with $k = 0$ was studied in [3] and [4].
There $v(x) \equiv v_0(x)$ and the problem reduces to solving $(2.1)_1$,
$(2.2)_1$, and $(2.3)_1$. In this case, even with segregated initial
data, solutions eventually mix, an apparent contradiction in
behavior. The limit $k \to 0$ in Problem (I) would therefore be
interesting.

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References

1. Gurney, W.S.C., Nisbet, R.M.: The regulation of inhomogeneous
    populations. J. Theoret. Biol. $\underset{\sim}{52}$, 441-457 (1975).

2. Gurtin, M. E., MacCamy, R. C.: On the diffusion of biological
    populations. Math. Biosc. $\underset{\sim}{33}$, 35-49 (1977).

3. Bertsch, M., Hilhorst, D.: A density dependent diffusion
    equation in population dynamics: stabilization to equilibrium.
    To appear.

4. Bertsch, M., Gurtin, M. E., Hilhorst, D., Peletier, L. A.:
    On interacting populations that disperse to avoid crowding:
    The effect of a sedentary colony. J. Math. Biol. $\underset{\sim}{19}$, 1-12
    (1984).

5. Gurtin, M. E., Pipkin, A. C.: A note on interacting populations
    that disperse to avoid crowding. Quart. Appl. Math.
    $\underset{\sim}{42}$, 87-94 (1984)

6. Busenberg, S. N., Travis, C. C.: Epidemic models with spatial
    spread due to population migration. J. Math. Biol. $\underset{\sim}{16}$,
    181-198 (1983).

7. Shigesada, N., Kawasaki, K., Teramoto, E.: Spatial segregation
    of interacting species. J. Theoret. Biol. $\underset{\sim}{79}$, 83-99 (1979).

8. Peletier, L. A.: The porous media equation, Application of
    nonlinear analysis in the physical sciences (Amann, H.,
    Bazley, N., Kirchgassner, K., eds.) pp. 229-241. Boston:
    Pitman 1981.

9. Aronson, D. G., Peletier, L. A.: Large time behavior of
    solutions of the porous media equation. J. Differential
    Eqns. $\underset{\sim}{39}$, 378-412 (1981).

10. Bertsch, M., Gurtin, M. E., Hilhorst, D.: A nonlinear
    diffusion equation in population dynamics: Segregation of
    populations. To appear.

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    This paper discusses the dispersal of two interacting biological species. The dispersal - a response to population pressure alone - is modelled by the degenerate parabolic system

$$u_t = [u(u+v)_x]_x$$

$$v_t = k[v(u+v)_x]_x$$

(continued)

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ABSTRACT (continued)

in conjunction with an initial prescription of the individual densities u
and v together with standard zero-flux boundary conditions. We demonstrate
here the following interesting feature of this model: <u>segregated</u> <u>initial</u> <u>data</u>
<u>give</u> <u>rise</u> <u>to</u> <u>solutions</u> <u>which</u> <u>are</u> <u>segregated</u> <u>for</u> <u>all</u> <u>time</u>.

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