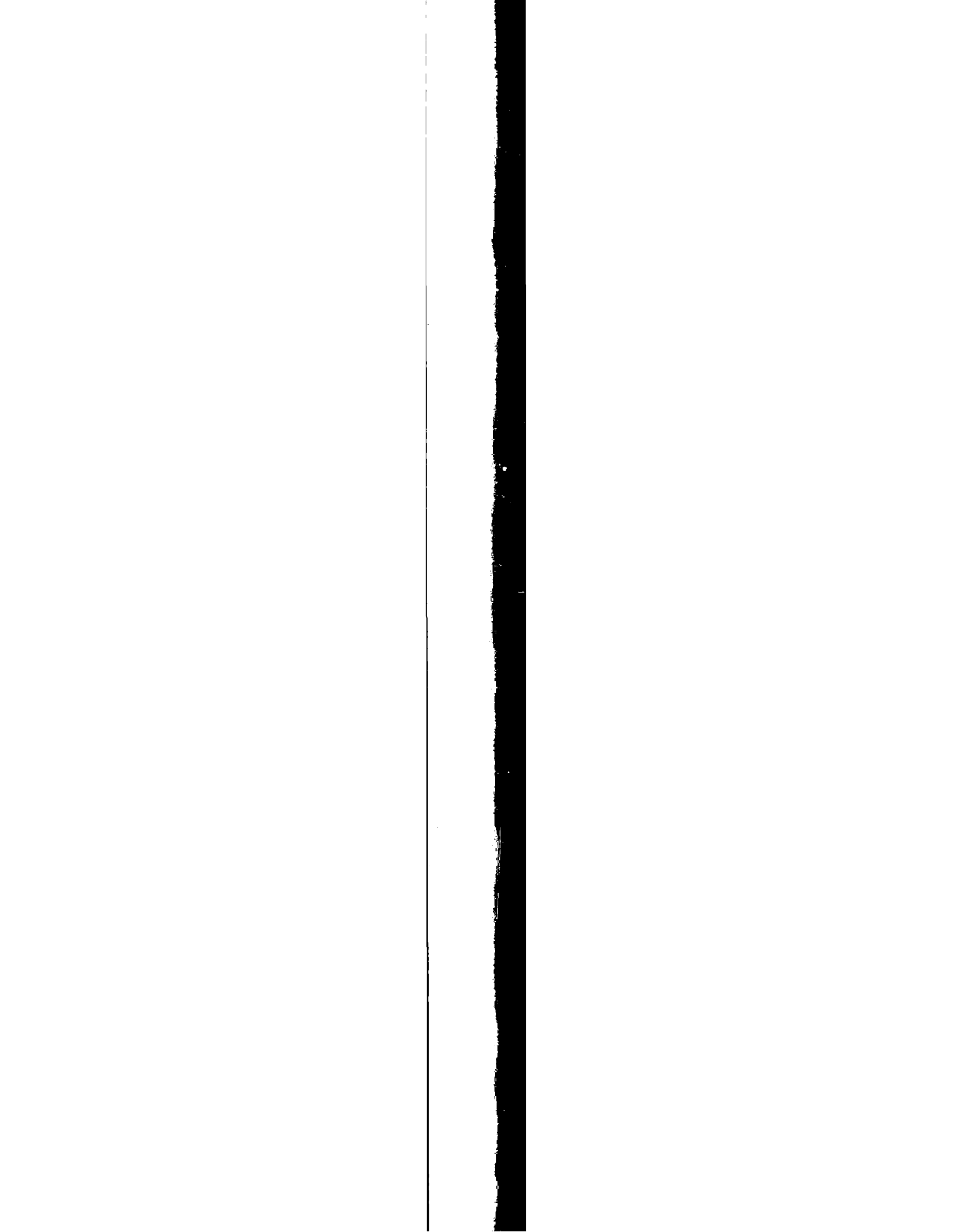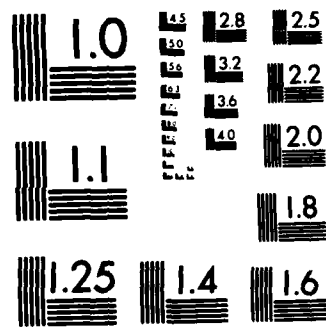MICROCOPY RESOLUTION TEST CHART

NATIONAL BUREAU OF STANDARDS-1963-A

AD-A149 093

MRC Technical Summary Report #2759

SYMMETRY BREAKING FOR A CLASS OF
SEMI-LINEAR ELLIPTIC PROBLEMS

Giovanna Cerami

Mathematics Research Center
University of Wisconsin—Madison
610 Walnut Street
Madison, Wisconsin 53705

October 1984

(Received August 24, 1984)

DTIC
ELECTE
JAN 16 1985

D

85   01   15   016

UNIVERSITY OF WISCONSIN - MADISON
MATHEMATICS RESEARCH CENTER

SYMMETRY BREAKING FOR A CLASS OF
SEMI-LINEAR ELLIPTIC PROBLEMS

Giovanna Cerami[*]

Technical Summary Report #2759

October 1984

ABSTRACT

Consider the nonlinear elliptic problem.

$$(*) \quad \begin{cases} -\Delta u = f(u) & |x| < R \\ u = 0 & |x| = R \end{cases}$$

Suppose this problem has a family of positive radial solutions parametrized by R, i.e., $u_R^o(|x|)$. In the paper we study the possibility of the existence of nonradial solutions of (*) bifurcating from the radial solutions family.

Answering a question posed by Smoller and Wasserman, we show this happens if f satisfies suitable assumptions. Therefore, we investigate the global structure of the nonradial solution set is investigated.

AMS (MOS) Subject Classifications: 58E07, 47H15, 35J25, 35B32

Key Words: Semilinear elliptic boundary value problem, Radial and nonradial solutions, Bifurcation, Symmetry breaking.

Work Unit Number 1 - Applied Analysis

*

Istituto di Matematica, Facoltá di Scienze della Universitá di Palermo, Via Archirafi n. 34, 90123 PALERMO, ITALY.

SIGNIFICANCE AND EXPLANATION

Consider the nonlinear elliptic PDE

$$\begin{cases} -\Delta u = f(u) & |x| < R \\ u = 0 & |x| = R \ . \end{cases}$$

Suppose this problem has a family of positive radial solutions parametrized by R, i.e., $u_R(|x|)$. We are interested in the way in which this family of solutions can bifurcate into nonradial solutions. When this happens we say that the (radial) symmetry breaks.

We give sufficient conditions for symmetry breaking to occur and we study the structure of the nonsymmetric solution set.

Accession For	
NTIS GRA&I	☒
DTIC TAB	☐
Unannounced	☐
Justification	

By
Distribution/

Availability Codes	
	Avail and/or
Dist	Special
A/1	

DTIC
COPY
INSPECTED
8

SYMMETRY BREAKING FOR A CLASS OF SEMI-LINEAR ELLIPTIC PROBLEMS

Giovanna Cerami[*]

1.

Consider the problem

$$(1.1) \qquad \begin{cases} -\Delta u = f(u) & \text{in } B_R \\ u = 0 & \text{on } \partial B_R \end{cases}$$

where $B_R \subset \mathbb{R}^n$, $n > 2$, is the open ball of radius R centered at the origin and $f : \mathbb{R} \to \mathbb{R}$ is a C^2 function.

Suppose that there exists an interval $(R_1, R_2) \subset \mathbb{R}^+$ such that $\forall R \in (R_1, R_2)$ the problem (1.1) has a radial solution u_R. A natural question to ask is whether or not for these values of R there are nonradial solutions, and, in particular, whether the nonradial solutions, if they exist, are close to the radial ones. In some cases it is easily possible to give a (negative) answer. If, for example, f is non negative (or non positive) a simple application of the maximum principle and of a well known theorem by Gidas-Ni-Nirenberg [5] permit us to conclude that every solution of (1.1) must be radially symmetric.

In this paper we are interested in the possibility of the existence of nonradial solutions of (1.1) bifurcating from the radial family.

In order to be more precise it is useful to rewrite the problem (1.1) in the form

$$(1.2) \qquad \Phi(R, u) = 0$$

where

$$\Phi : \mathbb{R}^+ \times C_0^{1+\alpha}(\overline{B}_1) \to C_0^{1+\alpha}(\overline{B}_1)$$

is the operator defined by

$$(1.3) \qquad (R, u) \to u - R^2 G\, f(u) \qquad G = (-\Delta)^{-1}$$

---

[*] Istituto di Matematica, Facoltá di Scienze della Universitá di Palermo, Via Archirafi n. 34, 90123 PALERMO, ITALY.

and $C_0^{1+\alpha}(\overline{B}_1)$ denotes the set of the continuously differentiable functions on $\overline{B}_1$ which vanish on ∂B_1 and whose first order derivatives are Hölder continuous in $\overline{B}_1$ with exponent α ($\alpha \in (0,1)$). $C_0^{1+\alpha}(\overline{B}_1)$ is a Banach space under the usual norm

$$\|u\|_{1+\alpha} = \max_{x \in B_1} |u(x)| + \max_{1 \le i \le n} \max_{x \in B_1} |u_{x_i}(x)| + \max_{1 \le i \le n} \max_{\substack{x,y \in B_1 \\ x \ne y}} \frac{|u_{x_i}(x) - u_{x_i}(y)|}{|x-y|^\alpha} .$$

Then to a radial solution $u_R \in C_0^{1+\alpha}(\overline{B}_R)$ of (1.1) there corresponds the solution of (1.2), $(R, \overline{u}_R) \in R^+ \times C_0^{1+\alpha}(\overline{B}_1)$, $\overline{u}_R(\cdot) \equiv u_R(R \cdot)$.

We will say that the symmetry breaks at R if $(R, \overline{u}_R)$ is a nonradial bifurcation point i.e. if every neighborhood of $(R, \overline{u}_p)$ contains solutions of (1.2) (R,u) with u nonradial.

It is known (see for example [4]) that a necessary condition to have symmetry breaking at R is that $\ker \Phi_u(R, \overline{u}_R)$ contains nonradial functions.

We are concerned with giving sufficient conditions to have nonradial bifurcation for (1.2) at a point (R, u_R) where u_R is a positive radial function.

Our main interest is in a problem posed by Smoller and Wasserman in [8], where they study symmetry breaking problems for positive solutions of semilinear elliptic equations. They close [8] with the following conjecture: suppose $f \in F$, F being the family of functions $f \in C^2(R)$ such that $f(0) < 0$, $(f(t)/t)' > 0$ and $f''(t) < 0$ $\forall\, t > 0$ and $f(t) > 0$ for some $t > 0$. Then by a result of [7] it is known that there is an interval $(R_1, R_2]$ such that $\forall\, R \in (R_1, R_2]$ there exists a unique positive radial solution of (1.1) and the necessary condition for the symmetry breaking is satisfied only if $R = R_2$. Does the symmetry break at R_2?

We will show this actually happens for f "generically" chosen in the subclass F' of F made by the function satisfying the slightly stronger condition $f''(t) < 0$ $\forall\, t > 0$. Here "generically" means that given $f \in F'$, it is possible to compose it with an arbitrarily small translation obtaining a function which is still in F' and for which there is symmetry breaking from the positive solutions.

-2-

We will prove, more, that this bifurcation phenomenon is global i.e. the set of nonradial solutions of (1.2) bifurcating from (R_2, u_{R_2}) is either unbounded in $R^+ \times C_0^{1+\alpha}(\bar{B}_1)$ or meets the connected component of the radial solutions of (1.2) containing the positive solutions in a point different from (R_2, u_{R_2}).

In our argument we have taken advantage of some of the ideas of [8]; indeed, using the method introduced in [8], Smoller and Wasserman could have obtained an analogous <u>local</u> result. However our proof is more straightforward and simpler. Moreover our point of view allows us to investigate and understand the global structure of the nonradial solutions set.

The paper is organized as follows: first, in section 2, the bifurcation problem from a family radial solutions is studied in a little more abstract framework. Then, in section 3, the result is applied to the Smoller-Wasserman problem to get a local symmetry breaking. Finally the global bifurcation result is proved in section 4.

2.

In what follows we will denote by $\tilde{C}_0^{1+\alpha}(\overline{B}_1)$ and $\hat{C}_0^{1+\alpha}(\overline{B}_1)$ the subspaces of $C_0^{1+\alpha}(\overline{B}_1)$ consisting respectively of radial functions and of functions even with respect to the first $n - 1$ coordinates, i.e.

$$\tilde{C}_0^{1+\alpha}(\overline{B}_1) = \{u \in C_0^{1+\alpha}(\overline{B}_1) : u(x) = u(|x|)\}$$

$$\hat{C}_0^{1+\alpha}(\overline{B}_1) = \{u \in C_0^{1+\alpha}(\overline{B}_1) : u(x_1,x_2,\ldots,x_{n-1},x_n) = u(-x_1,-x_2,\ldots,-x_{n-1},x_n)\} \quad .$$

Φ will be the operator defined by (1.3).

The aim of this section is to prove the following:

Theorem 2.1. **Suppose that there exists** $\overline{R}, \epsilon \in R^+, \epsilon < \overline{R}$ **and a** C^1 **map**
$$\gamma : (\overline{R}-\epsilon,\overline{R}+\epsilon) \to C_0^{1+\alpha}(\overline{B}_1)$$
such that if we put $\gamma(R) = u_R$ **the following conditions hold**

H_1) $\forall R \in (\overline{R}-\epsilon,\overline{R}+\epsilon)$, (R,u_R) **is a solution of (1.2),** $u_R \in \tilde{C}_0^{1+\alpha}(\overline{B}_1)$ **and there is a**

neighborhood of $(\overline{R},u_{\overline{R}})$ **in** $(\overline{R}-\epsilon,\overline{R}+\epsilon) \times \tilde{C}_0^{1+\alpha}(\overline{B}_1)$ **in which there are no**

solutions of (1.2) **except these;**

H_2) $u_{\overline{R}}$ **is positive and** $\left.\dfrac{d}{dr} u_{\overline{R}}\right]_{r=1} = 0.$

H_3) $\overline{R} f''(u_{\overline{R}})\gamma'(\overline{R}) + 2 f'(u_{\overline{R}}) \neq 0.$

Then $(\overline{R},u_{\overline{R}})$ **is a nonradial bifurcation point of** Φ.

Denote by $\Phi_u(R,u)$ the linear operator from $C_0^{1+\alpha}(\overline{B}_1)$ in $C_0^{1+\alpha}(\overline{B}_1)$ defined by

$$\Phi_u(R,u)v = v - R^2 G f'(u)v \quad .$$

In order to prove theorem 2.1 we start with a lemma which gives a characterization of $\ker \Phi_u(R,u)$ when (R,u) is a solution of (1.2) and u does not change sign in B_1. We recall that in this case u is, by the Gidas-Ni-Nirenberg theorem [5], a radial function.

-4-

Lemma 2.2. Let (R,u) be a solution of (1.2) where u is positive [negative]. If

$$K \equiv \{v \in C_0^{1+n}(\bar{B}_1) : v + R^2 G f'(u)v = 0\}$$

then K is

   (a) either $\{0\}$ or a 1 dimensional set made by radial functions if $\left.\dfrac{d}{dr} u\right]_{r=1} \neq 0$,

   (b) either an n-dimensional set made by nonradial functions or an (n+1)-dimensional set spanned by 1 radial function and n nonradial functions if $\left.\dfrac{d}{dr} u\right]_{r=1} = 0$.

Proof. Suppose u is positive. Then, as mentioned before, by the Gidas-Ni-Nirenberg theorem [5], u is a radial function and, moreover, $\dfrac{d}{dr} u(r) < 0$, $0 < r < 1$. It is a standard result [3] that every element of K can be written in the form

$$(2.1) \qquad v(r,\theta) = \sum_{k=0}^{\infty} a_k(r)\sigma_k(\theta) \qquad 0 < r < 1 \quad \theta \in S^{n-1}$$

where σ_0 is a constant and for $k \geq 1$, σ_k is an eigenfunction of the Laplacian on the $(n-1)$-sphere S^{n-1} corresponding to the k-th nonradial eigenvalue. Hence $a_k(r)$ is a solution of

$$(2.2) \qquad -r^{1-n}\frac{d}{dr}(r^{n-1}w'(r)) + r^{-2}\lambda_k w(r) = R^2 f'(u)w(r)$$

where $r \in (0,1)$ and $\lambda_k = k(k + n - 2)$ for $k \geq 1$. Moreover $a_k(1) = 0$ and by the continuity of v in B_1, $a_k(0) = 0$ $\forall k$. It is obvious then that any a_k and in particular a_0 is uniquely determined up to a constant.

Differentiating $-\Delta u = R^2 f(u)$ with respect to x_1, we deduce that $\dfrac{d}{dr} u$ is a solution of

$$(2.3) \qquad -r^{1-n}\frac{d}{dr}(r^{n-1}w'(r)) + r^{-2}(n-1)w(r) = R^2 f'(u)w(r)$$

$0 < r < 1$, satisfying the condition $w(0) = 0$.

Thus

$$a_1 = c\left(\frac{d}{dr} u\right) \qquad c = \text{const.}$$

because both solve (2.3) with the initial condition $w(0) = 0$. So in the case (a), $a_1(1) = 0$ implies $c = 0$.

If $k > 1$, $\lambda_k > n-1$ so a_k has at least $k - 1$ zeros in $(0,1)$. Using comparison arguments, since $\dfrac{d}{dr} u$ is never 0 in $(0,1)$, from (2.2) and (2.3) we deduce then

$$a_k \equiv 0 \qquad \forall k > 1 \quad .$$

Therefore we can conclude that in case (a), the set K is $\{c\, a_0(r)/c \in \mathbb{R}\}$ so is $\{0\}$ or a 1-dimensional set according to $a_0 \equiv 0$ or not. In case (b)

$$K \equiv \{c_1 a_0(r) + c_2(\frac{d}{dr}\, u)\varphi_1^{(q)}/c_1, c_2 \in \mathbb{R}\}$$

so, since $\varphi_1^{(q)}$ varies in an n-dimensional space (the space of the spherical harmonics of the first order in n variables) and $\frac{d}{dr}\, u \neq 0$ in $0 < r < 1$, the conclusion follows.

If u is negative the proof is the same after observing that Gidas-Ni-Nirenberg theorem implies u radial and $\frac{d}{dr}\, u > 0$ $\forall\, r \in (0,1)$. ∎

<u>Remark 2.3.</u> A result analogous to lemma 2.2 is proved in [7], but, for completeness, we have preferred to give the above proof here. It is slightly different from that of [7].

We recall now a bifurcation theorem by Crandall and Rabinowitz [1].

<u>Theorem 2.4.</u> <u>Let</u> X, Y <u>be Banach spaces,</u> V <u>a neighborhood of</u> 0 <u>in</u> X, $\overline{\lambda}$, $\varepsilon \in \mathbb{R}$, <u>and</u>

$$F : (\overline{\lambda}-\varepsilon, \overline{\lambda}+\varepsilon) \times V \to Y$$

<u>have the properties</u>

  a) $F(\lambda,0) = 0$ $\forall\, \lambda :\, |\lambda-\overline{\lambda}| < \varepsilon$

  b) <u>the partial derivatives</u> F_x, F_λ, $F_{\lambda x}$ <u>exist and are continuous</u>

  c) $\ker F_x(\overline{\lambda},0)$ <u>and</u> $Y \setminus \ker F_x(\overline{\lambda},0)$ <u>are one dimensional</u>

  d) $F_{\lambda x}(\overline{\lambda},0)x_0 \notin \text{Range}(F_x(\overline{\lambda},0))$ <u>where</u> $\ker(F_x(\overline{\lambda},0)) = \text{Span}\{x_0\}$.

<u>If</u> Z <u>is any complement of</u> $\ker(F_x(\overline{\lambda},0))$ <u>in</u> X <u>then there is a neighborhood</u> U <u>of</u> $(\overline{\lambda},0)$ <u>in</u> $\mathbb{R} \times X$, <u>an interval</u> $(-a,a)$ <u>and continuous functions</u> $\varphi : (-a,a) \to \mathbb{R}$ $\psi : (-a,a) \to Z$ <u>such that</u> $\varphi(0) = \overline{\lambda}$ $\psi(0) = 0$ <u>and</u>

$$F^{-1}(0) \cap U = \{(\varphi(\alpha), \alpha x_0 + \alpha\psi(\alpha)) :\, |\alpha| < a\} \cup \{(\lambda,0) : (\lambda,0) \in U\}$$

<u>Proof of Theorem 2.1</u>

Consider the operator

$$F : (\overline{R}-\varepsilon, \overline{R}+\varepsilon) \times C_0^{1+\alpha}(\overline{B}_1) \to C_0^{1+\alpha}(\overline{B}_1)$$

defined by

$$F(R,z) = z + u_R + R^2 G\, f(z+u_R)$$

where G was introduced in §1, (1.3). Note that G is a compact operator from $C_0^{1+\alpha}(\overline{B}_1)$

-6-

in $C_0^{1+\alpha}(\overline{B}_1)$.

Since u_R solves (1.2), $F(R,0) = 0$ $\forall R \in (\overline{R}-\varepsilon, \overline{R}+\varepsilon)$.

It is clear that proving that there are nonradial solutions of $F(R,z) = 0$ in any neighborhood of $(\overline{R},0)$ is equivalent to proving that $(\overline{R}, u_{\overline{R}})$ is a nonradial bifurcation point of Φ.

The operator
$$F_z(R,0) : C_0^{1+\alpha}(\overline{B}_1) \rightarrow C_0^{1+\alpha}(\overline{B}_1)$$
is defined by
$$v \rightarrow v + R^2 G \; f'(u_R)v$$
and using H_2) and lemma (2.2) we deduce that the set
$$K \equiv \{v \in C_0^{1+\alpha}(\overline{B}_1) : F_z(\overline{R},0)v = 0\}$$
is either n or $n+1$ dimensional.

Hypothesis H_1) and H_3) exclude the last possibility. Indeed consider the restriction of F to $(\overline{R}-\varepsilon, \overline{R}+\varepsilon) \times \tilde{C}_0^{1+\alpha}(\overline{B}_1)$, it defines an operator
$$\tilde{F} : (\overline{R}-\varepsilon, \overline{R}+\varepsilon) \times \tilde{C}_0^{1+\alpha}(\overline{B}_1) \rightarrow \tilde{C}_0^{1+\alpha}(\overline{B}_1)$$
such that $\tilde{F}(R,0) = 0$ $\forall R \in (\overline{R}-\varepsilon, \overline{R}+\varepsilon)$.

Using the regularity of γ it is easy to check that the condition (b) of theorem 2.4 is satisfied. Moreover
$$\tilde{K} = \{v \in \tilde{C}_0^{1+\alpha}(\overline{B}_1) : \tilde{F}_z(\overline{R},0)v = 0\} = K \cap \tilde{C}_0^{1+\alpha}(\overline{B}_1)$$

Thus if K is $(n+1)$-dimensional, $\tilde{K}$ is 1-dimensional, and the condition (c) of Theorem 2.4 is verified too. The last condition of Theorem 2.4 in our case becomes
$$\int_0^1 [\overline{R}f''(u_{\overline{R}})\gamma'(\overline{R}) + 2f'(u_{\overline{R}})]v^2 dr \neq 0$$

where $v \in \tilde{K}$, and, by H_3), it is fulfilled. Then we can deduce that in any neighborhood of $(\overline{R},0)$ there are nontrivial radial solutions of $\tilde{F}(R,0) = 0$. But this implies that in any neighborhood of $(\overline{R}, u_{\overline{R}})$ there are radial solutions of (1.2) different from those of the type $(R,\gamma(R))$ in contradiction to H_1). So K must be n-dimensional and consist of nonradial functions.

Since

$$\frac{\partial u_{\overline{R}}}{\partial x_i} = \frac{x_i}{r} \frac{d}{dr} u_{\overline{R}} \qquad \text{and} \qquad \frac{d}{dr} u_{\overline{R}} \Big]_{r=1} = 0$$

differentiating the equation $-\Lambda u = R^2 f(u)$, with respect to x_i we see that $\frac{\partial u_{\overline{R}}}{\partial x_i}$ is a

solution of $-\Lambda v = \overline{R}^2 f'(u_{\overline{R}})v$ in B_1, satisfying the 0-boundary condition. Then the set

$$B = \left\{ \frac{\partial u_{\overline{R}}}{\partial x_i}, \ i = 1,2\ldots n \right\}$$

is a basis for K. Therefore it is easy to check that

$$\hat{K} \equiv K \cap \hat{C}_0^{1+\alpha}(\overline{B}_1)$$

is a 1-dimensional set [spanned by $\frac{\partial u_{\overline{R}}}{\partial x_n}$, a nonradial function].

Now consider the restriction of F to $(\overline{R}-\epsilon, \overline{R}+\epsilon) \times \hat{C}_0^{1+\alpha}(\overline{B}_1)$. It defines an operator

$$\hat{F} : (\overline{R}-\epsilon, \overline{R}+\epsilon) \times \hat{C}_0^{1+\alpha}(\overline{B}_1) \to \hat{C}_0^{1+\alpha}(\overline{B}_1)$$

and arguing in the same way as before for $\tilde{F}$, we can apply Theorem 2.4 and conclude that,

in a suitable neighborhood V of $(\overline{R}, u_{\overline{R}})$ in $(\overline{R}-\epsilon, \overline{R}+\epsilon) \times \hat{C}_0^{1+\alpha}(\overline{B}_1)$, the set of nontrivial

solutions of $\hat{F}(R,z) = 0$ is a continuous branch of nonradial functions. $\blacksquare$

Remark 2.5. If we denote by $O(N)$ the group of the orthogonal matrix T acting on $\mathbb{R}^n$

and by Γ the representation of $O(N)$ in $C_0^{1+\alpha}(\overline{B}_1)$ defined by

$$\Gamma(T)z(x) = z(Tx) \quad , \quad z \in C_0^{1+\alpha}(\overline{B}_1), \ T \in O(N)$$

the operator F turns out to be equivariant, i.e.

$$\Gamma(T)F(R,z) \overset{\text{def}}{=} F(R,z(Tx)) = F(R,\Gamma(T)z)$$

and the set of the solutions of $F(R,z) = 0$ is invariant. In fact, if $F(R,z) = 0$

$$0 = \Gamma(T)F(R,z) = F(R,\Gamma(T)z)$$

Moreover it is not difficult to check that the group $O(N)$ acts transitively on K,

i.e.

$$\forall \ v_1, v_2 \in K, \ v_1 \neq 0 \quad \exists \xi \in \mathbb{R}^+ \text{ and } T \in O(N) \text{ such that } v_2 = \xi\Gamma(T)v_1 \quad .$$

So the manifold of the nontrivial and nonradial solutions of $\hat{F}(R,z) = 0$ in V

corresponds, via the group action, to an n-dimensional set in $(\overline{R}-\epsilon, \overline{R}+\epsilon) \times C_0^{1+\alpha}(\overline{B}_1)$ of

nonradial solutions of $F(R,z) = 0$.

Remark 2.6. A more precise description of the bifurcating set from $(\bar{R},0)$, of $F(R,z) = 0$ can be obtained using a theorem of Prodi ([6], Th. 1) or a theorem of Vanderbauwede ([11], Th. 6.2.6) concerning bifurcation for Fredholm operators of 0-index, subject to the action of a group of symmetries, when the dimension of the kernel of the linearized operator is bigger than 1. Both of them allow us to conclude that, under the hypothesis of theorem 2.1, the bifurcating set is locally an n-dimensional manifold.

But, since in our application we are more interested in the "global" structure of the bifurcating set, we have preferred to state the local bifurcation result in a way such that the proof is simpler.

3.

Consider the family F of functions $f \in C^2(\mathbb{R})$ satisfying the following assumption:

h_1) $f(0) < 0$; $\exists \bar{t} > 0$: $f(\bar{t}) > 0$

h_2) $f''(t) < 0$ $\forall t > 0$

h_3) $\left(\dfrac{f(t)}{t}\right)' > 0$ $\forall t > 0$

In [7] it was proved that

Theorem 3.1. For each $f \in F$, $\exists R_1, \bar{R} \in \mathbb{R}^+ : 0 < R_1 < \bar{R}$ such that $\forall R \in (R_1, \bar{R}]$ the problem (1.1) has a unique positive (and therefore radial) solution u_R and

$\dfrac{d}{dr} u_R \Big]_{r=R} \leq 0$ where the equality holds if and only if $R = \bar{R}$.

This section will be devoted to the study, locally, of the symmetry breaking problem for this class of functions.

For $\delta \in \mathbb{R}^+$ let $f_\delta(t) = f(t-\delta)$.

We have the following result:

Theorem 3.2. Let $f \in F$. Then $\exists \bar{\delta} \equiv \bar{\delta}(f) > 0$ such that $\forall \delta \in [0, \bar{\delta}]$, $f_\delta \in F$. Let $(R_1, \bar{R}]$ denote the interval of $\mathbb{R}^+$ such that $\forall R \in (R_1, \bar{R}]$ the problem

$$\begin{cases} -\Delta u = f_\delta(u) & \text{in } B_R \\ u = 0 & \text{on } \partial B_R \end{cases}$$

has a unique positive solution. Then for almost every $\delta \in [0, \bar{\delta}]$ the symmetry breaks at $\bar{R}$.

Remark 3.3. Observe that the interval $(R_1, \bar{R}]$ depends on δ and f.

We start the proof of theorem 3.2 with the

Lemma 3.4. $\forall f \in F$ $\exists \delta_0 \equiv \delta(f) : \forall \delta \in [0, \delta_0]$ $f_\delta \in F$.

Proof. We begin by observing that it is an easy exercise to verify that $h_1)$, $h_2)$, $h_3)$ imply

$$(3.1) \qquad\qquad f'(t) > 0 \quad \forall \, t > 0 \ .$$

By the continuity of f, f', f'' it is possible to find $\delta_0 > 0$ such that

$$\begin{cases} f(t) < 0 \\ f'(t) > 0 \qquad\qquad \forall \, t \in [-\delta_0, 0] \\ f''(t) < 0 \end{cases}$$

and

$$f(t) > 0 \qquad \forall \, t \in [\bar{t} - \delta_0, \bar{t}]$$

then $f_\delta \ \forall \, \delta \in [0, \delta_0]$ satisfies $h_1)$, $h_2)$ and $f_\delta(t) > 0 \ \forall \, t > 0$

In order to verify $h_3)$ we have to show that

$$f'_\delta(t)t - f_\delta(t) > 0 \qquad \forall \, \delta \in [0, \delta_0]$$

i.e.

$$f'(t-\delta)t - f(t-\delta) > 0 \qquad \forall \, t > 0$$

and this is obvious too because

$$\text{if } t > \delta \qquad f'(t-\delta)t - f(t-\delta) > f'(t-\delta)(t-\delta) - f(t-\delta)$$

while if $t < \delta \qquad f'(t-\delta) > 0 \qquad$ and $\qquad f(t-\delta) < 0.$ ■

We now turn to the <u>**proof of the symmetry breaking result**</u> as an application of Theorem 2.1.

In order to do this, first we need to give an idea of the way in which Theorem 3.1 is proved in [7].

We observe that since a positive solution of (1.1) is a radial function (by the Gidas-Ni-Nirenberg theorem), it must satisfy the boundary value problem

$$(3.2) \qquad \begin{cases} u_{rr} + \dfrac{n-1}{r}\, u_r + f(u) = 0 \qquad 0 < r < R \\ u_r(0) = u(R) = 0 \end{cases}$$

Then the initial value problem

$$(3.3) \quad \begin{cases} u_{rr} + \dfrac{n-1}{r} u_r + f(u) = 0 & 0 < r < M \\[2mm] u_r(0) = 0 \qquad u(0) = p \end{cases}$$

is considered (when M is a suitably big number). The solution of this IVP is denoted by $u(\cdot, p)$.

It is shown that if $f \in F$, $\exists \bar{p}$ such that $\forall p > \bar{p}$ the unique solution of (3.3), $u(\cdot, p)$ is such that $u(R, p) = 0$ for some R.

Precisely it is proved that, if we define

$$R(p) = \min\{R : u(R, p) = 0\}$$

the domain of $R(p)$ is $[\bar{p}, +\infty)$, $R(p)$ is continuous decreasing, $\lim\limits_{p \to +\infty} R(p) = R_1$, $0 < R_1 < R(\bar{p})$, $u_r(R(p), p) < 0 \quad \forall p > \bar{p}$ and $u_r(R(\bar{p}), \bar{p}) = 0$. So for $p > \bar{p}$, the solution $u(\cdot, p)$ of (3.3) will be the unique positive solution of (3.2) in $0 < r < R(p)$ (and, then, a radial solution of (1.1)) satisfying the boundary condition $u(R(p)) = 0$.

Of course $u(\cdot, p)$ can be considered as a function of both its arguments in $[0, M] \times \mathbb{R}^+$ ($M > R(\bar{p})$) and it can be proved [see for example [7] appendix] that if $f \in C^2$ so is u.

Suppose now that

$$(3.4) \qquad u_p(\bar{R}, \bar{p}) \neq 0 \quad \text{where} \quad \bar{R} = R(\bar{p}), \ u_p = \frac{\partial}{\partial p} u$$

Then by the implicit function theorem we deduce that there exists a neighborhood of $(\bar{R}, \bar{p})$ $\mathcal{N} \equiv (\bar{R}-\epsilon, \bar{R}+\epsilon) \times (\bar{p}-\eta, \bar{p}+\eta)$ such that all of the solutions in $\mathcal{N}$ of the equation $u(R, p) = 0$ are pairs $(R, p(R))$ where p is a C^1 function defined in $(\bar{R}-\epsilon, \bar{R}+\epsilon)$.

Thus we are able to define

$$\gamma : (\bar{R}-\epsilon, \bar{R}+\epsilon) \to C_0^{1+\alpha}(\bar{B}_1)$$

by

$$(\gamma(R))(x) \equiv u(R|x|, p(R)) \equiv u_R(|x|)$$

Moreover it is not difficult to see that there is a neighborhood of $(\bar{R}, u_{\bar{R}})$, $\mathcal{U} \equiv \{(R, u) \in \mathbb{R}^+ \times C_0^{1+\alpha}(\bar{B}_1) : |R-\bar{R}| < \epsilon, \ \|u-u_{\bar{R}}\| < \epsilon\}$ such that to each $R : |R-\bar{R}| < \epsilon$ there corresponds a unique radial solution of (1.1) in $\mathcal{U}$.

It is clear then that, since u is a C^1 function of r and p, Y is continuous differentiable and satisfies the conditions $H_1)$ and $H_2)$ of the Theorem 2.1. Moreover

$$(Y'(R))(x) = u_r(R|x|,p(r))|x| + u_p(R|x|,p(r))p'(R)$$

Note that $\forall\, x \in B_1$, $u(\overline{R}\cdot,p(\overline{R})) > 0$ and $u_r(\overline{R}\cdot,p(\overline{R})) < 0$, by the Gidas-Ni-Nirenberg theorem, and $p'(\overline{R}) = 0$.

Therefore the condition $H_3)$ of theorem 2.1 becomes

$$(3.5) \qquad \overline{R}f''(u_{\underset{R}{\_}})(u_r(\overline{R}|x|,p(\overline{R}))|x|) + 2f'(u_{\underset{R}{\_}}) \neq 0$$

and it is verified because if $f \in F$, $f''(t) < 0$ and $f'(t) > 0$ $\forall\, t > 0$.

Theorem (2.1), then, will give our statement if we prove that, in our hypothesis, (3.4) is verified.

So we have to show that for fixed $f \in F$, $\exists\, \overline{\delta} > 0$ s.t. for a.e. $\delta \in [0,\overline{\delta}]$, the solution $\tilde{u}$ of (3.3) corresponding to f_δ verifies (3.4).

In order to do this, consider u, the solution of (3.3) related to f, and observe that u takes negative values for $r \in (0,R(\overline{p}))$: in fact $\forall\, p > \overline{p}$, $u(R(p),p) = 0$ and $u_r(R(p),p) < 0$.

Let $\tilde{\chi} > 0$ be a number such that $-\tilde{\chi}$ is a value of u. By Sard's theorem [10] almost every number in $[-\tilde{\chi},0]$ will be a regular value of u. Let us take $\overline{\delta} = \min\{\hat{\delta},\delta_0\}$ where δ_0 is the number defined in lemma 3.4. Choose $\delta \in [0,-\overline{\delta}]$ such that $-\delta$ is a regular value of u and consider

$$\tilde{u} = u + \delta$$

Then

$$u_{rr} + \frac{n-1}{r}\, u_r + f(u) = \tilde{u}_{rr} + \frac{n-1}{r}\, \tilde{u}_r + f_\delta(\tilde{u})$$

and, since $u(\cdot,p)$ satisfies the initial conditions $u'_r(0,p) = 0$, $u(0,p) = p$, $\tilde{u}(\cdot,p+\delta)$ satisfies $\tilde{u}'_r(0,p+\delta) = 0$ and $\tilde{u}(0,p+\delta) = p+\delta$. Obviously 0 is a regular value of $\tilde{u}$ because $\tilde{u} = 0$ if and only if $u = -\delta$.

Now to get (3.4) observe that $f_\delta \in F$ so $\exists\, p_\delta$ and a continuous function $\tilde{R}(p) : \forall\, p > p_\delta$ $\tilde{u}(r,p)$ is the only positive solution of

$$\begin{cases} u_{rr} + \dfrac{n-1}{r}\, u_r + f_\lambda(u) = 0 & 0 < r < \widetilde{R}(p) \\[2mm] u_r(0) = 0 = u(\widetilde{R}(p)) \end{cases}$$

and $\widetilde{R}(p)$ is the only value for which

$$\widetilde{u}(\widetilde{R}(p_\lambda),p_\lambda) = 0 = \widetilde{u}_r(\widetilde{R}(p_\lambda),p_\lambda) \quad .$$

Since 0 is a regular value of $\widetilde{u}$, $\widetilde{u}_p(\widetilde{R}(p_\lambda),p_\lambda) \neq 0$. ∎

4.

This section will be devoted to the proof of a global result concerning the bifurcating set of nonradial solutions of (1.2).

In what follows f will be a fixed function in F and $\tilde{\Phi}$, $\hat{\Phi}$ the operators

$$\tilde{\Phi} : R^+ \times \tilde{C}_0^{1+\alpha}(\overline{B}_1) \to \tilde{C}_0^{1+\alpha}(\overline{B}_1)$$

$$\hat{\Phi} : R^+ \times \hat{C}_0^{1+\alpha}(\overline{B}_1) \to \hat{C}_0^{1+\alpha}(\overline{B}_1)$$

obtained restricting the operator Φ defined by (1.3) to $R^+ \times \tilde{C}_0^{1+\alpha}(\overline{B}_1)$ and to $R^+ \times \hat{C}_0^{1+\alpha}(\overline{B}_1)$ respectively.

We will suppose that f is an element of F for which the local result of symmetry breaking can be proved. We think it is useful to summarize the properties of f.

P_1) $\exists R_1, \overline{R}, 0 < R_1 < \overline{R}$: $\forall R \in (R_1, \overline{R}]$ there exists a unique $u_R \in C_0^{1+\alpha}(\overline{B}_1)$ positive such that $\Phi(R, u_R) = u_R - R^2 G \ f(u_R) = 0$, $\frac{d}{dr} u_R(r) < 0$, $0 < r < 1$, $\forall R \in (R_1, \overline{R})$ and $\max_{B_1} u_R \xrightarrow[R \to R_1]{} +\infty$

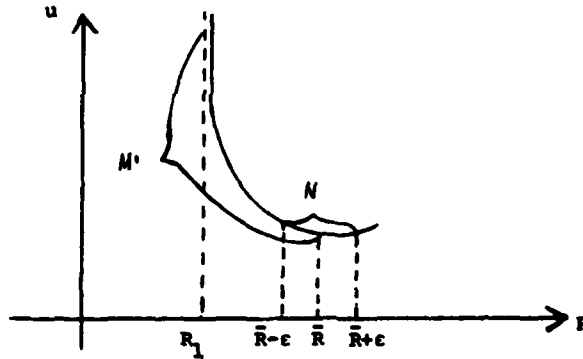P_2) $\exists \epsilon > 0$ such that in $U = \{(R, u) \in R^+ \times C_0^{1+\alpha}(\overline{B}_1) : |R - \overline{R}| < \epsilon, \|u - u_{\overline{R}}\| < \epsilon\}$ the solution set of $\tilde{\Phi}(R, u) = 0$, N, is made by pairs (R, u_R), $u_R = \gamma(R)$, where γ is a continuous differentiable function defined in $(\overline{R} - \epsilon, \overline{R} + \epsilon)$ and $\gamma'(\overline{R}) < 0$.

P_3) $(\overline{R}, u_{\overline{R}})$ is a nonradial bifurcation point for Φ.

Denote by

$$M' \equiv \{(R, u_R) \in (R_1, \overline{R}] \times \tilde{C}_0^{1+\alpha}(\overline{B}_1) : \tilde{\Phi}(R, u_R) = 0, \ u_R > 0\}$$

$$M \equiv M' \cup N .$$

Lemma 4.1. There exists a neighborhood V of M in $(R_1, \bar{R}+\varepsilon) \times \tilde{C}_0^{1+\alpha}(\bar{B}_1)$ such that the only solutions of $\tilde{\mathcal{T}}(R,u) = 0$ in V are the points of M. So $\forall R \in (R_1, \bar{R}+\varepsilon)$ there is a unique function $u_R \in \tilde{C}_0^{1+\alpha}(\bar{B}_1)$ such that $(R, u_R) \in V$ and $\tilde{\mathcal{T}}(R, u_R) = 0$.

Proof. The statement follows using the property $P_2)$ if $(R, u_R) \in N$ and observing that if $(R, u_R) \in M \setminus N$, u_R is in the interior part of the positive cone ($u_R > 0$ and $\frac{d}{dr} u_R < 0$, $0 < |x| < 1$) so, by the property $P_1)$ cannot be a bifurcation point for $\tilde{\mathcal{T}}$. ∎

In what follows if P is a subset of $\mathbb{R}^+ \times C_0^{1+\alpha}(\overline{B}_1)$ [resp. $\mathbb{R}^+ \times \widetilde{C}_0^{1+\alpha}(\overline{B}_1)$, $\mathbb{R}^+ \times \hat{C}_0^{1+\alpha}(\overline{B}_1)$] we will denote

$$P_\lambda \equiv \{u \in C_0^{1+\alpha}(\overline{B}_1) [\text{resp. } \widetilde{C}_0^{1+\alpha}(\overline{B}_1), \hat{C}_0^{1+\alpha}(\overline{B}_1)] : (\lambda,u) \in P\}$$

As norm in $\mathbb{R}^+ \times C_0^{1+\alpha}(\overline{B}_1)$ we will take

$$\|(\lambda,u)\| = |\lambda| + \|u\|_{C_0^{1+\alpha}(\overline{B}_1)}$$

and if A, B are subsets of $\mathbb{R}^+ \times C_0^{1+\alpha}(\overline{B}_1)$

$$\rho(A,B)$$

will be the distance between A and B.

Denote by $\mathcal{R}$ the connected component of the set of the solutions of $\mathcal{X}(R,u) = 0$ containing M.

Lemma 4.2. The set $\mathcal{R} \setminus M'$ _is either unbounded in_ $\mathbb{R}^+ \times \widetilde{C}_0^{1+\alpha}(\overline{B}_1)$ _or is bounded in which case it meets_ $(0,0)$.

Proof. First we observe that if $(R,u_R) \in M$ by lemma 4.1 the index

$$i(\mathcal{S}(R,\cdot), u_R, 0) \text{ is defined } \forall R \in (R_1, \overline{R}+\varepsilon)$$

moreover, by the homotopy invariance property of the topological degree, must be constant $\forall R \in (R_1, \overline{R}+\varepsilon)$. So, since the Fréchet derivative $\mathcal{S}_u(\overline{R},u_{\overline{R}})$ is nonsingular (see proof of Th. 2.1) and then $i(\mathcal{S}(\overline{R},\cdot), u_{\overline{R}},0)$ is equal to $+1$ or -1, we can put

$$i(\mathcal{S}(R,\cdot),u_R,0) = 1 \qquad \forall R \in (R_1,\overline{R}+\varepsilon) \ .$$

Suppose that $\mathcal{R} \setminus M'$ is bounded and does not meet $(0,0)$.

Then there exists an interval $[R',R''] \subset \mathbb{R}^+ : 0 < R' < \overline{R} < R''$, such that

$$\mathcal{R} \setminus M' \subset [R',R''] \times \widetilde{C}_0^{1+\alpha}(\overline{B}_1)$$

We can assume $R' < R_1$.

Since $\lim_{\substack{R \to R_1 \\ (R,u_R) \in M'}} \|u_R\|_{\widetilde{C}_0^{1+\alpha}} = +\infty$ and $\mathcal{R} \setminus M'$ is bounded and cannot meet any point $(R,u_R) \in M'$ (by lemma 4.1), an R^* must exist $: R^* > R_1$ such that $\forall R \in (R_1,R^*)$

$$(R,u) \in \mathcal{R} \setminus M' \implies \|u\|_{\widetilde{C}_0^{1+\alpha}(\overline{B}_1)} < \|u_R\|_{\widetilde{C}_0^{1+\alpha}(\overline{B}_1)}$$

-17-

[where $\forall$ R, u_R is s.t. $(R, u_R) \in M'$].

Consider then a number R_0 : $R_1 < R_0 < R^*$ and

$$R_0 = R \cap ([R_0, +\infty) \times \widetilde{C}_0^{1+\alpha}(\overline{B}_1)) \quad .$$

Of course R_0 can be chosen such that $R_0 \cap M'$ is bounded. Then since $R \setminus M'$ is bounded R_0 will be bounded. So, following the method described in [10] (Th. VIII.1) it is possible to construct an open bounded set O in $[R_0, +\infty) \times \widetilde{C}_0^{1+\alpha}(\overline{B}_1)$ containing R_0 and having no zeros of $\widetilde{\delta}(R, \cdot)$ on its boundary ∂O. Thus the topological degree

$$d(\widetilde{\delta}(R, \cdot), O_R, 0) \text{ is defined } \forall R > R_0$$

and, by the homotopy invariance property, must be constant. Since if we take R ($R > R^*$) big enough $O_R = \emptyset$, we deduce

$$d(\widetilde{\delta}(R, \cdot), O_R, 0) = 0 \quad \forall R > R_0$$

Moreover $\forall$ $\Gamma \in [R_0, R^*]$, $\exists s_R$ such that in the closed set
$\overline{B}(u_R, s_R) = \{u \in \widetilde{C}_0^{1+\alpha}(\overline{B}_1) : |u - u_R| < s_R\}$ there are no solutions of $\widetilde{\delta}(R, \cdot) = 0$ other than u_R. So $\forall R \in [R_0, R^*]$

$$d(\widetilde{\delta}(R, \cdot), O_R \setminus \overline{B}(u_R, s_R), 0) = -i(\widetilde{\delta}(R, \cdot), u_R, 0) = -1$$

Now consider the set

$$R^* \equiv (R \setminus M') \cap ([0, R^*] \times \widetilde{C}_0^{1+\alpha}(\overline{B}_1))$$

Since $R \setminus M'$ is bounded it is possible to construct (following [9] or [10]) an open set A in $[0, R^*] \times \widetilde{C}_0^{1+\alpha}(\overline{B}_1)$ containing R^* and having no zeros of $\widetilde{\delta}(R, \cdot)$ on its boundary ∂A. Then

$$d(\widetilde{\delta}(R, \cdot), A_R, 0) \text{ is defined } \forall R \in [0, R^*] \quad ,$$

and, since for $R < R'$ small enough $A_R = \emptyset$, using the homotopy invariance property we infer

$$d(\widetilde{\delta}(R, \cdot) A_R, 0) = 0 \quad \forall R \in [0, R^*]$$

On the other hand for $R \in [R_0, R^*]$, A can be constructed such that
$A_R \subseteq O_R \setminus \overline{B}(u_R, s_R)$, so, by excision, we have $\forall R \in [R_0, R^*]$

$$d(\widetilde{\delta}(R, \cdot), A_R, 0) = d(\widetilde{\delta}(R, \cdot), O_R \setminus \overline{B}(u_R, s_R), 0) = -1$$

and we get a contradiction. ∎

Denote now by

$$E = \{(R,u) \in R : \text{Ker } \hat{\phi}_z(R,u) \neq \{0\} \}$$

$(\overline{R},u_{\overline{R}}) \in E$ and it is not difficult to see (using for example Corollary 1.13 and Theorem 1.16 of [2]) that $(\overline{R},u_{\overline{R}})$ is an isolated point in E.

Let S be the closure in $\mathbb{R}^+ \times \hat{C}_0^{1+\alpha}(\overline{B}_1)$ of the set of the solutions of $\hat{\phi}(R,z) = 0$ that are not in R.

S is locally compact in $\mathbb{R}^+ \times \hat{C}_0^{1+\alpha}(\overline{B}_1)$.

From the proof of theorem 3.2 we easily deduce that $(\overline{R},u_{\overline{R}})$ is a nonradial bifurcation point of $\hat{\phi}$ and there exists a $\overline{\eta} > 0$ such that in $V \equiv \{(R,z) : |R-\overline{R}| < \overline{\eta},$ $|u - u_{\overline{R}}| < \overline{\eta}\}$ the bifurcating set is a continuous curve.

The following theorem gives a result about the global behavior of the set of nonradial solutions of $\phi(R,u) = 0$ bifurcating from $(\overline{R},u_{\overline{R}})$.

Theorem 4.3. **The connected component** C **of** $S \cup \{(\overline{R},u_{\overline{R}})\}$ **to which** $(\overline{R},u_{\overline{R}})$ **belongs is either unbounded or meets** E **outside of a neighborhood of** $(\overline{R},u_{\overline{R}})$.

Remark 4.4. Note that to C by the action of the symmetry, corresponds an n-dimensional set of solutions of $\phi(R,z) = 0$.

In order to prove theorem 4.3 we require a lemma.

Lemma 4.5. **If** C **is bounded and does not meet any point** $(R,u) \in E \setminus \{(\overline{R},u_{\overline{R}})\}$ **then there exists a bounded open set** $O \subset \mathbb{R}^+ \times \hat{C}_0^{1+\alpha}(\overline{B}_1)$ **such that**

i) $C \subset O$

ii) $\partial O \cap S = \emptyset$

iii) $O \cap R = \{(R,u) \in R : |R-\overline{R}| < \varepsilon_0 \text{ and } |u-u_{\overline{R}}| < \varepsilon_0\}$

where

$$\varepsilon_0 < \frac{1}{2} \min\{\overline{\eta}, \varepsilon, \rho((\overline{R},u_{\overline{R}}), E \setminus \{(\overline{R},u_{\overline{R}})\}), \rho(R, C \setminus (C \cap U))\}$$

-19-

iv) $\exists \alpha > 0 : \forall (R,u) \in O : |R-\bar{R}| > \epsilon_0$ <u>or</u> $\|u-u_{\bar{R}}\| > \epsilon_0 \implies \rho((R,u),\mathcal{R}) > \alpha$.

We will not give here the proof of this lemma which can be done in the same way, with obvious modifications, that in [10] (Lemma VIII.3) or in [9] (Lemma 1.2). ∎

Proof of theorem 4.3.

We will argue by contradiction.

Suppose that C is bounded and does not meet any point $(R,u_R) \in E \setminus \{(\bar{R},u_{\bar{R}})\}$. Then there exist O, ϵ_0, α as in lemma 4.5.

Let $(R,u) \in \mathcal{R}, (R,u) \neq (\bar{R},u_{\bar{R}})$.

If $0 < |R-\bar{R}| < \epsilon_0$ and $\|u-u_{\bar{R}}\| < \epsilon_0$, $(R,u) \in U \cap \mathcal{R}$, so, because of the ϵ_0 choice, u is uniquely determined as a function of $R : u = u_R$. Thus put $s_{(R,u_R)} = \frac{1}{2} \rho(u_R, S_R)$, $s_{(R,u_R)} > 0$ and there are no zeros of $\hat{\Phi}(R,\cdot)$ in $\{R\} \times \partial\{O_R \setminus \bar{B}(u_R, s_{(R,u_R)})\}$ where $\bar{B}(u_R, s_{(R,u_R)})$ is the set $\{z \in \hat{C}_0^{1+\alpha}(\bar{B}_1) : \|z-u_R\| < s_{(R,u_R)}\}$.

If $|R-\bar{R}| < \epsilon_0$ and $\|u_{\bar{R}}-u\| > \epsilon_0$ or if $|R-\bar{R}| > \epsilon_0$ put $s_{(R,u)} = \frac{1}{2} \alpha$, then
$$\bar{O}_R \cap \bar{B}(u,s_{(R,u)}) = \emptyset.$$

Take now $R : \bar{R} < R < \bar{R}+\epsilon_0$, and choose R^* big enough that $O_{R^*} = \emptyset$. Consider
$$s = \inf\{s_{(L,u)} : R < L < \bar{R}+\epsilon, (L,u) \in \mathcal{R}\}$$
$s > 0$, because of the choice of ϵ_0 and since $R > \bar{R}$. Consider the set
$$B = \{(L,u) \in \mathbb{R}^+ \times \hat{C}_0^{1+\alpha}(\bar{B}_1) : \rho((L,z),\mathcal{R}) < \bar{s}\}$$
$$\bar{s} = \min(s,\frac{1}{2}\alpha) ,$$
then
$$Q = (O \setminus \bar{B}) \cap ([R,R^*] \times \hat{C}_0^{1+\alpha}(\bar{B}_1))$$
is an open set in $[R,R^*] \times \hat{C}_0^{1+\alpha}(\bar{B}_1)$ and there are no zeros of $\hat{\Phi}$ on ∂Q.

Therefore
$$d(\hat{\Phi}(L,\cdot),Q_L,0) \text{ is defined and is constant } \forall L \in [R,R^*]$$
by homotopy invariance, so is equal to 0 since $Q_{R^*} = \emptyset$.

On the other hand since $R < \bar{R}+\epsilon_0$

-20-

$$Q_R = O_R \setminus \overline{B}(u_R, \overline{s})$$

and by excision we deduce

$$d(\hat{\phi}(R, \cdot), O_R \setminus \overline{B}(u_R, s_R), 0) = d(\hat{\phi}(R, \cdot), Q_R, 0) = 0 \quad .$$

Using the same argument we obtain the same result if R is such that $\overline{R} - \varepsilon_0 < R < \overline{R}$.

Moreover observe that, by the homotopy invariance property, $d(\hat{\phi}(R, \cdot), O_R, 0)$ is constant for $R \in (\overline{R} - \varepsilon_0, \overline{R} + \varepsilon_0)$.

Choose finally R_α and R_β : $\overline{R} - \varepsilon_0 < R_\alpha < \overline{R} < R_\beta < \overline{R} + \varepsilon_0$. Using the excision and additivity properties and the fact that u_{R_α} is the only zero of $\hat{\phi}(R_\alpha, \cdot)$ in $\overline{B}(u_{R_\alpha}, s_{R_\alpha})$ we infer

$$d(\hat{\phi}(R_\alpha, \cdot), O_{R_\alpha}, 0) = i(\hat{\phi}(R_\alpha, \cdot), u_{R_\alpha}, 0) + d(\hat{\phi}(R_\alpha, \cdot), O_{R_\alpha} \setminus \overline{B}(u_{R_\alpha}, s_{R_\alpha}), 0)$$

analogously

$$d(\hat{\phi}(R_\beta, \cdot), O_{R_\beta}, 0) = i(\hat{\phi}(R_\beta, \cdot), u_{R_\beta}, 0) + d(\hat{\phi}(R_\beta, \cdot), O_{R_\beta} \setminus \overline{B}(u_{R_\beta}, s_{R_\beta}), 0)$$

from which

$$(4.1) \qquad \qquad i(\hat{\phi}(R_\alpha, \cdot), u_{R_\alpha}, 0) = i(\hat{\phi}(R_\beta, \cdot), u_{R_\beta}, 0)$$

But

$$\hat{\phi}_u(R, u_R)v = v + R^2 \, G \, f'(u_R)v$$

and

$$\frac{d}{dR} R^2 \, G \, f'(u_R) \Big]_{R=\overline{R}} \neq 0 \qquad \qquad \text{(by (3.5))}$$

So an eigenvalue of $\hat{\phi}_u(R, u_R)$ crosses 0 when R crosses $\overline{R}$ and this eigenvalue is simple. Then the index $i(\hat{\phi}(R, \cdot), u_R, 0)$ must have opposite sign on opposite sides of $\overline{R}$ in $(\overline{R} - \varepsilon_0, \overline{R} + \varepsilon_0)$ in contradiction to (4.1). ∎

REFERENCES

[1] Crandall, M. G. - Rabinowitz, P. H., Bifurcation from simple eigenvalues. _J. Functional Analysis_, 8 (1971), pp. 321-340.

[2] Crandall, M. G. - Rabinowitz, P. H., Bifurcation, perturbation of simple eigenvalues and linearized stability, _Arch. Rat. Mech. Analysis_, 52 (1973), pp. 161-180.

[3] Courant, R. - Hilbert, D., _Methods of Mathematical Physics_, New York : Interscience, 1953.

[4] Dancer, E. N., On non-radially symmetric bifurcation, _J. London Math. Soc._ - (2) 20. (1979), pp. 287-292.

[5] Gidas, B. - Ni, W. M. - Nirenberg, L., Symmetry and related properties via the maximum principle, _Comm. Math. Phys._ 68 (1979), pp. 209-243.

[6] Prodi, G., _Biforcazione e simmetria_, Appunti Corso S.M.I., Cortona 1979.

[7] Smoller, J. A. - Wasserman, A. G., Existence, uniqueness and nondegeneracy of positive solutions of semilinear elliptic equations. To appear.

[8] Smoller, J. A. - Wasserman, A. G., Symmetry-breaking for positive solutions of semilinear elliptic equations. To appear.

[9] Rabinowitz, P. H., Some global result for nonlinear eigenvalue problems, _J. Functional Analysis_, 7 (1971), pp. 487-513.

[10] Rabinowitz, P. H., _Théorie du degré topologique et applications à des problèmes aux limites non lineaires_ - Notes, Lab. Analyse numerique - Un. Paris VI, 1975 (n. 75010).

[11] Vanderbauwhede, A., _Local bifurcation and symmetry_, Research Notes in Mathematics, 75, Pitman Advanced Publishing Program.

GC/jvs

-22-

REPORT DOCUMENTATION PAGE	READ INSTRUCTIONS BEFORE COMPLETING FORM

1. REPORT NUMBER #2759	2. GOVT ACCESSION NO. AD-A149093	3. RECIPIENT'S CATALOG NUMBER

4. TITLE (and Subtitle) Symmetry Breaking for a Class of Semi-Linear Elliptic Problems	5. TYPE OF REPORT & PERIOD COVERED Summary Report - no specific reporting period
	6. PERFORMING ORG. REPORT NUMBER

7. AUTHOR(s) Giovanna Cerami	8. CONTRACT OR GRANT NUMBER(s) DAAG29-80-C-0041

9. PERFORMING ORGANIZATION NAME AND ADDRESS Mathematics Research Center, University of 610 Walnut Street Wisconsin Madison, Wisconsin 53706	10. PROGRAM ELEMENT, PROJECT, TASK AREA & WORK UNIT NUMBERS Work Unit Number 1 - Applied Analysis

11. CONTROLLING OFFICE NAME AND ADDRESS U. S. Army Research Office P.O. Box 12211 Research Triangle Park, North Carolina 27709	12. REPORT DATE October 1984
	13. NUMBER OF PAGES 22

14. MONITORING AGENCY NAME & ADDRESS(if different from Controlling Office)	15. SECURITY CLASS. (of this report) UNCLASSIFIED
	15a. DECLASSIFICATION/DOWNGRADING SCHEDULE

16. DISTRIBUTION STATEMENT (of this Report)

Approved for public release; distribution unlimited.

17. DISTRIBUTION STATEMENT (of the abstract entered in Block 20, if different from Report)

18. SUPPLEMENTARY NOTES

19. KEY WORDS (Continue on reverse side if necessary and identify by block number)

Semilinear elliptic boundary value problem, Radial and nonradial solutions, Bifurcation, Symmetry breaking

20. ABSTRACT (Continue on reverse side if necessary and identify by block number)

Consider the nonlinear elliptic problem

(*)
$$\begin{cases} -\Delta u = f(u) & |x| < R \\ u = 0 & |x| = R \end{cases}$$

Suppose this problem has a family of positive radial solutions parametrized by R, i.e., $u_R(|x|)$. In the paper we study the possibility of the existence of nonradial solutions of (*) bifurcating from the radial solutions family.

DD FORM 1473 EDITION OF 1 NOV 65 IS OBSOLETE   (continued)

ABSTRACT (continued)

Answering a question posed by Smoller and Wasserman, we show this happe
if f satisfies suitable assumptions. Therefore, we investigate the global
structure of the nonradial solution set.

END

FILMED

2-85

DTIC