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In this report we use a simplified clutter return model for MTI simulation. We show how this model produces the same correlation function and spectral density function as the ones measured from an actual radar return. We then use this simulation program to compute the statistical distribution of the improvement factor of a MTI system with binomial weights.

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MTI System Simulation and Clutter Output Statistics

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Radar Division

December 31, 1984

NAVAL RESEARCH LABORATORY
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CONTENTS

MTI SYSTEM SIMULATION AND CLUTTER OUTPUT STATISTICS

INTRODUCTION

A radar return from a patch of clutter usually consists of a large number of echoes from individual scatterers. Each of these scatterers moves randomly and introduces a randomly distributed doppler spectrum. Therefore, to simulate the performance of a MTI system, we must sum a large number of randomly distributed samples from each radar pulse return. The computer-time required in this case is very lengthy, and therefore it is not feasible for us to use this approach in a large-scale simulation. In this report we propose to use a simplified version of the clutter model. Such a model, as we will show, will generate a correlation function and a spectral density function which are experimentally measured. Such a model reduces the required computer-time a great deal.

The second problem we will investigate is the improvement factor of the MTI system. This improvement factor is a function of the clutter output and will be used as a base for the design of an MTI system and its required performance. The clutter output is the weighted summation of a number of delayed radar returns. Since the spectral distribution of these radar returns is random, the improvement factor is also a random function. In the past, the expected value of this improvement factor was used for the radar MTI design; therefore it is possible that it accounts for the fact that a well-designed MTI system may not be adequate to eliminate clutter noises at all times. In this report, we present, as an example, the probability distribution of the improvement factor of a MTI system. This MTI system uses binomial weights, and the results are obtained through computer simulation.

CLUTTER SPECTRAL DENSITY FUNCTION

The radar clutter returns usually are characterized by two parameters:

a. The radar cross section and

b. The spectral density function.

The radar cross section is the measure of the reflection of a clutter patch when it is illuminated by radar emission. The spectral density function is the distribution of the random motion of the scatterers. In general, since a clutter patch consists of a large number of scatterers, and the radar echo is the summation of the reflections from each individual scatterer, the reflection from each scatterer is random and independent. Therefore, according to the central limit theory, the summation of these independent random variables is asymptotically normal as the number of random variable approaches infinity. Hence, the radar return changes into a Rayleigh distribution. This fact has been pointed out very early by Lawson and Uhlenbeck [1]; later, other distributions such as log normal have been reported for sea and land returns [2]. The radar cross section of clutter is a very important factor in the detection of target. If the clutter is strong enough, it might mask the target; however, some means must be provided to eliminate the clutter interference. Since the clutter correlation time usually is very long, the integration scheme does not work under this condition. To reduce clutter interferences we must avail ourselves of the evident fact that the clutter in general is stationary. We can use a kind of velocity filter which may filter out the stationary clutter and enhance the detection of a moving target. This has been proven and successfully carried out. To design such a velocity filter, it is very important that we possess knowledge of the characteristics of clutter movement.

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In reality, the clutter is not completely stationary. As we pointed out earlier, the clutter consists of a large number of scatterers, and since each scatterer moves randomly because of prevailing wind or other effects, the motion of the scatterers introduces doppler shift of the radar returns. The doppler shift is a random function that is characterized by the correlation function or the spectral density function. In this section, based on the model that a clutter patch consists of a large number of clutter scatterers moving randomly, we determine the correlation function.

Let us assume that there is a large number of scatterers located in a patch of clutter, and let us also assume that point 0 is the phase center (a reference point) of this clutter patch; if a radar signal is reflected from a point scatterer α, then this reflected wave can be represented as (see Fig. 1):

$$n_\alpha(t_1) = \gamma_\alpha \cos(\omega_0 t + 2kR + k\hat{R} \cdot \mathbf{r}_1 + \beta_\alpha)$$

where γ_α is the amplitude of the reflected wave.

ω_0 is the radar carrier frequency, for simplification we have assumed that the radar signal is narrow band.

R is the distance from radar site to the phase center point 0.

$\hat{R}$ is a unit vector of R.

$\mathbf{r}_1$ is position vector of the scatterer point α.

β_α is the phase due to reflection from the scatterer.

k is the wave number.

For this information we assumed that $R >> r_1$.

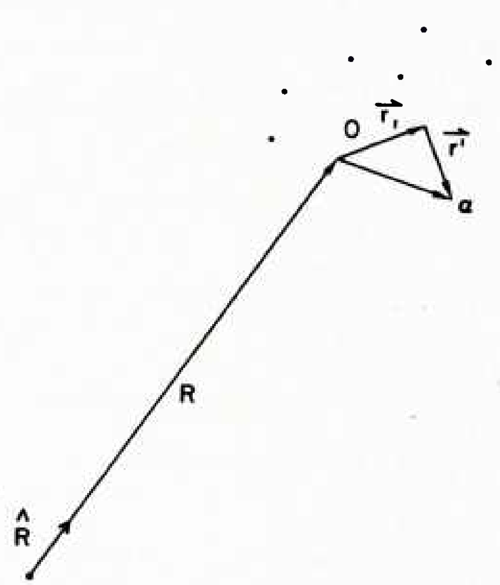

Fig. 1 — Scattering by randomly moving clutter particles

At time t_2 this particle moves to a new position; therefore, the radar return becomes

$$\eta_\alpha(t_2) = \gamma_\alpha \cos\,(\omega_0\, t_2 + 2kR + k\hat{R}\cdot \mathbf{r}_2 + \beta_\alpha).$$

Since $\mathbf{r}_2 = \mathbf{r}_1 + \mathbf{r}'$ we have

$$\eta_\alpha(t_2) = \gamma_\alpha \cos\,(\omega_0\, t_2 + 2kR + k\hat{R}\cdot (\mathbf{r}_1 + \mathbf{r}') + \beta_\alpha).$$

Since kR represents a constant phase delay for all scatterers, it can be ignored. Expression $k\hat{R}\cdot \mathbf{r}_1 + \beta_\alpha$ is a random phase which varies from scatterer to scatterer; therefore, we can represent it by a random variable ϕ_α. Furthermore, because the time elapse from t_1 to t_2 is very short, we can assume that the scatterer α moves at a constant velocity. Then $k\hat{R}\cdot \mathbf{r}' = (2\pi/\lambda)\ \mathrm{v}\ (t_2 - t_1)$, and v is the radial component of the velocity of scatterer α. We should notice that v is a random variable.

The correlation function is:

$$E[\eta(t_1)\,\eta(t_2)] = E\left[\sum_\alpha\sum_\beta \gamma_\alpha\,\gamma_\beta \cos\,(\omega_0\, t_1 + \phi_\alpha)\cos\,(\omega_0\, t_2 + \phi_\beta + (2\pi/\gamma)\ \mathrm{v}(t_2 - t_1))\right]$$

$$= E\left\{\sum_\alpha\sum_\beta \gamma_\alpha\,\gamma_\beta\left[\frac{1}{2}\cos\,(\omega_0\,(t_1 - t_2) - (2\pi/\lambda\mathrm{v})(t_2 - t_1) + (\phi_\alpha - \phi_\beta))\right.\right.$$

$$\left.\left. + \frac{1}{2}\cos\,(\omega_0\,(t_1 + t_2) + (\phi_\alpha - \phi_\beta) + (2\pi/\lambda\mathrm{v})(t_2 - t_1))\right]\right\}.$$

The reflection coefficient γ, phase angle ϕ, and velocity v are all independent random variables, and the process is stationary. The phase angle ϕ has a uniform distribution between $-\pi$ to π, because we assume that the scatterers are randomly distributed within the clutter patch. Because of this assumption, the correlation function becomes

$$R(\tau) = R_0\, E\left[\sum_\alpha \cos(w_0\tau - (2\pi/\lambda)\mathrm{v}\tau)\right]$$

where

$$\tau = t_2 - t_1$$

$$R_0 = \frac{1}{2}\sum_\alpha \gamma_\alpha^2.$$

The velocity v may contain two components. A constant component which is common to each scatterer, and which represents the velocity that the clutter patch moves as a single body. The second component, which varies randomly from scatterer to scatterer, represents the velocity of movement of each scatterer, thus

$$\mathrm{v} = \mathrm{v}_0 + \Delta\mathrm{v}.$$

The correlation function is

$$R(\tau) = R_0 E\left[\sum_\alpha \cos(2\pi(f + f_0 + \Delta f)\tau)\right].$$

Where f_0 is the doppler due to the mean velocity v_0, and Δf is the random component. Let us suppose that this signal has been converted into the base band, and let us assume that the probability density function Δf has a probability function $S(f)$, then

$$R(\tau) = R_0 \int S(\Delta f)\cos(2\pi(f_0 + \Delta f)\tau)\,d\Delta f$$

or

$$R(\tau) = \mathrm{Re}\left[R_0 e^{j2\pi f_0\tau}\int S(\xi)\,e^{j\xi\tau}d\xi\right]$$

which states that the correlation function is the Fourier transformation of the $S(f)$. The $S(f)$ function is commonly referred to as the power spectral density function of the random process $\eta(t)$. We may also notice that function $S(f)$ represents the probability density function of the doppler frequency of the clutter scatterer. The clutter echo can thus be simulated by the summation of many scatterer returns.

SIMULATION OF CLUTTER RETURN SIGNAL

As we have seen in the previous section, we can simulate the clutter return by summing a large number of scatterer returns, each of which has a random doppler frequency $S(f)$, as its probability density function. However, in the actual simulation, this approach requires too much computer-time and therefore is not feasible for a large-scale simulation. In a real radar system, most MTI or doppler filter is performed digitally. The returns of the radar echo at each range bin are sampled and processed. Therefore, if these samples were used for simulation, the results would represent an actual radar operation. It is evident that such an approach requires much less computation because one sample is all that is required instead of the summation of a large number of returns. Furthermore, the measured clutter data is usually characterized by the correlation function which was computed from these sampled data at each range bin. A numerical Fourier transformation of this correlation function is then performed to obtain the spectral density function [3]. It is evident that if we can reconstruct these sampled data from their computer spectral density function, these data can be used to accurately simulate the radar MTI operations. In this section we shall show how these sampled data can be reconstructed from the known spectral density function.

The clutter return is known to be a wide sense stationary process. Its mean value is independent of time, and its correlation function is only a function of the correlation time.

The appendix of this report shows a time function of the following form that can be used to represent such a random process:

$$x(t) = \cos(2\pi ft + \xi).$$

Both f and ξ are random variables. Random variable ξ has uniform distribution from $-\pi$ to π, while f has a probability density function

$$p(f) = 2S(f),$$

where $S(f)$ is the spectral density function which has a correlation function

$$R(\tau) = E[x(t)x(t+\tau)].$$

This time function can, then, be used as the radar return for simulation purposes. Frequency f represents the doppler shift, and ξ represents the initial phase. We notice that each $x(t)$ function for a given f and ξ represents a sample of the ensemble of the random stochastic function of the clutter return. For a different sample, the f and ξ vary according to each probability density function, as shown above.

MTI IMPROVEMENT FACTOR

An MTI system, as shown in Fig. 2, has a filter weight a_n and a delay of τ at each stage. The clutter output of such a system is equal

$$S_0 = \left|\sum_n a_n \exp\,(j2\pi f_i n\tau)\right|^2.$$

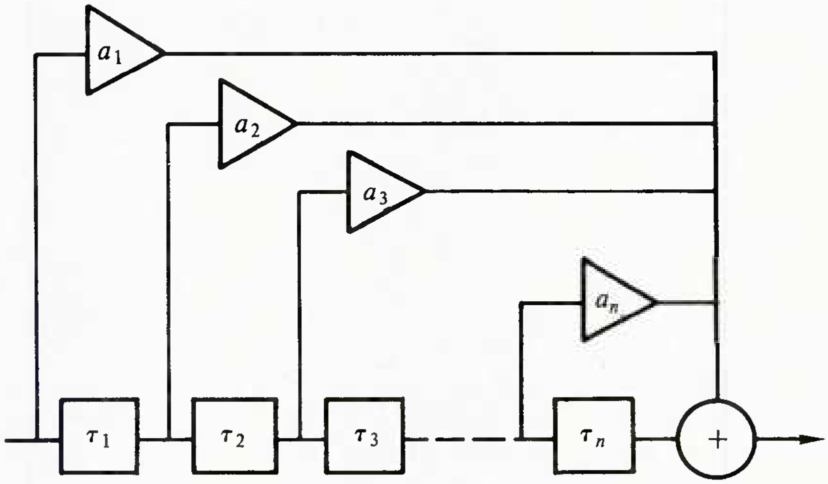

Fig. 2 — An MTI system

Where f_i is the doppler frequency of the target, τ is the delay, and it is assumed that the input signal has a unity amplitude. We notice that the pulse repetition frequency (PRF) f_r is equal to $1/\tau$. If the target doppler is an integer of f_r, then the input signal function S_0 repeats itself. S_0 is thus a periodic function. For convenience, we may introduce a normalized doppler frequency f_i', that

$$f_i' = f_i/f_r,$$

then the signal output becomes

$$S_0 = \sum_n \sum_m a_n a_m \, e^{j2\pi(n-m)f_i'}.$$

When the target velocity is not known a priori, we may assume that the target velocity has a uniform distribution. Under this condition, the average signal gain is thus

$$E(S_0) = \sum_n a_n^2.$$

Similarly the clutter output is

$$C = C_0^2 \sum_n \sum_m a_n a_m \, e^{j2\pi(n-m)f_i'},$$

where C_0 is amplitude of the input clutter. If the clutter has a spectral density function $G(f)$, then the average clutter output is

$$E(C) = C_0^2 \sum_n \sum_m a_n a_m R_{nm},$$

where R_{nm} is the correlation function and it is the Fourier transformation of the spectral density function.

The improvement factor is defined as

$$I.F = E\left[\frac{\text{Input target signal power/input clutter power}}{\text{Output target signal power/output clutter power}}\right]$$

$$= \frac{\sum_n a_n^2}{\sum_n \sum_m a_n a_m R_{nm}}.$$

If the filter weight is normalized so that

$$\sum_n a_n^2 = 1$$

then

$$I.F. = 1 \Big/ \sum_n \sum_m a_n a_m R_{nm}.$$

The correlation function, R_{nm}, is the Fourier transformation of the spectral density function. According to the data measured, most clutter spectral density has a normal distribution. Although some ground clutter has been reported to have a f^3 distribution, it has been shown that it has a general bell shape as that of a normal distribution, except for a long tail. We shall assume, however, that clutter spectral has a Gaussian distribution, and furthermore, that the clutter mean velocity, because of prevailing winds or platform movement, usually has to be removed by some other means. In our following discussion, this mean velocity is assumed to be zero. The clutter spectrum density function has the following form

$$G(f) = \frac{1}{\sqrt{2\pi^{\sigma}}}\, e \,\frac{-f^2}{2\alpha^2}.$$

For convenience, the variable f and standard deviation σ are both normalized with respect to PRF. We therefore, can consider a range of f from 0 to 1. The correlation function has the form

$$R_{nm} = \exp[-2\pi^2\sigma^2(n - m)^2].$$

A set of improvement factors for MTI filter with binomial weights is shown in Fig. 4. The MTI improvement is plotted in dB scale and it is plotted as a function of the normalized σ for a value less than 0.25, or a standard deviation of doppler no more than 1/4 of the PRF.

The binomial coefficients are used for MTI filter weights. It has been shown that binomial weight is only optimal at $\sigma = 0$ [4]. Nevertheless, it has been used since the beginning of the MTI system, and it is very convenient to use. There are other types of filter weights [4-6].

We have noticed in the previous formulation, that the target velocity is assumed to be uniformly distributed; therefore, if the target velocity is known, we might use some other filter weights to achieve a better signal gain. One such example is the doppler filter [6].

MTI SIMULATION

We assume that the clutter return signal for the MTI simulation has the following form:

$$I - \text{channel } x(t) = \cos(2\pi ft + \xi)$$

$$Q - \text{channel } x(t) = \sin(2\pi ft + \xi)$$

where f and ξ are random variables. Random variable f has a normal distribution with zero mean and a given σ. While ξ has a uniform distribution from $-\pi$ to π. The MTI output is then

$$I - \text{channel}$$

$$C_I = \sum_n a_n \cos(nf\tau + \xi)$$

$$Q - \text{channel}$$

$$C_Q = \sum_n a_n \sin(nf\tau + \xi)$$

where τ is interpulse delay time. The total clutter output is then

$$C^2 = C_I^2 + C_Q^2$$

$$= \sum_n \sum_m a_n a_m \cos(nft + \xi) \cos(mft + \xi)$$

$$+ \sum_n \sum_m a_n a_m \sin(nft + \xi) \sin(mft + \xi)$$

$$= \frac{1}{2} \sum_n \sum_m a_n a_m \{[\cos((n - m) f\tau) - \cos((m + n) f\tau + 2\xi)] + [\cos((n-m) f\tau) + \cos((m + n) f\tau + 2\xi)]\}$$

$$= \sum_n \sum_m a_n a_m \cos((n - m) f\tau).$$

If the weights a_n's are normalized, then the expected value of the reciprocal represents the improvement factor.

Figure 3 shows the results of one of such simulation. This figure shows that we have plotted the improvement factor as a function of the normalized σ, the variance of the spectrum density function. Also, this figure shows that we have plotted the expected value of the improvement factor of a three-pulse MTI system. Later at each σ value, we have computed a large sample of clutter output. Each sample has a different doppler frequency which is generated from a normal distribution with a variance of σ. The initial phase ξ was selected from a uniform distribution $-\pi$ to π. We computed the improvement factor of a total of 1000 samples for each σ in Fig. 3, and then we computed its average value which is also plotted in Fig. 3. We may see that the average improvement factor value is almost identical to that of the theoretical value. This may prove the validity of this simulation approach. Figure 4 shows the average improvement factors as a function of normalized σ for 2 to 6 pulse MTI systems.

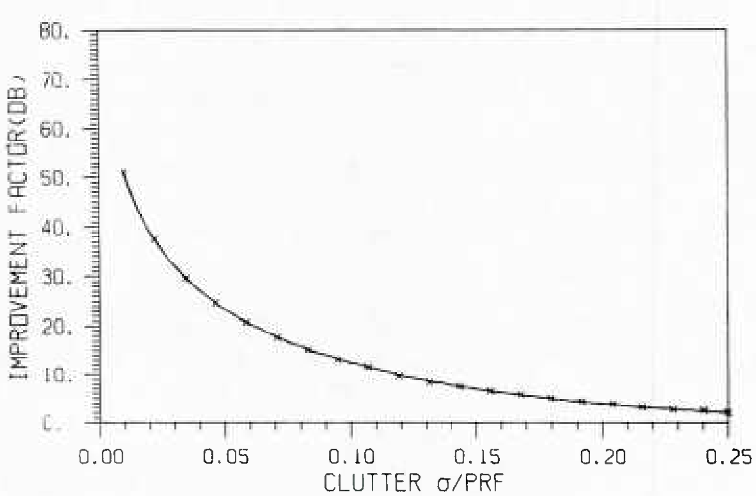

Fig. 3 — Improvement factor vs normalized clutter σ for theoretical and simulation results

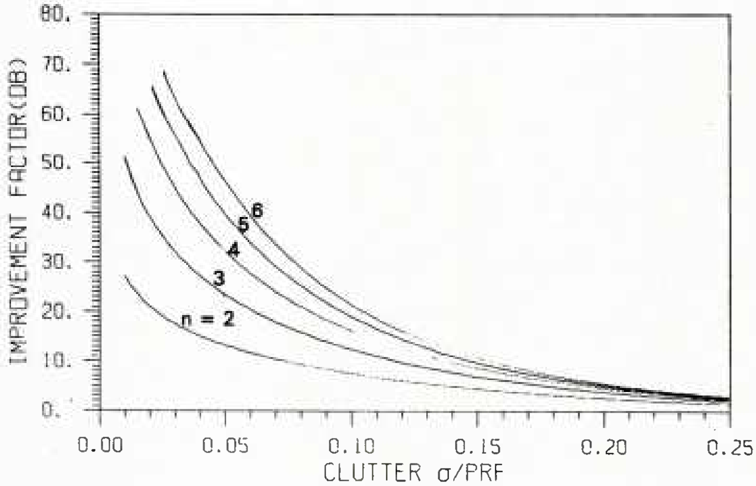

Fig. 4 — Average improvement factor as a function of clutter spectral standard deviation

STATISTICAL DISTRIBUTION OF THE IMPROVEMENT FACTOR

In the previous section we demonstrated how the improvement factor is computed. We also showed how the simulation of improvement factor is formed. This improvement factor usually is used as a base for the design of an MTI system. However, this improvement factor is a random function. In the past its expected value has been used for this purpose; however, in a real system, this value may fluctuate. There are two reasons that contribute to the fluctuation. First, in computing the improvement factor we used the average target velocity. In reality however, the target velocity may fluctuate therefore producing a fluctuation in achievable improvement factors. Second, the clutter doppler is a random variable; hence the clutter output is also a random function. It varies from clutter sample to clutter sample and it also varies at different sampling time (if sampling interval is greater than correlation time).

The variation of target velocity is not addressed here. In general, this problem can be easily solved by using some sort of doppler filter.

In this section we address the second problem. The clutter output is a summation of random variables. Usually we can assume that such a function has a normal distribution if the summed random variables are independent. Unfortunately, the clutter output is the summation of many MTI pulse returns which, because of long clutter correlation time, cannot be treated as independent. The assumption of a normal distribution is therefore invalid. There is no known probability density function to describe such a process; therefore, the following results are based on the computer simulations.

We have performed this simulation for a 3-pulse MTI system with binomial weights. We vary the normalized σ from 0.01 to 0.1. For each σ, we compute 10,000 samples. For each sample, we generate the doppler frequency f and phase ξ. We generate the doppler frequency according to a normal density function with zero mean and σ^2 as the variance. Then we generate the random number ξ as a uniform distribution from $-\pi$ to π. Finally we compute the clutter output. This computed clutter output is then normalized with respect to the average clutter output. We plot the cumulative probability of such samples as shown in Fig. 5. There are five such curves, each for a given σ value which varies from 0.01 to 0.1. One of the interesting results that we may notice is that these cumulative probability curves are not a function of the normalized σ. Similar results were obtained for 4-pulse and 5-pulse MTI system. The clutter level in this plot is normalized with respect to the average clutter level. In a

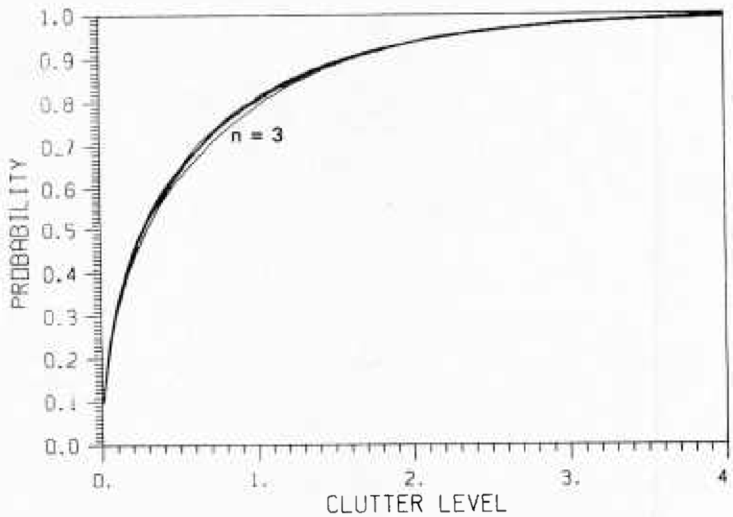

Fig. 5 — Cumulative probability of normalized clutter output level (normalized with respect to average clutter output) for a 3-pulse MTI system

three-pulse canceller, as shown in Fig. 4, the average clutter output amplitude at $\sigma = 0.05$ is 0.067 (or -23.48 dB). In Fig. 5 we see that 80% of the samples have less than this clutter output. However, if we wish to keep a 90% probability improvement factor, the allowable clutter becomes 0.134 (or -17.46 dB).

Figure 6 shows the same cumulative probability curve for 2-, 3-, 4-, 5-, and 6-pulse MTI systems. This figure shows the same normalized clutter as that shown in Fig. 5. For example, for $n = 2$, the probability that the clutter output is less than or equal to the average clutter is 0.66. Table 1 lists the probabilities of MTI filter which are less than or equal to its expected values for 2 to 6 pulses.

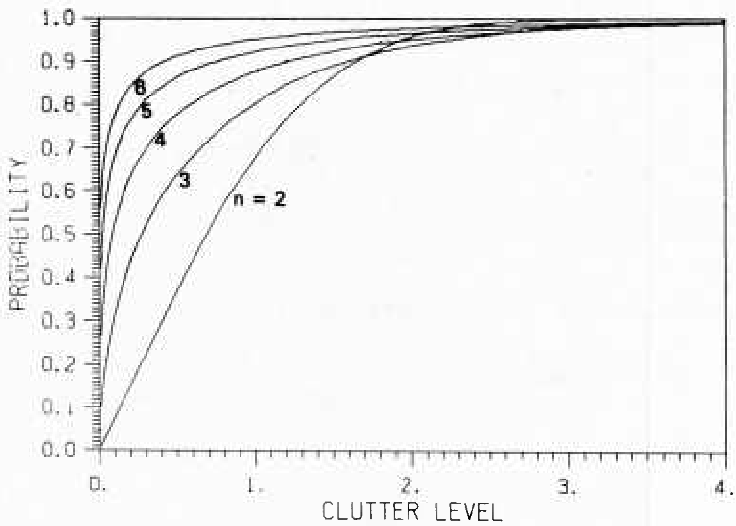

Fig. 6 — Cumulative probability of normalized clutter output level (normalized with respect to average clutter output)

Table 1 — Probabilities of MTI Filter

No. of Pulses	Probability
2	0.66
3	0.80
4	0.87
5	0.92
6	0.95

It is evident that the increase in the number of pulses not only reduces the average clutter output, but there is also a higher probability that the clutter output will not exceed that level.

The information contained in Fig. 6, and in Fig. 4, can be used to find the probabilistic improvement factor for a conventional binomial weighted MTI system. For example, at a normalized $\sigma = 0.05$, the average improvement factor of a 4-pulse canceller is 32 dB. From the information in Fig. 6 we may notice that 0.87 probability of all samples will achieve this improvement factor. However, if we want to achieve a 0.95 probability, the improvement factor decreases to 26.4 dB.

CONCLUSION

In this report we have reviewed how a clutter correlation function is formulated and how it relates to the spectral density function. To simplify MTI simulation, we have formulated a random time function that can produce a correlation function and spectral density function as the one measured experimentally. We have checked the validity of such simulation against the theoretical results; we have computed the improvement factors for binomial weighted MTI systems and have plotted its cumulative probabilities of staying below a certain clutter output level.

REFERENCES

1. *Threshold Signals*, edited by J.L. Lawson and G.E. Uhlenbeck (McGraw-Hill, New York) **24**, MTI Radiation Laboratory Series (1950).

2. Maurice W. Long, *Radar Reflectivity of Land and Sea* (Lexington Books, 1975).

3. V.W. Pidgeon, "Doppler Dependence of Radar Sea Return," Geophys. Res. **73**(4), 1968.

4. J.K. Hsiao, "On the Optimization of MTI Clutter Rejection," *IEEE Trans. AES*, **AES-10**(5), 622, Sep. 1974.

5. Alan V. Oppenheim and R.W. Schafer, *Digital Signal Processing* (Prentice Hall, Inc., Englewood Cliffs, NJ, 1975).

6. M.I. Skolnik, *Introduction to Radar Systems*, Sec. Ed. (McGraw-Hill Book Co., 1980).

APPENDIX

Given a random function $x(t)$, its expected function is

$$E[x(t)] = E[\cos(2\pi ft + \xi)]$$
$$= E[\cos 2\pi ft \cos \xi - \sin 2\pi\ ft \sin \xi].$$

Since f and ξ are independent,

$$E[x(t)] = E(\cos 2\pi ft) E(\cos\xi) - E(\sin 2\pi ft\ E(\sin\xi).$$

If $x(t)$ is stationary, $E[x(t)]$ must be a constant and not a functin of t. This will happen when

$$E(\cos\xi) = E(\sin\xi) = 0.$$

This implies that the characteristic function $\phi(k)$ of ξ

$$\phi(1) = 0,$$

where

$$\phi(k) = \int p(\xi)\ e^{jk\xi} d\xi$$

and $p(\xi)$ is the probability density function of random variable ξ.

The correlation of $x(t)$ is

$$R(\tau) = E(\cos(2\pi ft + \xi) \cos(2\pi f(t + \tau) + \xi)$$
$$= \frac{1}{2} E(\cos(2\pi f\tau)) + \frac{1}{2} E(\cos(4\pi ft + 2\pi f\tau + 2\xi))$$

if $E(\cos 2\xi) = E(\sin 2\xi) = 0$.

Then

$$R(\tau) = \frac{1}{2} E(\cos 2\pi f\tau)$$

or

$$R(\tau) = \frac{1}{2} \int p(f) \cos 2\pi f\tau df,$$

where $p(f)$ is the probability density function of the frequency f. Since $R(\tau)$ is the Fourier transformation of the spectrum density function $S(t)$, we have

$$S(f) = \frac{1}{2} p(f).$$

The probability density function of ξ must have a uniform distribution in the range from $-\pi$ to π. This can be shown as follows:

$$\phi(k) = \int_{-\pi}^{\pi} \frac{1}{2\pi} e^{jk\alpha_{\alpha x}}$$
$$= \frac{\sin k\pi}{k\pi}.$$
$$\phi(k) = 0.\ k = 1,\ 2.$$

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