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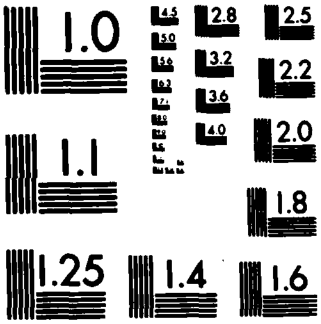

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NEUMANN TYPE BOUNDARY CONDITIONS FOR HAMILTON-JACOBI EQUATIONS

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ABSTRACT

In this paper, we present a notion of viscosity solutions of Hamilton-Jacobi equations for Neumann type boundary conditions (or more generally oblique derivative). In particular we prove the existence, uniqueness, stability of such solutions and we show that the vanishing viscosity method yields such solutions. Next, we check that value functions of control problems or differential games problems for reflected dynamical processes are solutions in that sense of the associated Bellman or Isaacs equations. Finally, we consider the ergodic problems.

AMS (MOS) Subject Classifications: 35F30, 49C99

Key Words: Hamilton-Jacobi equations, viscosity solutions, Neumann conditions, oblique derivative, vanishing viscosity method, optimal control, differential games, dynamic programming, reflected processes, ergodic problems.

Work Unit Number 1 - Applied Analysis

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NEUMANN TYPE BOUNDARY CONDITIONS FOR HAMILTON-JACOBI EQUATIONS

P. L. Lions[1]

Introduction:

In this paper, we consider the classical first order Hamilton-Jacobi equations

$$H(x,u(x),Du(x)) = 0 \quad \text{in} \quad \Omega \tag{1}$$

where u is a scalar function on Ω bounded smooth open set of $\mathbb{R}^N$, where Du denotes the gradient of u and H - the Hamiltonian - is a given continuous function on $\bar{\Omega} \times \mathbb{R} \times \mathbb{R}^N$.

We want to study how is is possible to define for solutions of (1) Neumann type boundary conditions that is

$$\frac{\partial u}{\partial n} = 0 \quad \text{on} \quad \partial\Omega \tag{2}$$

where n is the unit outward normal to $\partial\Omega$. However, as it is remarked in P. L. Lions [25], A. Sayah [35], such a boundary condition is not always possible and has to be relaxed somehow.

Recently, M. G. Crandall and the author [8], [9] introduced a general notion of solutions of (1) (requiring only $u \in C(\Omega)$) and proved various properties of these solutions - called <u>viscosity solutions</u> - including stability and uniqueness (provided boundary conditions of Dirichlet type are imposed). This led to a complete treatment of (1) with, possibly, Dirichlet boundary conditions and we refer to M. G. Crandall, L. C. Evans and P. L. Lions [7]; P. L. Lions [26]; P. E. Souganidis [37]; G. Barles [3]; H. Ishii [22], [23]; M. G. Crandall and P. L. Lions [10], [11], [12], [13](*)...

Our goal here is to adapt the notion of viscosity solutions of (1) in order to take into account boundary conditions of the form (2). Roughly speaking, we will present some

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(*) The reader should be aware that this list is by no means complete!

weak formulation (in "viscosity style") of an equation combining (1) and (2) on $\partial\Omega$ and this will be interpreted as the relaxed form of (2). The precise definition is given in section I where we also motivate and explain this definition in the light of the so-called vanishing viscosity method which here consists of finding u_ε solution of the equation (3) below and letting ε go to 0_+

$$-\varepsilon\Delta u_\varepsilon + H_\varepsilon(x,u_\varepsilon,Du_\varepsilon) = 0 \text{ in } \Omega, \frac{\partial u_\varepsilon}{\partial n} = 0 \text{ on } \partial\Omega \tag{3}$$

where $H_\varepsilon \to H$ as $\varepsilon \to 0_+$ (one can take $H_\varepsilon = H$ as well).

In section II, we give some properties of these viscosity solutions of (1) - (2) including stability, and adaptations to Cauchy problems like

$$\begin{cases} \frac{\partial u}{\partial t} + H(x,t,u,Du) = 0 \text{ in } \Omega\times]0,T[\\ \frac{\partial u}{\partial n} = 0 \text{ on } \partial\Omega\times]0,T[,\ u(x,0) = u_0(x) \text{ in } \Omega \end{cases} \tag{4}$$

Sections III and IV are devoted to uniqueness, comparison and existence results which will be of a comparable level of generality to the case of viscosity solutions of (1) with $\Omega = \mathbb{R}^N$ (no boundary conditions).

In section V, we adapt the preceding results to more general boundary conditions

$$\frac{\partial u}{\partial \gamma} = 0 \text{ on } \partial\Omega \tag{5}$$

where γ is a smooth vector field on $\partial\Omega$ pointing outward i.e.

$$\exists \nu > 0, \forall x \in \partial\Omega, (n(x),\gamma(x)) \geq \nu \tag{6}$$

As it was remarked in P. L. Lions [25] for exit problems in optimal deterministic control theory, the dynamic programming arguments easily yield the fact that value functions are viscosity solutions of the related Bellman (or Hamilton-Jacobi-Bellman) equations - see also P. L. Lions [27], [28]. This remark was also applied to differential games by P. E. Souganidis [38]; L. C. Evans and P. E. Souganidis [18]; N. E. Barron, L. C. Evans and R. Jensen [4]. We want to show in section VI that the value function of control problems (or differential games problems) for solutions of ordinary differential equations with reflection at the boundary are indeed the viscosity solutions of (1) - (2) (for the Hamiltonian H occurring in Bellman or Isaacs equations).

Finally, section VII is devoted to the study of the so-called ergodic problems: here, we study the limit as ε goes to 0 of, say, $(\varepsilon u_\varepsilon, u_\varepsilon - u_\varepsilon(x_0))$ where x_0 is any point in Ω and u_ε is the viscosity solution of

$$H(x,Du_\varepsilon) + \varepsilon u_\varepsilon = 0 \quad \text{in} \quad \Omega, \ \frac{\partial u_\varepsilon}{\partial n} = 0 \quad \text{on} \quad \partial\Omega \ . \tag{7}$$

We would like to conclude this introduction by explaining our motivation for studying (1) - (2): The first one concerns optimal control theory where state constraints are imposed on the system. Then reflection at the boundary of the domain defining the constraints is one possible way to "realize the constraints" and in many applications this is actually done (specially in optimal stochastic control problems which correspond to (3) and ε going to 0 corresponds to the intensity of the noise going to 0). But also from the PDE view point it is quite natural to try to analyze what happens when ε goes to 0 in (3). And this is very much related to the question of large deviations of reflecting diffusion processes (see Anderson and Orey [1] for some results on this problem and L. C. Evans and H. Ishii [17], W. H. Fleming and P. E. Souganidis [19] for relations between the vanishing viscosity method, large deviations and viscosity solutions).

We want also to emphasize that solutions of problems like (1) - (2) lead to solutions of hyperbolic systems of conservation laws and that boundary conditions like (2), (5) correspond then to some Dirichlet type condition. Indeed if u solves (formally)

$$\frac{\partial u}{\partial t} + H(x,t,Du) = 0 \quad \text{in} \quad \Omega\times]0,T[, \ \frac{\partial u}{\partial \gamma} = 0 \quad \text{on} \quad \partial\Omega\times]0,T[$$

- where γ may even depend on t if we wish -, then $p = Du$ solves

$$\begin{cases} \dfrac{\partial p_i}{\partial t} + \dfrac{\partial}{\partial x_i} \{H(x,t,p)\} = 0 \quad \text{in} \quad \Omega\times]0,T[\\ (p,\gamma) = 0 \quad \text{on} \quad \partial\Omega\times]0,T[\ ; \end{cases}$$

and since such boundary conditions for hyperbolic systems are natural, this motivates the study of Neumann boundary conditions for Hamilton-Jacobi equations. Let us mention at this stage that the case of one-dimensional scalar conservation laws is studied in C. Bardos, Le Roux and Nedelec [2].

Let us also mention that some particular cases of (1) - (2) are studied in Burch and Goldstein [6], P. L. Lions [25], A. Sayah [35].

Finally, we would like to point out that we restricted our attention to the case of bounded domains Ω but we could as well treat unbounded domains (as for example half-spaces) with similar ideas, combining (if necessary) the techniques below with those concerning unbounded viscosity solutions in $\mathbb{R}^N$ (see M. G. Crandall and P. L. Lions [10], [11], [12]; H. Ishii [22], [23]).

I. Definition and justification

Let Ω be a bounded smooth open set in $\mathbb{R}^N$ and let $H(x,t,p) \in C(\bar{\Omega} \times \mathbb{R} \times \mathbb{R}^N)$. We denote by n the vector field of unit outward normal vectors to $\partial\Omega$ and we are going to define "viscosity solution of (1) - (2)".

Definitions: Let $u \in C(\bar{\Omega})$. We say that

i) u is a vicosity subsolution of (1) - (2) if for all $\phi \in C^1(\bar{\Omega})$ the following property holds: let x_0 be a local maximum of $u - \phi$ in $\bar{\Omega}$ then we have

$$(8) \qquad \begin{cases} H(x_0,u(x_0),D\phi(x_0)) < 0 & \text{if } x_0 \in \Omega \\ H(x_0,u(x_0),D\phi(x_0)) < 0 & \text{if } x_0 \in \partial\Omega \text{ and } \frac{\partial\phi}{\partial n}(x_0) > 0 \ . \end{cases}$$

ii) u is a viscosity supersolution of (1) - (2) if for all $\phi \in C^1(\bar{\Omega})$ the following property holds: let x_0 be a local minimum point of $u - \phi$ in $\bar{\Omega}$ then we have

$$(9) \qquad \begin{cases} H(x_0,u(x_0),D\phi(x_0)) > 0 & \text{if } x_0 \in \Omega \\ H(x_0,u(x_0),D\phi(x_0)) > 0 & \text{if } x_0 \in \partial\Omega \text{ and } \frac{\partial\phi}{\partial n}(x_0) < 0 \ . \end{cases}$$

iii) u is a viscosity solution of (1) - (2) if u is both a viscosity sub and supersolution of (1) - (2).

■

Remarks: i) Of course, (8) - (9), whenever $x_0 \in \Omega$, are nothing but the usual viscosity formulation of (1).

ii) It is a straightforward exercise to check that one obtains equivalent formulations if we replace $\phi \in C^1$ by $\phi \in C^2$ or $\phi \in C^\infty$ and local maximum (resp. minimum) by local strict, global strict, or global maximum (resp. minimum).

iii) As we will see below, an equivalent formulation of (8) - (9) which allows $\frac{\partial\phi}{\partial n}(x_0) < 0$ (or > 0) is possible (in addition it is intrinsic in the sense that no test functions are necessary). ∎

<u>Theorem 1</u>: Let $u_\varepsilon \in C^2(\bar{\Omega})$ be a solution of (3), assume that H_ε converges uniformly to H on $\bar{\Omega} \times [-R,+R] \times \bar{B}_R$ $(\forall R < \infty)$ and that for some sequence ε_n going to 0 u_{ε_n} converges uniformly on $\bar{\Omega}$ to some u. Then, u is a viscosity solution of (1) - (2).

<u>Proof</u>: We already know from the usual properties of viscosity solutions of (1) (see [9]) that u is a viscosity solution inside Ω. Therefore, we only have to prove (8), (9) in the case when $x_0 \in \partial\Omega$. We are going to prove (8) with $x_0 \in \partial\Omega$, the proof of (9) being similar. Thus, let x_0 be a local strict maximum point of $u - \phi$ where $x_0 \in \partial\Omega$, $\phi \in C^2(\bar{\Omega})$, $\frac{\partial\phi}{\partial n}(x_0) > 0$. We then choose $\psi \in C^2(\bar{\Omega})$ satisfying

$$\frac{\partial\psi}{\partial n} = 1 \text{ on } \partial\Omega, \ \psi = 0 \text{ on } \partial\Omega, \ \psi > 0 \text{ in } \Omega \ .$$

Obviously, for any $\delta > 0$, $u - \phi - \delta\psi$ still has a local strict maximum point at x_0. Hence, for n large enough, $u_{\varepsilon_n} - \phi - \delta\psi$ has a local maximum point x_n in $\bar{\Omega}$ and $x_n \xrightarrow[n]{} x_0$. We claim that $x_n \in \Omega$. Indeed, if it were not the case, we would have

$$0 = \frac{\partial u_{\varepsilon_n}}{\partial n}(x_n) > \frac{\partial\phi}{\partial n}(x_n) + \delta$$

and the last quantity is strictly positive for n large. Next, since $x_n \in \Omega$, we deduce

$$H_{\varepsilon_n}(x_n, u_{\varepsilon_n}(x_n), D\phi(x_n) + \delta D\psi(x_n)) < \varepsilon_n(\Delta\phi(x_n) + \delta\Delta\psi(x_n))$$

where we used the relations $D\phi(x_n) + \delta D\psi(x_n) = Du_{\varepsilon_n}(x_n)$, $\Delta u_{\varepsilon_n}(x_n) < \Delta\phi(x_n) + \delta\Delta\psi(x_n)$. Letting n go to ∞, and then letting δ go to 0_+, we conclude. ∎

We now present some equivalent formulations of (8) - (9). To this end, we consider (following [7]) the subdifferential and the superdifferential of $v \in C(\bar{\Omega})$ at $x \in \bar{\Omega}$ given respectively by

$$D^-v(x) = \{\xi \in \mathbb{R}^N, \ \liminf_{y\to x\ y\in\Omega} \{v(y)-v(x) - (\xi,y-x)\}|y-x|^{-1} \geq 0\}$$

$$D^+v(x) = \{\xi \in \mathbb{R}^N, \ \limsup_{y\to x\ y\in\Omega} \{v(y)-v(x) - (\xi,y-x)\}|y-x|^{-1} \leq 0\} \quad .$$

Observe that (as in [7]), if $v - \phi$ has a local maximum at $x_0 \in \bar\Omega$ where ϕ is differentiable then $D\phi(x_0) \in D^+v(x_0)$ and that if $\xi \in D^+v(x_0)$, there exists $\phi \in C^1(\bar\Omega)$ such that $v - \phi$ has a global strict maximum at x_0 and $D\phi(x_0) = \xi$.

We have the

Theorem 2. Let $u \in C(\bar\Omega)$. Then, u is a viscosity subsolution (resp. supersolution) of (1) - (2) if and only if

$$(8')\qquad \begin{cases} \forall x \in \Omega,\ \forall\xi \in D^+u(x),\ H(x,u(x),\xi) \leq 0 \\ \forall x \in \partial\Omega,\ \forall\xi \in D^+u(x),\ \inf\limits_{0\leq\theta\leq 1} H(x,u(x),\xi+\theta(\xi,n)^-n) \leq 0 \end{cases}$$

(resp.

$$(9')\qquad \begin{cases} \forall x \in \Omega,\ \forall\xi \in D^-u(x),\ H(x,u(x),\xi) \geq 0 \\ \forall x \in \partial\Omega,\ \forall\xi \in D^-u(x),\ \sup\limits_{0\leq\theta\leq 1} H(x,u(x),\xi-\theta(\xi,n)^+n) \geq 0 \quad . \end{cases}$$

Equivalently, u is a viscosity subsolution (resp. supersolution) of (1) - (2) if and only if we have for all $\phi \in C^1(\bar\Omega)$

$$(8'')\qquad \begin{cases} \text{at any local maximum point } x_0 \text{ of } u - \phi, \text{ we have} \\ H(x_0,u(x_0),D\phi(x_0)) \leq 0 \quad \text{if } x_0 \in \Omega \\ \inf\limits_{0\leq\theta\leq 1} H(x_0,u(x_0),D\phi(x_0) + \theta(\frac{\partial\phi}{\partial n}(x_0))^-n) \leq 0 \quad \text{if } x_0 \in \partial\Omega \end{cases}$$

(resp.

$$
(9'')\qquad \begin{cases} \text{at any local minimum point } x_0 \text{ of } u-\phi, \text{ we have} \\ H(x_0,u(x_0),D\phi(x_0)) \geq 0 \text{ if } x_0 \in \Omega \\ \sup\limits_{0\leq\theta\leq 1} H(x_0,u(x_0),D\phi(x_0) - \theta(\frac{\partial\phi}{\partial n}(x_0))^{+}n) \geq 0 \text{ if } x_0 \in \partial\Omega) \ . \end{cases}
$$

■

Remarks:

i) As usual, we may replace $\phi \in C^1$ by $\phi \in C^2$, C^∞ and local by global strict, local strict, or global.

ii) In what follows, we will obtain use of the function $d(x) = \mathrm{dist}(x,\partial\Omega)$ which is smooth - say C^2 - near $\partial\Omega$ and which satisfies $\nabla d = -n$ on $\partial\Omega$ - see for instance J. Serrin [36], D. Gilbarg and N. S. Trudinger [20]. When we deal with points x of $\partial\Omega$ the fact that d is not smooth globally on $\bar{\Omega}$ will never create any difficulty since one can always smoothe d in the interior, while keeping it positive.

iii) Let us observe that if $\xi \in D^+u(x_0)$, $x_0 \in \partial\Omega$ then $\xi-\lambda n \in D^+u(x_0)$ for all $\lambda \geq 0$.

■

The proof of Theorem 2 relies on a general extension lemma of viscosity solutions of (1)

Lemma 3: Let $u \in C(\bar{\Omega})$ be a viscosity subsolution (resp. supersolution) of (1). Let $x_0 \in \partial\Omega$ and let $\xi \in D^+u(x_0)$ (resp. $D^-u(x_0)$). We then set

$$
(10)\qquad \lambda_0 = \sup\{\lambda \geq 0 \ / \ \xi+\lambda n(x_0) \in D^+u(x_0)\}
$$

and thus $0 \leq \lambda_0 \leq +\infty$

(resp.

$$
(11)\qquad \lambda_0 = \sup\{\lambda \geq 0, \ \xi-\lambda n(x_0) \in D^-u(x_0)\} \ .
$$

Then, if $\lambda_0 < \infty$, we have

$$
H(x_0,\ u(x_0),\ \xi + \lambda_0 n(x_0)) \leq 0
$$

(resp.

$$
H(x_0,\ u(x_0),\ \xi - \lambda_0 n(x_0)) \geq 0) \ .
$$

■

We first apply Lemma 3 to prove Theorem 2, and then prove Lemma 3. It is clear that (8") (resp. (9")) is equivalent to (8') (resp. (9')). Hence, we just have to prove that (8) implies (8'). Thus, let $x_0 \in \partial\Omega$ and let $\xi \in D^+u(x_0)$. If $(\xi,n) > 0$, we have nothing to prove, hence we assume $(\xi,n) < 0$. Two cases are then possible: first, if $\lambda_0 > (\xi,n)^-$, we see that $\xi + (\xi,n)^-\xi \in Du(x_0)$ and we conclude applying (8). Notice that in this case (8') holds with $\theta = 1$. If $\lambda_0 < (\xi,n)^-$, we apply Lemma 3 and we conclude since in this case

$$\xi + \lambda_0 n(x_0) = \xi + \theta(\xi,n)^- n$$

where $\theta \in [0,1[$ is given by $\theta = \lambda_0/(\xi,n)^-$.

We now prove Lemma 3: as $\lambda_0 < \infty$ and $D^+u(x_0)$ is closed, $\xi+\lambda_0 n(x_0) \in D^+u(x_0)$. Let $\psi \in C^1(\bar{\Omega})$ be such that

$$\psi(x_0) = u(x_0),\ D\psi(x_0) = \xi + \lambda_0 n(x_0) \quad , \quad \psi(x) > u(x) \quad \forall x \neq x_0 \quad .$$

Then, for $\delta > 0$ small we set

$$\Omega_\delta = B(x_0,\delta) \cap \Omega,\ \mu(\delta) = \inf[\psi(x)-u(x)/|x-x_0| = \delta,\ x \in \bar{\Omega}] \quad .$$

Choosing $\alpha(\alpha) = \mathrm{Min}(\delta,\ \mu(\delta)/2\delta)$, we claim that $u - \psi + \alpha(\delta)d$ has a local minimum inside Ω_δ. Indeed let x_δ be a maximum point of $u - \psi + \alpha(\delta)d$ over $\bar{\Omega}_\delta$. If $x_\delta \in \partial\Omega$, then $u(x_\delta)-\psi(x_\delta) > u(x_0)-\psi(x_0)$ and thus $x_\delta = x_0$. But this would yield that $D\psi(x_0) + \alpha(\delta)n(x_0) \in D^+u(x_0)$ contradicting the choice of λ_0. Therefore, $x_\delta \notin \partial\Omega$. If $|x_\delta-x_0| = \delta$, this would imply

$$u(x_0) - \psi(x_0) < u(x_\delta) - \psi(x_\delta) + \alpha(\delta)d(x_\delta) < -\mu(\delta) + \delta\alpha(\delta) < 0$$

again a contradiction. And we have proved that $x_\delta \in \Omega_\delta$. Since u is a viscosity subsolution of (1) we deduce

$$H(x_\delta,u(x_\delta),D\psi(x_\delta) - \alpha(\delta)\nabla d(x_\delta)) < 0$$

and we conclude letting δ go to 0_+.

■

II. Properties and extensions.

First of all, we would like to mention that many of the properties of viscosity solutions proved in M. G. Crandall and P. L. Lions [9]; M. G. Crandall, L. C. Evans and P. L. Lions [7]; P. L. Lions [25] have their counterparts in our setting. We will only make two remarks: the first one concerns differentiability points of a viscosity subsolution lying on $\partial\Omega$. More precisely assume $u \in C(\bar{\Omega})$ is a viscosity subsolution of (1) - (2) and that u is differentiable at $x_0 \in \partial\Omega$. We then observe that

$$D^+u(x_0) = \{Du(x_0) - \lambda n(x_0) \,/\, \lambda > 0\} \quad .$$

Therefore if we denote by $D_T u(x_0) = Du(x_0) - \frac{\partial u}{\partial n}(x_0)n(x_0)$, we deduce from condition (8)

$$(12) \quad \begin{cases} \text{if } \frac{\partial u}{\partial n}(x_0) > 0, \; H(x_0,u(x_0),\, Du(x_0)) \leq 0 \\ \text{if } \frac{\partial u}{\partial n}(x_0) < 0, \; \inf\{H(x_0,u(x_0),D_T u(x_0) + \lambda n(x_0)/\lambda \in [\frac{\partial u}{\partial n}(x_0),0]\} \leq 0 \quad . \end{cases}$$

Clearly, if u is a viscosity subsolution of (1), $u \in C(\bar{\Omega})$, u is differentiable at each point x_0 of $\partial\Omega$ and if (12) holds for all $x_0 \in \partial\Omega$, u is a viscosity subsolution of (1) - (2).

We now turn to a stability result

<u>Proposition 4</u>: Let $(u_k)_k \in C(\bar{\Omega})$ be viscosity subsolutions (resp. supersolutions) of

$$(13) \qquad H_k(x,u_k,Du_k) = 0 \text{ in } \Omega, \; \frac{\partial u_k}{\partial n} = 0 \text{ on } \partial\Omega \quad .$$

Assume that u_k converges unformly on $\bar{\Omega}$ to u and that H_k converges uniformly on $\bar{\Omega} \times [-R,+R] \times \bar{B}_R (\forall R < \infty)$ to H. Then u is a viscosity subsolution (resp. supersolution) of (1) - (2).

■

<u>Proof</u>: It is basically the same as in [9], [7]. We just have to prove (8") when $x_0 \in \partial\Omega$ is a local strict maximum point in $\bar{\Omega}$ of $u - \phi$ with $\phi \in C^1(\bar{\Omega})$. For k large enough, $u_k - \phi$ has a local maximum point x_k in $\bar{\Omega}$ and $x_k \xrightarrow[k]{} x_0$. Therefore, we find

$$(14) \quad \begin{cases} H_k(x_k,u_k(x_k),D\phi(x_k)) \leq 0 \quad \text{if } x_k \in \Omega \\ H_k(x_k,u_k(x_k),D\phi(x_k) + \theta_k(\frac{\partial\phi}{\partial n}(x_k))^- n(x_k)) \leq 0 \quad \text{if } x_k \in \partial\Omega \end{cases}$$

for some $\theta_k \in [0,1]$. Without loss of generality we may assume that $\theta_k \xrightarrow[k]{} \theta \in [0,1]$ and we find (8") passing to the limit in (14).

■

We next want to explain how one adapts to the definitions to cover situations like problem (4): let $H(x,t,s,p) \in C(\bar{\Omega} \times [0,T] \times \mathbb{R} \times \mathbb{R}^N)$, we wish to define viscosity solutions of

$$(15) \qquad \frac{\partial u}{\partial t} + H(x,t,u,Du) = 0 \quad \text{in} \quad \Omega \times]0,T[\ , \ \frac{\partial u}{\partial n} = 0 \quad \text{on} \quad \partial\Omega \times]0,T[\ .$$

Before giving the easy analogues of the preceding definitions, let us point out that (15) is a very special case of (1) coupled with a Neumann type boundary condition on some part only of the boundary, while on other parts Dirichlet boundary conditions are assumed (here initial conditions). Let us mention that we could treat in much greater generality these mixed problems but we will skip here these straightforward extensions.

<u>Definitions</u>: Let $u \in C(\bar{\Omega} \times]0,T[)$. We will say that u is a

i) viscosity subsolution of (15) if for all $\phi \in C^1(\bar{\Omega} \times]0,T[)$ the following property holds: at any local maximum point (x_0,t_0) of $u - \phi$ on $\bar{\Omega} \times]0,T[$ then we have

$$(16) \begin{cases} \frac{\partial \phi}{\partial t}(x_0,t_0) + H(x_0,t_0,u(x_0,t_0),D\phi(x_0,t_0)) \le 0 & \text{if} \quad x_0 \in \Omega \\ \frac{\partial \phi}{\partial t}(x_0,t_0) + H(x_0,t_0,u(x_0,t_0),D\phi(x_0,t_0) + \theta(\frac{\partial \phi}{\partial n}(x_0,t_0))^- n(x_0)) \le 0 & \text{if} \quad x_0 \in \partial\Omega \end{cases}$$

for some $\theta \in [0,1]$.

ii) viscosity supersolution of (15) if for all $\phi \in C^1(\bar{\Omega} \times [0,T])$ the following property holds: at any local minimum point (x_0,t_0) of $u - \phi$ on $\bar{\Omega} \times]0,T[$ then we have

$$(17) \begin{cases} \frac{\partial \phi}{\partial t}(x_0,t_0) + H(x_0,t_0,u(x_0,t_0),D\phi(x_0,t_0)) \ge 0 & \text{if} \quad x_0 \in \Omega \\ \frac{\partial \phi}{\partial t}(x_0,t_0) + H(x_0,t_0,u(x_0,t_0),D\phi(x_0,t_0) - \theta(\frac{\partial \phi}{\partial n}(x_0,t_0))^+ n(x_0)) \ge 0 & \text{if} \quad x_0 \in \partial\Omega \end{cases}$$

for some $\theta \in [0,1]$.

iii) viscosity solution of (15) if it is both a viscosity subsolution and supersolution of (15).

■

<u>Remark</u>: Exactly as before we may replace C^1 by C^2, C^∞, or $C^1(\bar{\Omega} \times [0,T])$; local by global, global strict or local strict.. . We could also use the analogues of (8) - (9). Finally, one can give a definition in terms of sub and super differentials only as in (8') - (9')... .

■

Exactly as in [9], [7], its is useful to extend (16), (17) on $\bar{\Omega} \times \{T\}$ as follows

<u>Proposition 5</u>: Let $u \in C(\bar{\Omega} \times]0,T])$ be a viscosity subsolution (resp. supersolution) of

(15). Then for any $\phi \in C^1(\overline{\Omega}\mathrm{x}]0,T])$, if (x,T) is a local maximum (resp. minimum) point of $u - \phi$ in $\Omega\mathrm{x}]0,T]$ then have

$$(18) \quad \begin{cases} \frac{\partial\phi}{\partial t}(x,T) + H(x,T,u(x,T),D\phi(x,T) > 0 & \text{if } x \in \Omega \\ \frac{\partial\phi}{\partial t}(x,T) + H(x,T,u(x,T),D\phi(x,T) + \theta(\frac{\partial\phi}{\partial n}(x,T))^- n(x)) < 0 & \text{if } x \in \partial\Omega \end{cases}$$

for some $\theta \in [0,1]$ (resp.

$$(19) \quad \begin{cases} \frac{\partial\phi}{\partial t}(x,T) + H(x,T,u(x,T),D\phi(x,T)) > 0 & \text{if } x \in \Omega \\ \frac{\partial\phi}{\partial t}(x,T) + H(x,T,u(x,T),D\phi(x,T) - \theta(\frac{\partial\phi}{\partial n}(x,T))^+ n(x)) > 0 & \text{if } x \in \Omega \end{cases}$$

for some $\theta \in [0,1]$). ■

<u>Proof:</u> Again, it is almost the same proof as in [9], [7] so we will just sketch it. Without loss of generality we may assume that $(x,T) \in \partial\Omega \times \{T\}$ is a local strict maximum point of $u - \phi$ on $\overline{\Omega}\mathrm{x}]0,T]$ where $\phi \in C^1(\overline{\Omega}\mathrm{x}]0,T])$. Then for ε small enough $u - \phi - \frac{\varepsilon}{T-t}$ has a local maximum point $(x_\varepsilon,t_\varepsilon)$ in $\overline{\Omega}\mathrm{x}]0,T[$ such that $x_\varepsilon \underset{\varepsilon}{\to} x$, $t_\varepsilon \underset{\varepsilon}{\to} T$. Using (16), we find

$$\frac{\varepsilon}{(T-t_\varepsilon)^2} + \frac{\partial\phi}{\partial t}(x_\varepsilon,t_\varepsilon) + H(x_\varepsilon,t_\varepsilon,u(x_\varepsilon,t_\varepsilon),D\phi(x_\varepsilon,t_\varepsilon)) < 0 \quad \text{if} \quad x_\varepsilon \in \Omega$$

$$\frac{\varepsilon}{(T-t_\varepsilon)^2} + \frac{\partial\phi}{\partial t}(x_\varepsilon,t_\varepsilon) + H(x_\varepsilon,t_\varepsilon,u(x_\varepsilon,t_\varepsilon),D\phi(x_\varepsilon,t_\varepsilon) + \theta_\varepsilon(\frac{\partial\phi}{\partial n}(x_\varepsilon,T))^- n(x_\varepsilon)) < 0 \quad \text{if} \quad x_\varepsilon \in \partial\Omega$$

for some $\theta_\varepsilon \in [0,1]$. and we conclude easily letting ε go to 0. ■

<u>Remark:</u> *Exactly as in section I one may prove* that if there exists $u_\varepsilon \in C^{2,1}(\overline{\Omega}\mathrm{x}]0,T[)$ solution of

$$\frac{\partial u_\varepsilon}{\partial t} - \varepsilon\Delta u_\varepsilon + H_\varepsilon(x,t,u_\varepsilon,Du_\varepsilon) = 0 \quad \text{in} \quad \Omega\mathrm{x}]0,T[, \quad \frac{\partial u_\varepsilon}{\partial n} = 0 \quad \text{on} \quad \partial\Omega\mathrm{x}]0,T[$$

where H_ε converges uniformly on compact subsets of $\overline{\Omega}\mathrm{x}]0,T[\times\mathbb{R}\times\mathbb{R}^N$ to H and if u_{ε_n} converges uniformly to u on compact subsets of $\overline{\Omega}\mathrm{x}]0,T[$ for some sequence $\varepsilon_n \underset{n}{\to} 0$ then u is a viscosity solution of (15). ■

III. Uniqueness results.

We begin with uniqueness results concerning viscosity solutions of (1) - (2). We will use the following assumptions

(20) $$H(x,t,\lambda(x-y)) - h(y,t,\lambda(x-y)) \geq -\omega_R(\lambda|x-y|^2 + |x-y|)\forall x,y \in \bar{\Omega},$$

for $|t| \leq R$, $\lambda \geq 1$, and where $\omega_R(s) \to 0$ if $s \to 0_+$;

(21) $$\forall R < \infty,\ \gamma_R > 0,\ H(x,t,p) - h(x,s,p) \geq \gamma_R(t-s) \quad \text{if} \quad -R \leq s \leq t \leq R$$

for all $x \in \bar{\Omega}$, $p \in \mathbb{R}^N$;

(22) $$\sup\{|H(x,t,p) - H(x,t,q)|/x \in \partial\Omega,\ |t| \leq R,\ |p-q| \leq \varepsilon\} \to 0 \quad \text{as} \quad \varepsilon \to 0$$

for all $R < \infty$.

Then our main uniqueness and comparison result is the

Theorem 6: Let $H \in C(\bar{\Omega} \times [-R,+R] \times \bar{B}_R)(\forall R < \infty)$ satisfy (21). Let $u, v \in C(\bar{\Omega})$ be respectively viscosity subsolution of (1) - (2), viscosity supersolution of (1') - (2) where (1') is the equation given by

(1') $$H(x,v,Dv) + f(x) = 0 \quad \text{in} \quad \Omega$$

and $f \in C(\bar{\Omega})$. Then, if we assume either that (20) holds and Ω is convex, or that (20), (22) hold or that u (or v) $\in W^{1,\infty}(\Omega)$, we have

$$\max_{\bar{\Omega}} (u-v)^+ \leq \frac{1}{\gamma} \max_{\bar{\Omega}} f^+$$

where $\gamma = \gamma_{R_0}$ and $R_0 = \max(\|u\|_\infty, \|v\|_\infty)$. ■

Proof: Of course the proof follows the corresponding proofs in [7], [9] the main changes being at the boundary. Hence, we consider as in [7], [9]: $M > R_0$; $\beta \in C^\infty(\mathbb{R})$, $0 \leq \beta \leq 1$, $\beta(0) = 1$, $\beta(t) = 1 - \frac{t^2}{2}$ for t small, $\beta(t) < 1$ if $t \neq 0$, Supp $\beta \subset [-1,+1]$; $\beta_\varepsilon(p) = \beta(|p|/\varepsilon)$ for $p \in \mathbb{R}^N$, $\varepsilon > 0$; $w(x,y) = u(x) - v(y) + 3M\,\beta_\varepsilon(x-y)$ for $x,y \in \bar{\Omega}$. We may assume that $L = \max_{\bar{\Omega}} (u-v) > 0$ so that $\max_{\bar{\Omega}\times\bar{\Omega}} w \geq 3M + L > 3M$. Hence, if $(\bar{x},\bar{y})$ is a maximum point of $w(x,y)$ on $\bar{\Omega} \times \bar{\Omega}$ we deduce

$$\bar{x}-\bar{y} \in \text{Supp}\ \beta_\varepsilon \quad \text{and thus} \quad |\bar{x}-\bar{y}| \leq \varepsilon \ .$$

In fact, we have

$$3M\,\beta_\varepsilon(\bar{x}-\bar{y}) + u(\bar{x}) - v(\bar{x}) + \omega_v(\varepsilon) > \underset{\bar{\Omega}\times\bar{\Omega}}{\mathrm{Max}}\, w > 3M + L \ ;$$

where ω_v is a modulus of continuity of v, and thus we deduce easily from the property of β_ε that

(23) $$|\bar{x}-\bar{y}| < \varepsilon\delta(\varepsilon),\ \nabla\beta_\varepsilon(\bar{x}-\bar{y}) = -(\bar{x}-\bar{y})/\varepsilon^2 \ .$$

As in the usual uniqueness proofs, we observe freezing y at $\bar{y}$, resp. x at $\bar{x}$ that $\xi_\varepsilon = -3M\nabla\beta_\varepsilon(\bar{x}-\bar{y}) \in D^+u(\bar{x}) \cap D^-v(\bar{y})$ (even if $\bar{x}$ or $\bar{y} \in \partial\Omega$). Therefore applying the definitions and assumptions

(24) $$\begin{cases} H(\bar{x},u(\bar{x}),\xi_\varepsilon) < 0 & \text{if } \bar{x} \in \Omega \\ H(\bar{x},u(\bar{x}),\xi_\varepsilon + \theta(\xi_\varepsilon,n(\bar{x}))^- n(\bar{x})) < 0 & \text{if } \bar{x} \in \partial\Omega, \text{ for some } \theta \in [0,1] \end{cases}$$

(25) $$\begin{cases} H(\bar{y},v(\bar{y}),\xi_\varepsilon) > 0 & \text{if } \bar{y} \in \Omega \\ H(\bar{y},v(\bar{y}),\xi_\varepsilon - \theta(\xi_\varepsilon,n(\bar{y}))^+ n(\bar{y})) > 0 & \text{if } \bar{y} \in \partial\Omega, \text{ for some } \theta \in [0,1] \ . \end{cases}$$

Next, if Ω is convex, we observe that

$$(\xi_\varepsilon,n(\bar{x})) = 3M(\bar{x}-\bar{y},n(\bar{x}))\varepsilon^{-2} > 0 \quad \text{if } \bar{x} \in \partial\Omega,\ \bar{y} \in \bar{\Omega}$$

$$(\xi_\varepsilon,n(\bar{y})) = 3M(\bar{x}-\bar{y},n(\bar{y}))\varepsilon^{-2} < 0 \quad \text{if } \bar{x} \in \bar{\Omega},\ \bar{y} \in \partial\Omega \ .$$

Hence the cases when $\bar{x}$ or $\bar{y}$ belong to $\partial\Omega$ do not modify the usual proofs and we conclude.

On the other hand if Ω is arbitrary, then as it was observed in P. L. Lions [29], P. L. Lions and A. S. Sznitman [33] there exists $C_0 > 0$ such that for all $z_1,z_2 \in \bar{\Omega}$

(26) $$(z_1-z_2,n(z_1)) > -C_0|z_1-z_2|^2 \quad \text{if } z_1 \in \partial\Omega \ .$$

Using this remark we deduce from (23)

$$\varepsilon^{-2}(\xi_\varepsilon,n(\bar{x})) > -C_0\delta(\varepsilon)^2 \ \text{if } \bar{x} \in \partial\Omega,\ (\xi_\varepsilon,n(\bar{y}))\varepsilon^{-2} < C_0\delta(\varepsilon)^2 \ \text{if } \bar{y} \in \partial\Omega \ .$$

Therefore we see that the additional terms in the Hamiltonians due to the possibility of finding $\bar{x}$ or $\bar{y}$ on $\partial\Omega$ go to 0 and using (22), the usual uniqueness proofs still apply.

Finally, if u (for example) is Lipschitz on $\bar{\Omega}$, then we observe that

$$L + 3M\beta_\varepsilon(\bar{x}-\bar{y}) + C|\bar{x}-\bar{y}| > w(\bar{x},\bar{y}) = \operatorname*{Max}_{x,y} w > \operatorname*{Max}_{x\in\bar{\Omega}} w(x,x) > 3M + L$$

and this combined with the properties of β_ε yields

$$|\bar{x}-\bar{y}| < C\varepsilon^2 \quad .$$

Therefore $\xi_\varepsilon = 3M(\bar{x}-\bar{y})/\varepsilon^2$ remains bounded while $(\xi_\varepsilon,n(\bar{x}))^-$ (resp. $(\xi_\varepsilon,n(\bar{y}))^+$) go to 0 if $\bar{x} \in \partial\Omega$ (resp. $\bar{y} \in \partial\Omega$) as ε goes to 0 as we saw before. It is then easy to complete the proof. ∎

Remark: It is not surprising to see that in such problems the convexity of Ω simplifies matters. Since (1) - (2) is intimately connected with control problems of reflected processes (see section VI below) such simplications have to be expected in view of the works of A. Bensoussan and J. L. Lions [5]; H. Tanaba [39]; P.L. Lions, J. L. Menaldi and A. S. Sznitman [34]. ∎

We have proved the comparison result under three sets of assumptions: it is possible, however, to unite them in a single statement involving and rather technical condition. With the notations of Theorem 4, let ω be a modulus of continuity of u (or v, choose the best one!), denote by t_ε the maximum solution in $]0,\infty[$ of

(27) $$\omega(t_\varepsilon) = \frac{1}{2\varepsilon}\, t_\varepsilon^2 \quad ;$$

observe that $t_\varepsilon \varepsilon^{-1} \to 0$ as ε goes to 0. Then we will assume

(28) $$\begin{aligned}\limsup_{\varepsilon\downarrow 0}\{H(y,t,\tfrac{x-y}{\varepsilon} - \theta'(\tfrac{x-y}{\varepsilon},n(y))^+n(y)) - H(x,t,\tfrac{x-y}{\varepsilon} &+ \\ + \theta(\tfrac{x-y}{\varepsilon},n(x))^-n(x))/(x,\theta) \in \Omega\times(0) \text{ or } (x,\theta) &\in \partial\Omega\times[0,1]; \\ (y,\theta') \in \Omega\times(0) \text{ or } (y,\theta') \in \partial\Omega\times[0,1];\ |x-y| < t_\varepsilon,\ |t| < R\} &= 0\end{aligned}$$

for all $R < \infty$. The proof above gives then

Corollary 7: Let $H \in C(\bar{\Omega}\times[-R,+R]\times\bar{B}_R)(\forall R < \infty)$ satisfy (21). Let $u,v \in C(\bar{\Omega})$ be respectively viscosity subsolution of (1) - (2), viscosity supersolution of (1') - (2). Let $f \in C(\bar{\Omega})$ and set $R_0 = \max(\|u\|_\infty,\|v\|_\infty)$, $\gamma = \gamma_{R_0}$ and let ω be a modulus of continuity of u (or v). Then, if (28) holds, we have

$$\max_{\overline{\Omega}}(u-v)^+ \leq \frac{1}{\gamma}\max_{\overline{\Omega}} f^+ \quad . \tag{29}$$

■

Remarks: Of course (28) is awkward. On the other hand it holds if (20) holds (condition which was introduced by R. Jensen) and Ω is convex, or if (20), (22) hold, or if u is Lipschitz since in that case $|t_\varepsilon| \leq C\varepsilon$. In addition if $u \in C^{0,\alpha}$ for some $\alpha \in]0,1[$ then $|t_\varepsilon| \leq C\varepsilon^{1/(2-\alpha)}$: for example if $H(x,t,p) = \phi(x)|p|^m + \gamma t$ with $m > 1$, $\phi \in W^{1,\infty}(\Omega)$ then (22) holds only if $\phi \equiv 0$ on $\partial\Omega$ while if $u \in C^{0,\alpha}$, (28) holds if $\alpha > (m-1)/m$.

■

We will not state any results on Cauchy problems like (15): let us mention that if $u,v \in C(\overline{\Omega} \times [0,T])$ are respectively viscosity subsolution of (15), viscosity supersolution of

$$\frac{\partial v}{\partial t} + H(x,t,v,Dv) + f(x,t) = 0 \quad \text{in} \quad \Omega \times]0,T[, \quad \frac{\partial v}{\partial n} = 0 \quad \text{on} \quad \partial\Omega \times]0,T[\tag{30}$$

then provided the analogues of (20), (21) (with now $\gamma_R > -\infty$), (22) (or even (28), where the inequalities are uniform in $t \in [0,T)$, hold then the following inequality holds

$$\max_{\overline{\Omega}}(u-v)^+(t) \leq e^{\gamma t}\max_{\overline{\Omega}}(u-v)^+(0) + \int_0^t \max_{\overline{\Omega}} f^+(s)e^{\gamma s}ds \quad .$$

IV. Existence results.

For problem (1) - (2), the main existence result is the following:

Theorem 8: Let $H \in C(\overline{\Omega} \times [-R,+R] \times \overline{B}_R)$, assume there exist $\overline{u},\underline{u} \in C(\overline{\Omega})$ viscosity supersolution, resp. subsolution of (1) - (2) and assume that H satisfies (21) and either (20) and Ω is convex, either (20) and (22), or that

$$H(x,t,p) \to +\infty \quad \text{as} \quad |p| \to +\infty, \quad \text{unformly in} \quad x \in \overline{\Omega},\ t \quad \text{bounded} \ . \tag{31}$$

Then there exists a unique viscosity solution of (1) - (2).

■

Remarks: i) If in (21), γ_R is bounded away from 0 independently of R then one may choose $\overline{u} = c$, $\underline{u} = -c$ for some large constant c.

ii) The uniqueness part of the above result is contained in Theorem 4 since (31) yields that any viscosity subsolution (in Ω) of (1) belongs to $W^{1,\infty}(\Omega)$ (see [9], [25] for a proof of this fact).

■

Proof: To simplify the presentation, we will make the proof only in the case when $H(x,t,p) = H(x,p) + \lambda t$, with H satisfying (20) (or (20 - (22), or (31)...) and $\lambda > 0$.

Our first observation concerns a priori estimates on solutions u of (1) - (2). By comparison with $\bar{u}$ and $\underline{u}$ we obtain uniform bounds. Now, exactly as in H. Ishii [23] and M. G. Crandall and P. L. Lions [11], one may obtain an estimate of the modulus of continuity of u: indeed one checks easily that $v(x,y) = (u(x) - u(y))^+$ is a viscosity subsolution of

$$\begin{cases} (H(x,D_x v) - H(y,-D_y v)) \wedge 0 + \lambda v \leq 0 \quad \text{in} \quad \Omega \times \Omega \\ \dfrac{\partial v}{\partial n} = 0 \quad \text{on} \quad \partial(\Omega \times \Omega) \quad . \end{cases}$$

Then we claim that under the assumptions of Theorem 6, we can find for all $\varepsilon > 0$ constants $\bar{C} = \bar{C}(\varepsilon) > 0$, $\gamma = \gamma(\varepsilon) \in]0,1[$ such that

$$z_\varepsilon(x,y) = \varepsilon + \bar{C}|x-y|^\gamma$$

is a viscosity supersolution of (32), where $\gamma \in]0,1]$, $\bar{C}$ depend only on the moduli involved by (20), (22). Formally, one checks this claim by computing

$$(H(x,D_x z_\varepsilon) - H(y,-D_y z_\varepsilon))^- + \lambda z_\varepsilon \geq \lambda\varepsilon + \lambda\bar{C}|x-y|^\gamma +$$

$$- \omega(\bar{C}\gamma|x-y|^\gamma + |x-y|) \quad , \quad \forall\, x,y \in \Omega$$

and if for example $x \in \partial\Omega$, $y \in \Omega$

$$\inf_{\theta \in [0,1]} \{0 \wedge (H(x,D_x z_\varepsilon + \theta(\frac{\partial z_\varepsilon}{\partial n})^- n(x)) - H(y,-D_y z_\varepsilon))\} + \lambda z_\varepsilon \geq$$

$$\geq \lambda\varepsilon + \lambda\bar{C}|x-y|^\gamma - \omega(\bar{C}\gamma|x-y|^\gamma + |x-y|), \quad \text{if} \quad \Omega \quad \text{is convex}$$

$$\geq \lambda\varepsilon + \lambda\bar{C}|x-y|^\gamma - \omega(\bar{C}\gamma|x-y|^\gamma + |x-y|) - \mu(C_0\gamma|x-y|^\gamma)$$

where C_0 is given by (26) and μ is the modulus given by (22). The remaining cases $x \in \Omega$, $y \in \partial\Omega$ or $x,y \in \partial\Omega$ are estimated in a similar way. Then, one concludes easily as in M. G. Crandall and P. L.Lions [11]. In conclusion, if Ω is convex and (20) holds, or if (20) and (22) hold, we have obtained bounds and a modulus of continuity for any solution of (1) - (2) which depend only on the moduli in (20), (22).

Therefore, by easy approximation arguments, we may assume that $H(x,p)$ is smooth and that H is Lipschitz on $\bar{\Omega} \times \mathbb{R}^N$. If (31) holds, since one deduces from [9], [25] easy Lipschitz estimates, getting existence in that case is also enough to conclude (as usual for existence results in Hamilton-Jacobi equations). Then, the particular case is treated via the vanishing viscosity method

$$-\varepsilon \Delta u_\varepsilon + H(x, Du_\varepsilon) + \lambda u_\varepsilon = 0 \quad \text{in} \quad \Omega, \; u_\varepsilon \in C^2(\bar{\Omega}), \; \frac{\partial u_\varepsilon}{\partial n} = 0 \quad \text{on} \quad \partial\Omega \quad . \tag{32}$$

The existence of u_ε is insured by standard results on quasilinear equations (see for example [20]): recall indeed that H has bounded derivatives in (x,p) on $\bar{\Omega} \times \mathbb{R}^N$. Using maximum principle, one *obtains uniform bounds on* u_ε. To obtain $W^{1,\infty}(\Omega)$ bounds, we may use the methods of P. L. Lions [30], [31] based on Bernstein ideas: indeed, if $v \in C^2(\bar{\Omega})$ satisfies (2) then

$$\frac{\partial}{\partial n} |\nabla v|^2 \le C_1 |\nabla v|^2 \quad \text{on} \quad \partial\Omega \tag{33}$$

where C_1 depends only on Ω and $C_1 = 0$ if Ω is convex. Then, we consider a function $\phi \in C^2(\bar{\Omega})$ satisfying

$$\phi > 0 \quad \text{in} \quad \bar{\Omega}, \; \frac{\partial \phi}{\partial n} = -C_1 \quad \text{on} \quad \partial\Omega \tag{34}$$

(take for example $\phi = e^{C_1 d}$ *where* $d = \mathit{dist}(x, \partial\Omega)$ *nearby* $\partial\Omega$). We finally set $w = \phi |\nabla u_\varepsilon|^2$ and we compute

$$\begin{cases} -\varepsilon \Delta w + \frac{\partial H}{\partial p} \nabla w + 2\lambda w \le -2\varepsilon \phi |D^2 u_\varepsilon|^2 - 4\varepsilon \phi_i \, \partial_k u_\varepsilon \partial^2_{ki} u_\varepsilon + \\ \quad + 2Mw \qquad \text{in} \quad \Omega, \; \frac{\partial w}{\partial n} \le 0 \quad \text{on} \quad \partial\Omega \end{cases}$$

where M depends only on Ω, $\|\frac{\partial H}{\partial p}\|_\infty$ and C depends only on Ω and $\|\frac{\partial H}{\partial x}\|_\infty$. Applying Cauchy-Schwarz inequalities and using the maximum principle, we see that for $\lambda > \lambda_1$

$$w \le K \quad \text{on} \quad \bar{\Omega}$$

where K is independent of ε and λ_1 depends only on Ω, $\|\frac{\partial H}{\partial p}\|_\infty$. Using Theorem 1, we deduce from these estimates the existence of a viscosity solution u of

$$H(x,Du) + (\lambda+\lambda_1)u = \lambda_1 f \quad \text{in} \quad \Omega, \ \frac{\partial u}{\partial n} = 0 \quad \text{on} \quad \partial\Omega$$

where $f \in W^{1,\infty}(\Omega)$. But then Theorem 6 yields that if u_1, u_2 are the solutions corresponding to f_1, f_2 we have

$$\operatorname*{Max}_{\overline{\Omega}} |u_1 - u_2| < \frac{\lambda_1}{\lambda+\lambda_1} \operatorname*{Max}_{\overline{\Omega}} |f_1 - f_2| \quad .$$

Therefore, by an easy application of the usual iteration method, we finally obtain the existence of a solution u of (1) - (2).

■

We now turn to some regularity results:

<u>Corollary 9</u>: Let $H \in C(\overline{\Omega} \times [-R,+R] \times \overline{B}_R)(\forall R < \infty)$ satisfy (21), let $u \in C(\overline{\Omega})$ be a viscosity solution of (1) - (2). Set $R_0 = \|u\|_\infty$, $\gamma = \gamma_{R_0}$. We finally assume that H satisfies

(35) $$|H(x,t,p) - H(y,t,p)| < C_1|x-y| \ |p| + C|x-y|, \ \forall x, \ y \in \overline{\Omega}, \ \forall p$$

for all $|t| < R_0$, for some constants C_1, $C > 0$; and that Ω is convex or that H satisfies

(36) $$|H(x,t,p) - H(x,t,q)| < C_2|p-q|, \ \forall x \in \partial\Omega, \ \forall p, \ q \in \mathbb{R}^N, \ \forall |t| < R_0 \quad .$$

In the first case we set $\theta = \gamma/C_1$ if $\gamma < C_1$, θ arbitrary in $]0,1[$ if $\gamma = C_1$, $\theta = 1$ if $\gamma > C_1$ while in the second case we set $\theta = \gamma/(C_1+C_2C_0)$ if $\gamma < C_1+C_2C_0$, θ arbitrary in $]0,1[$ if $\gamma = C_1+C_2C_0$, $\theta = 1$ if $\gamma > C_1+C_2C_0$ - where C_0 is given by (26).

<u>Proof</u>: One just checks that $C|x-y|^\theta$ is a viscosity supersolution of

$$\begin{cases} (H(x,u(y),D_x v) - H(y,u(y), - D_y v) \wedge 0 + \gamma v > 0 & \text{in} \quad \Omega \times \Omega \\ \dfrac{\partial v}{\partial n} = 0 & \text{on} \quad \partial(\Omega\times\Omega) \end{cases}$$

while $(u(x) - u(y))^+$ is a viscosity subsolution of the same problem. We then conclude by an application of Theorem 4.

■

We now conclude by stating the corresponding results for the Cauchy problem (15). Let $T \in]0,\infty[$, we will say that $H(x,t,s,p) \in C(\overline{\Omega} \times [0,T] \times \mathbb{R} \times \mathbb{R}^N)$ satisfies (20), (22) if (20), (22) are satisfied uniformly in $t \in [0,T]$. Finally, we will replace (21) by

$$\exists \gamma > -\infty,\ H(x,t,s_2,p) - H(x,t,s_1,p) > \gamma\,(s_2-s_1) \tag{21'}$$

for all $x \in \bar{\Omega}$, $t \in [0,T]$, $s_1 < s_2$, $p \in \mathbb{R}^N$ and we will use the assumptions

$$|H(x,t,s,p) - H(y,t,s,p)| < C_1^R|x-y|\ |p| + C^R,\ \forall x,y,t,p \tag{35'}$$

for $|s| < R$,

$$|H(x,t,s,p) - H(x,t,s,q)| < C_2^R|p-q|,\ \forall x \in \partial\Omega\ \forall p,q,t \tag{36'}$$

for $|s| < R$, where C_1^R, C^R, C_2^R are various positive constants.

Theorem 10: Let $u_0 \in C(\bar{\Omega})$, let $H \in C(\bar{\Omega} \times [0,T] \times R \times R^N)$ satisfy (21'). We assume in addition either that (20) holds and Ω is convex, or that (20), (22) hold, or that H satisfies

$$H \to +\infty \text{ as } |p| \to \infty \text{ uniformly in } x \in \bar{\Omega},\ t \in [0,T],\ s \text{ bounded} \tag{37}$$

$$H(x,t_1,s,p) - H(x,t_2,s,p) > -C_R(t_1-t_2)^+ \text{ for } |s| < R \tag{38}$$

for all $x \in \bar{\Omega}$, $p \in \mathbb{R}^N$, $t \in [0,T]$. Then there exists a unique solution u of (15) in $C(\bar{\Omega} \times [0,T])$ satisfying: $u(x,0) = u_0(x)$ in $\bar{\Omega}$. In addition, if we assume either (35') and Ω convex, or (35') and (36'), or (37) and (38), and if $u_0 \in W^{1,\infty}(\Omega)$ then $u \in W^{1,\infty}(\Omega \times]0,T[)$.

■

V. More general boundary conditions.

We consider now the case of the general boundary condition (5) where γ is smooth (say C^3) and γ satisfies (6). We first define viscosity solutions of (1) - (5).

Definition: $u \in C(\bar{\Omega})$ is said to be a viscosity subsolution (resp. supersolution) of (1) - (5) if we have for all $\phi \in C^1(\bar{\Omega})$

$$\begin{cases} \text{at each local maximum point } x_0 \text{ of } u - \phi \text{ in } \bar{\Omega}, \text{ we have} \\ H(x_0,u(x_0),D\phi(x_0)) < 0 \quad \text{if } x_0 \in \Omega \\ H(x_0,u(x_0),D\phi(x_0)) < 0 \quad \text{if } x_0 \in \partial\Omega \text{ and } \frac{\partial\phi}{\partial\gamma}(x_0) > 0 \end{cases} \tag{39}$$

(resp.

$$(40)\quad \begin{cases} \text{at each local minimum point } x_0 \text{ of } u - \phi \text{ in } \overline{\Omega}, \text{ we have} \\ H(x_0,u(x_0),D\phi(x_0)) > 0 \quad \text{if } x_0 \in \Omega \\ H(x_0,u(x_0),D\phi(x_0)) > 0 \quad \text{if } x_0 \in \partial\Omega \text{ and } \frac{\partial\phi}{\partial\gamma}(x_0) < 0) \quad . \end{cases}$$

Finally, u is a viscosity solution if it is a viscosity sub and supersolution.

■

Remarks: i) One obtains equivalent formulations replacing $D\phi$ by $\xi \in D^{+}u(x_0)$ (resp. $D^{-}u(x_0)$), or $\phi \in C^1$ by $\phi \in C^2$, $\phi \in C^\infty$, or local by global, global strict or local strict. Finally, one may consider only $\phi \in C^1(\overline{\Omega})$ such that $\frac{\partial\phi}{\partial\gamma} > 0$ on $\partial\Omega$ (resp. $\frac{\partial\phi}{\partial\gamma} < 0$ on $\partial\Omega$). Arguing as in Theorem 2, we also remark that u is a viscosity subsolution of (1) - (5) (resp. supersolution) if and only if we have

$$(39')\quad \begin{cases} \forall x \in \Omega,\ \forall \xi \in D^{+}u(x),\ H(x,u(x),\xi) < 0 \\ \forall x \in \partial\Omega,\ \forall \xi \in D^{+}u(x),\ \operatorname*{Inf}_{0<\theta<1} H(x,u(x),\ \xi + \theta(\xi,\gamma)^{-} \frac{n}{(n,\gamma)}) < 0 \end{cases}$$

(resp.

$$(40')\quad \begin{cases} \forall x \in \Omega,\ \forall \xi \in D^{-}u(x),\ H(x,u(x),\xi) > 0 \\ \forall x \in \partial\Omega,\ \forall \xi \in D^{-}u(x),\ \operatorname*{Sup}_{0<\theta<1} H(x,u(x),\xi - \theta(\xi,\gamma)^{+} \frac{n}{(n,\gamma)}) > 0) \quad . \end{cases}$$

(ii) Exactly as in sections I, II, one may prove stability results and the relations of the above definition with the vanishing viscosity method.

■

We now turn to existence and uniqueness results: first of all, following P. L. Lions [29], P. L. Lions and A. S. Sznitman (33), we introduce $a_{ij}(x) = a_{ji}(x)$ (smooth on $\mathbb{R}^N$, say $C_b^3(\mathbb{R}^N)$) satisfying

(41) $$\nu > 0,\ \forall x \in \mathbb{R}^N,\ (a_{ij}(x)) > \nu I_n$$

(42) $$\forall x \in \partial\Omega \quad a_{ij}(x)\gamma_j(x) = n_i(x) \quad \text{for} \quad 1 < i < N \ .$$

Clearly if we had $\gamma = n$, we would just take $a_{ij}(x) = \delta_{ij}$. Next, the matrices $a_{ij}(x)$ induce a metric on $\mathbb{R}^N$ defined by

(43) $$\begin{gathered} d(x,y) = \inf\{\textstyle\int_0^1 [a_{ij}(\xi(t))\dot{\xi}_i(t)\dot{\xi}_j(t)]^{1/2}\, dt/\xi \in C^1([0,1];\mathbb{R}^N) \\ \xi(0) = y,\ \xi(1) = x\} \end{gathered}$$

and $L(x,y) = d^2(x,y)$ satisfies

(43') $$\begin{gathered} L(x,y) = \inf\{\textstyle\int_0^1 a_{ij}(\xi(t))\dot{\xi}_i\dot{\xi}_j\, dt/\xi \in C^1([0,1];\mathbb{R}^N), \\ \xi(0) = y,\ \xi(1) = x\} \ . \end{gathered}$$

Then it is well-known that for $|x-y|$ small (say $|x-y| < \varepsilon_0$), L is C^1, there exists a unique minimizer in (43) or (43') ξ_0 and

(44) $$\begin{cases} \nabla_x\{\frac{1}{2} L(x,y)\} = a_{ij}(x)\dot{\xi}_0(1) \\ |\nabla_x\{\frac{1}{2} L(x,y)\} - a_{ij}(x)(x_j-y_j)| < C|x-y|^2 \\ \alpha,\beta > 0,\ \alpha|x-y| < L(x,y) < \beta|x-y| \ . \end{cases}$$

With these notations, we introduce the following assumptions:

(45) $$\begin{gathered} H(x,t,\lambda\nabla_x L(x,y)) - H(y,t,-\lambda\nabla_y L(x,y)) > \\ -\omega_R(\lambda|x-y|^2 + |x-y|) \quad \textit{for} \quad x,y \in \bar{\Omega},\ |x-y| \ \text{small},\ \lambda > 0,\ |t| < R \end{gathered}$$

where $\omega_R(s) \to 0$ if $s \to 0_+$;

(46) $$\exists\, \lambda > 0,\ H(x,t,p) - H(x,s,p) > \lambda(t-s),\ \forall x \in \bar{\Omega},\ \forall t > s,\ \forall p \in \mathbb{R}^N \ .$$

We then have the

<u>Theorem 11</u>: 1) <u>Uniqueness</u>. Assume that H satisfies (46). Let $u, v \in C(\bar{\Omega})$ be respectively viscosity subsolution, resp. supersolution, of (1) - (5), resp. (1') - (5). In addition assume that either (45) and (22) hold, or u(or v) $\in W^{1,\infty}(\Omega)$. In both cases,

(29) holds.

2) Existence. Assume that H satisfies (46) and that either (45) and (22) hold or (31) holds. Then there exists a unique solution $u \in C(\overline{\Omega})$ of (1) - (5).

■

Remarks: i) Analogous results holds for the Cauchy problem. One just makes similar uniformly in $t \in [0,T]$ (replacing $\lambda > 0$ in (46) by $\lambda > -\infty$). In addition, we may consider as well vector fields depending on t.

ii) We could treat in a similar way more general boundary conditions such as

$$\frac{\partial u}{\partial \gamma} + f(x,u) = 0 \quad \text{on} \quad \partial\Omega$$

where $f(x,t) \in C(\partial\Omega \times \mathbb{R})$ is nondecreasing with respect to t.

iii) Of course, when (31) holds, the solution u belongs to $W^{1,\infty}(\Omega)$. And if H satisfies (35), (36), one may prove that $u \in C^{0,\theta}(\overline{\Omega})$ where θ depends only γ, Ω, C_1, C_2, λ. In particular $\theta = 1$ if λ is large.

iv) Clearly if $\gamma = n$, choosing $a_{ij}(x) = \delta_{ij}$, we find $d(x,y) = |x-y|$, $L(x,y) = |x-y|^2$ and (45) reduces to (20).

■

Proof: The proof of this result is very much similar to the ones of Theorems 4 and 6. The uniqueness is proved using $\beta_\varepsilon(x,y) \in C^1(\overline{\Omega}\times\overline{\Omega})$ satisfying for $|x-y| \leq \frac{1}{2}\varepsilon$, $\beta_\varepsilon(x,y) = 1 - \frac{1}{2\varepsilon^2} L(x,y)$, $\beta_\varepsilon \equiv 0$ if $|x-y| \geq \varepsilon$, $0 \leq \beta_\varepsilon < 1$ if $x \neq y$. We then observe that in view of (44)

$$(\nabla_x L(x,y), \gamma(x)) \geq a_{ij}(x)\gamma_j(x)(x_i-y_i) - C|x-y|^2$$

$$\geq (n(x), x-y) - C|x-y|^2 \geq -(C+C_0)|x-y|^2$$

for $x \in \partial\Omega$, $y \in \overline{\Omega}$, $|x-y|$ small. This allows us to mimick the proof of Theorem 4.

For the existence, we also observe that replacing $|x-y|^\gamma$ by $L(x,y)^{\gamma/2}$ we obtain exactly as in the proof of Theorem 6 estimates on the modulus of continuity of a solution u of (1) - (5). Therefore, we just have to show uniform $W^{1,\infty}(\Omega)$ estimates on the solution u_ε of

$$-\varepsilon\Delta u_\varepsilon + H(x,u_\varepsilon,Du_\varepsilon) = 0 \text{ in } \Omega, \ \frac{\partial u_\varepsilon}{\partial\gamma} = 0 \text{ on } \partial\Omega$$

where H is smooth, Lipschitz in (x,t,p) and satisfies (46). Again, this is achieved as in the proof of Theorem 6 using the ideas of P L. Lions [30], [31]: if $v \in C^2(\Omega)$, $\frac{\partial v}{\partial\gamma} = 0$ then

$$\begin{aligned}\frac{\partial}{\partial\gamma}(g_{ij}\partial_i v\partial_j v) &\le C|\nabla v|^2 + 2a_{ij}\partial_\gamma(\partial_i v)\partial_j v \\ &\le C|\nabla v|^2 + 2a_{ij}\partial_i(\partial_\gamma v)\partial_j v \\ &\le C|\nabla v|^2 + 2g_{ij}n_i\partial_j v\,\frac{\partial v}{\partial n}\end{aligned}$$

where C denotes various constants independent of v. If we choose $g_{ij}(x) = (a_{ij}(x))^{-1}$, we obtain $g_{ij}n_i = \gamma_j$ and thus

$$\frac{\partial}{\partial\gamma}(g_{ij}\partial_i v\partial_j v) \le C|\nabla v|^2 \le C(g_{ij}\partial_i v\partial_j v) \quad .$$

This allows us to argue as in the proof of Theorem 6. ∎

In fact, it is possible to extend Theorem 11 (and Theorems 6, 8, Corollary 7) by considering the distances relative to $\widetilde{\Omega}$ i.e.

$$L'(x,y) = \inf\{\int_0^1 a_{ij}(\xi(t))\dot\xi_i\dot\xi_j\,dt \mid \xi \in C^1([0,1];\mathbb{R}^N), \xi(0) = y, \xi(1) = x \ ,$$

$$\xi(t) \in \overline{\Omega} \ \ \forall t \in [0,1]\} \quad .$$

Replacing L by L' in (45) enables us to get rid of (22); on the other hand checking (45) then become difficult.

VI. Applications to optimal control and differential games.

We begin with optimal control problems of reflected deterministic problems: let A be a metric space, we consider systems whose state is governed by the solution X_t of the following ordinary differential equation with reflection on the boundary

$$(47) \quad \begin{cases} X_t = x + \int_0^t b(X_s,\alpha_s)ds - \int_0^t \gamma(X_s)dA_s \quad \text{for } t \ge 0 \\ \text{with } X_t \in \overline{\Omega}, \ \forall t \ge 0, \ A_t \text{ is continuous, nondecreasing and} \\ \quad A_t = \int_0^t 1_{\partial\Omega}(X_s)dA_s \quad \text{for } t \ge 0 \end{cases}$$

here and below α_t is the control process i.e. any measurable function from $[0,\infty[$ into A. Heuristically, this dynamic problem corresponds to a usual controlled ordinary differential equation (with dynamics determined by $b(x,\alpha)$) while X_t lies in Ω, when X_t crosses $\partial\Omega$ X_t is "pushed back in $\bar{\Omega}$" along the direction $\gamma(X_t)$ with a "force" dA_t. This is one way of realizing state constraints (here $X_t \in \bar{\Omega}$ $\forall t \geq 0$) by specific boundary actions on the system: the above one is probably the simplest possible.

Provided convenient Lipschitz conditions on b are assumed (see below) problem (47) admits a unique solution (X_t, A_t) - see for instance [33]. We then introduce the cost function and the value function

$$J(x,\alpha_t) = \int_0^\infty f(X_t,\alpha_t)e^{-\lambda t}dt, \ \forall x \in \bar{\Omega} \tag{48}$$

$$u(x) = \inf_{\alpha_t} J(x,\alpha_t), \ \forall x \in \bar{\Omega} \tag{49}$$

where the infimum is taken over all possible control processes. We will assume that $\lambda > 0$ and that f, b satisfy

$$\left\{\begin{array}{l} |b(x,\alpha) - b(y,\alpha)| \leq C|x-y|, \ \forall x,y \in \bar{\Omega}, \ \forall \alpha \in A \ ; \\ |b(x,\alpha)| + |f(x,\alpha)| \leq C, \forall (x,\alpha) \in \bar{\Omega} \times A; \ b, f \text{ are continuous on } \bar{\Omega} \times A \ ; \\ |f(x,\alpha) - f(y,\alpha)| \leq C\, m(|x-y|), \ \forall x,y \in \bar{\Omega}, \ \forall \alpha \in A, \text{ and } m(t) \to 0 \text{ as } t \to 0_+ \ . \end{array}\right. \tag{50}$$

The above control problem is an infinite horizon problem; we could treat as well time-dependent finite horizon problems (which in some sense are simpler but involve heavier notations!).

The usual argument of dynamic programming yields that

$$u(x) = \inf_{\alpha_t}\left\{\int_0^t f(X_s,\alpha_s)e^{-\lambda s}ds + u(X_t)e^{-\lambda t}\right\}, \ \forall x \in \bar{\Omega} \tag{51}$$

where $t \geq 0$ (we could even choose t depending on the control process). In addition, $u \in C(\bar{\Omega})$. Both statements are proved exactly as in P. L. Lions [25] (see [29] for a proof); let us just mention that the continuity is easily derived from the following observation: let X_t^1, X_t^2 be two solutions of (47) corresponding to $x^1, x^2 \in \bar{\Omega}$ and let $a_{ij}(x)$ be the matrix introduced in the preceding section. We consider $\phi \in C^1(\bar{\Omega})$ such that $\frac{\partial\phi}{\partial\gamma} = 1$ on $\partial\Omega$, and we set

$$\psi_t = \exp M\{\phi(X_t^1) + \phi(X_t^2)\} \quad \text{for} \quad t > 0$$

where M is to be determined. Then for $t > 0$

$$d[\psi_t(a_{ij}(X_t^1) + a_{ij}(X_t^2))(X_t^1 - X_t^2)_i(X_t^1 - X_t^2)_j] <$$

$$C(M)|X_t^1 - X_t^2|^2 dt - M\nu\psi_t|X_t^1 - X_t^2|^2(dA_t^1 + dA_t^2) +$$

$$+ C\psi_t|X_t^1 - X_t^2|^2(dA_t^1 + dA_t^2) - 2\psi_t a_{ij}(X_t^1)\gamma_j(X_t^1)(X_t^1 - X_t^2)_i dA_t^1$$

$$-2\psi_t a_{ij}(X_t^2)\gamma_j(X_t^2)(X_t^2 - X_t^1)_i dA_t^2$$

where C does not depend on α_t, x^1, x^2, t, M. Next since dA_t^α charges only the set where $X_t^\alpha \in \partial\Omega$, we see that the last two terms may be bounded by

$$2C_0|X_t^1 - X_t^2|^2\phi_t(dA_t^1 + dA_t^2) \quad .$$

Therefore, choosing M large enough so that: $M\nu > C + 2C_0$, we deduce easily from Grönwall's lemma that

$$|X_t^1 - X_t^2|^2 < Ce^{\lambda_0 t}|x^1 - x^2| \quad , \quad \forall t > 0$$

where λ_0 depends only on the Lipschitz constant of b and on Ω, γ.

Once we have the continuity of u, the following result is to be expected

<u>Theorem 12</u>: Assume (50). Then the value function $u \in C(\bar{\Omega})$ *and* u is the unique viscosity solution of (1) - (5) where the Hamiltonian $H(x,t,p)$ is given by

(52) $$H(x,t,p) = \sup_{\alpha \in A} [-b(x,\alpha).p - f(x,\alpha)] + \lambda t \quad .$$

Furthermore, we have

(53) $$\begin{cases} \forall x \in \partial\Omega, \ \forall \xi \in D^+u(x) \ (\text{resp.} \ \ \forall \xi \in D^-u(x)) \\ \sup_{\alpha \in A}[-b(x,\alpha).\xi + (b(x,\alpha),n(x))^+(\gamma(x),n(x))^{-1}(\xi,\gamma(x)) - f(x,\alpha)] + \lambda u(x) < 0 \\ \qquad\qquad\qquad\qquad\qquad\qquad (\text{resp.} \ > 0) \ . \end{cases}$$

■

<u>Proof</u>: We already know from [25] that since u satisfies (51), u is a viscosity solution of (1) in Ω. Hence, we just have to check that u satisfies the viscosity properties on $\partial\Omega$. To do this, we will first check (53). We will only prove the case when $x \in \partial\Omega$, $\xi \in D^-u(x)$ (the other case being simpler). Let $\phi \in C^1(\bar{\Omega})$, $\phi(x) = u(x)$,

$\nabla\phi(x) = \xi$, $\phi(y) < u(y)$ for $y \in \bar{\Omega}$, $y \neq x$. Following the proof in [25], we deduce from (51)

$$\phi(x) \geq \inf_{\alpha_t}\{\int_0^t f(X_s,\alpha_s)e^{-\lambda s}ds + \phi(X_t)e^{-\lambda t}\},\ \forall t \geq 0 \quad .$$

And we deduce easily as in [25]

$$(54)\qquad \begin{cases} \sup_{\alpha_t}\{-(\xi, \frac{1}{t}\int^t{}_0\, b(x,\alpha_s)ds) + (\xi, \frac{1}{t}\int_0^t \gamma(X_s)dA_s) - \frac{1}{t}\int_0^t f(x,\alpha_s)ds\} \\ \qquad\qquad\qquad\qquad + \lambda u(x) \geq -\varepsilon(t) \to 0 \quad \text{as} \quad t \to 0_+ \quad . \end{cases}$$

In addition, from the results of P. L. Lions and A. S. Sznitman [33] we obtain

$$(55)\qquad 0 \leq dA_t \leq (b(X_t,\alpha_t),\ n(X_t))^+(n(X_t),\gamma(X_t))^{-1}dt \quad .$$

Now if $(\xi,\gamma(x)) \geq 0$, it is easy to deduce (53) combining (54) and (55). On the other hand if $(\xi,\gamma(x)) < 0$, we argue by contradiction and we assume that there exist $\delta > 0$, $\alpha \in A$ such that

$$(53')\quad -(b(x,\alpha),\xi) + (b(x,\alpha),n(x))^+(n(x),\gamma(x))^{-1}(\xi,\gamma(x)) - f(x,\alpha) + \lambda u(x) \leq -\delta < 0 \quad .$$

We may assume that $(b(x,\alpha),n(x)) > 0$ since if this is not true (54) and (55) easily yield a contradiction. Now, if we choose $\alpha_t \equiv x$, and if Y_t is the solution of

$$\begin{cases} \dot{Y}_t = b(Y_t,\alpha) - (b(Y_t,\alpha),n(Y_t))(n(Y_t),\gamma(Y_t))^{-1}\gamma(Y_t),\ t \geq 0 \\ Y_0 = x \end{cases}$$

then $Y_t \in \partial\Omega$ for all $t \geq 0$ and setting $B_t = \int_0^t (b(Y_s,\alpha),n(Y_s))(n(Y_s),\gamma(Y_s))^{-1}ds$ we see that B_t is increasing for t small and thus by the uniqueness of the solution of (47) we have for t small: $X_t = Y_t$, $A_t = B_t$. Then, (53') yields for t small

$$-(\xi,b(x,\alpha)) + (\xi, \frac{1}{t}\int_0^t \gamma(X_s)dA_s) - f(x,\alpha) + \lambda u(x) \leq -\frac{\delta}{s}$$

and this contradicts (54). Therefore, (53) is proved.

To prove (39), we consider $x \in \partial\Omega$, $\xi \in D^+u(x)$ and we introduce $\lambda_0(\xi) = \sup(\lambda \geq 0,\ \xi + \lambda n(x) \in D^+u(x))$: recall that if $\lambda_0(\xi) < \infty$ then $H(x,u(x),\ \xi + \lambda_0(\xi)n) \leq 0$. Therefore we may assume that $\lambda_0(\xi) > (\xi,\gamma(x))^-(n(x),\gamma(x))^{-1} = \lambda_1$. Of course, if $(\xi,\gamma(x)) \geq 0$ i.e. $\lambda_1 = 0$, then (53) immediately yields (39). Now if $(\xi,\gamma) < 0$, we observe that $\xi + \lambda_1 n(x) \in D^+u(x)$ and using the fact that u is a viscosity solution of (1) one deduces from the extension technique to the boundary of M. G. Crandall and P. L. Lions [9],

M. G. Crandall and R. Newcomb [14] (see also [27])

$$\sup[-(b(x,\alpha),\xi + \lambda_1 n(x)) - f(x,\alpha)/\alpha \in A,\ (b(x,\alpha),n(x)) < 0] + \lambda u(x) < 0 \quad ;$$

on the other hand, (53) yields

$$\sup_{\alpha\in A}[-b(x,\alpha).\xi - (b(x,\alpha),n(x))^{+}\lambda_1 - f(x,\alpha)] + \lambda u(x) < 0 \quad .$$

Combining these two inequalities we conclude.

We now turn to differential games: we will consider differential games for reflected processes and we will use Elliott-Kalton's formulation [15], [16], thus following the approach of L. C. Evans and P. E. Souganidis [18]. Let A, B two compact metric spaces, we will controls and strategies for both players by

$\mathcal{A} = \{\alpha_t$ measurable from $[0,\infty[$ to $A\}$

$\mathcal{B} = \{\beta_t$ measurable from $[0,\infty[$ to $B\}$

$\overline{A} = \{\alpha : \mathcal{B} \to \mathcal{A},\ \alpha$ nonanticipating$\}$

$\overline{B} = \{\beta : \mathcal{A} \to \mathcal{B},\ \beta$ nonanticipating$\}$

where α nonanticipating means: $\alpha[\beta^1_t] = \alpha[\beta^2_t]$ a.e. on $[0,T]$ if $\beta^1_t = \beta^2_t$ a.e. on $[0,T]$. For $\alpha_t \in \mathcal{A}$, $\beta \in \overline{B}$ (resp. $\beta_t \in \mathcal{B}$, $\alpha \in \overline{A}$) we define the state of the system by the solution of

$$\begin{cases} X_t = x + \int_0^t b(X_s,\alpha_s,\beta[\alpha_s])ds - \int_0^t \gamma(X_s)dK_s \ ; \\ X_t \in \overline{\Omega}, \forall t > 0;\ K_t \text{ is continuous, nondecreasing on } [0,\infty[\ ; \\ K_t = \int_0^t 1_{\partial\Omega}(X_s)dK_s \quad \text{for } t > 0 \end{cases}$$

(resp.

$$\begin{cases} Y_t = x + \int_0^t b(X_s,\alpha[\beta_s],\beta_s)ds - \int_0^t \gamma(X_s)dL_s \ ; \\ Y_t \in \overline{\Omega}, \forall t > 0;\ L_t \text{ is continuous, nondecreasing on } [0,\infty[\ ; \\ L_t = \int_0^t 1_{\partial\Omega}(Y_s)dL_s \quad \text{for } t > 0\ .\ )\ . \end{cases}$$

We next define the upper value and the lower value functions by

$$\overline{u}(x) = \sup_{\beta\in B} \inf_{\alpha_t\in A} \int_0^\infty f(X_t,\alpha_t,\beta[\alpha_t])e^{-\lambda t}dt \quad , \quad \forall x \in \overline{\Omega}$$

$$\underline{u}(x) = \inf_{\alpha\in A} \sup_{\beta_t\in B} \int_0^\infty f(Y_t,\alpha[\beta_t],\beta_t)e^{-\lambda t}dt \quad , \quad \forall x \in \overline{\Omega} \quad .$$

And we assume the analogue of (50) on $f(x,\alpha,\beta)$, $b(x,\alpha,\beta)$. Combining the methods introduced above and those of L. C. Evans and P. E. Souganidis [18] we obtain

Theorem 13: The value function $\overline{u}$, $\underline{u} \in C(\overline{\Omega})$ and are the unique solutions of (1) - (5) where H is given respectively by

$$H_1(x,t,p) = \sup_{\alpha\in A} \inf_{\beta\in B} [-b(x,\alpha,\beta).p - f(x,\alpha,\beta)] + \lambda t$$

$$H_2(x,t,p) = \inf_{\beta\in B} \sup_{\alpha\in A} [-b(x,\alpha,\beta).p - f(x,\alpha,\beta)] + \lambda t \quad .$$

Furthermore, $\overline{u}$, $\underline{u}$ satisfy the analogues of (53).

■

VII. Ergodic problems.

In this section we consider an Hamiltonian $H(x,p)$ satisfying

(31') $$H(x,p) \to \infty \quad \text{as} \quad |p| \to +\infty, \quad \text{uniformly in} \quad x \in \overline{\Omega}$$

(and $H \in C(\overline{\Omega} \times \mathbb{R}^N)$). Let γ be a vector field satisfying (6). We know from the preceding sections there exist unique viscosity solutions $u_\varepsilon \in W^{1,\infty}(\Omega)$, $u \in W^{1,\infty}(\Omega \times]0,T[)(\forall T < \infty)$ of

(56) $$H(x,Du_\varepsilon) + \varepsilon u_\varepsilon = 0 \quad \text{in} \quad \Omega, \quad \frac{\partial u_\varepsilon}{\partial\gamma} = 0 \quad \text{on} \quad \partial\Omega$$

(57) $$\begin{cases} \dfrac{\partial u}{\partial t} + H(x,Du) = 0 \quad \text{in} \quad \Omega \times]0,\infty[, \quad \dfrac{\partial u}{\partial\gamma} = 0 \quad \text{on} \quad \partial\Omega \times]0,\infty[\\ u(x,0) = u^0(x) \quad \text{in} \quad \overline{\Omega} \end{cases}$$

where $u^0 \in W^{1,\infty}(\Omega)$ (for example).

We want to explain in what follows the behaviour of $\varepsilon u_\varepsilon$, u_ε as ε goes to 0, or $u(\cdot,t)$, $\frac{\partial u}{\partial t}$ as t goes to $+\infty$.

Theorem 14: Under assumption (31'), $\varepsilon u_\varepsilon$ converges uniformly to the unique $u_0 \in \mathbb{R}$ such that there exists $v \in C(\overline{\Omega})$ viscosity solution of

$$H(x,Dv) + u_0 = 0 \ \text{in} \ \Omega, \ \frac{\partial v}{\partial \gamma} = 0 \ \text{on} \ \partial\Omega \ . \tag{58}$$

In addition, if $x_0 \in \overline{\Omega}$, $v_\varepsilon = u_\varepsilon - u_\varepsilon(x_0)$ is bounded in $W^{1,\infty}(\Omega)$ and any convergent subsequence of v_ε (in $C(\overline{\Omega})$) converges to a viscosity solution of (58) satisfying $v(x_0) = 0$. Furthermore $\frac{1}{t} u(x,t)$ converges uniformly on $\overline{\Omega}$ to u_0 as $t \to +\infty$.

■

Remarks: i) We do not know if v_ε converges.

ii) In general, there is no uniqueness of solutions of (58) even up to the addition of a constant. Indeed, consider $H(x,p) = (|p|-1)^+$. Then clearly $u_0 = 0$ and $v \equiv 0$ is a solution of (52). But so is any $C^1(\overline{\Omega})$ function v satisfying: $|Dv| < 1$, $\frac{\partial v}{\partial \gamma} = 0$ on $\partial\Omega$.

iii) Similar ergodic problems are considered in F. Gimbert [21], J. M. Lasry [24], P. L. Lions and B. Perthame [32] but they all involve elliptic equations or inequalities.

iv) If we keep the notations of the preceding sections, assuming that $H(x,p)$ is given by one of the formulas in Theorems 12 - 13, we obtain the following formulas for u_0

$$u_0 = \lim_{\varepsilon \to 0} \varepsilon \inf_{\alpha_t} \int_0^\infty f(X_t,\alpha_t)e^{-\varepsilon t}dt$$

(resp.

$$\overline{u}_0 = \lim_{\varepsilon \to 0} \varepsilon \sup_{\beta \in B} \inf_{\alpha_t \in A} \int_0^\infty f(X_t,\alpha_t,\beta[\alpha_t])e^{-\varepsilon t}dt \quad ,$$

$$\underline{u}_0 = \lim_{\varepsilon \to 0} \varepsilon \inf_{\alpha \in A} \sup_{\beta_t \in B} \int_0^\infty f(X_t,\alpha[\beta_t],\beta_t)e^{-\varepsilon t}dt) \quad ;$$

$$u_0 = \lim_{T \to \infty} \frac{1}{T} \inf_{\alpha_t} \int_0^T f(X_t,\alpha_t)dt$$

$$= \inf_{\alpha_t} \varliminf_{T \to \infty} \frac{1}{T} \int_0^T f(X_t,\alpha_t)dt$$

(resp.

$$\overline{u}_0 = \lim_{T \to \infty} \frac{1}{T} \sup_{\beta \in B} \inf_{\alpha_t \in A} \int_0^T f(X_t,\alpha_t,\beta[\alpha_t])dt \quad ;$$

$$\underline{u}_0 = \lim_{T \to \infty} \frac{1}{T} \inf_{\alpha \in A} \sup_{\beta_t \in B} \int_0^T f(X_t,\alpha[\beta_t],\beta_t)dt) \quad .$$

■

Proof: By a straightforward use of the comparison result (Theorem 6) we see that

$$|\varepsilon u_\varepsilon| < \|H(x,0)\|_\infty \quad \text{in} \quad \bar{\Omega} \ .$$

Then using (56) and (31'), one deduces

$$|Du_\varepsilon| < c \quad \text{in} \quad \Omega$$

(in viscosity sense) and thus v_ε is bounded in $W^{1,\infty}(\Omega)$. Now, if for some sequence $\varepsilon_n \xrightarrow[n]{} 0$, v_{ε_n}, $\varepsilon_n u_{\varepsilon_n}$ converge uniformly to v, u_0; clearly u_0 does not depend on x and by the stability results for viscosity solutions v is a viscosity solution of (58).

To prove the uniqueness of u_0: we argue as follows. Let $u_0, \bar{u}_0 \in \mathbb{R}$ be such that there exist $v, \bar{v}$ viscosity solutions of (58) corresponding to $u_0, \bar{u}_0$ respectively. Since $v, \bar{v}$ are clearly defined up to a constant we may always assume if $u_0 \neq \bar{u}_0$

$$u_0 < \bar{u}_0 \quad , \quad v < \bar{v} \quad \text{in} \quad \bar{\Omega} \ .$$

Thus, for ε small enough so that $u_0 - \varepsilon v < \bar{u}_0 - \varepsilon \bar{v}$ on $\bar{\Omega}$, we see that v is a viscosity supersolution of

$$H(x,Dv) + \bar{u}_0 + \varepsilon v = \varepsilon \bar{v} \quad \text{in} \quad \Omega, \ \frac{\partial v}{\partial \gamma} = 0 \quad \text{on} \quad \partial\Omega \ .$$

Since $\bar{v}$ is clearly a viscosity solution of this problem, Theorem 6 yields that $v > \bar{v}$ and this contradicts our choice. Thus u_0 is unique.

Finally, observing that $\frac{\partial u}{\partial t}$ is bounded on $\Omega \times]0,\infty[$, we see that $Du, \frac{\partial u}{\partial t}$ are bounded on $\Omega \times]0,\infty[$. Next, we consider $w(x,t) = u(x,t) - u_0 t$: w is a viscosity solution of

$$\frac{\partial w}{\partial t} + H(x,Dw) + u_0 = 0 \quad \text{in} \quad \Omega \times]0,\infty[, \ \frac{\partial w}{\partial \gamma} = 0 \quad \text{on} \quad \partial\Omega \times]0,\infty[, \ w|_{t=0} = u^0 \ .$$

On the other hand, if v is a solution of (58), $v \pm C$ are respectively viscosity super and subsolutions of this problem and they satisfy for large C: $v + C > u^0 > v - C$ in $\bar{\Omega}$.

Thus, by the comparison results, we deduce that $w \in W^{1,\infty}(\Omega \times]0,\infty[)$. In particular

$$\frac{1}{t} u(x,t) - u_0 = \frac{1}{t} w(x,t) \to 0 \quad \text{as} \quad t \to \infty \quad \text{in} \quad C(\bar{\Omega}) \ .$$

■

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In this paper, we present a notion of viscosity solutions of Hamilton-Jacobi equations for Neumann type boundary conditions (or more generally oblique derivative). In particular we prove the existence, uniqueness, stability of such solutions and we show that the vanishing viscosity method yields such solutions. Next, we check that value functions of control problems or differential games problems for reflected dynamical processes are solutions in that sense of the associated Bellman or Isaacs equations. Finally, we consider the ergodic problems.

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