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FREQUENCY/PERIOD ESTIMATION AND ADAPTIVE REJECTION OF  
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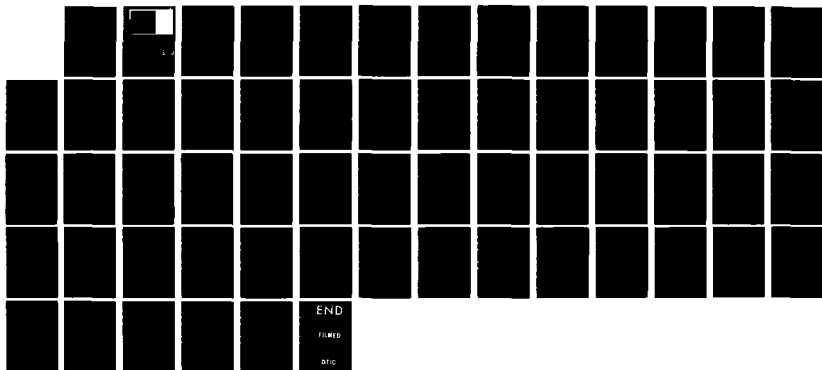
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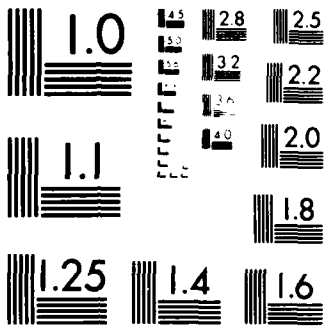
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DISTURBANCES

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D. L. Russell\*

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ABSTRACT

We discuss a method for suppressing the oscillations of a linear system subject to an external periodic disturbance of fixed, but unknown, period. The method entails augmentation of the original plant with a compensator and parameter identifier. The near equilibrium dynamics of the resulting system are analyzed and shown to be related to a linear delay equation with infinite delay and periodic coefficients. A corresponding Floquet theory is indicated. A FORTRAN program approximately realizing the period identifier is included and numerical results obtained with this program are graphically displayed and analyzed.

MOS: 93B30, B40, C40, 45A05, D05.

Key Words: Adaptive Control, Decoupling, Delay Equations

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SIGNIFICANCE AND EXPLANATION

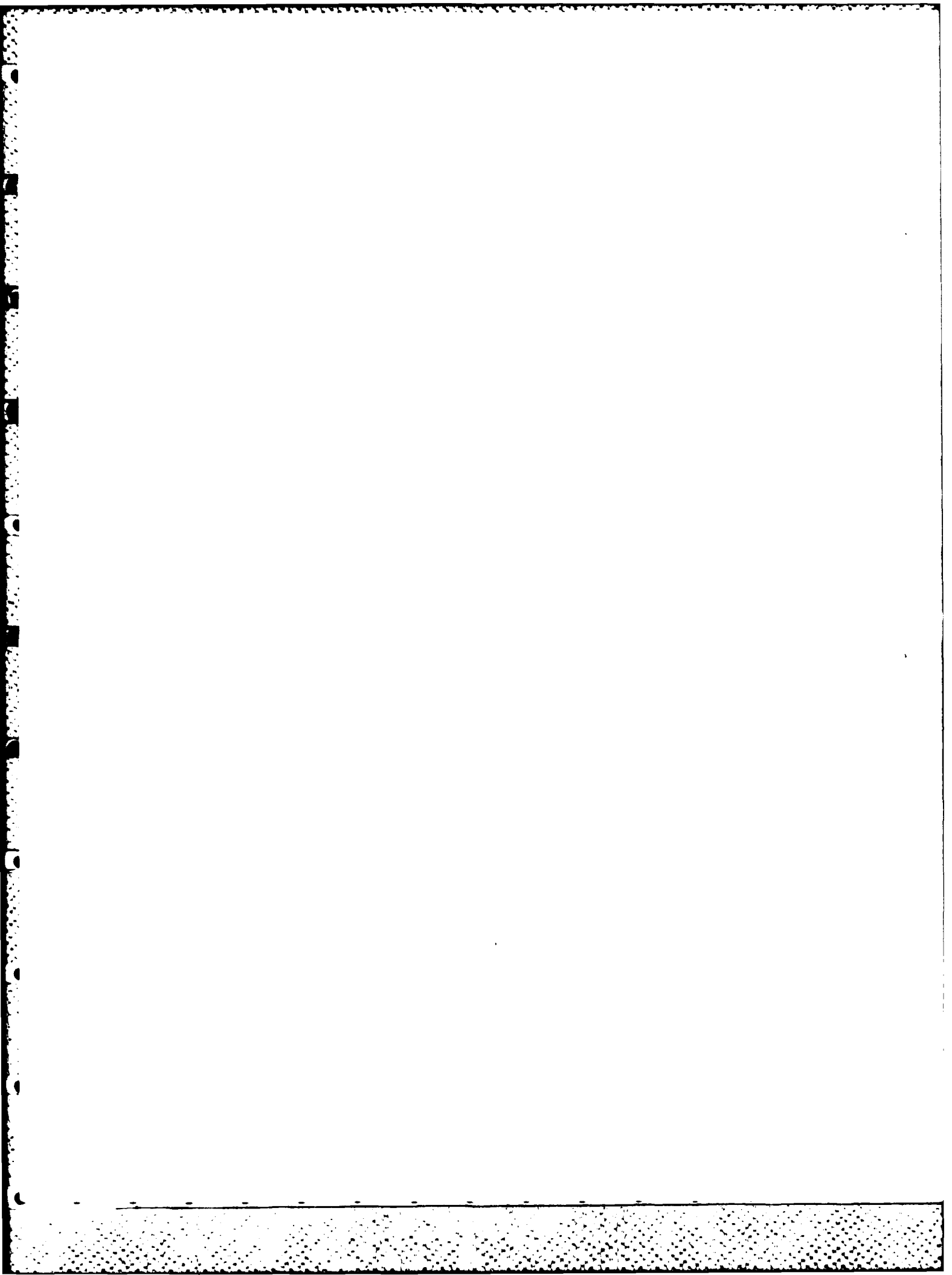
For a wide variety of systems, including sighting devices, weapons, machine tool arms, etc., operation under conditions which involve significant oscillatory disturbances is necessary. Often it is desirable to dynamically decouple the system from the disturbances by means of the intervention of active control. In many cases this must be done without a prior knowledge of the period (equivalently, the frequency) of the incoming disturbance. In this paper we propose a method for such vibration suppression using a compensator and frequency/period identifier. The stability of the resulting complex is analyzed and numerical studies are presented to indicate the potential effectiveness of the procedure.

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FREQUENCY/PERIOD ESTIMATION AND ADAPTIVE  
REJECTION OF PERIODIC DISTURBANCES

D. L. Russell\*

0. INTRODUCTION

In a wide variety of applications one encounters a system of the form

$$\dot{x} = Ax + Cu + v, \quad \left(\dot{x} = \frac{dx}{dt}\right) \quad (0.1)$$

wherein  $x$  is the  $n$ -dimensional state vector,  $u$  is the  $m$ -dimensional control vector and  $v$  is a periodic  $n$ -dimensional vector disturbance function with least positive period  $T$ :

$$v(t) = v(t + T). \quad (0.2)$$

In many cases  $\dot{x} = Ax$  by itself represents the dynamics of an elastic system, the disturbance  $v$  arises from the environment in which the elastic system is placed, and the control  $u$  is used to mitigate the effects of this disturbance. Examples include sighting devices (cameras, telescopes, etc.), weapons, and machine tool arms, operated under conditions which involve significant oscillatory disturbances, such as would be the case for a telescope operated from an aircraft, e.g.. Another important application arises in connection with the measurement and active suppression of aerodynamic flutter in aircraft wings, tail structures, etc.

The approach taken in this paper is to suppose that  $v(t)$  can be adequately modelled by

$$v(t) = Bz(t), \quad B \quad n \times 2r, \quad (0.3)$$

where  $z(t)$  satisfies a linear system

$$\dot{z} = Fz, \quad (0.4)$$

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$$F = \begin{pmatrix} 0 & \alpha_1 & 0 & 0 & \dots & 0 & 0 \\ -\alpha_2 & 0 & 0 & 0 & \dots & 0 & 0 \\ 0 & 0 & 0 & \alpha_2 & \dots & 0 & 0 \\ 0 & 0 & \alpha_2 & 0 & \dots & 0 & 0 \\ \equiv & \equiv & \equiv & \equiv & & \equiv & \equiv \\ 0 & 0 & 0 & 0 & \dots & 0 & \alpha_r \\ 0 & 0 & 0 & 0 & \dots & -\alpha_r & 0 \end{pmatrix}, \quad (0.5)$$

$$\alpha_j = \frac{2k_j \pi}{T} \equiv k_j \alpha, \quad j = 1, 2, \dots, r, \quad (0.6)$$

the  $k_j$  being positive integers. These need not necessarily be  $1, 2, \dots, r$ ; in some cases it is known, e.g., that only odd order harmonics occur so that we would use  $k_1 = 1$ ,  $k_2 = 3$ , ...,  $k_r = 2r - 1$ .

Assuming  $F$  known, and this will bring us to the subject of frequency estimation later on, we can construct a compensator

$$\dot{y} = Sx + Fy \quad (0.7)$$

where  $y$  is the  $2r$ -dimensional compensator state, and consider the combined system

$$\begin{aligned} \dot{x} &= Ax && + Bz + Cu \\ \dot{y} &= Sx + Fy && \\ \dot{z} &= && Fz. \end{aligned} \quad (0.8)$$

We will suppose that the range of  $C$  includes the range of  $B$ . This means that, in principle, one could solve

$$Cu = -Bz \quad (0.9)$$

and cancel the effect of the disturbance altogether. For a telescope operated from a moving vehicle, neglecting translational motion and considering only the angular displacements, this would be the case if the controls, acting through the mounting, have both azimuth and elevation correctional capability. In practice the direct cancellation (0.9) is rarely feasible due to noise, measurement delays, limited measurement capability, etc.



Let  $\phi$  be any nonsingular  $2r \times 2r$  matrix which commutes with  $F$ ; in most cases we would use the identity matrix. We may then find an  $m \times 2r$  matrix  $L$  such that

$$CL = -B\phi^{-1}. \quad (0.10)$$

Assuming additionally that  $(A,C)$  is stabilizable, let  $K$  be an  $m \times n$  matrix such that  $A + CK$  is a stability matrix and let  $u$  be generated by the feedback control law

$$u = Kx + Ly. \quad (0.11)$$

Using this in (0.8) we have

$$\begin{pmatrix} \dot{x} \\ y \\ z \end{pmatrix} = \begin{pmatrix} A + CK - B\phi^{-1} & B \\ S & F & 0 \\ 0 & 0 & F \end{pmatrix} \begin{pmatrix} x \\ y \\ z \end{pmatrix}. \quad (0.12)$$

We will see in Section 1 that it is possible to select  $S$  in such a way that the control law (0.11) dynamically decouples the plant state  $x$  from the periodic disturbance  $v(t) = Bz(t)$ .

The foregoing scheme, to be developed more fully in the next section, clearly amounts to the construction of a reduced order observer for the disturbance state  $z(t)$  (see [10]) and assumes that the plant state  $x(t)$  is completely accessible. If this is not the case, dynamic decoupling is probably best realized with the construction of a full  $n + 2r$  dimensional state observer. Assuming an observation

$$w = H_0x + H_1z \quad (0.13)$$

available such that the pair

$$(H_0, H_1) \begin{pmatrix} A + CK & B \\ 0 & F \end{pmatrix} \quad (0.14)$$

is observable, compatible matrices  $L_0, L_1$  are selected (see [8], e.g.) such that

$$\begin{pmatrix} A + CK - L_0H_0 & B - L_0 \\ -L_1H_0 & F - L_1H_1 \end{pmatrix} \quad (0.15)$$

is a stability matrix. We then adjoin to the plant disturbance system

$$\dot{x} = (A + CK)x + Bz + Cu \quad (0.16)$$

$$\dot{z} = Fz \quad (0.17)$$

the estimator system

$$\begin{aligned}\dot{\xi} &= (A + CK)\xi + L_0(H_0x - H_0\xi) + L_0(H_1z - H_1\zeta) \\ \dot{\zeta} &= Fz + L_1(H_0x - H_0\xi) + L_1(H_1z - H_1\zeta) .\end{aligned}\tag{0.18}$$

Then, choosing  $u$  such that

$$Cu = -BF\tag{0.19}$$

and letting  $e = x - \zeta$ ,  $F = z - F$  we find that

$$\begin{pmatrix} \dot{e} \\ \dot{f} \end{pmatrix} = \begin{pmatrix} A + CK - L_0H_0 & B - L_0H_1 \\ -L_1H_0 & F - L_1H_1 \end{pmatrix} \begin{pmatrix} e \\ f \end{pmatrix}$$

and we conclude, since (0.15) is a stability matrix, that

$$\lim_{t \rightarrow \infty} e(t) = \lim_{t \rightarrow \infty} f(t) = 0 .$$

Since, with (0.19), (0.16), (0.17) become

$$\dot{x} = (A + CK)x + Bf\tag{0.20}$$

$$\dot{z} = Fz\tag{0.21}$$

we conclude that

$$\lim_{t \rightarrow \infty} x(t) = 0$$

and thus  $x(t)$  is decoupled from  $z$ . This is a standard procedure, such as described in [10], for example.

Whether decompling is carried out as in Section 1 or as above, it is clear that the estimator system requires knowledge of the matrix (0.5) and hence the parameter  $\alpha = 2\pi/T$  in (0.6). When the period,  $T$ , and hence  $\alpha$ , is unknown it is necessary to adjoin a parameter estimator to supply the system with an estimate for  $T$ . Such a parameter estimator is described in Section 2. Stability considerations in connection with the period estimator lead to examination of a related functional equation of retarded type in Section 3. A numerical realization of the estimator of Section 2 is developed in Section 4 and examples of its use are presented in Section 5.

1. COMPENSATOR DESIGN FOR A KNOWN DISTURBANCE FREQUENCY

If the period,  $T$ , or, equivalently, the frequency  $\nu = 1/T$ , of the disturbance  $v$  is known, then we may assume that  $F$  is known and the only problem in constructing the compensator (0.7) is the selection of the  $2r \times n$  matrix  $S$ . Let us note that the matrix equation

$$\begin{pmatrix} A + CK & -B\phi^{-1} \\ S & F \end{pmatrix} \begin{pmatrix} 0 \\ \phi \end{pmatrix} - \begin{pmatrix} 0 \\ \phi \end{pmatrix} F + \begin{pmatrix} B \\ 0 \end{pmatrix} = 0 \quad (1.1)$$

is clearly valid whatever  $S$  may be. This means that if we define  $F, n$  by

$$\begin{pmatrix} \dot{x} \\ y \end{pmatrix} = \begin{pmatrix} 0 \\ \phi \end{pmatrix} z + \begin{pmatrix} F \\ n \end{pmatrix}, \quad \dot{z} = Fz \quad (1.2)$$

we shall have

$$\begin{pmatrix} \dot{F} \\ n \end{pmatrix} = \begin{pmatrix} A + CK & -B\phi^{-1} \\ S & F \end{pmatrix} \begin{pmatrix} F \\ n \end{pmatrix}, \quad (1.3)$$

as is easily checked. If the matrix in (1.3) is a stability matrix, then  $x(t) = F(t)$  will have the property

$$\lim_{t \rightarrow \infty} \|x(t)\| = 0$$

so that the periodic disturbance  $v(t) = Bz(t)$  has only a transient effect on  $x(t)$ ; the range of the transfer function matrix from  $z$  to  $\begin{pmatrix} x \\ y \end{pmatrix}$  includes only vectors of the form  $\begin{pmatrix} 0 \\ \phi \end{pmatrix}$ . Thus the plant state vector  $x$  is dynamically decoupled from  $z$ . If the matrix in (1.3) is not a stability matrix no such inferences are valid. Our proof that  $S$  can be selected so as to satisfy this stability requirement begins with

THEOREM 1 Let  $F$  be antihermitian, (as in (0.5)), so that

$$F^* = -F.$$

Then the  $n \times m$  linear matrix equation

$$(A + CK)P_0 - P_0F - B\phi^{-1} = 0 \quad (1.4)$$

has a unique  $n \times 2r$  solution  $P_0$ . If the pair  $(P_0, F)$  is observable, then the  $2r \times n$

matrix S can be chosen in such a way that

$$M = \begin{pmatrix} A + CK & -B\phi^{-1} \\ S & F \end{pmatrix} \quad (1.5)$$

is a stability matrix.

Proof Since  $F^* = -F$  implies that  $F$  has only purely imaginary eigenvalues, the existence of a unique solution  $P_0$  of (1.4) is assured by a familiar theorem in matrix theory (see, e.y., [2]). An easy application of the implicit function theorem then shows that the cubic matrix equation

$$(A + CK)P - PF - B\phi^{-1} + \epsilon PP^*P \equiv Q(P, \epsilon) = 0 \quad (1.6)$$

has a unique solution  $P = P(\epsilon)$  defined for small  $\epsilon \rightarrow 0$  with

$$\lim_{\epsilon \rightarrow 0} P(\epsilon) = P_0 \quad (1.7)$$

Setting

$$S = S(\epsilon) = -\epsilon P(\epsilon)^* \quad (1.8)$$

we note that  $M$  in (1.5) is similar to

$$\begin{aligned} \tilde{M}(\epsilon) &= \begin{pmatrix} I_n & -P(\epsilon) & A + CK & -B\phi^{-1} \\ 0 & I_{2r} & -\epsilon P(\epsilon)^* & F \end{pmatrix} \begin{pmatrix} I_n & P(\epsilon) \\ 0 & I_{2r} \end{pmatrix} \\ &= \begin{pmatrix} A + CK + \epsilon P(\epsilon)P(\epsilon)^* & Q(P(\epsilon), \epsilon) \\ -\epsilon P(\epsilon)^* & F - \epsilon P(\epsilon)^*P(\epsilon) \end{pmatrix} \\ &= \begin{pmatrix} A + CK + \epsilon P(\epsilon)P(\epsilon)^* & 0 \\ -\epsilon P(\epsilon)^* & F - \epsilon P(\epsilon)^*P(\epsilon) \end{pmatrix}. \end{aligned}$$

Since  $K$  has been chosen so that  $A + CK$  is a stability matrix,

$$M_n(\epsilon) \equiv A + CK + \epsilon P(\epsilon)P(\epsilon)^*$$

is an  $n \times n$  stability matrix for sufficiently small  $\epsilon > 0$ . From the antihermitian property of  $F$  we can see that

$$(F - \epsilon P(\epsilon)^*P(\epsilon))^* I_{2r} + I_{2r} (F - \epsilon P(\epsilon)^*P(\epsilon)) + 2\epsilon P(\epsilon)^*P(\epsilon) = 0.$$

Applying a well known modification of Liapounov's Theorem (See, e.g., [8]) we conclude that

$$M_{2r}(\epsilon) \equiv F - \epsilon P(\epsilon) * P(\epsilon)$$

is a stability matrix for  $\epsilon > 0$  if  $(P(\epsilon), F)$  is observable. Since we have assumed  $(P_0, F)$  observable and (1.7) is true,  $(P(\epsilon), F)$  is observable for  $\epsilon > 0$  sufficiently small and  $M_{2r}(\epsilon)$  is thus a stability matrix for these values of  $\epsilon$ , at least. Since  $\tilde{M}(\epsilon)$  is lower block triangular with blocks  $M_n(\epsilon)$ ,  $M_{2r}(\epsilon)$ , its stability, and hence that of  $M = M(\epsilon)$  in (1.5) is assured with the choice (1.8) for  $S = S(\epsilon)$  for sufficiently small  $\epsilon > 0$ .

It will be noted that the choice of the feedback matrix  $K$  is important in at least two ways. Improvement of the convergence of  $\Phi^{-1}y$  to  $z$ , i.e., reduction of the transient effect of the disturbance  $v = Bz$ , dictates choosing  $\epsilon$  larger to improve the stability properties of  $F - \epsilon P(\epsilon) * P(\epsilon)$ . But, since  $A + CK + \epsilon P(\epsilon) P(\epsilon) *$  suffers, stability-wise, as  $\epsilon$  is increased,  $K$  must be used to offset this effect. In Section 3 we will find even further considerations to take into account in the selection of  $\epsilon$  and  $K$ .

Since  $P(\epsilon)$  satisfies a cubic equation, which may entail some difficulty of solution, the following corollary is useful in applications.

**COROLLARY 2** If  $\epsilon$  is sufficiently small, then

$$\hat{M}(\epsilon) = \begin{pmatrix} A + CK - B\Phi^{-1} \\ -\epsilon P_0^* & F \end{pmatrix}, \quad (1.9)$$

corresponding to

$$S = \hat{S}(\epsilon) = -\epsilon P_0^* \quad (1.10)$$

in (1.5) is also a stability matrix.

**Proof** With the indicated choice of  $S$  the matrix  $\hat{M}(\epsilon)$  is similar to

$$\begin{pmatrix} I & -P_0 \\ 0 & I \end{pmatrix} \begin{pmatrix} A + CK & -B\Phi^{-1} \\ -\epsilon P_0^* & F \end{pmatrix} \begin{pmatrix} I & P_0 \\ 0 & I \end{pmatrix}$$

$$\begin{aligned}
&= \begin{pmatrix} A + CK + \epsilon P_0 P_0^* & (A + CK)P_0 - P_0 F - B\phi^{-1} + \epsilon P_0 P_0^* P_0 \\ -\epsilon P_0^* & F - \epsilon P_0^* P_0 \end{pmatrix} \\
&= (\text{c.f. (1.4)}) = \begin{pmatrix} A + CK + \epsilon P_0 P_0^* & \epsilon P_0 P_0^* P_0 \\ -\epsilon P_0^* & F - \epsilon P_0^* P_0 \end{pmatrix}. \tag{1.11}
\end{aligned}$$

Let  $\mu = \epsilon^{1/2}$  for  $\epsilon > 0$ . With  $P_0(\mu) = \mu P_0$ , the matrix (1.11) becomes

$$\begin{pmatrix} A + CK + P_0(\mu)P_0(\mu)^* & \frac{1}{\mu} P_0(\mu)P_0(\mu)^* P_0(\mu) \\ -\mu P_0(\mu)^* & F - P_0(\mu)^* P_0(\mu) \end{pmatrix}$$

which is similar to

$$\begin{aligned}
&\begin{pmatrix} A + CK + P_0(\mu)P_0(\mu)^* & P_0(\mu)P_0(\mu)^* P_0(\mu) \\ -P_0(\mu)^* & F - P_0(\mu)^* P_0(\mu) \end{pmatrix} \\
&\equiv \begin{pmatrix} A_1(\mu) & P_3(\mu) \\ -P_0(\mu)^* & F_1(\mu) \end{pmatrix}. \tag{1.12}
\end{aligned}$$

The corresponding lower triangular matrix

$$\begin{pmatrix} A_1(\mu) & 0 \\ -P_0(\mu)^* & F_1(\mu) \end{pmatrix}$$

is a stability matrix, using essentially the same argument as in Theorem 1, provided

$\mu > 0$  is sufficiently small. Consider the equation

$$\begin{aligned}
&\begin{pmatrix} A_1(\mu)^* & -P_0(\mu) \\ 0 & F_1(\mu)^* \end{pmatrix} \begin{pmatrix} Q & R \\ R^* & T \end{pmatrix} + \begin{pmatrix} Q & R \\ R^* & T \end{pmatrix} \begin{pmatrix} A_1(\mu) & 0 \\ -P_0(\mu)^* & F_1(\mu) \end{pmatrix} \\
&+ \begin{pmatrix} I & 0 \\ 0 & 2P_0(\mu)^* P_0(\mu) \end{pmatrix} = 0. \tag{1.13}
\end{aligned}$$

Solving this, we find that  $T = I_{2r}$  and

$$A_1(\mu)^* Q - P_0(\mu) R^* + Q A_1(\mu) - R P_0(\mu)^* + I = 0 \tag{1.14}$$

$$A_1(\mu)^* R - P_0(\mu) + R F_1(\mu) = 0. \tag{1.15}$$

For small  $\mu$  (equivalently, small  $\varepsilon$ ) the eigenvalues of  $A_1(\mu)$  and  $-F_1(\mu)$  are uniformly separated and solution of (1.15) shows that

$$R = O(\|P_0(\mu)\|) = O(\mu)$$

and then a similar analysis of the first equation shows that

$$Q = Q_0(\mu) + O(\mu^2)$$

where

$$A_1(\mu)Q_0(\mu) + Q_0(\mu)A_1(\mu) + I_n = 0.$$

Thus  $Q_0(\mu)$ , and hence  $Q$ , remains bounded for  $\mu > 0$  small. Since (1.13) is satisfied, using the matrix of (1.12) instead, we have

$$\begin{aligned} & \begin{pmatrix} A_1(\mu)^* & -P_0(\mu) \\ P_3(\mu)^* & F_1(\mu)^* \end{pmatrix} \begin{pmatrix} Q & R \\ R^* & T \end{pmatrix} + \begin{pmatrix} Q & R \\ R^* & T \end{pmatrix} \begin{pmatrix} A_1(\mu) & P_3(\mu) \\ -P_0(\mu)^* & F_1(\mu) \end{pmatrix} \\ &= - \begin{pmatrix} I & \frac{-\mu^2}{\sqrt{2}} Q P_0 P_0^* \\ 0 & \sqrt{2} \mu P_0^* \end{pmatrix} \begin{pmatrix} I & 0 \\ \frac{-\mu^2}{\sqrt{2}} P_0 P_0^* Q & \sqrt{2} \mu P_0 \end{pmatrix} \\ &+ \begin{pmatrix} \frac{\mu^4}{2} Q P_0 P_0^* P_0 P_0^* Q & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}. \end{aligned}$$

From this it is easy to see that the matrix on the right hand side is nonpositive for small  $\mu > 0$  and the result follows by the familiar Liapounov theorem, provided that whenever

$$\begin{pmatrix} \dot{\zeta} \\ \dot{\eta} \end{pmatrix} = \begin{pmatrix} A_1(\mu) & P_3(\mu) \\ -P_0(\mu)^* & F_1(\mu) \end{pmatrix} \begin{pmatrix} \zeta \\ \eta \end{pmatrix}$$

the quadratic form

$$\begin{aligned}
 & (\hat{z}(t)^*, \hat{n}(t)^*) \left[ - \begin{pmatrix} I & \frac{-\mu^2}{\sqrt{2}} Q P_0 P_0^* \\ 0 & \sqrt{2} \mu P_0^* \end{pmatrix} \begin{pmatrix} I & 0 \\ \frac{-\mu^2}{\sqrt{2}} P_0 P_0^* Q & \sqrt{2} \mu P_0 \end{pmatrix} + \begin{pmatrix} \mu^4 Q P_0 P_0^* P_0 P_0^* Q & 0 \\ 0 & 0 \end{pmatrix} \right] \\
 & \quad \times \begin{pmatrix} \hat{z}(t) \\ \hat{n}(t) \end{pmatrix}
 \end{aligned}$$

cannot vanish on any interval of positive length. For small  $\mu > 0$  this question reduces very quickly to the observability of the pair  $(P_0, F)$ , which has already been assumed. This completes the proof.



## 2. PERIOD ESTIMATION AND A RELATED STABILITY PROBLEM

Whether the submatrix  $S$  in the definition (1.5) of  $M$  is selected as in Theorem 1 or as in Corollary 2, or by some other procedure, it is clear that the overall matrix  $M$  will depend on the period,  $T$ , of the disturbance  $v$  so that, supposing now that the design parameter  $\epsilon$  has been fixed, we have

$$M = M(\alpha) = \begin{pmatrix} A + CK & -B\Phi(\alpha)^{-1} \\ S(\alpha) & F(\alpha) \end{pmatrix} \quad (2.1)$$

where  $\alpha = \frac{2\pi}{T}$  (if (0.6)).

It would be possible to estimate  $\alpha$  directly using various well known parameter estimation procedures ([6], [9],). However, in these procedures one tends to encounter either instability or slow convergence, or other difficulties. For example, the model reference algorithm of [6] cannot be applied because, in the complete system (0.12) the portion  $\dot{z} = Fz$  of that system is not controllable with respect to  $u$ .

We have elected to use a very simple procedure to estimate the period,  $T$ , directly. Assuming that an output, or observation

$$w(t) = H_1 x(t) + H_2 y(t) = (H_1, H_2) \begin{pmatrix} x(t) \\ y(t) \end{pmatrix} \equiv Hw(t), \quad (2.2)$$

where (cf. (0.3), (0.12), (1.5)).

$$\dot{w} = M(\alpha)w + R \quad (R = \begin{pmatrix} v \\ 0 \end{pmatrix}) \quad (2.3)$$

is available, from the assumed stability property of the matrix  $M(\alpha)$  it follows that a periodic disturbance input  $v$  will result in an output  $w(t)$  which, except for transient behavior, is also periodic with the same period. It therefore makes sense, in the continuous framework which we use here for analysis, to consider the cost functional, for  $\gamma > 0$ ,

$$\begin{aligned} C_0(T, t) &= \int_0^t e^{-\gamma(s-t)} (w(s) - w(s-T))^* (w(s) - w(s-T)) ds \\ &= \int_0^t e^{-\gamma(s-t)} (w(s) - w(s-T))^* H^* H (w(s) - w(s-T)) ds \end{aligned}$$

and select, as our estimate for the period  $T$  at time  $t$ , that value  $T(t)$  which minimizes  $C(T,t)$  within a given range  $T_1 < T < T_2$ . (The range must be restricted in order to avoid the trivial period  $T = 0$  and multiples of the minimum period of the disturbance.) Then

$$\alpha(t) = \frac{2\pi}{T(t)} \quad (2.5)$$

is the estimate at time  $t$  for  $\alpha$  in (0.6), (2.3). A numerical procedure approximating this optimization process is described in Section 4 and is used to obtain the computational results of Section 5. There it will be seen that certain steps do have to be taken in order to ensure the stability of the combined control/estimation system. Our purpose here is to provide a framework for the stability analysis by developing a linearized variational equation for that system about the nominal time trajectory in the case where the true period, which we will call  $T_0$ , lies in the interior of the interval  $T_1 < T < T_2$ .

For our analysis of the combined use of (2.3) and (2.4) we will consider, instead of  $C_0(T,t)$ , as given by (2.4), the cost

$$C(T,t) = \int_0^t e^{\gamma(s-t)} (w(s) - w(s-T))^* H^* H (w(s) - w(s-T)) ds, \quad (2.6)$$

wherein we assume that the trajectory  $w(s)$  is defined in the indefinite past. The justification for this lies in the fact that if (2.3) and (2.6) together yield a stable process, the difference between the use of (2.6) and (2.4) will be transient.

To carry out this program we begin by supposing that when the correct value  $\alpha_0 = \frac{2\pi}{T_0}$  is used in (2.3) the steady state  $T_0$ -periodic solution resulting from the  $T_0$ -periodic input  $v(t)$  is  $w_0(t)$  and  $\dot{w}_0(t) = H w_0(t)$ . Since our estimate,  $\alpha(t)$ , will vary from  $\alpha_0$ , we suppose that the actual solution of (2.3) which we obtain is  $w(t)$ . Thus

$$\dot{w}(t) = \alpha(t - T_0) w(t) \quad (2.7)$$

$$w(t) = w_0(t) + \Delta w(t), \quad (2.8)$$

$$\alpha(t) = \alpha_0 + \Delta\alpha(t) , \quad (2.9)$$

$$T(t) = T_0 + \Delta T(t) . \quad (2.10)$$

A necessary condition in order that  $T(t)$  should minimize  $C(T,t)$  is obtained by differentiating  $C(T,t)$  with respect to  $T$  and setting the result at  $T = T(t)$  equal to zero. Thus (cf. (2.2), (2.3), (2.5))

$$\begin{aligned} 0 &= \frac{1}{2} \frac{\partial C(T,t)}{\partial T} \Big|_{T = T(t)} \\ &= \int_{-\infty}^t e^{\gamma(s-t)} (w(s) - w(s - T(t))) * H * \dot{w}(s - T(t)) ds \\ &= \int_{-\infty}^t e^{\gamma(s-t)} (w(s) - w(s - T(t))) * H * [A(\alpha(s - T(t)))w(s - T(t)) \\ &\quad + R(s - T(t))] ds . \end{aligned} \quad (2.11)$$

Noting (2.5), we see that (2.11) is, implicitly, an equation for  $T(t)$  which is coupled with the system (2.3) satisfied by  $w(t)$ . The resulting coupled system is clearly a nonlinear functional equation of delay type. We are concerned with the (at least local with respect to  $\alpha_0$  and  $w_0(t)$ ) existence, uniqueness and asymptotic stability of solutions.

**LEMMA 3.** For fixed  $t$  and a trajectory  $w(s)$   $-\infty < s < t$ , for (2.3), corresponding to a continuous  $T_0$  - periodic  $R(s)$ ,  $-\infty < s < t$ , the equation (2.11) is solvable for  $T(t)$  near  $T_0$  if

$$\begin{aligned} &- (w(t) - w(t - T_0)) * H * \dot{w}(t - T_0) \\ &+ \int_{-\infty}^t e^{\gamma(s-t)} \dot{w}(s) * H * \dot{w}(s - T_0) ds \\ &+ \gamma \int_{-\infty}^t e^{\gamma(s-t)} (w(s) - w(s - T_0)) * H * \dot{w}(s - T_0) ds \end{aligned}$$

$$\begin{aligned}
&= \int_0^t e^{\gamma(s-t)} \dot{w}(s) * H * \dot{H}w(s - T_0) ds \\
&- \frac{\lambda}{\partial t} \int_0^t e^{\gamma(s-t)} (w(s) - w(s - T_0)) * H * \dot{w}(s - T_0) ds \neq 0
\end{aligned} \tag{2.12}$$

Within the class of w which satisfy

$$|w(s)| < M_0, \quad -\infty < s < t, \tag{2.13}$$

$$|w(s) - w_0(s)| < \epsilon, \quad t - \tau < s < t \tag{2.14}$$

this is true for sufficiently large  $\tau$  are sufficiently small  $\epsilon$  if

$$\int_{-\infty}^t e^{\gamma(s-t)} \dot{w}_0(s) H * \dot{H}w_0(s) ds \neq 0 \tag{2.15}$$

and this, in turn, is true if the  $T_0$  - periodic function  $Hw_0(s) = w_0(s)$  is not constant.

Proof Assume for the moment that  $R(t)$  is a function in  $C^1$ . Differentiating the second line of (2.11) with respect to  $T$  at  $T = T_0$  and retaining only zero and first order terms in  $\Delta T(t)$  we obtain the equation, linearized with respect to  $\Delta T$ ,

$$\begin{aligned}
0 &= \int_{-\infty}^t e^{\gamma(s-t)} \dot{w}(s - T_0) * H * \dot{H}w(s - T_0) ds \Delta T(t) \\
&- \int_{-\infty}^t e^{\gamma(s-t)} (w(s) - w(s - T_0)) * H * \dot{H}w(s - T_0) ds \Delta T(t) \\
&+ \int_{-\infty}^t e^{\gamma(s-t)} (w(s) - w(s - T_0)) * H * \dot{H}w(s - T_0) ds + \dots
\end{aligned} \tag{2.16}$$

Integrating the second term by parts we have

$$0 = [-(w(t) - w(t - T_0)) * H * \dot{H}w(t - T_0)] + \int_{-\infty}^t e^{\gamma(s-t)} \dot{w}(s) * H * \dot{H}w(s - T_0) ds +$$

$$\begin{aligned}
& + \alpha \int_{-\infty}^t e^{\gamma(s-t)} (w(s) - w(s - T_0)) * H * H \dot{w}(s - T_0) ds \Delta T(t) \\
& + \int_{-\infty}^t e^{\gamma(s-t)} (w(s) - w(s - T_0)) * H * H \dot{w}(s - T_0) ds + \dots
\end{aligned} \tag{2.17}$$

This expression no longer depends on  $\ddot{w}(s - T_0)$ , hence we may relax the requirement  $R \in C^1$  since  $R \in C^0$  can be uniformly approximated in the  $C^0$  norm by  $R \in C^1$ . Examination of the remainder term shows that the same argument applies there and we conclude that (2.16) is indeed valid to first order in  $\Delta T(t)$ . The first statement of our lemma then follows immediately from the implicit function theorem.

The last statement follows from the property

$$w_0(s) - w_0(s - T_0) \equiv 0, \quad -\infty < s < t \tag{2.18}$$

together with (2.3), (2.7), (2.17), which enable one to make the first term in (2.16) arbitrarily close to the left hand side of (2.15) and the second term arbitrarily close to zero. This completes the proof of Lemma 3.

The linearization with respect to  $\Delta T, \Delta w$  is obtained by using (2.18) in (2.16). Because  $w_0(s) - w_0(s - T_0) \equiv 0$  and because only zero order terms are retained as coefficients of the first order term  $\Delta T(t)$ , the result is

$$\begin{aligned}
& \left[ \int_{-\infty}^t e^{\gamma(s-t)} \dot{w}_0(s) * H * H \dot{w}_0(s - T_0) ds \right] \Delta T(t) \\
& + \int_{-\infty}^t e^{\gamma(s-t)} (\Delta w(s) - \Delta w(s - T_0)) * H * \dot{w}_0(s - T_0) ds = 0,
\end{aligned}$$

and using (2.18),  $\dot{w}_0(s - T_0) \equiv \dot{w}_0(s)$ , and the assumption (2.15) we have

$$\Delta T(t) = \frac{- \int_{-\infty}^t e^{\gamma(s-t)} (\Delta w(s) - \Delta w(s - T_0)) * H * H \dot{w}_0(s) ds}{\int_{-\infty}^t e^{\gamma(s-t)} \dot{w}_0(s) * H * H \dot{w}_0(s) ds}, \tag{2.19}$$

which is a delay type functional equation relating  $\Delta T(t)$  and the time history of  $\Delta w$ .

Then from (2.3) we have, to first order,

$$\Delta \dot{w} + \dot{w}_0 = M(\alpha_0)w_0 + \beta + M(\alpha_0)\Delta w + \left(\frac{\partial M}{\partial \alpha}(\alpha_0)\Delta \alpha\right)w_0 + \dots$$

and, since  $\dot{w}_0 = M(\alpha_0)w_0 + \beta$  and (2.5) applies, we have as our linearized system (2.19)

and

$$\Delta \dot{w}(t) = M(\alpha_0)\Delta w(t) - \left(\frac{2\pi}{T_0} \frac{\partial M}{\partial \alpha}(\alpha_0)w_0(t)\right)\Delta T(t). \quad (2.20)$$

A single equation may be obtained by noting that for  $\tau > T_0$

$$\begin{aligned} & \Delta \dot{w}(s) - \Delta \dot{w}(s - T_0) \\ &= M(\alpha_0)(\Delta w(s) - \Delta w(s - T_0)) - \frac{2\pi}{T_0} \frac{\partial M}{\partial \alpha}(\alpha_0)w_0(s)\Delta T(s) \\ & \quad + \frac{2\pi}{T_0} \frac{\partial M}{\partial \alpha}(\alpha_0)w_0(s - T_0)\Delta T(s - T_0) \\ &= M(\alpha_0)(\Delta w(s) - \Delta w(s - T_0)) - \frac{2\pi}{T_0} \frac{\partial M}{\partial \alpha}(\alpha_0)w_0(s)(\Delta T(s) - \Delta T(s - T_0)), \end{aligned}$$

where we have used  $w_0(s) = w_0(s - T_0)$ . Then

$$\begin{aligned} & \Delta w(t) - \Delta w(t - T_0) \\ &= \frac{-2\pi}{T_0} \int_{-\infty}^t e^{M(\alpha_0)(t-\sigma)} \frac{\partial M}{\partial \alpha}(\alpha_0)w_0(\sigma)(\Delta T(\sigma) - \Delta T(\sigma - T_0))d\sigma \end{aligned}$$

and thus

$$\begin{aligned} \Delta T(t) &= \frac{2\pi/T_0^2}{\int_{-\infty}^t e^{\gamma(s-t)} \dot{w}_0(s) * H * \dot{w}_0(s) ds} \\ & \quad \times \int_{-\infty}^t e^{\gamma(s-t)} \int_{-\infty}^s w_0(\sigma) * \frac{\partial M}{\partial \alpha}(\alpha_0) * e^{M(\alpha_0)*(s-\sigma)} \\ & \quad (\Delta T(\sigma) - \Delta T(\sigma - T_0)) d\sigma H * H w_0(s) ds. \end{aligned}$$

Letting

$$w_0(\gamma, t) = \frac{2\pi/T_0^2}{\int_{-\infty}^t e^{\gamma(s-t)} \dot{w}_0(s) * H * \dot{H} w_0(s) ds} \quad (2.21)$$

we have

$$\begin{aligned} \Delta T(t) &= w_0(\gamma, t) \int_{-\infty}^t w_0(\sigma) * \frac{\partial M}{\partial \alpha}(\alpha_0) * \int_{\sigma}^t e^{\gamma(s-t)} e^{M(\alpha_0) * (s-\sigma)} H * \dot{H} w_0(s) ds \\ &\quad \times (\Delta T(\sigma) - \Delta T(\sigma - T_0)) d\sigma, \end{aligned}$$

which has the form

$$\Delta T(t) = \int_{-\infty}^t w_1(\gamma, t, \sigma, M(\alpha_0)) (\Delta T(\sigma - T_0)) d\sigma \quad (2.22)$$

with

$$w_1(\gamma, t, \sigma, M(\alpha_0)) = w_0(\gamma, t) w_0(\sigma) * \frac{\partial M}{\partial \alpha}(\alpha_0) * \int_{\sigma}^t e^{\gamma(s-t)} e^{M(\alpha_0) * (s-\sigma)} H * \dot{H} w_0(s) ds \quad (2.23)$$

LEMMA 4.  $w_1(\gamma, t, \sigma, M(\alpha_0))$  is periodic in  $t$  and  $\sigma$  with period  $T_0$ , in the sense

$$w_1(\gamma, t, \sigma, M(\alpha_0)) = w_1(\gamma, t + T_0, \sigma + T_0, M(\alpha_0)).$$

Proof. Using the formula (2.23) directly we see that

$$\begin{aligned} w_1(\gamma, t + T_0, \sigma + T_0, M(\alpha_0)) &= w_0(\gamma, t + T_0) w_0(\sigma + T_0) * \frac{\partial M}{\partial \alpha}(\alpha_0) * \\ &\quad \times \int_{\sigma + T_0}^{t + T_0} e^{\gamma(s-t-T_0)} e^{M(\alpha_0) * (s-\sigma-T_0)} H * \dot{H} w_0(s) ds. \end{aligned}$$

From the  $T_0$ -periodicity of  $w_0(t)$  the same periodicity of  $w_0(\gamma, t)$  follows easily.

Then with  $r = s - T_0$

$$\begin{aligned} w_1(\gamma, t + T_0, \sigma + T_0, M(\alpha_0)) &= w_0(\gamma, t) w_0(\sigma) * \frac{\partial M}{\partial \alpha}(\alpha_0) * \\ &\quad \times \int_{\sigma}^t e^{\gamma(r-t)} e^{M(\alpha_0) * (r-\sigma)} H * \dot{H} w_0(r + T_0) dr \end{aligned}$$

and, using the  $T_0$  - periodicity of  $\dot{w}_0$  we have

$$w_1(\gamma, t + T_0, \sigma + T_0, M(\alpha_0)) = w_1(\gamma, t, \sigma, M(\alpha_0)) ,$$

as claimed.

The equation (2.22) can be rewritten as

$$\Delta T(t) = \int_{-\infty}^t w(\gamma, t, \sigma, M(\alpha_0)) \Delta T(\sigma) d\sigma \quad (2.24)$$

with

$$w(\gamma, t, \sigma, M(\alpha_0)) = w_1(\gamma, t, \sigma, M(\alpha_0)) , \quad t - T_0 < \sigma \leq t$$

$$w(\gamma, t, \sigma, M(\alpha_0)) = w_1(\gamma, t, \sigma, M(\alpha_0)) - w_1(\gamma, t, \sigma + T_0, M(\alpha_0)) , \quad -\infty < \sigma \leq t - T_0 .$$

Since, for  $\sigma = t - T_0$ ,  $w_1(\gamma, t, \sigma + T_0, M(\alpha_0)) = w_1(\gamma, t, t, M(\alpha_0)) = 0$  we see that

$w(\gamma, t, \sigma, M(\alpha_0))$  is continuous as a function of  $\sigma$  and, clearly,

$$w(\gamma, t + T_0, \sigma + T_0, M(\alpha_0)) = w(\gamma, t, \sigma, M(\alpha_0)) .$$

We have shown  $\gamma, M(\alpha_0)$  directly as arguments of  $w$  because  $\gamma, \varepsilon$  and  $K$ , the last two involved in the construction of  $M(\alpha_0)$  (see (2.1)) are the parameters which we have to work with in order to influence  $w$ , and hence the solutions of (2.24). It is clear from (2.22) that  $w$  depends on  $T_0$  as well.



### 3. ANALYSIS OF FLOQUET TYPE SOLUTIONS

The fact that the equation (2.24) involves an infinite time delay places it in a class of functional differential equations with periodic coefficients whose properties have not been fully explored. From the behavior of solutions of such equations with finite delays ([3], [4]) we expect that, with some restrictions on the kernel

$W(\gamma, t, \sigma, M(\alpha_0))$ , the dominant solutions should be solutions of "Floquet type", i.e., solutions of the form

$$\Delta T(t) = e^{\lambda t} P(t) , \quad (3.1)$$

where  $P(t)$  is a continuous  $T_0$  - periodic function:

$$P(t + T_0) = P(t) .$$

The main point of this section is to indicate that this is, indeed, the case for kernels satisfying a uniform decay condition

$$|W(\gamma, t, \sigma, M(\alpha_0))| < C e^{-c(t-\sigma)} , \quad \sigma < t ,$$

for positive  $C, c$ .

Before entering upon the proof of this, let us note some rather transparent results which, however superficial, give some indication of the factors which are likely to play a role in our analysis. Suppose an inequality (3.2) is satisfied for positive  $c, C$ . Supposing a solution of the form (3.1) to exist, we normalize  $P(t)$  so that

$$\sup_{s \in [t, t - T_0]} |P(s)| = 1 .$$

Then we let  $t$  be such that  $|P(t)| = 1$ . Multiplying by a constant, if need be, we may assume  $P(t) = 1$ . Then

$$e^{\lambda t} = \int_{-\infty}^t W(\gamma, t, \sigma, M(\alpha_0)) e^{\lambda \sigma} P(\sigma) d\sigma = 1$$

or

$$\int_{-\infty}^t W(\gamma, t, \sigma, M(\alpha_0)) e^{\lambda(\sigma-t)} P(\sigma) d\sigma = 1$$

But

$$|w(\gamma, t, \sigma, M(\alpha_0)) e^{\lambda(\sigma-t)} P(\sigma)| < C e^{(c + \operatorname{Re}(\lambda))(\sigma-t)}$$

So that

$$C \int_{-\infty}^t e^{(c + \operatorname{Re}(\lambda))(\sigma-t)} d\sigma > 1,$$

yielding an upper bound on  $\operatorname{Re}(\lambda)$ :

$$\frac{C}{c + \operatorname{Re}(\lambda)} > 1 \Rightarrow \operatorname{Re}(\lambda) < c - c.$$

Under what circumstances could a bound of the type (3.2) be expected? Recalling that

$$w_1(\gamma, t, \sigma, M(\alpha_0)) = w_0(\gamma, t) w_0(\sigma)^* \frac{\partial M}{\partial \alpha}(\alpha_0)^* e^{\gamma(\sigma-t)}$$

$$\times \int_{\sigma}^t e^{\gamma(s-\sigma)} e^{M(\alpha_0)^*(s-\sigma)} H^* \dot{H} w_0(s) ds$$

we note that with  $r = s - \sigma$ ,

$$\int_{\sigma}^t e^{\gamma(s-\sigma)} e^{(M(\alpha_0)^* + \gamma I)r} H^* \dot{H} w_0(r + \sigma) ds = \int_0^{t-\sigma} e^{(M(\alpha_0)^* + \gamma I)r} e^{\gamma r} H^* \dot{H} w_0(r + \sigma) dr$$

Since  $\dot{w}_0$  is periodic, if the eigenvalues  $\mu$  of  $M(\alpha_0)$  satisfy

$$\operatorname{Re}(\mu) < -\delta,$$

for some  $\delta > 0$ , we will have, for some  $M_0 > 0$

$$\| e^{(M(\alpha_0)^* + \gamma I)r} H^* \dot{H} w_0(r + \sigma) \| < M_0 e^{(\gamma-\delta)r}$$

so that

$$\begin{aligned} & \left\| \int_0^{t-\sigma} e^{(M(\alpha_0)^* + \gamma I)r} H^* \dot{H} w_0(r + \sigma) dr \right\| \\ & < M_0 \int_0^{t-\sigma} e^{(\gamma-\delta)r} dr = \frac{M_0}{\gamma-\delta} [e^{(\gamma-\delta)(t-\sigma)} - 1]. \end{aligned} \quad (3.3)$$

We expect  $w_0(t, \gamma)$  to be  $O(\gamma)$  from (2.21); write

$$|w_0(t, \gamma)| < M_1 \gamma$$

and then, since  $w_0(\sigma)$  is periodic,

$$\begin{aligned} |w_0(t, \gamma)w_0(\sigma)| &= \frac{\partial M}{\partial \alpha}(\alpha_0) e^{\gamma(\sigma-t)} \\ &< M_2 \gamma e^{\gamma(\sigma-t)}. \end{aligned} \quad (3.4)$$

Combining this with (3.3), (2.23)

$$|w_1(\gamma, t, \sigma, M(\alpha_0))| < M_0 M_2 \frac{\gamma}{\gamma - \delta} e^{\delta(\sigma-t)}$$

giving  $c = M_0 M_2 \frac{\gamma}{\gamma - \delta}$ ,  $c = \delta$ .

A comparable estimate will then apply to  $w(\gamma, t, \sigma, M(\alpha_0))$ .

From this we see that if we are to control the identifier stability properties, this must be done through  $\gamma$  and through the system matrix  $M(\gamma_0)$ , by choice of  $\gamma$ ,  $\epsilon$  and  $K$  (or through choice of  $\gamma$ ,  $K$ ,  $L_0$ ,  $L_1$  if we use the full system estimator as described in Section 0. Further, we see from (2.21) that  $w_0(\gamma, t)$ , and hence  $w_1(\gamma, t, \sigma, M(\alpha_0))$ ,  $w(\gamma, t, \sigma, M(\alpha_0))$  increase rapidly as the frequency parameter  $\alpha = \frac{2\pi}{T_0}$  increases, i.e., as  $T_0$  decreases. Thus, to be able to reject higher frequencies while maintaining stability we must expect to find it necessary to increase the damping in the system (2.3) by use of higher gains  $\epsilon$  and  $K$  (of (1.5), 61.8)). We will also see in Section 5 that this expectation is realized.

We proceed now to state a theorem to the effect that if a bound of the form (3.2) applies, then all solutions of (2.24) which do not satisfy

$$|\Delta T(t)| < B e^{-\beta t}, \quad 0 < t < \infty, \quad (3.5)$$

where  $B$  is positive and  $\beta > 0$  is less than  $c$  by an arbitrarily small amount, must be linear combinations of Floquet type solutions.

THEOREM 5. Consider the vector functional equation

$$z(t) = \int_{-\infty}^t W(t,s)z(s)ds, \quad z \in R^m, \quad (3.6)$$

where  $W(t,s)$  is a (piecewise continuous, at least)  $m \times m$  matrix function satisfying

$$\|W(t,s)\| < Ce^{-c(t-s)}, \quad -\infty < s < t, \quad (3.7)$$

$$W(t+T, s+t) = W(t,s) \quad (3.8)$$

for positive numbers  $C, c, T$ . Then, given any  $\beta < c$ , and any solution  $z(t)$  with locally square integrable initial function satisfying

$$\int_{-\infty}^0 e^{2cs} \|z(s)\|_{R^m}^2 ds < \infty, \quad (3.9)$$

we can write

$$z(t) = z_F(t) + z_B(t), \quad t > 0 \quad (3.10)$$

where, for some positive  $B$ ,

$$\|z_B(t)\| < Be^{-\beta t}, \quad t > 0 \quad (3.11)$$

and  $z_F(t)$  is a linear combination of Floquet type solutions, i.e., solutions of the form

$$\zeta(t) = e^{\lambda t} P(t), \quad P(t+T) = P(t), \quad P \in C([0, \infty), R^m), \quad (3.12)$$

or, in some cases (multiple "eigenvalues")

$$\zeta(t) = e^{\lambda t} t^p P(t), \quad (3.13)$$

where  $p$  is a positive integer and  $P(t)$  is as in (3.12).

A complete proof of Theorem 5 is beyond the scope of the present work but a sketch of the proof will be given in Section 6.

From this result we see that whenever an inequality of the type (3.2) is valid with  $c > 0$ , then all solutions of (2.24) decay at a uniform exponential rate unless there are actually solutions (3.1) of Floquet type for which  $\text{Re}(\lambda) > 0$ . The question arises, of course, as to how such Floquet exponents might actually be computed. It seems almost certain that the most efficient procedure involves actual solution of (2.24) or (2.19),

(2.20), assuming an adequate approximation procedure is available. The procedure is essentially the same one as is used to compute the dominant (pairs of) root(s) of an ordinary polynomial.

Returning to  $\Delta T(t)$  as the name for the solution, we select a more or less arbitrary initial state  $\Delta T(t)$  on some interval  $[-\tau, 0]$  (in terms of Section 6 this should be a  $\tilde{z}$  such that the residue of  $(I - Q(\lambda))^{-1} q(\lambda, \tilde{z})$  at  $\lambda = \lambda z$  is not zero, which is generically true). The resulting solution  $\Delta T(t)$ ,  $t > 0$ , is computed and we examine successive segments of length  $T_0$

$$\Delta T_k(s) = \Delta T(kT_0 + s), \quad 0 < s < T_0, \quad k = 0, 1, 2, 3, \dots$$

If the largest multiplier

$$\mu_v = e^{\lambda_v T_0}$$

is a unique real number, then generically with respect to the choice of initial function  $\Delta T(t)$ ,  $t \in [-\tau, 0]$ , we shall have (using the least squares approach)

$$\mu_v = \lim_{k \rightarrow \infty} \frac{\int_0^T \Delta T_k(s) \Delta T_{k-1}(s) ds}{\int_0^T (\Delta T_{k-1}(s))^2 ds}.$$

In the case of a dominant complex conjugate pair the procedure is only slightly more complicated. We solve

$$\Delta T_k(s) + \alpha \Delta T_{k-1}(s) + \beta \Delta T_{k-2}(s) = 0$$

for  $\alpha$  and  $\beta$  in the least squares (least  $L^2$  norm) sense, which amounts to

$$\begin{pmatrix} \int_0^T \Delta T_{k-1}(s)^2 ds & \int_0^T \Delta T_{k-1}(s) \Delta T_{k-2}(s) ds \\ \int_0^T \Delta T_{k-2}(s) \Delta T_{k-1}(s) ds & \int_0^T \Delta T_{k-2}(s)^2 ds \end{pmatrix} \begin{pmatrix} \alpha_k \\ \beta_k \end{pmatrix} +$$

$$+ \begin{pmatrix} T \\ \int_0^T \Delta T_k(s) \Delta T_{k-1}(s) ds \\ T \\ \int_0^T \Delta T_k(s) \Delta T_{k-2}(s) ds \end{pmatrix} = 0$$

The pair  $\mu_v, \bar{\mu}_v$  is then approximated at the  $k$ -th stage by the roots  $\mu_k, \bar{\mu}_k$  of

$$\mu^2 + \alpha_k \mu + \beta_k = 0 .$$

It seems likely that while (2.24) is nicer from the viewpoint of mathematical simplicity, it is better to solve (2.19), (2.20) rather than (2.24) because the formula for the kernel  $W(\gamma, t, \sigma, M(\alpha_0))$  in (2.24) is rather complicated.

If a simulation routine combining the period estimator, compensator and a mathematical model of the plant to be controlled is already in hand, as was the case for the writer, approximate solutions of the variational equation can be obtained by running the simulator with slightly different initial conditions and forming the appropriate difference quotient of the resulting solutions. This does not test the validity of our derivation of the variational equation but, as we will see in Section 5, it does provide results consistent with the proposed functional equation model for error propagation.

#### 4. NUMERICAL REALIZATION OF THE PERIOD ESTIMATOR

If  $x(t)$  is a solution of

$$\dot{x}(t) = (A + CK) x(t) + v(t), \quad t > 0,$$

and the disturbance  $v(t)$  is periodic with period  $T$ :

$$v(t) = v(t + T),$$

then an observation on  $x(t)$ ,

$$\omega(t) = H x(t)$$

will tend exponentially to a period observation, i.e.,

$$\lim_{t \rightarrow \infty} (\omega(t) - \omega(t + T)) = 0.$$

In this section we develop a numerical procedure for estimation of  $T$  which is a realization of the continuous procedure described in Section 2. We will take  $\omega$  to be scalar here but the extension to vector observations is quite immediate.

We will suppose that  $\omega(t)$  is not available continuously. Rather, we have discrete samples

$$\omega_k = \omega(t_k), \quad t_{k+1} - t_k = h > 0, \quad k = 0, 1, 2, \dots$$

For computational purposes we define the interpolated observation on  $t_k < t < t_{k+1}$  by

$$\tilde{\omega}(t_k + \sigma h) \equiv \eta_k(\sigma) = \sigma \omega_{k+1} + (1 - \sigma) \omega_k, \quad 0 < \sigma < 1. \quad (4.1)$$

We note that  $\tilde{\omega}(t_k) = \omega_k$ ,  $\tilde{\omega}(t_{k+1}) = \omega_{k+1}$ . We define  $\eta_k = \omega_k$ ,  $k = 0, 1, 2, \dots$ . Our method for estimating  $T$  is to form, at each instant  $t_k$ , and for a range  $L_0 < l < L_1$ , the functions

$$\rho_{k,l}(\sigma) = \eta_k - \eta_{k-l}(\sigma) \quad (4.2)$$

and determine values  $l_k, \sigma_k$  of  $l, \sigma$  which minimize

$$C_{k,l}(\sigma) = \sum_{j=0}^k v_j \rho_{k-j,l}(\sigma),$$

which should be compared with (2.6). The functions (4.2), of course, require only the values  $\eta_k = \omega_k$ ,  $\eta_{k+1} = \omega_{k+1}$  for this description and the  $\rho_{k,l}$  admit a comparable

finite characterization. Once  $\ell_k, \sigma_k$  have been determined, the estimate for  $T$  at the instant  $t_k$  is

$$T_k = (\ell_k - \sigma_k)h. \quad (4.3)$$

If  $\gamma$  is close to 1 this estimate may be expected to change only slowly, as  $k$  varies, in response to varying periodic behavior of  $\omega(t)$  while values of  $\gamma$  closer to zero provide more rapid updating capability. The use of the parameter  $\sigma$ , allowing for interpolation between recorded discrete data, permits one to obtain accurate results without an excessively fast sampling rate.

Let us now examine the computational considerations applying to the method. For  $0 < \sigma < 1$  we have

$$\rho_{k,\ell}(\sigma) = \eta_k - [\sigma \eta_{k-\ell+1} + (1-\sigma)\eta_{k-\ell}]$$

and thus

$$\begin{aligned} \rho_{k,\ell}(\sigma)^2 &= \eta_k^2 + \sigma^2 \eta_{k-\ell+1}^2 + (1-\sigma)^2 \eta_{k-\ell}^2 \\ &\quad - 2\sigma \eta_k \eta_{k-\ell+1} - 2(1-\sigma) \eta_k \eta_{k-\ell} + 2\sigma(1-\sigma) \eta_{k-\ell+1} \eta_{k-\ell}. \end{aligned}$$

Defining

$$S_k = \sum_{j=0}^{\infty} \gamma^j \eta_{k-j}^2, \quad S_{k-\ell+1} = \sum_{j=0}^{\infty} \gamma^j \eta_{k-\ell+1-j}^2,$$

$$S_{k-\ell} = \sum_{j=0}^{\infty} \gamma^j \eta_{k-\ell-j}^2,$$

$$P_{k,k-\ell} = \sum_{j=0}^{\infty} \gamma^j \eta_{k-j} \eta_{k-\ell-j},$$

$$P_{k,k-\ell+1} = \sum_{j=0}^{\infty} \gamma^j \eta_{k-j} \eta_{k-\ell+1-j},$$

$$P_{k-\ell+1,k-\ell} = \sum_{j=0}^{\infty} \gamma^j \eta_{k-\ell+1-j} \eta_{k-\ell-j},$$



we see that

$$C_{k,l}(\sigma) = \sigma^2 [S_{k-l+1} - S_{k-l} - 2P_{k-l+1,k-l}] \\ + 2\sigma [S_{k-l} - P_{k,k-l+1} + P_{k,k-l} + P_{k-l+1,k-l}] \\ + [S_k + S_{k-l} - 2P_{k,k-l}] .$$

The numbers  $S_k, S_{k-l}, S_{k-l+1}$  are included in  $S_k, \dots, S_{k-L}$ , and these are stored in a "push-down" mode and updated via

$$S_{k+1} = \eta_{k+1}^2 + \gamma S_k, \quad S_{(k+1)-1} = S_k, \dots \\ S_{(k+1)-L_2} = S_{k-(L_2-1)} .$$

Similarly  $P_{k,k-l}, P_{k,k-l+1}$  are stored among  $P_{k,k-1}, \dots, P_{k,k-L_2}$  and are updated via

$$P_{k+1,k+1-l} = \eta_{k+1} \eta_{k+1-l} + \gamma P_{k,k-l}, \text{ etc.} \quad (4.4)$$

Finally, it is necessary to store

$$P_{k,k-1}, P_{k-1,k-2}, \dots, P_{k-L_2+1,k-L_2} .$$

The numbers  $P_{k,k-1}$  are also updated via (4.4) and

$$P_{(k+1)-l,(k+1)-(l+1)} = P_{k-(l-1),k-l} .$$

defines the "push-down" operation.

With the above numbers available we clearly have

$$\frac{1}{2} \frac{\partial C_{k,l}}{\partial \sigma} = \sigma [S_{k-l+1} - S_{k-l} - 2P_{k-l+1,k-l}] \\ + [S_{k-l} - P_{k,k-l+1} + P_{k,k-l} + P_{k-l+1,k-l}] . \quad (4.5)$$

In particular,

$$\left. \frac{1}{2} \frac{\partial C_{k,l}}{\partial \sigma} \right|_{\sigma=0} = S_{k-l} - P_{k,k-l+1} + P_{k,k-l} + P_{k-l+1,k-l} , \\ \left. \frac{1}{2} \frac{\partial C_{k,l}}{\partial \sigma} \right|_{\sigma=1} = S_{k-l+1} - P_{k,k-l+1} + P_{k,k-l} - P_{k-l+1,k-l} .$$

Each pair  $l, \sigma$  corresponds to a delay  $T(l, \sigma) = (l - \sigma)k$ . Thus  $C_{k,l}(\sigma)$  can be associated with a function  $C_k(t)$  defined for  $t_k - L_2 h < t < t_k - L_1 h$ , the values of  $l$  corresponding, when  $\sigma = 0$ , to the points  $t = t_k - lh$ . As  $\sigma$  increases from 0 to 1 we pass from  $t = t_k - lh$  to  $t = t_k - (l - 1)h$ . Thus we have

$$\left. \frac{\partial C_{k,l}}{\partial \sigma} \right|_{\sigma=0} = \left. \frac{\partial C_k}{\partial t} \right|_{t=(t_k-lh)+}$$

$$\left. \frac{\partial C_{k,l}}{\partial \sigma} \right|_{\sigma=1} = \left. \frac{\partial C_k}{\partial t} \right|_{t=(t_k-(l-1)h)-}$$

candidate for the minimizing value  $T_k$  just in case

$$\frac{1}{2} \left. \frac{\partial C_{k,l}}{\partial \sigma} \right|_{\sigma=0} > 0, \quad \frac{1}{2} \left. \frac{\partial C_{k,l+1}}{\partial \sigma} \right|_{\sigma=1} < 0,$$

i.e.,

$$S_{k-l} - P_{k,k-l+1} + P_{k,k-l} + P_{k-l+1,k-l} > 0,$$

$$S_{k-l} - P_{k,k-l} + P_{k,k-l+1} - P_{k-l,k-l-1} < 0.$$

On the other hand, the interval  $[(l-1)h, lh]$  is a candidate for containing the minimizing value of  $T_k$  just in case

$$\frac{1}{2} \left. \frac{\partial C_{k,l}}{\partial \sigma} \right|_{\sigma=1} > 0, \quad \frac{1}{2} \left. \frac{\partial C_{k,l}}{\partial \sigma} \right|_{\sigma=0} < 0,$$

i.e.,

$$S_{k-l+1} - P_{k,k-l+1} + P_{k,k-l} - P_{k-l+1,k-l} > 0, \quad (4.6)$$

$$S_{k-l} - P_{k,k-l+1} + P_{k,k-l} + P_{k-l+1,k-l} < 0. \quad (4.7)$$

If (4.6), (4.7) are true for a given  $l$ , we compute the corresponding  $\sigma$  by setting (4.5) equal to zero, i.e.,

$$\sigma = - \frac{[S_{k-l} - P_{k,k-l+1} + P_{k,k-l} + P_{k-l+1,k-l}]}{[S_{k-l+1} - S_{k-l} - 2P_{k-l+1,k-l}]}$$

Once the finitely many possible candidates for  $T_k$  have been selected by this process,  $T_k$  is chosen from these as the one yielding the smallest value of  $C_{k,l}(\sigma)$ .

It is possible to economize on memory space by using slightly modified quantities.

With

$$\rho_{k,l} = \eta_k - \eta_{k-l} ,$$

$$\hat{S}_{k,l} = \sum_{j=0}^{\infty} \gamma^j (\rho_{k-j,l})^2 ,$$

$$\hat{P}_{k,l,l-1} = \sum_{j=0}^{\infty} \gamma^j \rho_{k-j,l} \rho_{k-j,l-1} ,$$

$$\hat{S}_{k,l-1} = \sum_{j=0}^{\infty} \gamma^j (\rho_{k-j,l-1})^2 ,$$

updated via

$$\rho_{k+1,l} = (\eta_{k+1} - \eta_k) + \rho_{k,l-1}, \quad l = 1, 2, \dots, L_2,$$

$$\hat{S}_{k+1,l} = (\rho_{k+1,l})^2 + \gamma \hat{S}_{k,l}, \quad l = 1, 2, \dots, L_2$$

$$\hat{P}_{k+1,l,l-1} = \rho_{k+1,l} \rho_{k+1,l-1} + \gamma \hat{P}_{k,l,l-1}, \quad l = 2, \dots, L_2 ,$$

it may be seen that we have

$$C_{k,l}(\sigma) = \sigma^2 [\hat{S}_{k,l} - 2\hat{P}_{k,l,l-1} + \hat{S}_{k,l-1}] - 2\sigma [\hat{S}_{k,l} - \hat{P}_{k,l,l-1}] + \hat{S}_{k,l-1}$$

so

$$\frac{1}{2} \frac{\partial C_{k,l}}{\partial \sigma} = \sigma [\hat{S}_{k,l} - 2\hat{P}_{k,l,l-1} + \hat{S}_{k,l-1}] - [\hat{S}_{k,l} - \hat{P}_{k,l,l-1}]$$

and this vanishes when

$$\sigma = [\hat{S}_{k,l} - \hat{P}_{k,l,l-1}] / [\hat{S}_{k,l} - 2\hat{P}_{k,l,l-1} + \hat{S}_{k,l-1}] .$$

The other aspects of the analysis remain as above. This procedure is the one actually used in Fortran SUBROUTINE PERIOD (L2, L1, GAMMA, H, PER, Y), whose listing follows and which forms the basis for the numerical experiments carried out in Section 5. Here PER is the

returned period estimate while  $Y$  is the supplied observation at each instant  $t_k$ . The other arguments are parameters whose identification is clear.

```

SUBROUTINE PERIOD(L2, L1, GAMMA, H, PER, Y)
DIMENSION S(40), P(40), RHO(40)
INTEGER L, L1, L2, L1P1, LM1
L2M1 = L2 - 1
ETAOLD = ETANEW
ETANEW = Y
DO 15 L = 1, L2M1
KL = L2 - L + 1
KLM1 = KL - 1
15 RHO(KL) = ETANEW - ETAOLD + RHO(KLM1)
RHO(1) = ETANEW - ETAOLD
DO 16 L = 1, L2
16 S(L) = GAMMA*S(L) + (RHO(L))**2
DO 17 L = 1, L2M1
LP1 = L+1
17 P(L) = GAMMA*P(L) + RHO(L)*RHO(LP1)
PER = FLOAT(L1)*H
SMINI = S(L1)
L1P1 = L1+1
DO 26 L = L1P1, L2
LM1 = L-1
IF(S(LM1).LT. SMINI)GO TO 21
GO TO 22
21 PER = FLOAT(LM1)*H
SMINI = S(LM1)
22 IF(P(LM1).GT. S(LM1))GO TO 24
IF(P(LM1).GT. S(L))GO TO 24
QUO = S(L)+S(LM1)-2.*P(LM1)
IF(QUO.LT. .000001)GO TO 26
SIG = (S(L)-P(LM1))/QUO
VALSIG = SIG*SIG*S(LM1)+(1.-SIG)*(1.-SIG)*S(L)+2.*SIG*
1(1.-SIG)*P(LM1)
IF(VALSIG.LT. SMINI)GO TO 23
GO TO 24
23 PER = (FLOAT(L)-SIG)*H
SMINI = VALSIG
24 IF(S(L).LT. SMINI)GO TO 25
GO TO 26
25 PER = FLOAT(L)*H
SMINI = S(L)
26 CONTINUE
RETURN
END

```

If there is some danger of confusing the minimal period  $T$  with one of its multiples  $2T$ ,  $3T$ , etc., this can usually be overcome by specifying  $L1$ ,  $L2$  correctly.

## 5. SOME NUMERICAL EXPERIENCE WITH PERIOD

We have carried out extensive computer based simulations using SUBROUTINE PERIOD described in the preceding section. Here we will describe results obtained in connection with the plant

$$\begin{pmatrix} \dot{x}^1 \\ \dot{x}^2 \end{pmatrix} = \begin{pmatrix} 0 & 1 \\ 0 & -.2 \end{pmatrix} \begin{pmatrix} x^1 \\ x^2 \end{pmatrix} + \begin{pmatrix} 0 \\ 28 \end{pmatrix} (u + v) \quad (5.1)$$

with

$$v(t) = z^1(t) \quad (5.2)$$

$$\begin{pmatrix} \dot{z}^1 \\ \dot{z}^2 \end{pmatrix} = \begin{pmatrix} 0 & \alpha \\ -\alpha & 0 \end{pmatrix} \begin{pmatrix} z^1 \\ z^2 \end{pmatrix}. \quad (5.3)$$

The rather nondescript parameters appearing in (5.1) result from the fact that this damped inertial system is a model for a certain physical plant of interest. A compensator was constructed in the form (0.7) using  $S = -\epsilon P_0^*$  with  $P_0$  as in (1.10). To provide a more or less standard basis of comparison the feedback coefficients in all cases were chosen to achieve critical damping (i.e., multiple real eigenvalues) at various rates in the closed loop matrix

$$A + CK = \begin{pmatrix} 0 & 1 \\ 28k_1 & -.2 + 28k_2 \end{pmatrix} \quad (5.4)$$

In this very simple example

$$B = C = \begin{pmatrix} 0 \\ 28 \end{pmatrix},$$

$\Phi$  and  $-L$  (cf. (0.1)) were both chosen to be  $2 \times 2$  identity matrices. For the values of  $k_1$  and  $k_2$  which were used  $P_0$  turns out to be a very small matrix and we used  $\epsilon = 1000$ .

The output used for the period estimation was  $w(t) = x^1(t) + y^1(t)$  and the output shown on the diagrams is  $x^1(t)$ . It will be seen that the initial estimates for the period,  $T(t)$ , are wildly inaccurate but, in the cases when the complete plant/compensator/identifier system is asymptotically stable, the estimate  $T(t)$

converges to the correct value  $T_0$  (within the accuracy permitted by the approximations inherent in PERIOD, as described in the preceding section). In all cases we selected  $z^1(0) = 1$ ,  $z^2(0) = 0$ , so

$$z^1(t) = \cos(at), \quad z^2(t) = -a \sin(at).$$

In Figures 1-16 the odd numbered Figures show the output  $x^1(t)$  while the next, even-numbered figure in each case shows the period estimate  $T(t)$  for the same run.

Figures 1 through 4 correspond to choices of  $k_1$  and  $k_2$  such that the matrix (5.4) has a double eigenvalue  $\lambda = -5$ . In Figs. 1, 2  $\alpha_0 = 20\pi$  (10 Hertz) corresponding to  $T_0 = .1$ , indicated by the dotted line. Figures 3, 4 illustrate the corresponding experience for  $\alpha_0 = 30\pi$  (15 Hertz), or  $T_0 = .0667$ . Here the period identifier diverges from the correct value and, as seen in Fig. 3, no significant reduction of the oscillation of  $x^1(t)$  is realized. We believe that this is accounted for by the fact that the term  $2\pi/(T_0)^2$  in (2.21) changes from  $200\pi$  in the 10 Hz case to  $450\pi$  in the 15 Hz case.

Figures 5-8 show the 10 Hz case with  $k_1$  and  $k_2$  chosen so that (5.4) has a double eigenvalue  $\lambda = -8$  (Figs. 5 and 6) and with  $k_1$  and  $k_2$  chosen so that  $\lambda = -9$ . These cases seem quite satisfactory with rapid attenuation of the oscillation in  $x^1(t)$ , better in the second case than in the first, and rapid convergence of the period estimate  $T(t)$  to  $T_0 = .1$ . The corresponding experience in the 15 Hz case is not nearly so satisfactory. Figs. 9 and 10 show the performance for  $\lambda = -8.5$  while Figs. 11 and 12 show  $\lambda = -10$ . We see from Figs. 10 and 12 that, although the value  $T_0$  is unstable, the estimate  $T(t)$  tends to undergo a self-excited oscillation about the equilibrium value indicated by the dotted lines. The evidence favors the conjecture that in these cases the nonlinear equation (2.11) may exhibit a Hopf type bifurcation as the parameter  $T_0$  passes from .1 (10 Hz.) to .0667 (15 Hz.). Detailed analysis of this possibility must await later treatment.

Figures 13 through 16 show experience in the 15 Hz case with  $k_1$  and  $k_2$  selected so that  $\lambda = -14$  (Figs. 13 and 14) and so that  $\lambda = -20$  (Figs. 15 and 16). We see that

the performance improves as (5.4) is made progressively more stable, in agreement with the conjectures of Section 3. The small residual oscillation evident in Fig. 15 is probably due to the fact that PERIOD does not provide an exact estimate even when the corresponding continuous process associated with minimization of (2.4) or (2.6) is asymptotically stable.

Figures 17 and 18 show variational  $\Delta T$  solutions, obtained in the manner described at the end of Section 3, for the 15 Hz case with  $\lambda = -9$  and  $\lambda = -14$ , respectively. Because  $T(t)$  does not converge to  $T_0$  in the first case (cf. Fig. 12), even the variational solutions are not sinusoidal.

In Fig. 18, corresponding to  $\lambda = -14$ ,  $T(t)$  converges to  $T_0 = .0667$  (cf. Fig. 14) and the corresponding variational solution tends to zero in a convincing exponentially damped sinusoidal manner, this behavior becoming more convincing as  $t$  increases. It is of interest to estimate the frequencies and damping factors here and compare them with the eigenvalues of  $M(\alpha_0)$ ,  $F(\alpha_0)$ . Analyzing the data plotted in Fig. 18 one obtains the estimate  $T = .66$  and, comparing the successive amplitudes, we see that the oscillation there is approximated by

$$c_+ e^{(-.525+19.52)t} + c_- e^{(-.525-19.52)t} . \quad (5.5)$$

Here

$$M(\alpha_0) = \begin{pmatrix} 0 & 1 & 0 & 0 \\ -190.12 & -27.188 & -28 & 0 \\ -2.966 & 82.417 & 0 & 1 \\ -.009 & -2.966 & -8873.8 & 0 \end{pmatrix}$$

and its eigenvalues may be computed to be

$$-2.77 \pm i105.32, \quad -10.82 \pm i5.9 .$$

None of these correspond to (5.5) and we conclude that the dynamics exhibited in the identification process arise from a different source - which, on the basis of our earlier investigations in this paper, we believe to be the functional equation (2.22) (equivalently (2.19), (2.20)).

In Figures 19-26 we indicate the effect of varying the parameters  $\gamma$  and  $\epsilon$  (cf. (1.9), (2.4)) while leaving the feedback parameters in  $K$  fixed at the values which

produced Figures 13 and 14 with  $\gamma = .9$  and  $\epsilon = 1000$ . This corresponds to the double eigenvalue  $\lambda = -14$  for (5.4).) Here it needs to be explained that the value "GAMMA" referred to on the figure heading corresponds to  $e^{-\gamma h}$ , where  $h$  is the length of the sampling interval used by PERIOD. For all cases studied here  $h$  is ten times the  $H$  value shown in the figure heading; thus  $h = .01$  and

$$\text{GAMMA} = .8 = e^{-.01\gamma}$$

$$\rightarrow \gamma = \frac{\log_e .8}{-.01} = -22.3 ,$$

$$\text{GAMMA} = .9 \rightarrow \gamma = -10.5$$

$$\text{GAMMA} = .95 \rightarrow \gamma = -5.13 .$$

As we see by comparison of Figs. 19 and 20 with 13 and 14, performance is substantially degraded by discounting past values too much, corresponding to  $\gamma = -5.13$  on the other hand, gives substantially better performance than  $\gamma = -10.5$ .

For Figs. 23 and 24 we have set  $\text{GAMMA} = .9$  again but have increased  $\epsilon$  to 2000 rather than the earlier 1000. The improvement over the results in Figs. 13 and 24 is again quite marked. To obtain the results of Figs. 25 and 26 we "pulled out all the stops" and set  $\text{GAMMA} = .95$ ,  $\epsilon = 2000$  and chose feedback parameters corresponding to a double eigenvalue  $\lambda = -20$  for (5.4). Here we finally obtain the sort of sinusoidal disturbance rejection one would hope for.

It is clear from these numerical studies that the procedure we have described here can be effective for vibration suppression under certain circumstances. We have also made a case in this article for the proposition that the near equilibrium plant/compensator/identifier dynamics are governed by a functional equation involving an infinite delay and periodic coefficients. It remains for other investigations to develop the detailed relationships between the design parameters and the dynamics of solutions of this functional equation upon which a design methodology for effective performance might be based.



Fig. 1

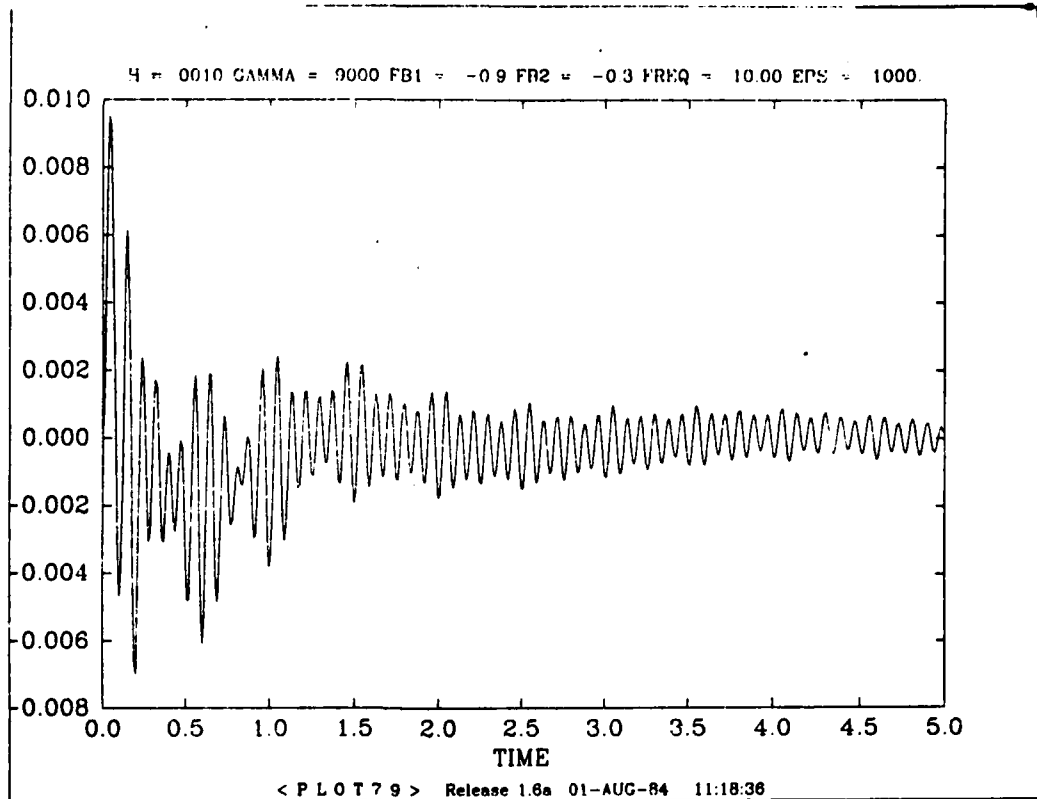


Fig. 2

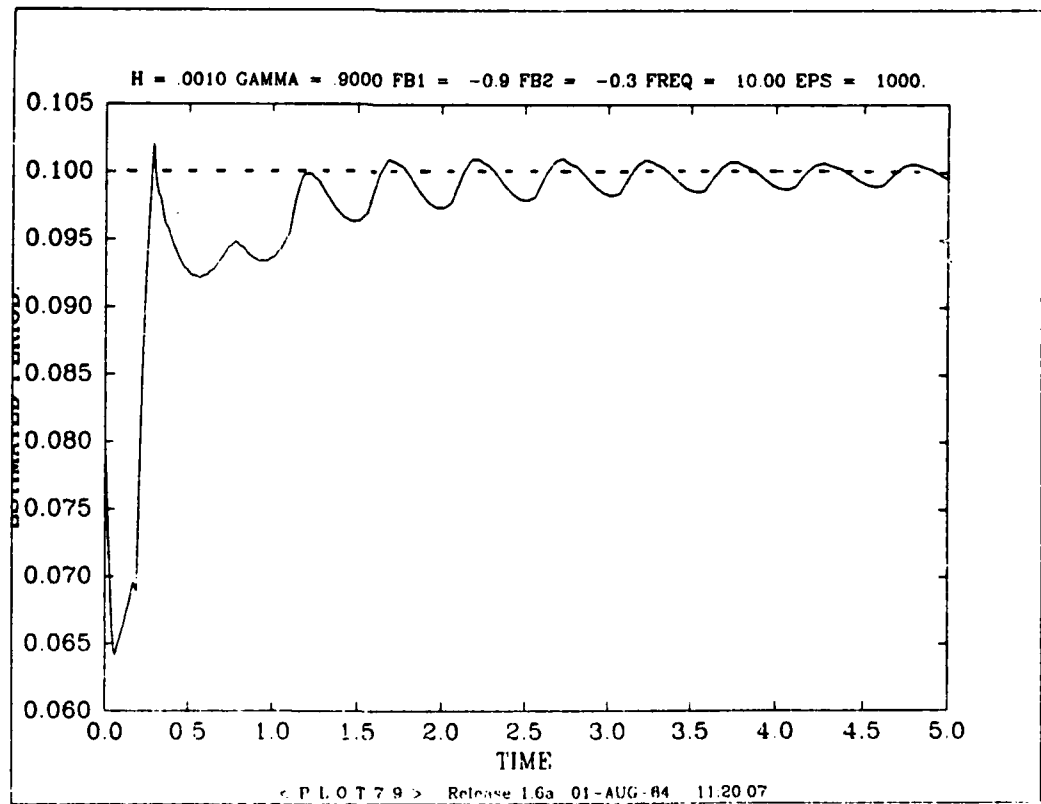


Fig. 3

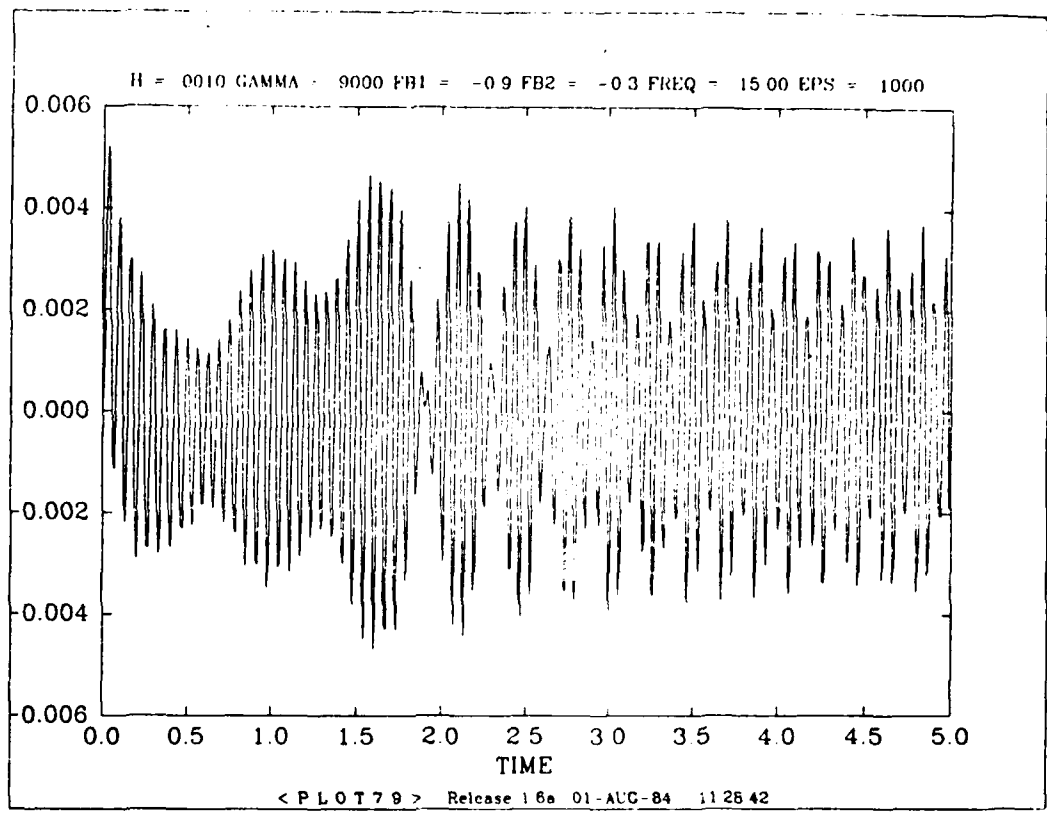


Fig. 4

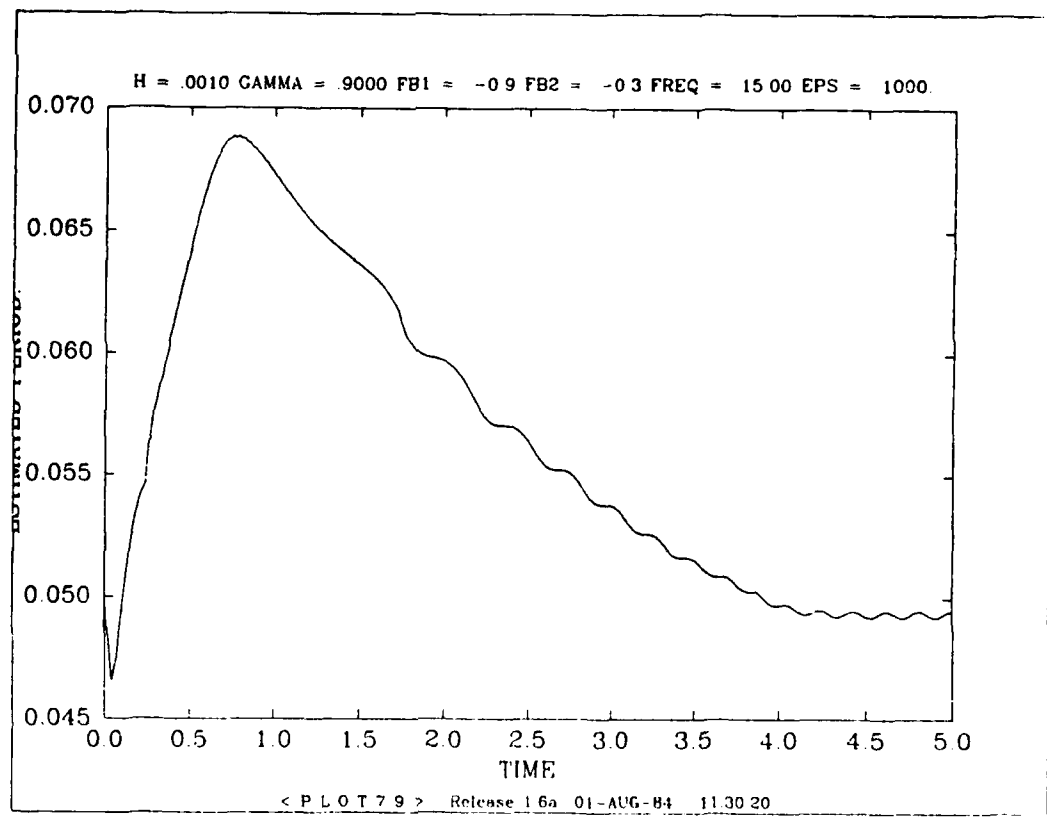


Fig. 5

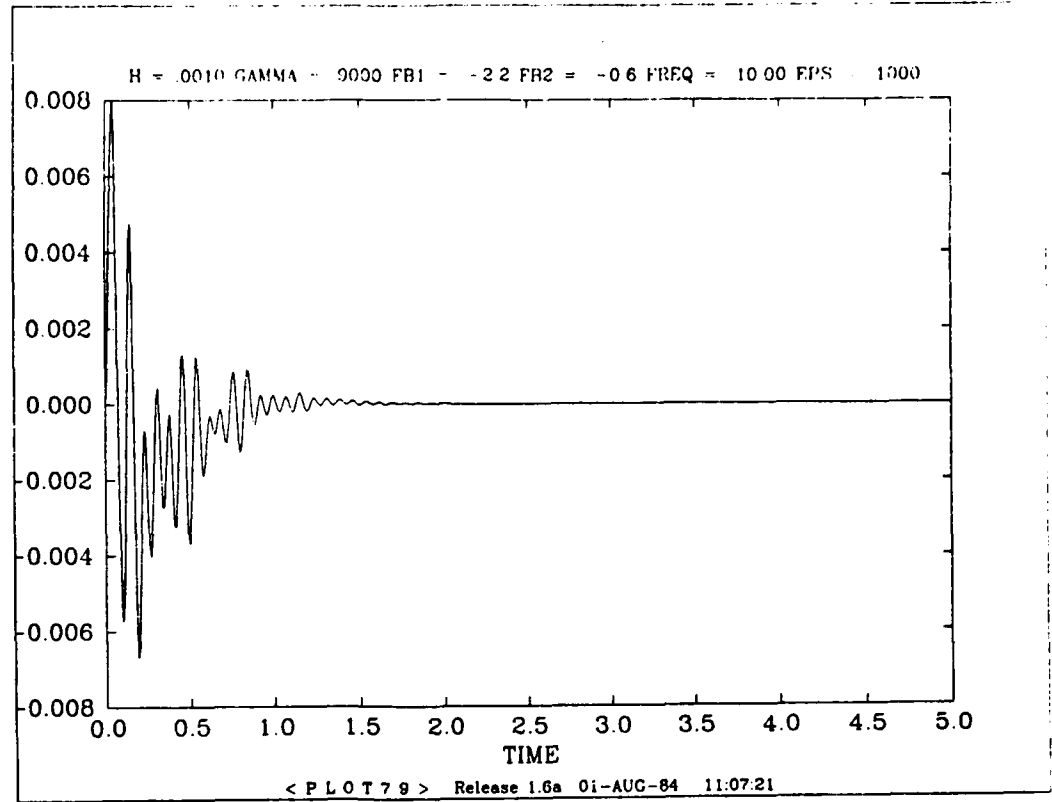


Fig. 6

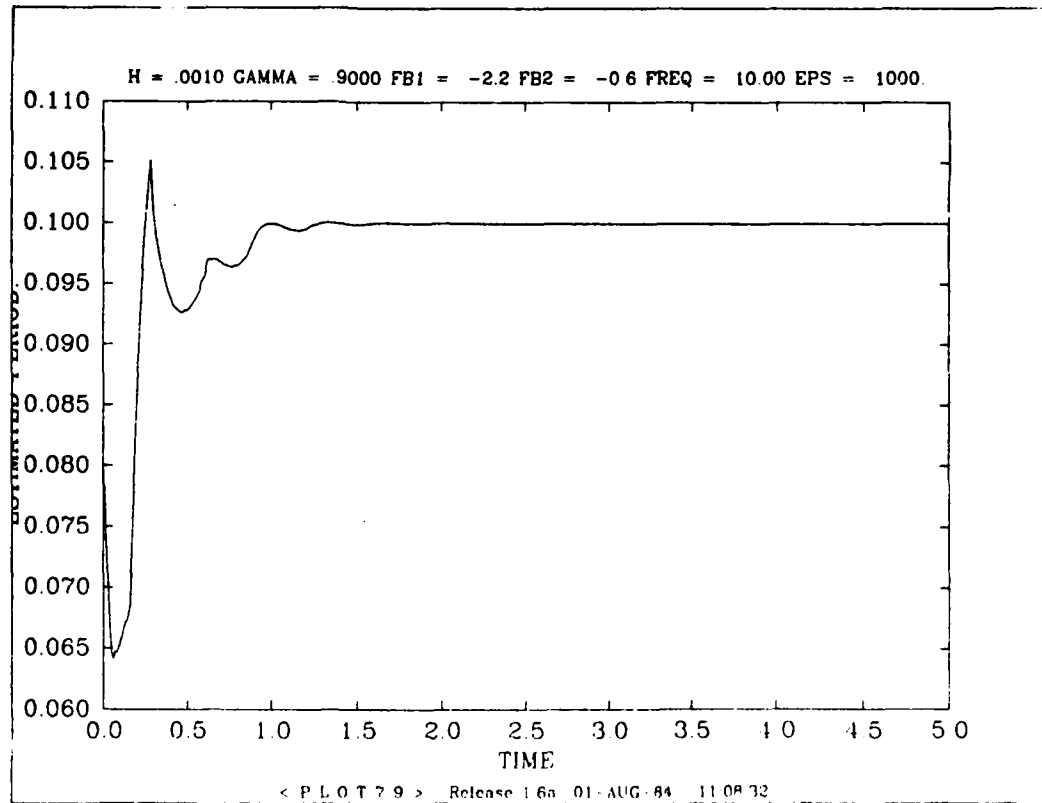


Fig. 7

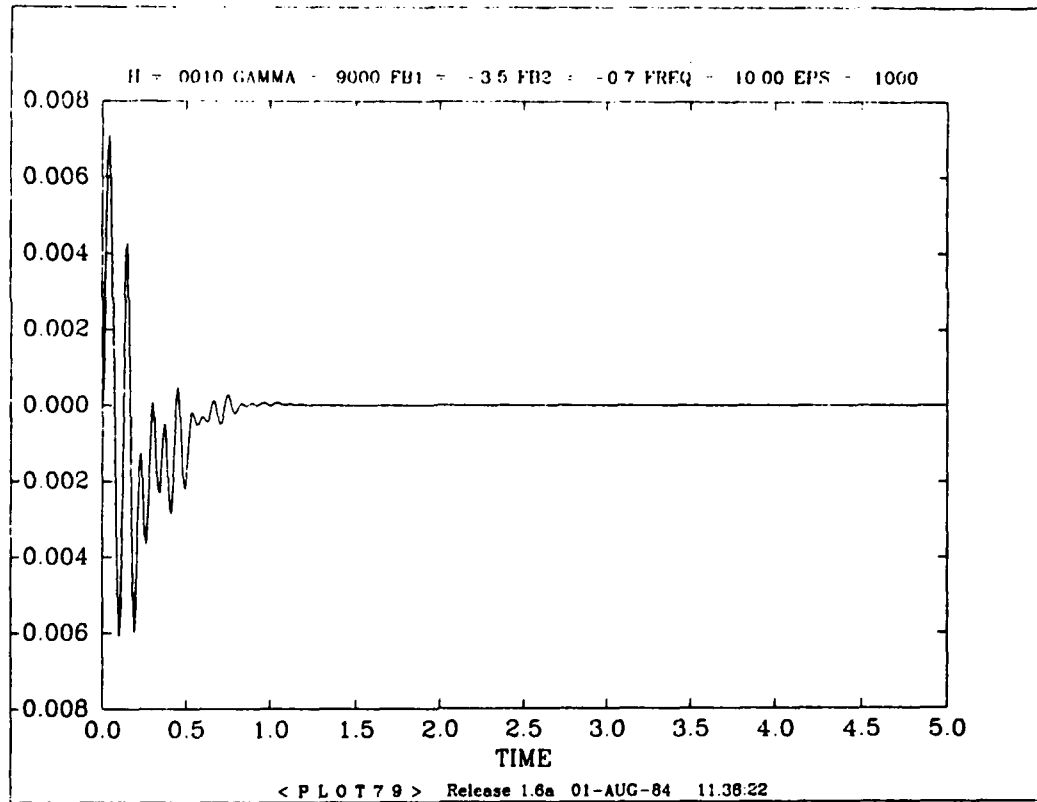


Fig. 8

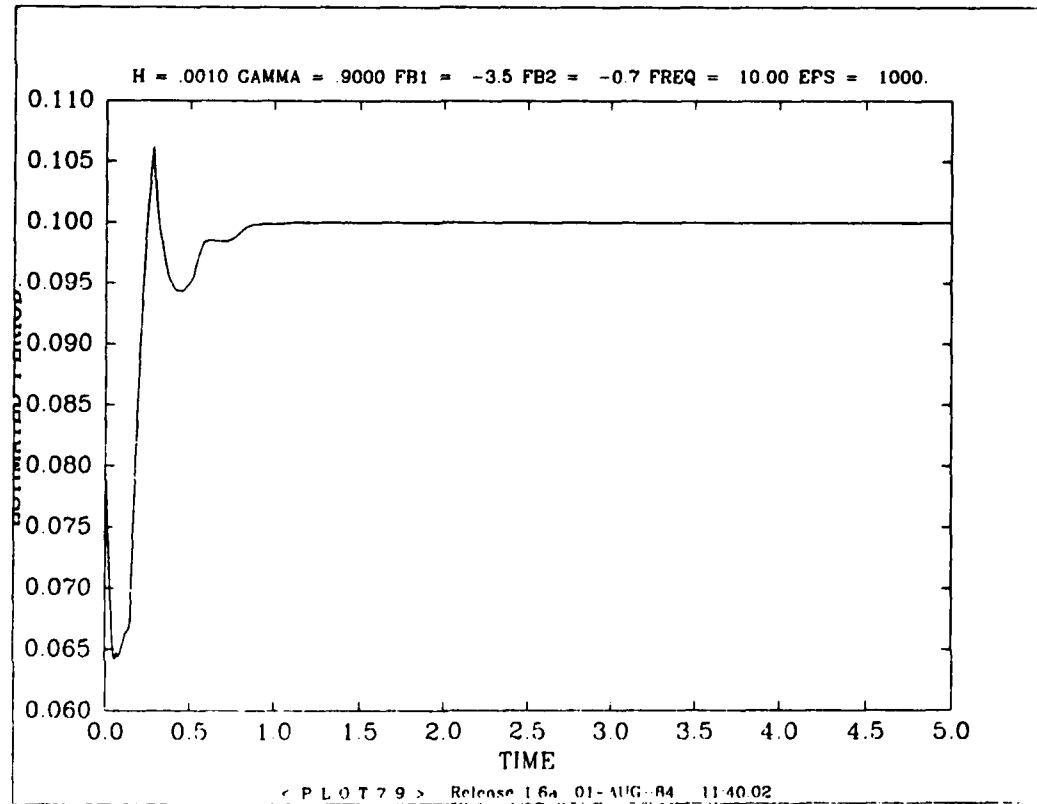


Fig. 9

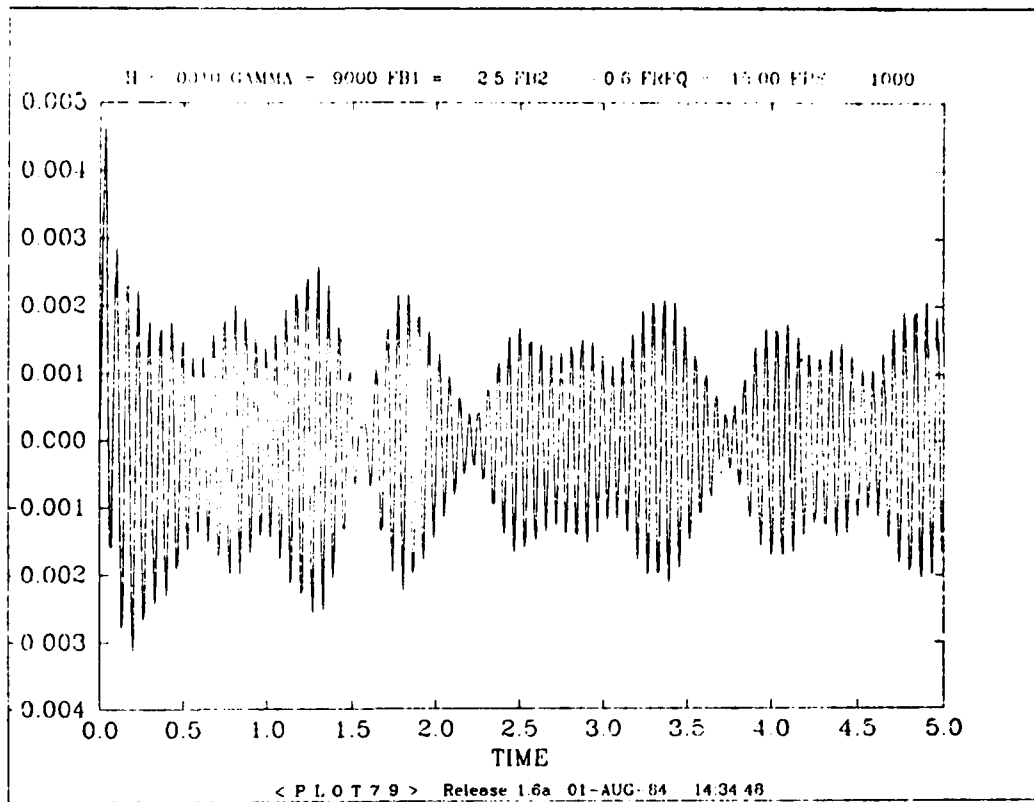


Fig. 10

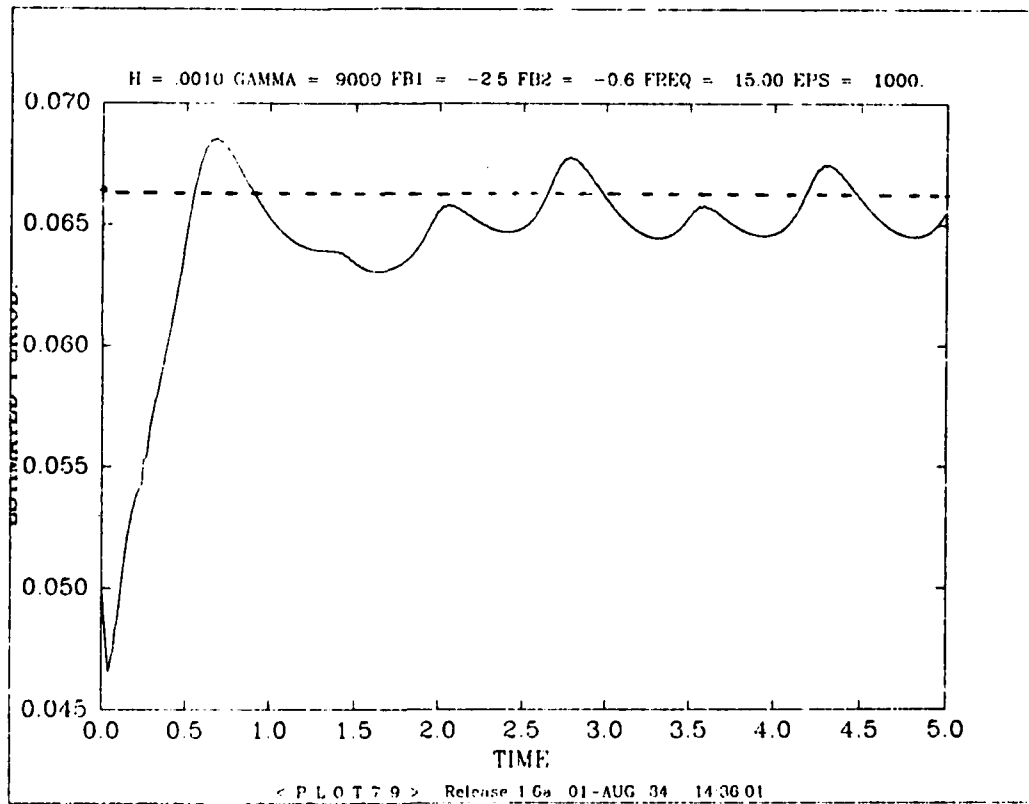


Fig. 11

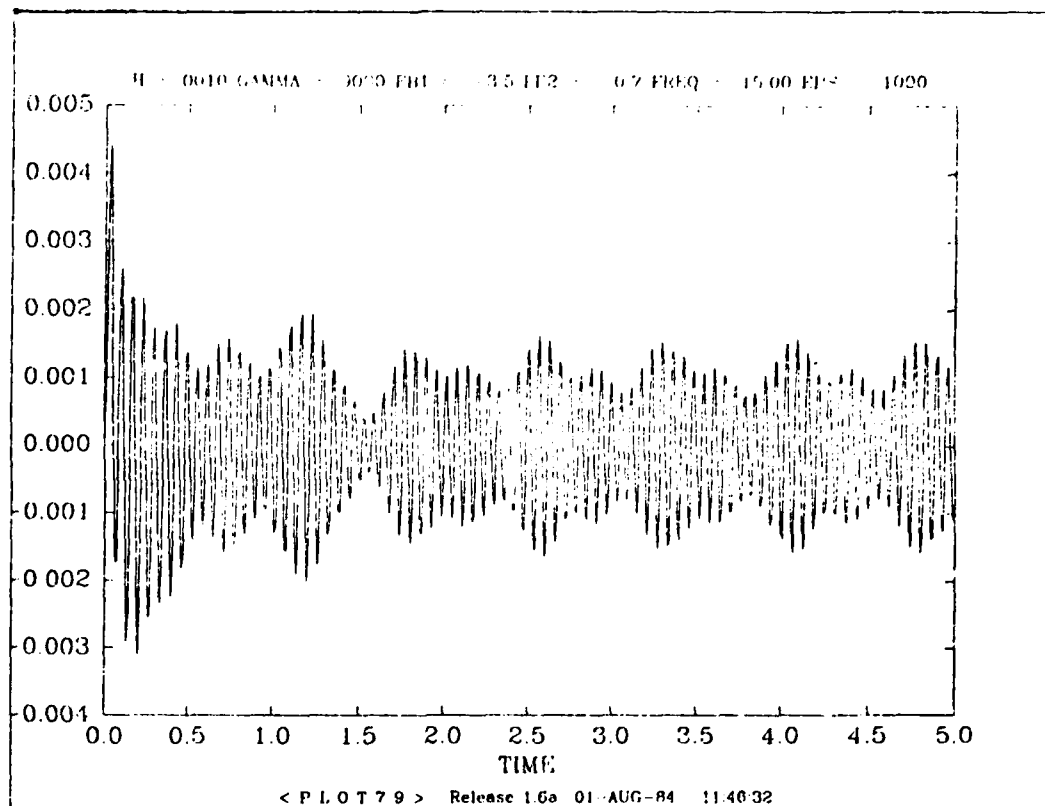


Fig. 12

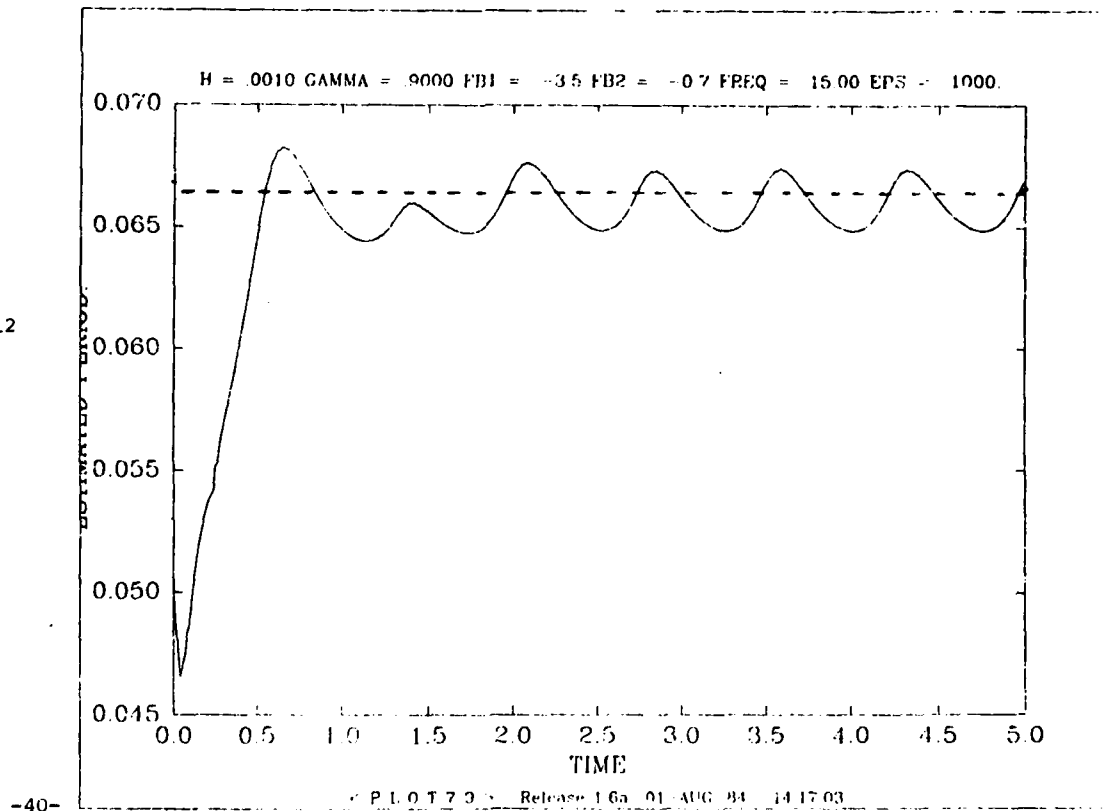


Fig. 13

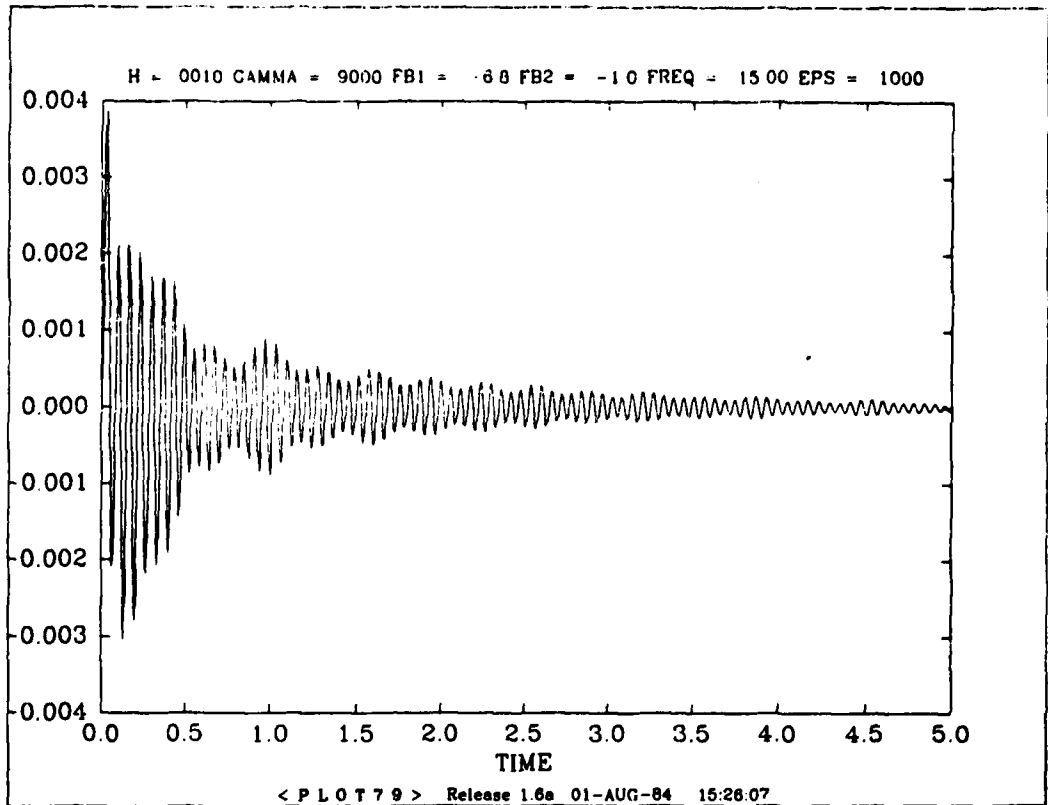


Fig. 14

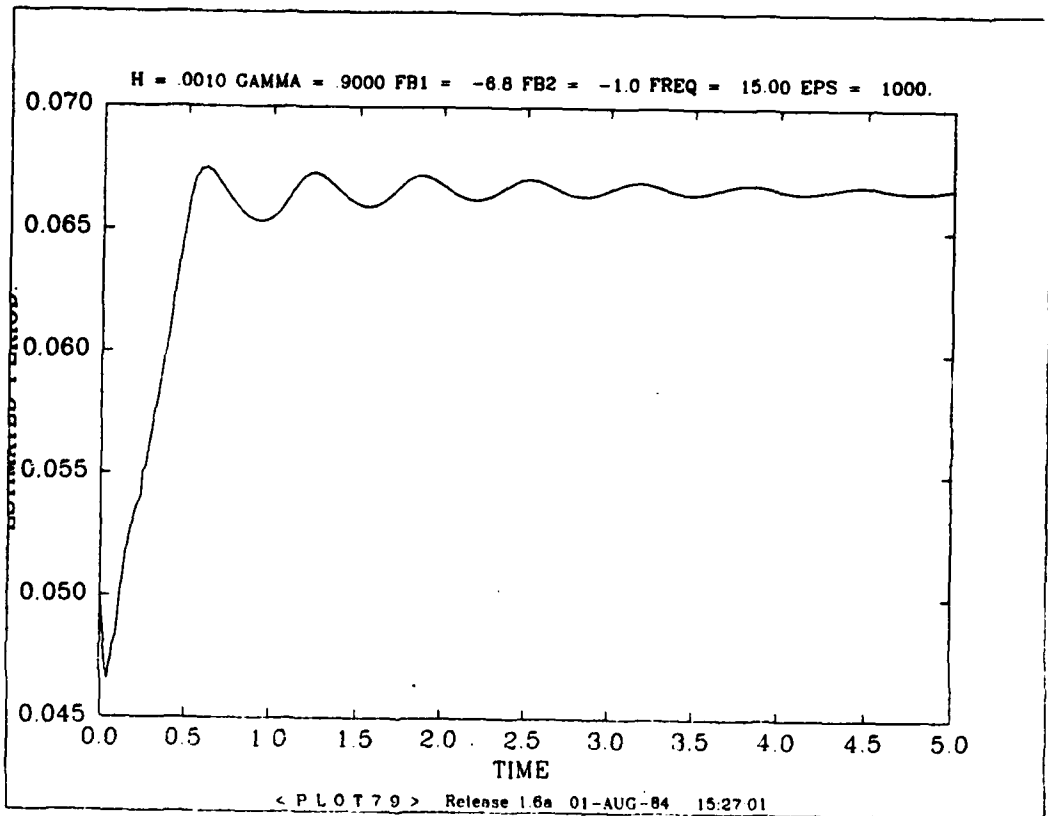


Fig. 15

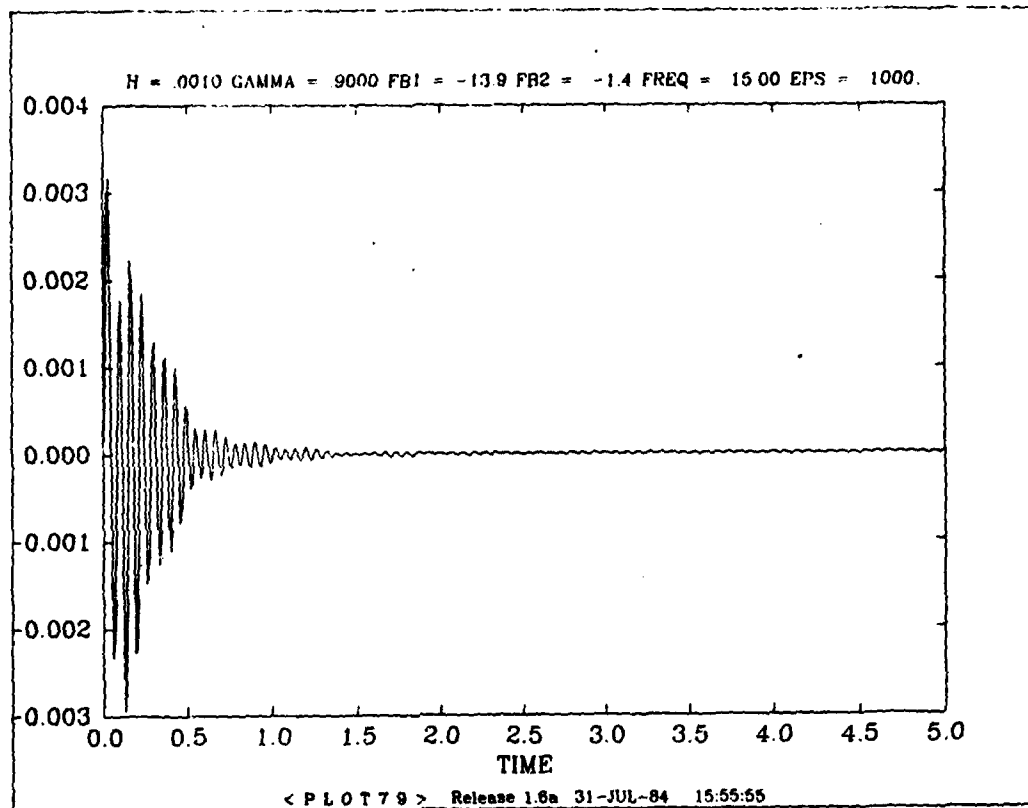


Fig. 16

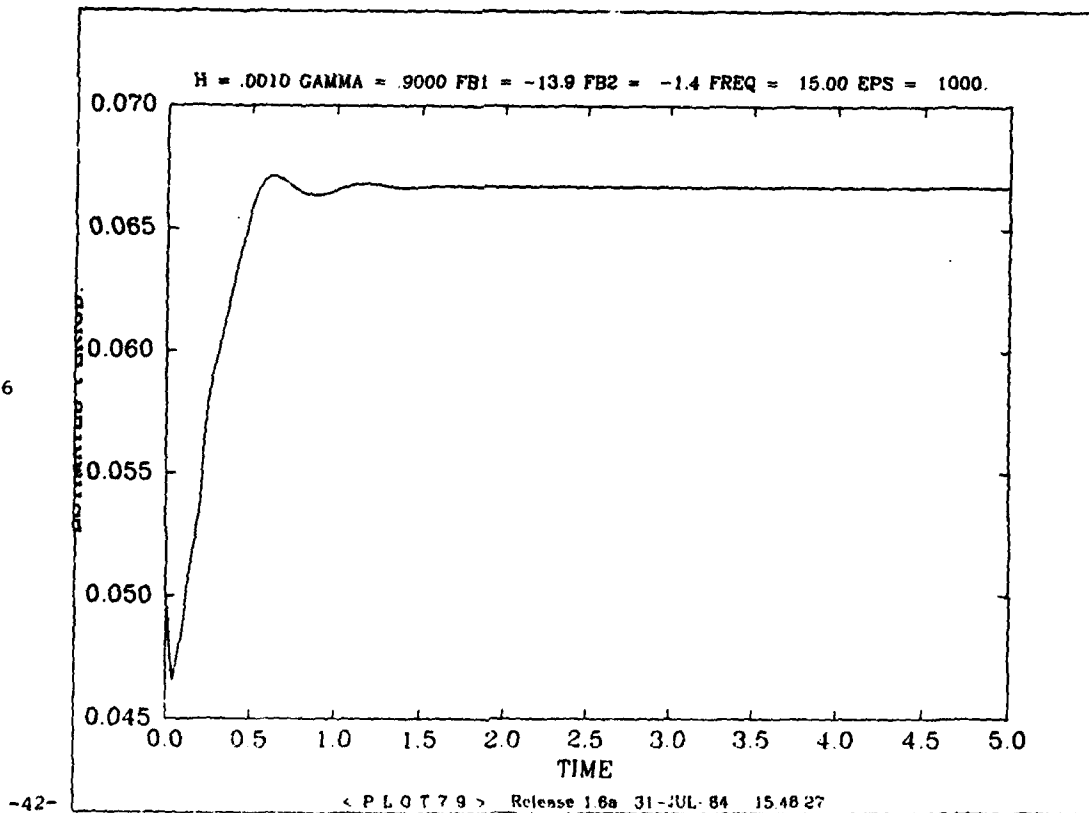




Fig. 17

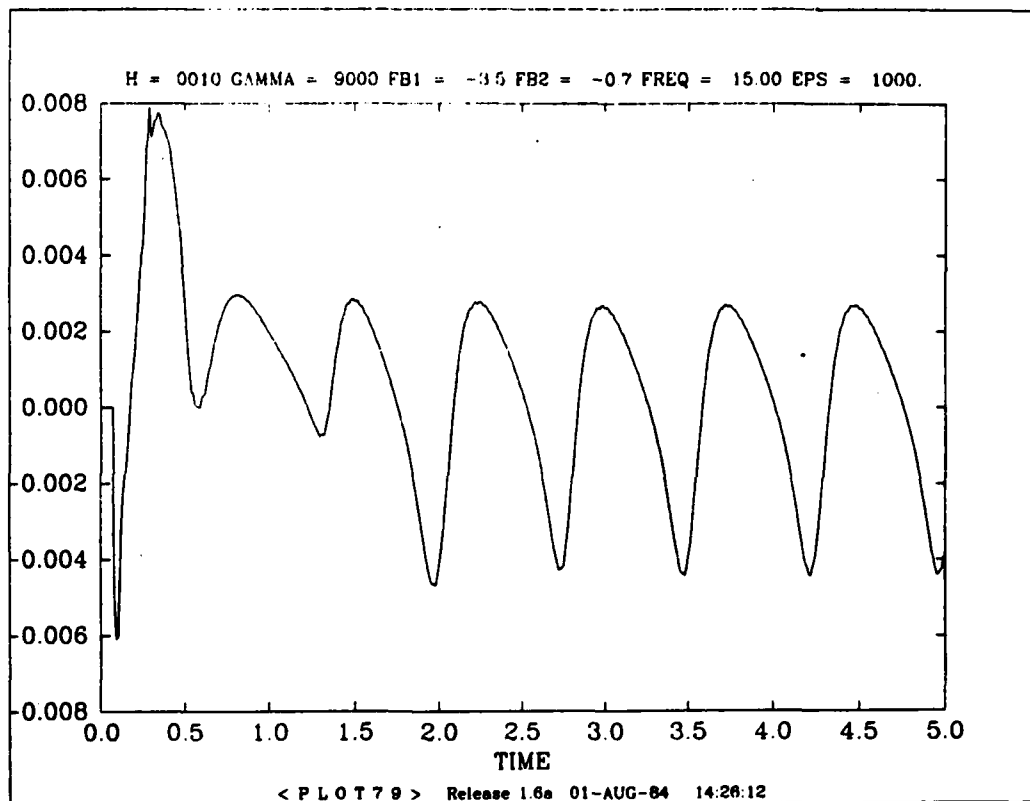


Fig. 18

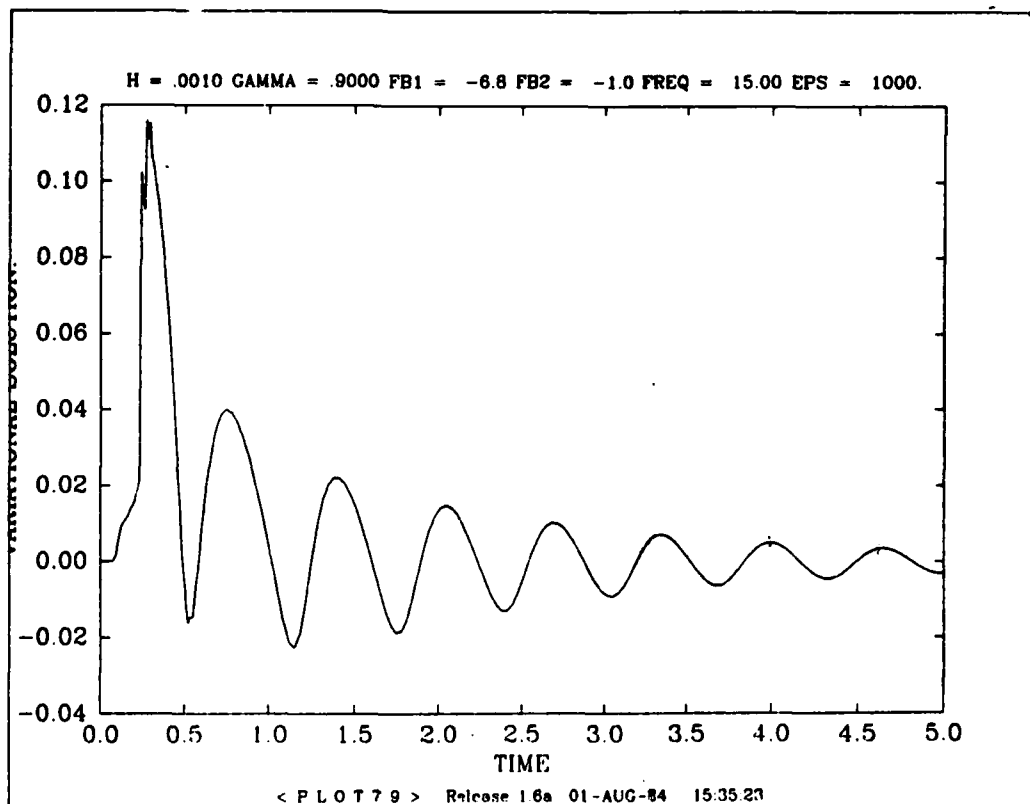


Fig. 19

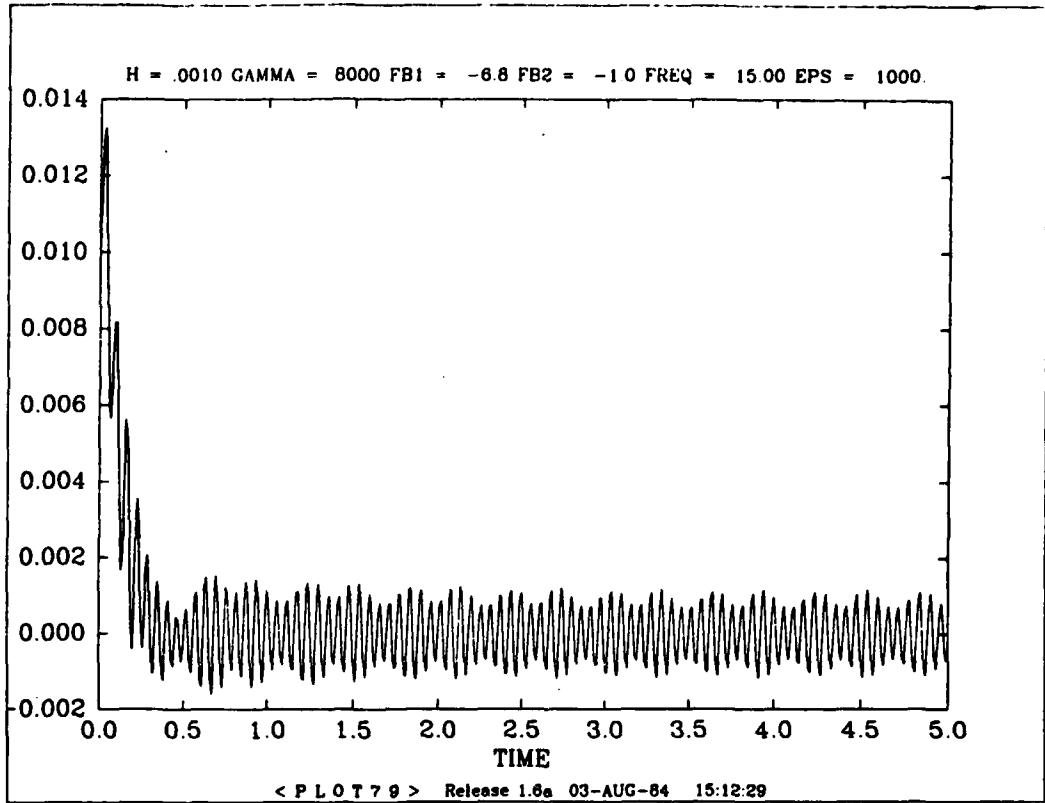


Fig. 20

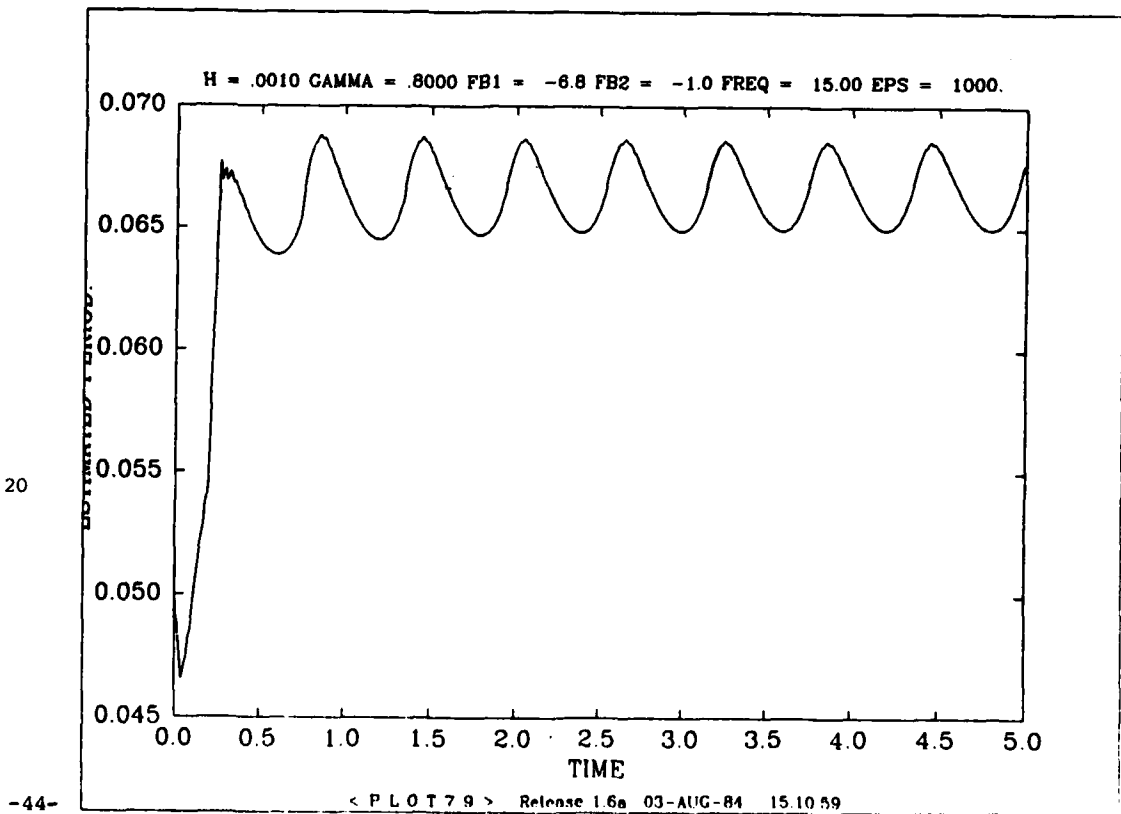


Fig. 21

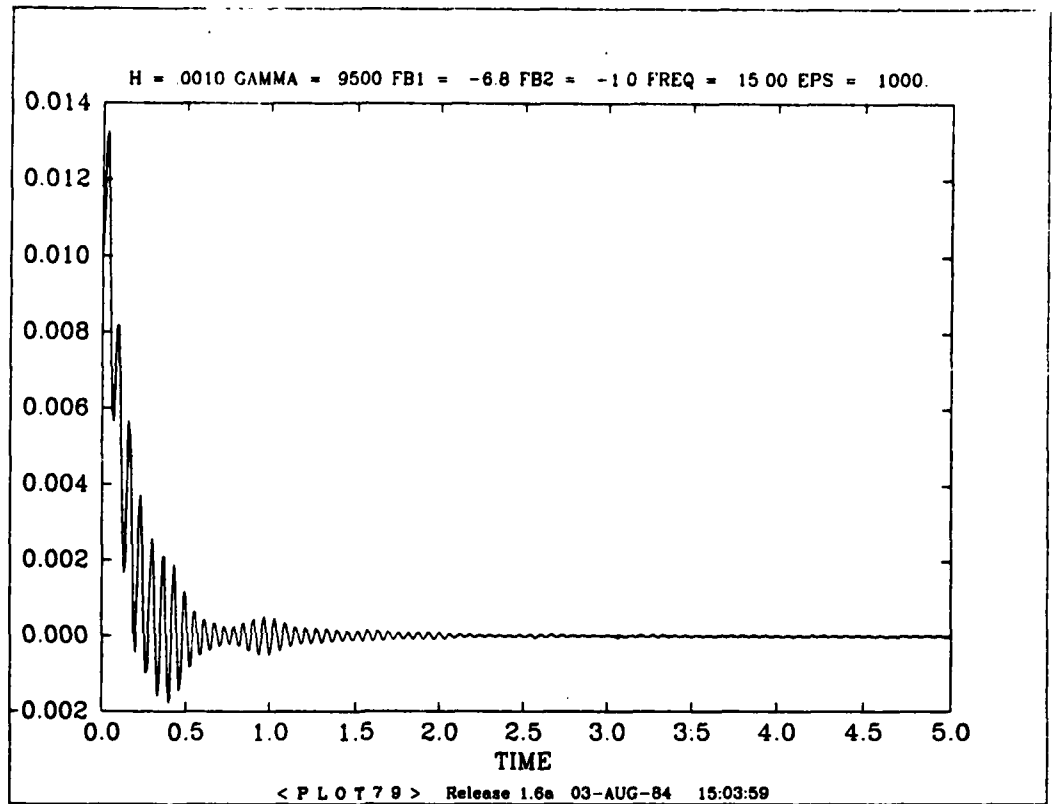


Fig. 22

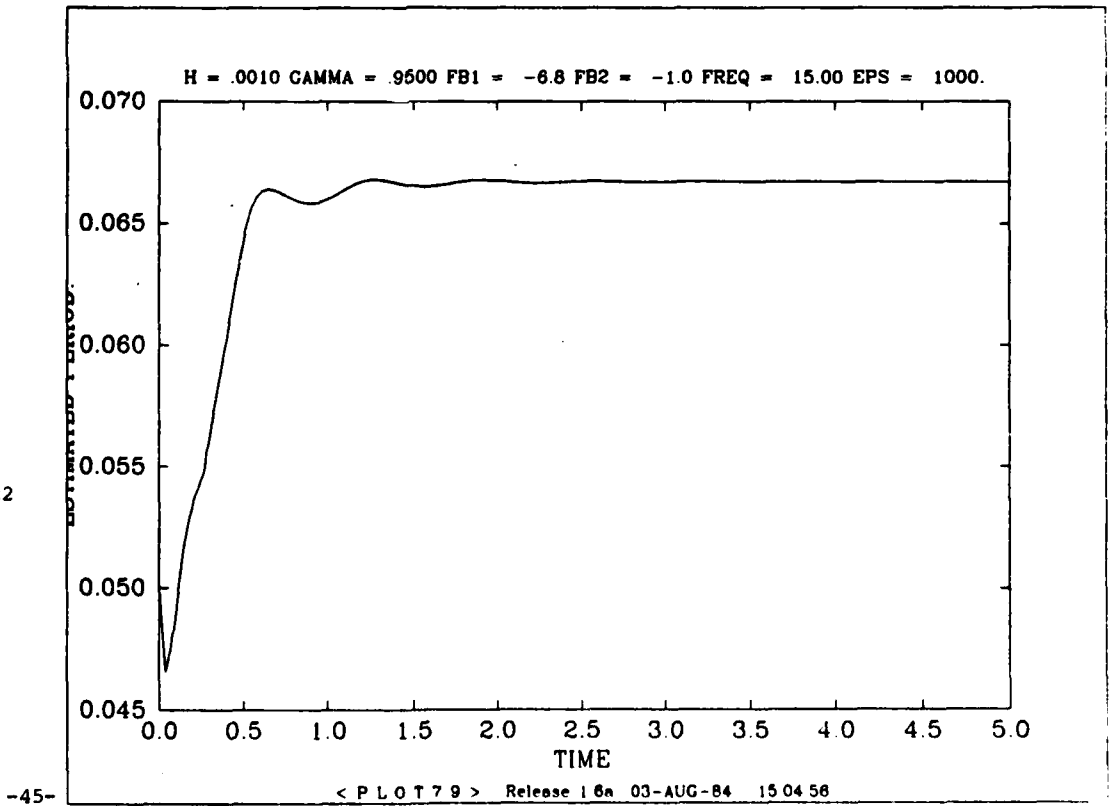


Fig. 23

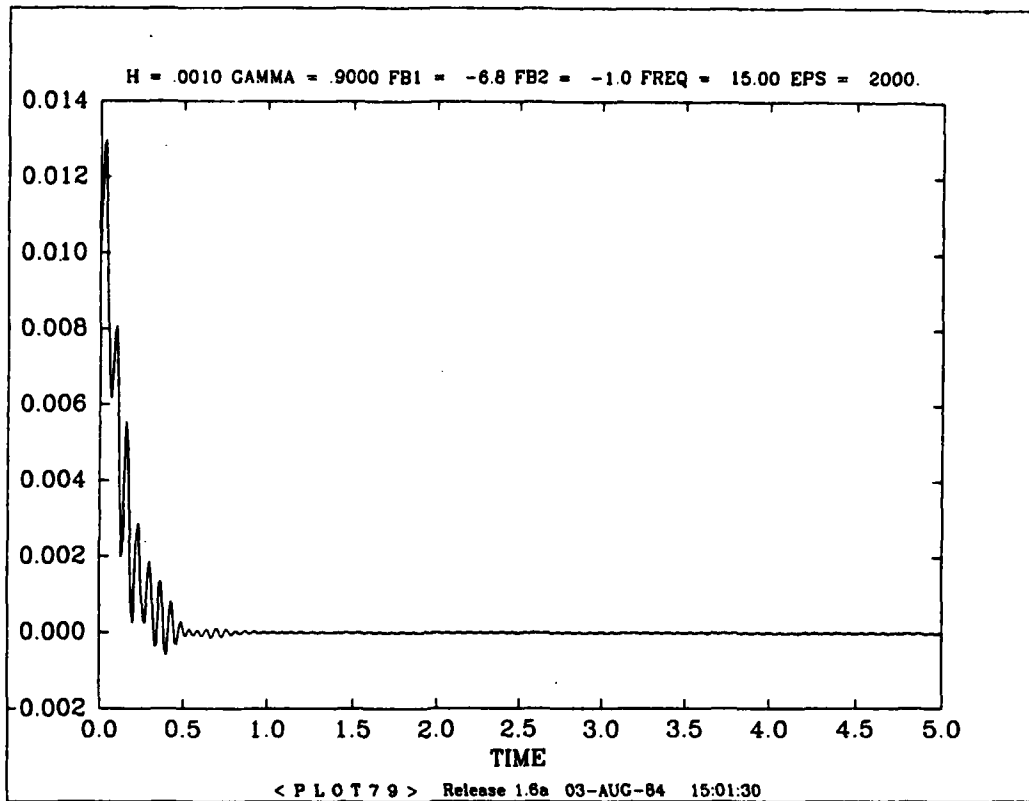


Fig. 24

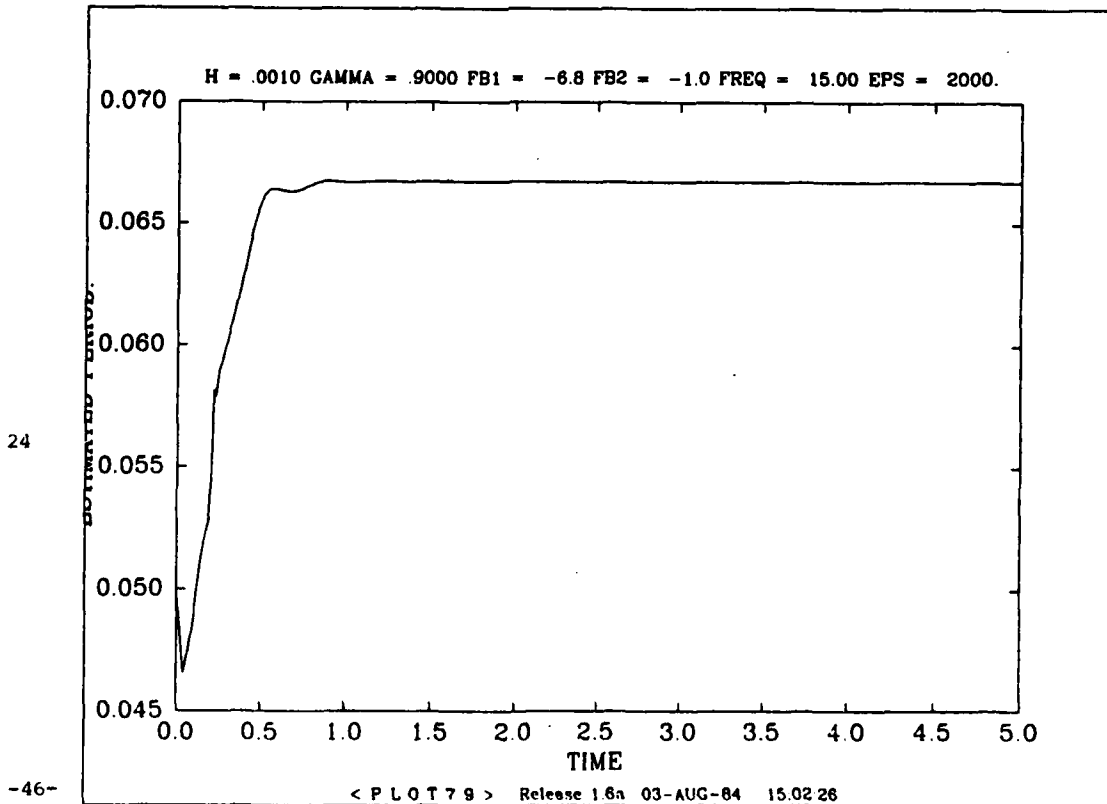


Fig. 25

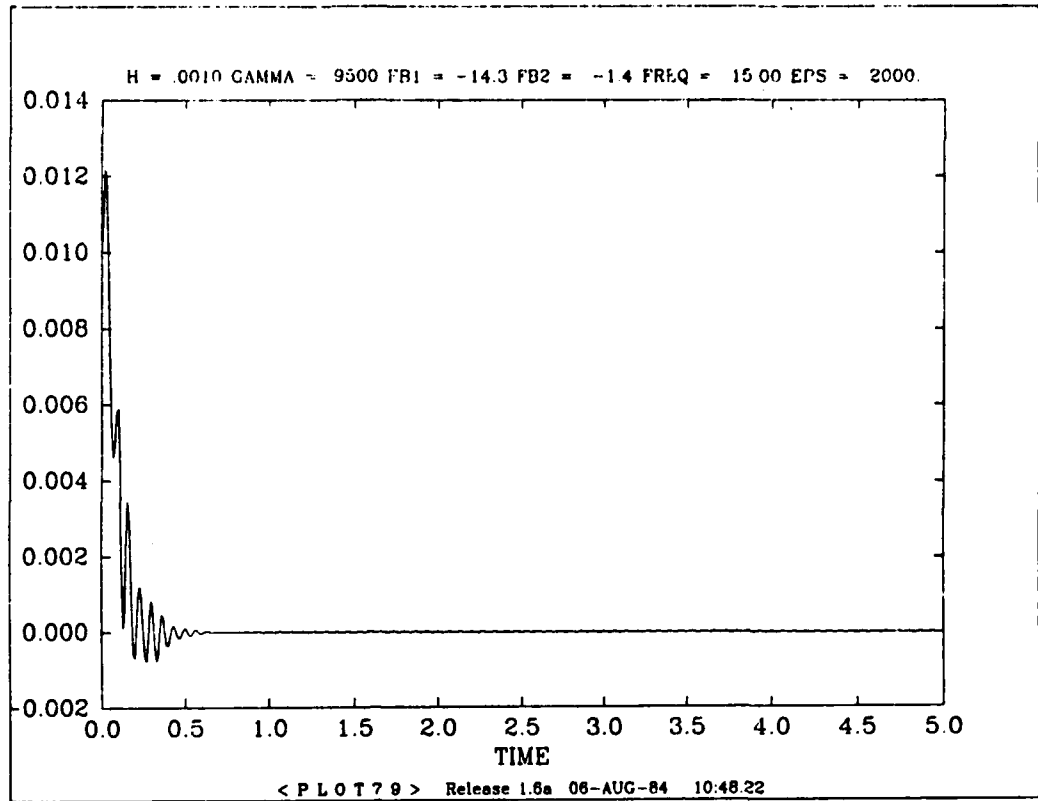
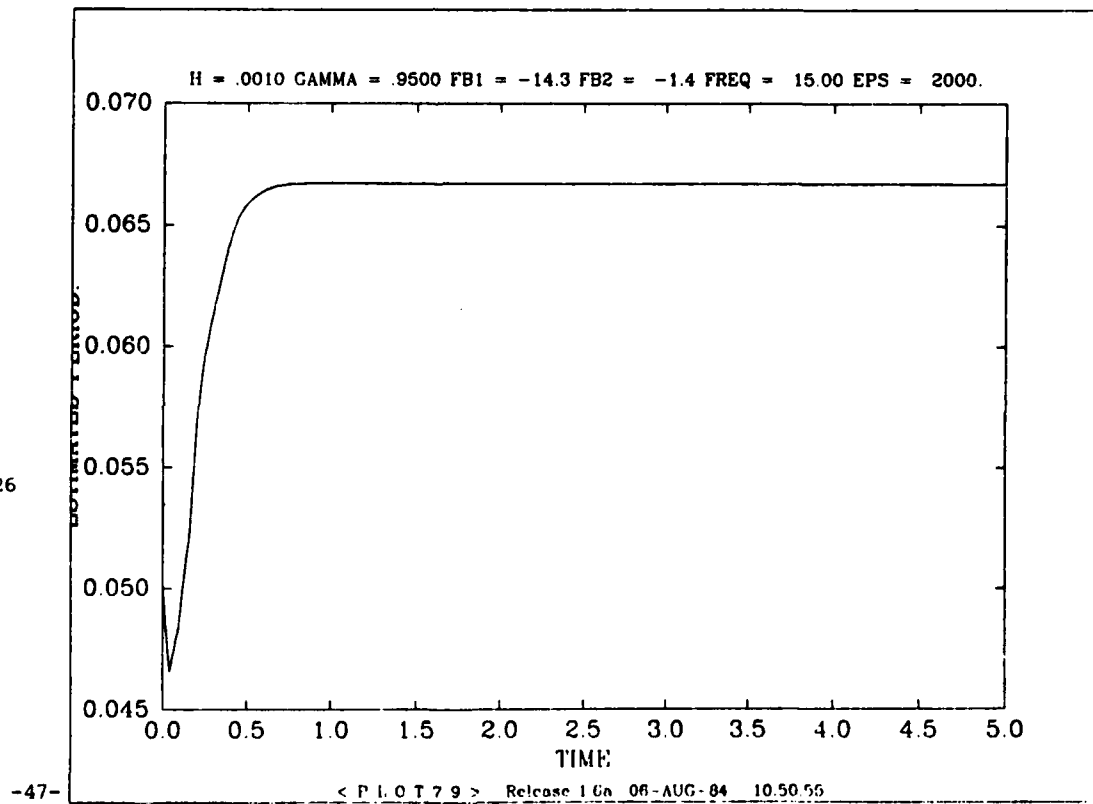


Fig. 26



6. SKETCH OF THE PROOF OF THEOREM 5.

Let  $z(t)$ ,  $W(t,s)$ , etc., be as in the statement of Theorem 5. For any  $t > 0$  we have

$$z(t) = \int_{-\infty}^t W(t,s)z(s)ds = \int_{kT}^t W(t,s)z(s)ds + \int_0^{kT} W(t,s)z(s)ds + \int_{-\infty}^0 W(t,s)\tilde{z}(s)ds, \quad (6.1)$$

where  $k$  is the largest integer such that  $kT < t$ . We define  $z_\ell \in L_m^2[0,T]$  by

$$z_\ell(\tau) = z((\ell-1)T + \tau), \quad \tau \in [0,T], \quad \ell > 1,$$

and we define  $\tilde{z}_{-\ell} \in L_m^2[0,t]$  by

$$\tilde{z}_{-\ell}(\tau) = \tilde{z}(-(\ell+1)T + \tau), \quad \tau \in [0,T], \quad \ell > 0.$$

Then, with

$$t = kT + \tau, \quad s = \ell T + \sigma, \quad s \in [\ell T, (\ell+1)T],$$

(6.1) yields

$$\begin{aligned} z_{k+1}(\tau) &= \int_0^\tau W(kT + \tau, kT + \sigma)z_{k+1}(\sigma)d\sigma - \sum_{j=1}^k \int_0^T W(kT + \tau, (j-1)T + \sigma)z_j(\sigma)d\sigma \\ &= \sum_{\ell=0}^{\infty} \int_0^T W(kT + \tau, -(\ell+1)T + \sigma)\tilde{z}_{-\ell}(\sigma)d\sigma. \end{aligned} \quad (6.2)$$

Using the periodicity relation (3.7) we have

$$W(kT + \tau, (j-1)T + \sigma) = W(\tau, -(k-j)T + \sigma) \equiv W_{k-j}(\tau, \sigma)$$

for any integer  $j < k$ . Then (6.2) becomes

$$\begin{aligned} z_{k+1}(\tau) &= \int_0^\tau W_0(\tau, \sigma)z_{k+1}(\sigma)d\sigma - \sum_{j=1}^k \int_0^T W_{k+1-j}(\tau, \sigma)z_j(\sigma)d\sigma \\ &= \sum_{\ell=0}^{\infty} \int_0^T W_{k+1+\ell}(\tau, \sigma)\tilde{z}_{-\ell}(\sigma)d\sigma. \end{aligned} \quad (6.3)$$

The conditions (3.7), (3.8) show that the last sum converges in  $L_m^2[0, T]$ . We may write (6.3) as a vector linear recursion equation in  $L_m^2[0, T]$ :

$$(I - P_0)z_{k+1} - \sum_{j=1}^k P_{k+1-j} z_j = \sum_{\ell=0}^{\infty} P_{k+1+\ell} \tilde{z}_{\ell} \quad (6.4)$$

where

$$(P_0 z)(\tau) = \int_0^{\tau} W_0(\tau, \sigma) z(\sigma) d\sigma,$$

$$(P_k z)(\tau) = \int_0^{\tau} W_k(\tau, \sigma) z(\sigma) d\sigma, \quad k = 1, 2, 3, \dots$$

As is well known ([3])  $I - P_0$  is boundedly invertible,  $P_0, P_1, P_2, \dots$  are all compact. Then

$$Q_{-k} = -(I - P_0)^{-1} P_k, \quad k = 1, 2, 3, \dots \quad (6.5)$$

are all compact and, keeping (3.7) in mind, it may be seen that for some positive number  $D$

$$\|Q_{-k}\| < D e^{-kCT}, \quad k = 1, 2, 3, \dots \quad (6.6)$$

Then (6.4) can be written, with an obvious re-indexing, as

$$z_k + \sum_{\ell=-k+1}^{-1} Q_{\ell} z_{\ell+k} + \sum_{\ell=-\infty}^{-k} Q_{\ell} \tilde{z}_{\ell+k}, \quad k = 1, 2, 3, \dots \quad (6.7)$$

Given a sequence  $\{y_k\}_{-\infty < k < \infty} \subset H$ , where  $H$  is a Hilbert space, and supposing that

$$\|y_k\| < M^+(\gamma^+)^k, \quad k = 1, 2, 3, \dots, \quad (6.8)$$

$$\|y_k\| < M^-(\gamma^-)^k, \quad k = 0, -1, -2, -3, \dots, \quad (6.9)$$

where  $M^+, M^-, \gamma^+, \gamma^-$  are all positive numbers and  $\gamma^+ > \gamma^-$ , we define the bilateral "Z-transform" (discrete Laplace transform) of  $\{y_k\}$  by

$$\begin{aligned} n(\gamma) &= \sum_{k=1}^{\infty} y_k \lambda^{-k} \equiv n^+(\lambda), \quad |\lambda| > \gamma^+, \\ & - \sum_{k=0}^{\infty} y_{-k} \lambda^k \equiv n^-(\lambda), \quad |\lambda| < \gamma^-. \end{aligned} \quad (6.10)$$

Clearly,  $\eta(\lambda)$  is analytic in neighborhoods of both 0 and  $\infty$ . In many cases

$\eta^+(\lambda)$  and  $\eta^-(\lambda)$  are analytic continuations of each other. For example, if for all integers  $k$

$$y_k = \mu^k y_0, \quad y_0 \in H, \quad \mu \neq 0 \in \mathbb{C}$$

then with  $\gamma^+ = \gamma^- = |\mu|$ ,  $m^+ = m^- = 1$ , all of the above are valid for

$$\eta^+(\lambda) = \eta(\lambda), \quad |\lambda| > |\mu|, \quad \eta^-(\lambda) = \eta(\lambda), \quad |\lambda| < |\mu|, \quad \eta(\lambda) = \frac{1}{\lambda - \mu}.$$

If, correspondingly,  $\{Q_k | -\infty < k < \infty\}$ , is a sequence of bounded operators on  $H$  such that

$$\|Q_k\| < B^+(\rho^+)^{-k}, \quad k = 1, 2, 3, \dots \quad (6.11)$$

$$\|Q_{-k}\| < B^-(\rho^-)^{-k}, \quad k = 0, -1, -2, \dots, \quad (6.12)$$

where  $B^+$ ,  $B^-$ ,  $\rho^+$ ,  $\rho^-$  are all positive (and  $\rho^+ > \gamma^+$ ,  $\rho^- < \gamma^-$  in our application), we may define the discrete Fourier transform of  $\{Q_k\}$  by

$$Q(\lambda) = \sum_{k=-\infty}^{\infty} Q_k \lambda^k. \quad (6.13)$$

Clearly the series converges and  $Q(\lambda)$  is a holomorphic operator valued function for  $\rho^- < |\lambda| < \rho^+$ . If  $Q_k = 0$  for all positive  $k$ , then  $\rho^+$  may be taken to be  $\infty$  and  $Q(\lambda)$  will be holomorphic for  $|\lambda| > \rho^-$ , including  $\lambda = \infty$ .

The convolution of  $\{Q_k\} \equiv Q$  and  $\{y_k\} \equiv y$  is defined by

$$f_\ell \equiv (Q^* y)_\ell = \sum_{k=-\infty}^{\infty} Q_k y_{k+\ell}, \quad (6.14)$$

the sum being convergent when (6.8), (6.9), (6.11) and (6.12) apply and

$\rho^+ > \gamma^+$ ,  $\rho^- < \gamma^-$ , as we suppose. To anyone familiar with transforms of this type the first question occurring concerns the relationship of the transform  $\phi(\lambda)$  of  $\{f_\ell\}$  to the transforms  $Q(\lambda)$ ,  $\eta(\lambda)$ . The answer is easy but not completely obvious. Let

$\Gamma^+$  and  $\Gamma^-$  be positively oriented circles centered at  $\lambda = 0$  with radii  $r^+$ ,  $r^-$ ,

$\gamma^+ < r^+ < \rho^+$ ,  $\rho^- < r^- < \gamma^-$ . Then, as we show in the more complete discussion [7], if



$\Gamma = \Gamma^+ - \Gamma^-$  and  $\lambda$  lies in the exterior of the annular regions between  $\Gamma^+$  and  $\Gamma^-$ ,

$$\phi(\lambda) = \frac{1}{2\pi i} \int_{\Gamma} \frac{Q(\zeta)\eta(\zeta)d\zeta}{\lambda-\zeta} \quad (6.15)$$

This condition is necessary and sufficient for (6.14) to be true.

When  $Q_k = 0$  for  $k$  positive and  $Q_0$  has a bounded inverse, (6.14) becomes

$$\sum_{k=-\infty}^0 Q_k Y_{k+l} = f_l \quad (6.16)$$

and, given the initial values

$$Y_{-l} = \tilde{y}_{-l}, \quad l = 0, 1, 2, 3, \dots \quad (6.17)$$

and  $f_1, f_2, f_3, \dots$ , we can compute  $y_1, y_2, y_3, \dots$ . The case of interest to us is the homogeneous case  $f_1 = f_2 = f_3 = \dots = 0$ . Here it may be seen that with

$$\tilde{\eta}(\lambda) = \sum_{l=0}^{\infty} \tilde{y}_{-l} \lambda^l, \quad \eta(\lambda) = \sum_{k=1}^{\infty} y_k \lambda^{-k}, \quad (6.18)$$

and for  $|\lambda| > r^-$ ,  $\lambda$  not a singularity of  $Q(\lambda)$

$$\eta(\lambda) = \frac{1}{2\pi i} Q(\lambda)^{-1} \int_{\Gamma^-} \frac{Q(\zeta)\tilde{\eta}(\zeta)d\zeta}{\lambda-\zeta}.$$

Since  $Q(\lambda)$  is analytic at  $\lambda = \infty$  and  $Q(\infty) = Q_0$ , which is nonsingular, if we take  $r^+$ , the radius of  $\Gamma^+$ , so large that all singularities of  $Q(\lambda)$  are included in the interior of  $\Gamma^+$ , then the individual  $y_k$ ,  $k = 1, 2, 3, \dots$  may be recovered via

$$y_k = \frac{1}{2\pi i} \int_{\Gamma^+} \eta(\lambda) \lambda^{k-1} d\lambda, \quad k = 1, 2, 3, \dots \quad (6.20)$$

With the use of formula (6.16) the proof of Theorem 5 may be completed. In our application  $H = L_m^2[0, T]$ ,  $Q_k = 0$  for  $k$  positive,  $Q_0 = I$ ,  $Q_{-1}, Q_{-2}, Q_{-3}, \dots$  are all compact and the series (6.13) converges uniformly for  $|\lambda| > r^- + \delta$  for any  $\delta > 0$ . It is known ([1], [5]) that the singularities of  $Q(\lambda)$  must be isolated in any such region and, for each such singularity  $\lambda_k$  the null space of  $Q(\lambda_k)$  must be finite dimensional. Let  $\Gamma_{\delta}^-$  be the circle centered at 0 with radius  $r^- + \delta$ . We may assume

that  $\Gamma_\delta^-$  meets no singularities of  $Q(\lambda)$ . Then, applying (6.16) to the  $z_k$  of (6.7) we have

$$z_k = \frac{1}{2\pi i} \int_{\Gamma_\delta} \eta(\lambda) \lambda^{k-1} d\lambda + \frac{1}{2\pi i} \int_{\Gamma_\delta^-} \eta(\lambda) \lambda^{k-1} d\lambda \quad (6.21)$$

$$= z_{k,F} + z_{k,r^- + \delta}, \quad k = 1, 2, 3, \dots,$$

where  $\Gamma_\delta = \Gamma^+ - \Gamma_\delta^-$ . From (6.19) and (6.21) it is clear that

$$\|z_{k,r^- + \delta} + S\| \leq \tilde{M}(r^- + \delta)^k$$

where  $\tilde{M}$  is a constant which may be bounded in terms of  $\tilde{\eta}$ , hence in terms of the  $\tilde{z}_{-\ell}$ ,  $\ell = 0, 1, 2, \dots$ . On the other hand

$$z_{k,F} = \lambda_2 e^{\sum \text{Int} \Gamma_\delta \lambda_j^k} \text{Res } \eta(\lambda_j).$$

In the case of a simple eigenvalue  $\lambda_j$  with one dimensional null space, which is all we will study here, from the formula (6.19) for  $\eta$  it may be seen that

$$\text{Res } \eta(\lambda_j) = \frac{1}{(\psi_j, Q^*(\lambda_j) \phi_j)} \int_{\Gamma^-} \frac{(\psi_j, Q(\zeta) \tilde{\eta}(\zeta) ds}{\lambda_2 - \zeta} \phi_j \quad (6.22)$$

where  $\phi_j$  is a non-zero vector in the one dimensional null space of  $Q(\lambda_j)$  and  $\psi_j$  is a corresponding vector in the null space of  $Q(\lambda_j)^*$  such that  $(\psi_j, \phi_j)_{L_m^2[0,T]} = 0$ . We see in any case that  $z_{k,F}$  is a sum of the form

$$z_{k,F} = \lambda_j e^{\sum \text{Int} \Gamma_\delta \lambda_j^k} C_j \phi_j, \quad k = 1, 2, 3, \dots \quad (6.23)$$

The corresponding solutions  $z_F(t)$ ,  $z_\beta(t)$  of (3.6) (or (6.1)) are obtained by inverting the transformations which follow (6.1). The term  $e^{-\beta T}$  of (3.11) is identified with  $r^- + \delta$ . It is greater than  $e^{-CT}$  which is identified with  $r^-$ . Thus  $z_\beta(t)$  satisfies (3.11).

Since  $z_p(t)$  is a solution of (6.2), the form of that equation shows that  $z_p(t)$  must be a continuous function. The form (6.23) then implies that  $z_p$  on any interval  $[kT, (k+1)T]$  is  $\lambda_j$  times the corresponding value of  $z_p$  on  $[(k-1)T, kT]$ . From this is clear that

$$\phi_j(t) = \lambda_j \phi_j(0) .$$

We identify  $e^{\lambda T}$  in (3.12) with  $\lambda_j$  and  $P(T)$  with the  $T$ -periodic extension of

$e^{-\lambda t} \phi_j(t)$  (which satisfies

$$P(T) = e^{-\lambda T} \phi_j(T) = \lambda_j^{-1} \phi_j(T) = \phi_j(0) = e^{-\lambda 0} \phi_j(0) = P(0))$$

Thus, modulo the usual remarks which most apply to non-simple poles of  $Q(\lambda)^{-1}$ , which lead to solutions of the form (3.13), we have completed the proof of Theorem 5. Further details may be found in [7]. The main point of the theorem is that the dominant solutions of (3.6) (or (6.1)) are those associated with the larger singularities of  $Q(\lambda)$  and those solutions are of the type (3.12) or (3.13).

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