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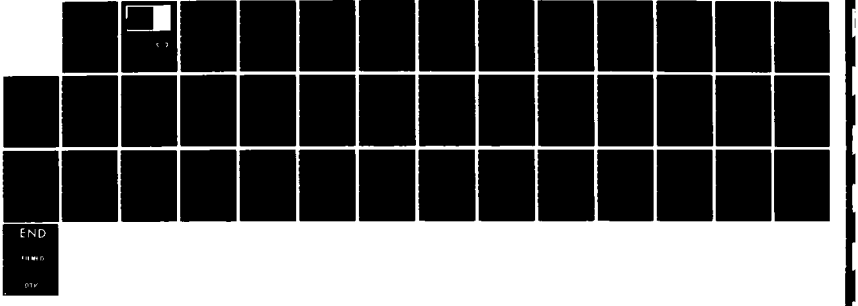

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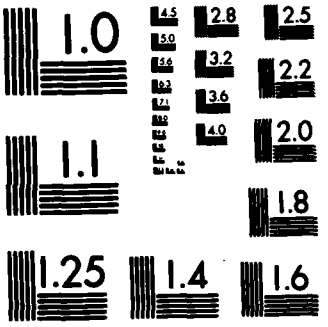

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PERTURBATION OF A MULTIPLE EIGENVALUE IN THE BÉNARD PROBLEM FOR TWO FLUID LAYERS

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ABSTRACT

In a recent paper, Y. Renardy and D. D. Joseph study the Bénard problem for two layers of different fluids lying on top of each other and bounded by walls. Their study shows that, in contrast to the Bénard problem for one fluid, the onset of instability can be oscillatory. The number of parameters involved in the problem is large, and there is yet no comprehensive picture of when the instability is oscillatory and when it is not. The study of limiting cases, accessible by perturbation methods, may be helpful in this respect. In this document, an analysis is given for the case when the properties of the two fluids are nearly equal and the fluids are allowed to slip at the boundaries.

AMS (MOS) Subject Classifications: 76E15, 76E20, 76T05, 76V05

Key Words: Overstability, Bénard instability, Two-component flow, Convective instability

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SIGNIFICANCE AND EXPLANATION

Flows involving two incompressible viscous fluids exhibit nonuniqueness in the sense that many interface positions are allowed when their densities are equal. Two-fluid flows also have quite different dynamical features from one-fluid flows. The one-fluid Bénard problem in which the fluid, lying between parallel horizontal plates, is heated from below has a static solution for which a linear stability analysis yields no complex eigenvalues. On the other hand, the two-fluid problem yields complex eigenvalues. In this paper, we use perturbation methods to examine the conditions under which such time-periodic instabilities occur. This may have application to the theory of convection in the Earth's mantle, which is sometimes based on the assumption that convection takes place in chemically uniform layers.

The responsibility for the wording and views expressed in this descriptive summary lies with MRC, and not with the authors of this report.

PERTURBATION OF A MULTIPLE EIGENVALUE IN THE BÉNARD PROBLEM FOR TWO FLUID LAYERS

Yuriko Renardy and Michael Renardy

1. INTRODUCTION

In the Bénard problem for one fluid, "exchange of stabilities" holds for a variety of boundary conditions[1], whether the fluid is bounded by walls or by stress-free surfaces, or by a wall below and by a gas above. Critical eigenvalues are not oscillatory in time. The consideration of additional effects can, however, introduce "overstability", that is, critical eigenvalues that are oscillatory in time. Overstability occurs, for example, if there are temperature-dependent surface tension gradients[2], if there is a temperature-dependent solute gradient[3], in mixtures of superfluids[4], or if two fluids lie in layers with different solute gradients[5,6]. On ther other hand, the Bénard problem with two layers of fluids with different thermal and mechanical properties, without surface tension gradients or solutes, has only recently been examined for the possibility of overstability[7]. Moreover, Busse has suggested that such a model may explain certain features of mantle convection, such as the size and aspect ratio of the convection cells [8] and the possibility of time-periodic [9] flows.

In a recent study, Renardy and Joseph [7] show that, in contrast to Bénard convection for one fluid, the linear stability problem for two fluids in layers is not self-adjoint, so that complex eigenvalues are possible. They then solve the eigenvalue problem numerically and find an oscillatory onset of instability in a situation where the two fluids are only slightly

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different. It would be interesting to know what determines whether the onset of Bénard convection is stationary or oscillatory. However, an attempt to answer this question by numerical calculations is not feasible because of the large number of parameters involved. We think that it is therefore helpful to first find out what happens in certain limiting situations accessible by perturbation methods.

In this paper, we look at two horizontal fluid layers, lying between parallel boundaries which are kept at constant but different temperatures. In order that the solution to the unperturbed problem be available in closed form,we require that slip boundary conditions apply, that is, the normal velocity and shear stress vanish. When the properties of the two fluids are the same, the eigenfunctions at criticality and the critical Rayleigh number can be determined exactly [1] and there is a double eigenvalue. When the properties of the fluids are slightly different, we obtain a regular perturbation expansion for the double eigenvalue, for which we calculate the two leading terms. This allows us to investigate the following two questions:

1. Which perturbations lead to eigenvalues with nonvanishing imaginary parts?

2. Which perturbations stabilize and which destabilize the flow?

2. GOVERNING EQUATIONS

We consider a fluid, filling the space between two parallel boundaries of infinite extent in the (x^*, z^*)-plane. Asterisks denote dimensional variables. The upper boundary at $z^* = l^*$ is kept at a constant temperature T_0^*, and the lower plate is kept at a higher constant temperature $T_0^* + \Delta T^*$. At temperature T_0^*, the fluid has a coefficient of cubical expansion $\hat{\alpha}$, thermal diffusivity κ, thermal conductivity k, viscosity μ, kinematic viscosity ν and density ρ. We use the Boussinesq approximation in the Navier-Stokes equations, that is, the density in the buoyancy term is approximated by $\rho(1 - \hat{\alpha}(T^* - T_0^*))$.

Following Drazin and Reid [1], we introduce dimensionless variables (without asterisks) as follows:

$$(x, z) = (x^*, z^*)/l^*,$$

$$t = \kappa t^*/l^{*2}, \tag{1}$$

$$\underline{u} = \underline{u}^* l^*/\kappa,$$

$$T = T^*/\Delta T^*,$$

$$p = p^* l^{*2}/(\rho\kappa^2).$$

Here, $\underline{u}^* = (u^*, w^*)$ is the velocity, p^* the pressure and T^* the temperature. We define a Rayleigh number

$$R = g\hat{\alpha}\Delta T^* l^{*3}/(\kappa\nu),$$

where g denotes gravitational acceleration, and a Prandtl number

$$P = \nu/\kappa.$$

We study the linear stability of the static solution

$$\underline{u} = \underline{0},\ T = T_0 + 1 - z.$$

If disturbances are proportional to $\exp(\sigma t + i\alpha x)$, then the following eigenvalue problem arises for the velocity $\underline{u}$ and perturbation Θ to the temperature:

$$\sigma\Theta = \triangle\Theta + w,$$

$$\sigma\underline{u} = -\nabla p + RP\Theta\underline{e}_z + P\triangle\underline{u}\ , \tag{2}$$

$$\nabla\cdot\underline{u} = 0.$$

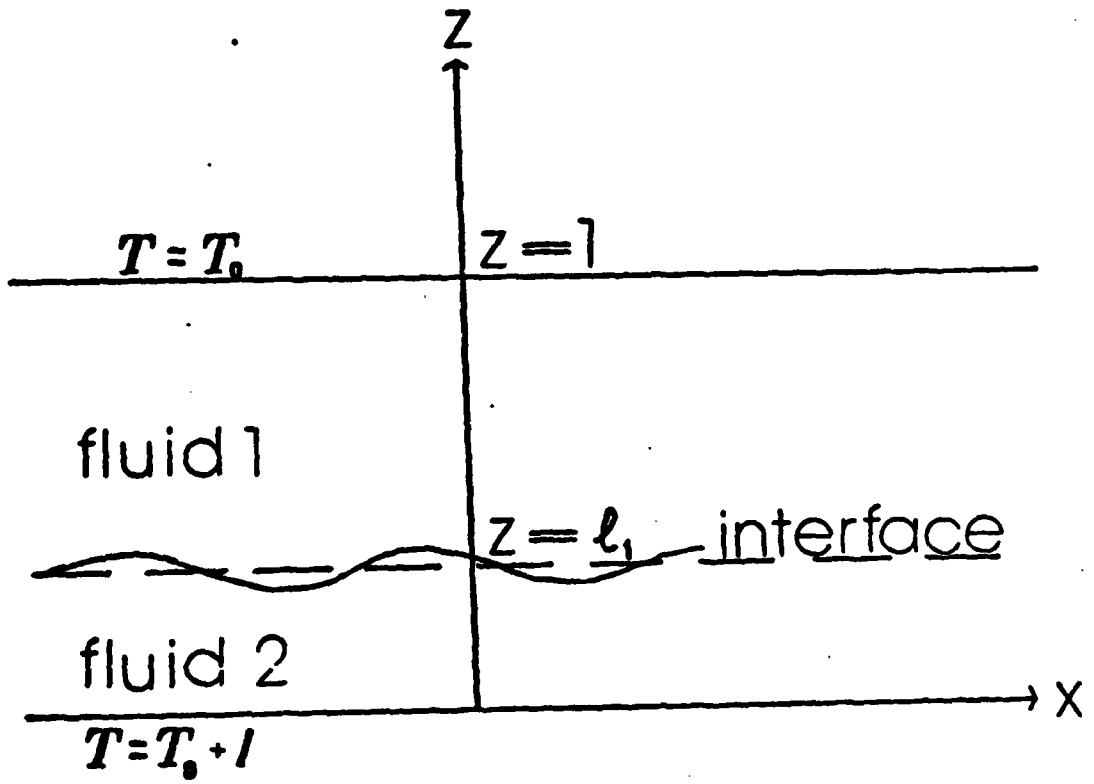
At the boundaries, $w = 0$ and Θ=0. At solid boundaries, we would, in addition, have $u = 0$. However, we prefer to look at the case of slip boundary conditions, where it is required that the shear stress be zero, or equivalently, $\partial u/\partial z = 0$. Although this is not physically realistic, this has the advantage that the eigenvalue and eigenfunction at the critical Rayleigh number are known in closed form. It is well known [1] that all eigenvalues are real, and the critical case σ=0 occurs first at $R = 27\pi^4/4$. The wavenumber of the critical mode in the x-direction is $\alpha = 2^{-1/2}\pi$.

We perturb this marginally stable eigenvalue by the introduction of a second fluid, whose properties are slightly different. At rest, this introduces an interface at $z = l_1$ as indicated in Fig. 1. Subscripts 1 or 2 on the physical parameters will now denote fluid 1 or 2, respectively. There are 6 dimensionless ratios:

$$m = \frac{\mu_1}{\mu_2},$$

$$r = \frac{\rho_1}{\rho_2},$$

Fig. 1



Sketch of Bénard problem perturbed by a second fluid.

$$\gamma = \frac{\kappa_1}{\kappa_2},$$

$$\varsigma = \frac{k_1}{k_2}, \tag{3}$$

$$\beta = \frac{\hat{\alpha}_1}{\hat{\alpha}_2},$$

$$l_1 = \frac{l_1^*}{l^*} = 1 - l_2.$$

The non-dimensionalization of the equations and the definitions of Rayleigh and Prandtl numbers will be based on fluid 1; for example, $R = g\hat{\alpha}_1 \Delta T^* l^{*3}/(\kappa_1 \nu_1)$. We include a surface tension between the fluids, described by a dimensionless parameter $S = S^* l^*/(\kappa_1 \mu_1)$, where S^* is the dimensional surface tension coefficient.

The linearized eigenvalue problem reads now as follows:

For $0 \leq z \leq l_1$,

$$\sigma\Theta = wA_1 + \triangle\Theta,$$

$$\sigma u = -\partial p/\partial x + P \triangle u,$$

$$\sigma w = -\partial p/\partial z + RP\Theta + P \triangle w, \tag{4}$$

$$\frac{\partial u}{\partial x} + \frac{\partial w}{\partial z} = 0,$$

where $A_1 = \frac{1}{(l_1 + \varsigma l_2)}$. For $l_1 \leq z \leq 1$,

$$\sigma\Theta = wA_2 + \frac{1}{\gamma} \triangle \Theta,$$

$$\sigma u = -r\frac{\partial p}{\partial x} + \frac{r}{m} P \triangle u,$$

$$\sigma w = -r\frac{\partial p}{\partial z} + \frac{RP}{\beta}\Theta + \frac{r}{m} P \triangle w, \tag{5}$$

$$\frac{\partial u}{\partial x} + \frac{\partial w}{\partial z} = 0,$$

where $A_2 = \varsigma A_1$.

The boundary conditions at z=0 and 1 are as before:

$$\Theta = w = \frac{\partial u}{\partial z} = 0. \tag{6}$$

The interface is perturbed to the position $z = l_1 + h(x,t)$. The interface conditions linearized about z=l_1 are:

continuity of temperature: $[[\Theta\ ]] = \mathrm{h}\ [[\mathrm{A}\ ]]$,

continuity of heat flux: $[[k\frac{\partial \Theta}{\partial z}]] = 0$,

continuity of velocity: $[[w\ ]] = [[u\ ]] = 0$, (7)

continuity of shear stress: $[[\mu(\frac{\partial u}{\partial z} + \frac{\partial w}{\partial z})\ ]] = 0$,

balance of normal stress:

$$p_2 - p_1 + 2P(\frac{\partial w_1}{\partial z} - \frac{1}{m}\frac{\partial w_2}{\partial z}) + M_1 h - PS\frac{\partial^2 h}{\partial x^2} = 0.$$

Here, $[[.\ ]]$ denotes the jump of a quantity across the interface, for example,

$$[[A]] = A_1 - A_2,$$

and we have set

$$M_1 = RP\{\frac{(1 - \frac{1}{r})}{\hat{\alpha}_1 \triangle T^*} + l_2 A_2(\frac{1}{r\beta} - 1)\}.$$

The kinematic free surface condition yields, at $z = l_1$,

$$\sigma h = w_1. \tag{8}$$

We now proceed as follows. We fix R=$\frac{27}{4}\pi^4$ and keep P and l_1 fixed but arbitrary. We consider only solutions to (4) - (8) proportional to $\exp(i\alpha x)$ with α fixed at $2^{-1/2}\pi$. We introduce a small parameter ϵ and regard $1-m, 1-r, 1-\gamma, 1-\varsigma, 1-\beta, M_1$ and S as small quantities proportional to ϵ; that is, we set

$$1 - m = \bar{m}\epsilon,$$

$$1 - r = \bar{r}\epsilon,$$

$$1 - \gamma = \bar{\gamma}\epsilon,$$

$$1 - \varsigma = \bar{\varsigma}\epsilon,$$

$$1 - \beta = \bar{\beta}\epsilon,$$

$$M_1 = \bar{M}_1\epsilon, \quad \bar{M}_1 = \frac{27\pi^4 P}{4}\left(-\frac{\bar{r}}{\hat{\alpha}_1 \Delta T^*} + l_2(\bar{r} + \bar{\beta})\right),$$

$$S = \bar{S}\epsilon.$$

At $\epsilon = 0$, there is an algebraically two-fold, but geometrically simple eigenvalue $\sigma = 0$. One eigenvalue arises from the first criticality of the Bénard problem for one fluid and the other arises from the presence of the interface. For small ϵ, this eigenvalue is perturbed into two eigenvalues which can be expanded in powers of $\epsilon^{1/2}$. The purpose of the following analysis is to find the coefficients of $\epsilon^{1/2}$ and ϵ in this expansion.

3. PERTURBATION OF MULTIPLE EIGENVALUES

The perturbation expansion for a *double* eigenvalue is not a simple series expansion in ϵ. In addition, eigenfunctions need not have series expansions in powers of ϵ. The perturbation of multiple eigenvalues is discussed in Ref. 10, chapter IV , §1. The procedure involves the generalized eigenspaces belonging to the eigenvalue $\sigma = 0$ for the *unperturbed* problem and its adjoint, and does *not* require finding the eigenspaces of the perturbed $O(\epsilon)$ problem at all.

We now quote the pertinent results from Ref. 10. Suppose σ_0 is an algebraically 2-fold eigenvalue of a matrix L_0. Let $\{a_1, a_2\}$ be a basis for the generalized eigenspace of L_0 with eigenvalue σ_0, and let $\{b_1, b_2\}$ be a basis for the generalized eigenspace of L_0^* (the adjoint of L_0) with eigenvalue $\bar{\sigma}_0$ (the overbar here denotes the complex conjugate). Let L_0 be perturbed into $L(\epsilon) = L_0 + \epsilon L_1(\epsilon) + O(\epsilon^2)$ with L_1 depending smoothly on ϵ. Then the perturbed eigenvalues σ are given by the zeros of the determinant of a matrix $\Psi_{ij}(\epsilon, \sigma), i, j, = 1, 2$, which represents to O($\epsilon$) the projection of $L(\epsilon) - \sigma$, first onto the eigenspace of the unperturbed problem and then onto the adjoint eigenspace:

$$\Psi_{ij}(\epsilon, \sigma) = \langle b_i, (L_0 + \epsilon L_1(\epsilon) - \sigma) a_j \rangle + O(\epsilon^2). \tag{9}$$

Some care must be taken when this result is applied to unbounded operators in infinite dimensional spaces, for example, differential operators. Such an operator has a *"domain"* that is specified not only by smoothness requirements on the function but also by the boundary conditions. If the domain of the operator that is being perturbed depends on ϵ, we cannot apply (9); the domains of $L(\epsilon)$ and L_0 may be different, and their combination would not make sense. We can, however, circumvent this problem by not looking at the

differential operator itself, but at its resolvent $(L(\epsilon) - \lambda \underline{I})^{-1}$ where λ is not an eigenvalue of $L(\epsilon)$. The domain of this does not depend on ϵ and we will need to redefine Ψ_{ij} accordingly. As will be seen from the following, the resolvent itself does not ever need to be computed.

In order to make these ideas more precise, we introduce some notation. Let X denote the set of functions (Θ, u, w, h). We introduce an inner product by

$$\langle X_1, X_2 \rangle = \int_0^{2\pi/\alpha} \int_{z=0}^{l_1} \bar{\Theta}_1 \Theta_2 + \bar{u}_1 u_2 + \bar{w}_1 w_2 \, dz dx$$

$$+ \int_0^{2\pi/\alpha} \int_{z=l_1}^{1} \bar{\Theta}_1 \Theta_2 + \bar{u}_1 u_2 + \bar{w}_1 w_2 \, dz dx$$

$$+ \int_0^{2\pi/\alpha} \bar{h}_1 h_2 \, dx \tag{10}$$

to generate a Hilbert space. In this Hilbert space, we consider the subspace determined by the "Hodge projection" (see space H in Theorem 1.4, Ref. 11), that is, by the conditions that the velocity field be divergence-free, that the vertical velocity vanish at the boundaries, and be continuous across the interface. By L(ϵ)X we denote the right hand sides of equations (4),(5) and (8). We regard L(ϵ) as an operator in the subspace so that the conditions on w in (6) and (7) and the normal stress balance in (7) are an integral part of the definition of L(ϵ). The domain of definition of L(ϵ) is determined by the rest of the boundary conditions in (6) and (7), which we write in the form B(ϵ)X=0. The range of the operator L(ϵ) must satisfy the following conditions in order for the pressure p occuring on the right sides of (4) and (5) to be determined as a function of X: The "velocity part" of L(ϵ)X must be divergence free, the vertical velocity must vanish on the walls and be continuous across the interface, and the jump in p across the interface must be given by the normal stress balance. Thus, the problem we wish to solve is: for small ϵ, find σ satisfying

$$L(\epsilon)X = \sigma X, \tag{11}$$

$$B(\epsilon)X = 0,$$

$$L(\epsilon) = L_0 + \epsilon L_1 + O(\epsilon^2),$$

$$B(\epsilon) = B_0 + \epsilon B_1 + O(\epsilon^2).$$

Explicitly,

$$L_0X = \begin{pmatrix} \triangle\Theta + w \\ -\frac{\partial p}{\partial z} + P\triangle u \\ -\frac{\partial p}{\partial z} + RP\Theta + P\triangle w \\ w \end{pmatrix}$$

in fluids 1 and 2, and

$$L_1X = \begin{pmatrix} l_2\tilde{\zeta}w \\ -\frac{\partial \tilde{p}}{\partial x} \\ -\frac{\partial \tilde{p}}{\partial z} \\ 0 \end{pmatrix}$$

in fluid 1 and

$$\begin{pmatrix} \bar{\gamma}\triangle\Theta - l_1\tilde{\zeta}w \\ \bar{r}\frac{\partial p}{\partial x} + (\bar{m}-\bar{r})P\triangle u - \frac{\partial \tilde{p}}{\partial x} \\ \bar{r}\frac{\partial p}{\partial z} + \bar{\beta}RP\Theta + (\bar{m}-\bar{r})P\triangle w - \frac{\partial \tilde{p}}{\partial z} \\ 0 \end{pmatrix}$$

in fluid 2, where $\tilde{p}$ denotes the $O(\epsilon)$-perturbation to the pressure,

$$B_0X = \begin{pmatrix} \Theta_1 - \Theta_2 \text{ at } z = l_1 \\ \frac{\partial \Theta_1}{\partial z} - \frac{\partial \Theta_2}{\partial z} \text{ at } z = l_1 \\ u_1 - u_2 \text{ at } z = l_1 \\ \frac{\partial u_1}{\partial z} - \frac{\partial u_2}{\partial z} \text{ at } z = l_1 \\ \Theta \text{ at } z = 0,1 \\ \frac{\partial u}{\partial z} \text{ at } z = 0,1 \end{pmatrix},$$

and

$$B_1X = \begin{pmatrix} -h\bar{\zeta} \text{ at } z = l_1 \\ -\bar{\zeta}\frac{\partial \Theta_1}{\partial z} \text{ at } z = l_1 \\ 0 \text{ at } z = l_1 \\ -\bar{m}\left(\frac{\partial u_1}{\partial z} + \frac{\partial w_1}{\partial x}\right) \text{ at } z = l_1 \\ 0 \text{ at } z = 0,1 \\ 0 \text{ at } z = 0,1 \end{pmatrix}.$$

With the above definitions, we are now ready to look at the resolvent of L(ϵ) and then to redefine the matrix in (9). Since ϵ is small, the eigenvalues of L(ϵ) are close to those of L_0, so that the λ in the resolvent should be chosen well away from zero. We choose $\lambda = 1$. Hence, instead of looking at (11) directly, we study the equivalent problem

$$(L(\epsilon) - 1)^{-1}X = (\sigma - 1)^{-1}X =: \hat{\sigma}X$$

and perturb around $\hat{\sigma} = -1$. We note that the definition of $(L(\epsilon) - 1)^{-1}$ already incorporates the boundary conditions. The relation (9) is applied to this problem. The determinant of the matrix

$$\Psi_{ij}(\epsilon, \hat{\sigma}) = \langle b_i,\ ((L(\epsilon) - 1)^{-1} - \hat{\sigma})a_j \rangle + O(\epsilon^2),\ i, j = 1, 2, \tag{12}$$

where the b_i and a_j are as before, is set equal to zero. We will require an expansion of the resolvent in powers of ϵ in order to carry out the calculation of Ψ_{ij}. We note that the inverse of $L_0 - 1$ is defined. Since parts of the calculation are rather lengthy, they will be organized into several appendices of this paper.

We first have to find the boundary value problem adjoint to (11). This is done in Appendix A. Then we determine the generalized eigenvectors at ϵ=0 for both (11) and the adjoint problem. These are denoted by a_1, a_2 and b_1, b_2, respectively, and satisfy:

$$\begin{aligned} L_0 a_1 &= 0,\ B_0 a_1 = 0, \\ L_0 a_2 &= a_1,\ B_0 a_2 = 0, \\ L_0^* b_1 &= 0,\ B_0^* b_1 = 0, \\ L_0^* b_2 &= b_1,\ B_0^* b_2 = 0. \end{aligned} \tag{13}$$

The generalized eigenvectors are determined in Appendix B. In order to apply formula (12), we must determine the expressions

$$\langle b_i, (L(\epsilon) - 1)^{-1} a_j \rangle \tag{14}$$

to first order in ϵ. To facilitate this calculation, we introduce x_j^0 and x_j^1 defined by

$$(L(\epsilon) - 1)^{-1} a_j = x_j^0 + \epsilon x_j^1 + O(\epsilon^2). \tag{15}$$

Equating the coefficients of equal powers of ϵ, we find the equations governing x_j^0 and x_j^1

$$(L_0 - 1)x_j^0 = a_j,$$

$$B_0 x_j^0 = 0, \tag{16}$$

and

$$L_1 x_j^0 + (L_0 - 1)x_j^1 = 0,$$

$$B_1 x_j^0 + B_0 x_j^1 = 0. \tag{17}$$

From (16), we immediately find $x_1^0 = -a_1$, $x_2^0 = -a_2 - a_1$. We will not need the solutions x_j^1 to the perturbation problem (17) but only certain inner products involving them, namely $\langle b_i, x_j^1 \rangle$. This is seen from (12) and (15):

$$\Psi_{ij}(\epsilon, \hat{\sigma}) = \langle b_i, x_j^0 \rangle + \epsilon \langle b_i, x_j^1 \rangle - \hat{\sigma} \langle b_i, a_j \rangle + O(\epsilon^2). \tag{18}$$

We calculate $\langle b_i, x_j^1 \rangle$ from (17):

$$\langle b_i, x_j^1 \rangle = \langle b_i, L_1 x_j^0 \rangle + \langle b_i, L_0 x_j^1 \rangle, \tag{19}$$

and an integration by parts:

$$\langle b_i, L_0 x_j^1 \rangle = \langle L_0^* b_i, x_j^1 \rangle + \textit{boundary integrals}, \tag{20}$$

where the boundary integrals are evaluated using the second part of (17). (The boundary integrals would vanish if $B_0 x_j^1$ were zero.) Details of these calculations are in Appendix C.

In the following section, we discuss the solution of the eigenvalue perturbation problem, which now reads $det\Psi_{ij} = 0$, where Ψ is given by (18).

4. RESULTS AND DISCUSSION

We set $\hat{\sigma} = -1 + \tau_1\epsilon^{1/2} + \tau_2\epsilon + O(\epsilon^{3/2})$. Since $\sigma = 1 + 1/\hat{\sigma}$,

$$\sigma = -\tau_1\sqrt{\epsilon} - (\tau_1^2 + \tau_2)\epsilon + O(\epsilon^{3/2}).$$

For the first-order term, we find (see equations C2, C5, C7 and C10 of Appendix C)

$$\tau_1^2 = -\frac{P\pi^2 \sin^2 \pi l_1}{3(P+1)}\left(9\pi\bar{\zeta}\cot \pi l_1 + \frac{27}{2}\pi^2[-\frac{\bar{r}}{\hat{\alpha}_1\Delta T^*} + l_2(\bar{\beta} + \bar{r})] + \bar{S}\right). \qquad (21)$$

The sign of this quantity determines whether the eigenvalue splits into real values or complex conjugates. We note that, in the situation studied by Renardy and Joseph [7], $\bar{\zeta}$, $\bar{r}$ and $\bar{S}$ were 0, and $\bar{\beta}$ was positive. (Their boundary conditions are different from ours, nevertheless, we expect some qualitative similarity). They find complex eigenvalues, in agreement with the prediction of (21). It is also worth noting that (21) does not involve the viscosity ratio, and that the sign does not depend on the Prandtl number.

If τ_1^2 is positive, we always have instability. However, if τ_1^2 is negative, we have to find $\tau_1^2 + \tau_2$ in order to determine whether the perturbations are stabilizing or destabilizing. Unfortunately, the formula for τ_2 is not nearly as simple as that for τ_1 (see equations C5, C6, C11, and C15 of Appendix C). We find

$$\tau_1^2 + \tau_2 = \frac{\sin \pi l_1}{3(1+P)}\Big(P\pi\cos\pi l_1(|-3 + 9\frac{\pi^2}{2}|\bar{\zeta} - \bar{m}) + (\bar{M}_1 + \frac{\pi^2}{2}P\bar{S})\sin\pi l_1$$

$$+\frac{\pi}{4}(1-11P)\bar{\zeta}\cos\pi l_1 + P\bar{\zeta}\frac{4}{\pi^2}Real|c_1(Q_1^2 - \pi^2/2)^2 Q_1\cosh Q_1 l_1|$$

$$+(1+P)\bar{\zeta}\frac{3}{4}\pi^2(\frac{\cos^2\pi l_1}{\sin\pi l_1} + l_1\sin\pi l_1)$$

$$+(\bar{M}_1 + \frac{\pi^2}{2}P\bar{S})(2Real|c_1\sinh Q_1 l_1| + (1+\frac{1}{P})\frac{l_2}{6\pi}\cos\pi l_1)$$

$$+\frac{9\pi P}{4\sin\pi l_1}\left(\pi l_2(\bar{\gamma}+\bar{m}-\bar{r}-\bar{\beta})+\frac{\sin 2\pi l_1}{2}(\bar{\zeta}+\bar{\gamma}-\bar{r}-\bar{\beta}-\frac{\bar{m}}{3})\right)$$

$$+\frac{\tau_1^2}{2}\left(1-\frac{2}{3\pi(1+P)}\left(\frac{P}{\pi}+\frac{(1+P)^2}{4P}(l_1\cot\pi l_2+\frac{1}{2\pi})\right)\right). \qquad (22)$$

where $\bar{M}_1$ is defined after equation (8), c_1 and Q_1 are defined in Appendix B.

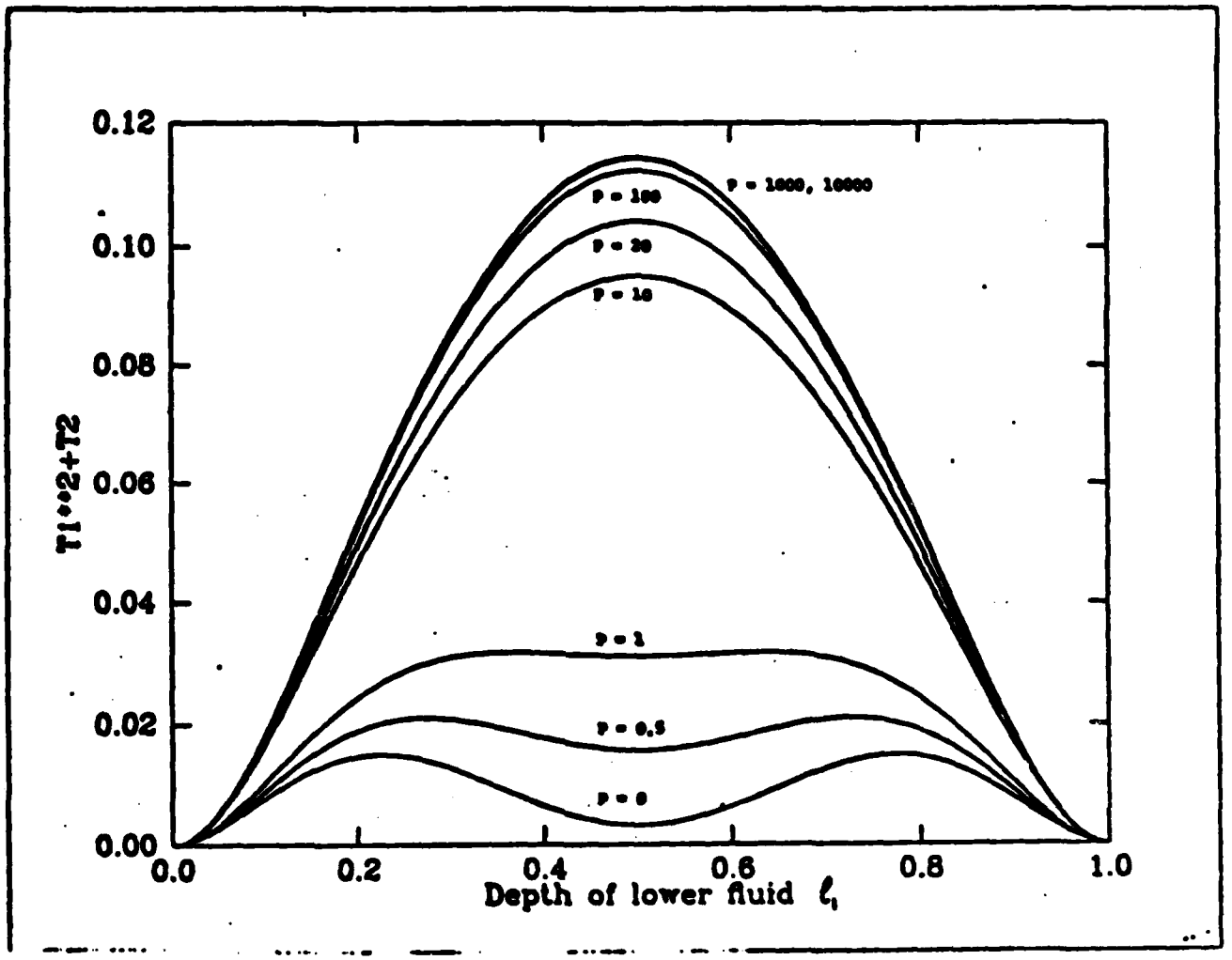
We examine some limiting cases below. If the Boussinesq approximation is to be justified, then $\hat{\alpha}_1\Delta T^*$ must be small[1], e.g. $\hat{\alpha} = 5.10^{-4}K^{-1}$ and $\Delta T^* \leq 10K$. For the purpose of computing graphs, we have taken $\hat{\alpha}_1\Delta T^* = 0.001$. As $P \to \infty$, both τ_1^2 and $\tau_1^2 + \tau_2$ approach finite limits.

Case i. $\bar{S} \neq 0, \bar{\zeta} = \bar{r} = \bar{\beta} = \bar{m} = 0$. If $\bar{S} < 0$, then $\tau_1^2 > 0$ so instabilities are exponential in time and leads to mixing. If $\bar{S} > 0$, then $\tau_1^2 < 0$ so the surface tension opposes the tendency for convection. However, whether this leads to a growth or not depends on the sign of (22) which depends nonlinearly on l_1 and the Prandtl number. Fig.2 shows this dependence to be $\tau_1^2 + \tau_2 > 0$ so that σ is stable and oscillatory.

Case ii. $\bar{r} \neq 0, \bar{S} = \bar{\zeta} = \bar{\beta} = \bar{m} = 0$. Since $\hat{\alpha}_1\Delta T^*$ is small, τ_1^2 and $\bar{r}$ have the same signs: if Fluid 1 is the less dense fluid, a convective instability takes place; if Fluid 2 is the more dense fluid, stable wavy modes occur. The graph of $\tau_1^2 + \tau_2$ versus l_1 for $\bar{r} < 0$ is similar to Fig.2.

Case iii. $\bar{\beta} \neq 0, \bar{S} = \bar{r} = \bar{\zeta} = \bar{m} = 0$. The signs of τ_1^2 and $\bar{\beta}$ are opposite. A heuristic explanation is as follows. In the Boussinesq approximation, the density at any depth l_1 in the buoyancy term of the momentum equations is approximated by $\rho_i(1-\hat{\alpha}_i(T^*-T_0^*))$, $i = 1,2$ where $(T^* - T_0^*)/\Delta T^* = 1 - z + \Theta$. Hence, at a depth z which is close to the interface $z = l_1$, the densities are approximately $\rho_i(1 - \hat{\alpha}_i l_2)$. Therefore, if $\bar{\beta} < 0$, the lower fluid is less dense than the upper fluid, locally at the interface, so that $\tau_1^2 > 0$. On the other

Fig. 2



Graph of $\tau_1^2 + \tau_2$ versus l_1 for various Prandtl numbers; $\bar{S} = 1, \bar{\zeta} = \bar{r} = \bar{\beta} = \bar{m} = \bar{\gamma} = 0$, $\hat{\alpha}_1 \Delta T^* = 0.001$.

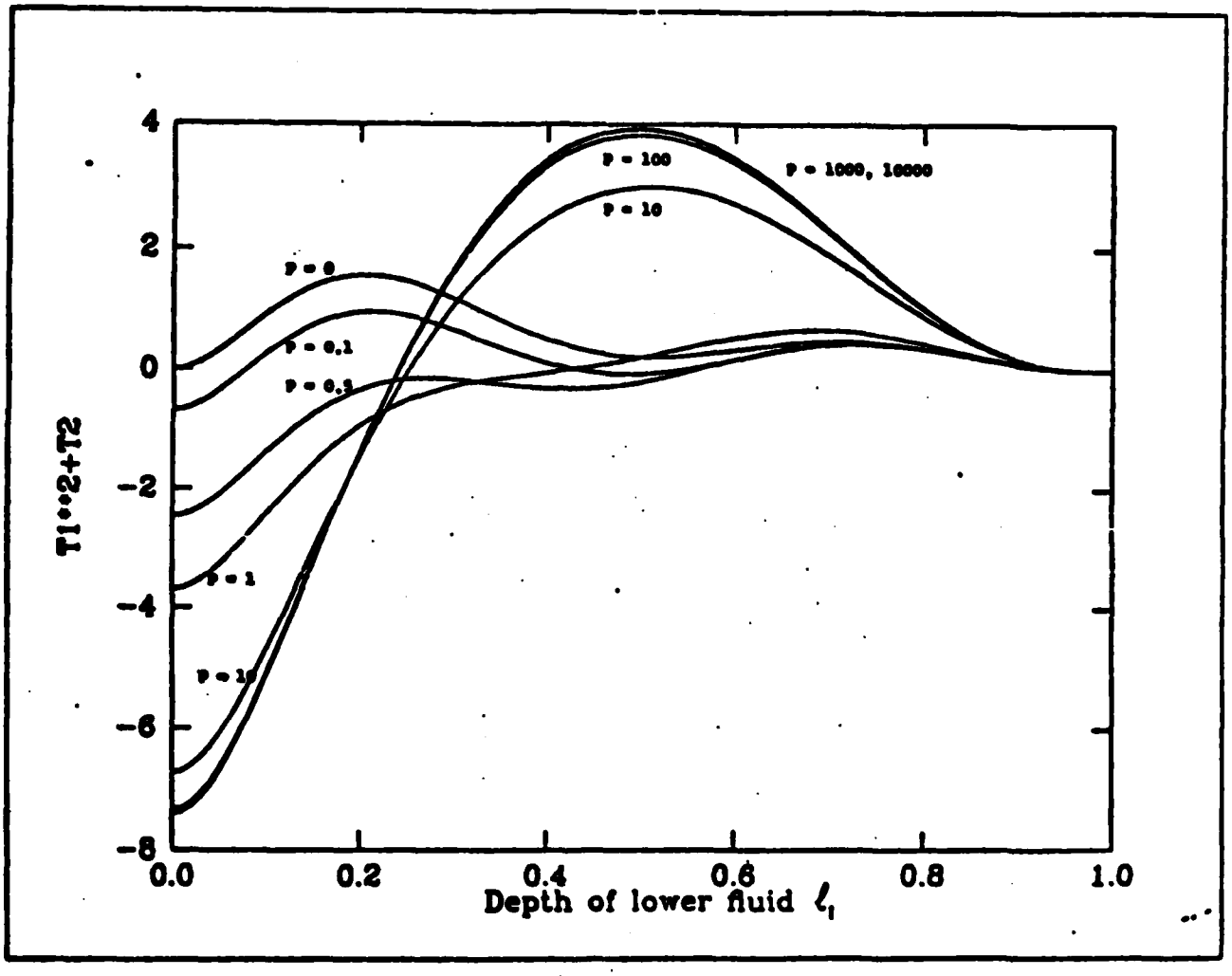
hand, if $\bar{\beta} > 0$, then the lower fluid is the more dense and $\tau_1^2 < 0$. Fig.3 presents $\tau_1^2 + \tau_2$ versus l_1 for $\bar{\beta} = 1$, showing that for a wide range of l_1 close to 0 and of Prandtl numbers greater than 0, there are unstable oscillatory modes. These modes are Hopf bifurcations.

Case iv. $\bar{\zeta} \neq 0, \bar{S} = \bar{r} = \bar{\beta} = \bar{m} = 0$. The dependence of τ_1^2 to l_1 is through $-\bar{\zeta} \sin 2\pi l_1$. Hence, if the thicker layer has the lesser coefficient of conductivity, (i.e. if $\bar{\zeta} > 0$ and $0.5 < l_1 < 1$; or if $\bar{\zeta} < 0$ and $0 < l_1 < 0.5$) then convective instability results. If the thicker layer has the greater coefficient of conductivity, then time-periodic modes occur, but whether these would be stable or not depends on the sign of $\tau_1^2 + \tau_2$. Fig.4 shows a wide range of unstable oscillatory modes for $0 < l_1 < 0.5$ when $\bar{\zeta} = 1$. This graph is antisymmetric about $l_1 = 0.5$, as can be deduced non-trivially from the equations. We note that when the fluid with the greater conductivity occupies most of the flowfield, the stability of the time-periodic eigenvalue σ depends on the Prandtl number: if the Prandtl number is very small, the oscillatory modes are stable, and if the Prandtl number is well away from 0, the oscillatory modes are unstable.

Case v. $\bar{m} \neq 0, \bar{\zeta} = \bar{S} = \bar{r} = \bar{\beta} = 0$. We find that $\tau_1^2 = 0$, and Fig.5 shows $\tau_1^2 + \tau_2$ versus l_1 for various Prandtl numbers. If Fluid 1 is the less viscous fluid, then there is stability, and if Fluid 1 is the more viscous fluid then there is convective instability, regardless of the depth ratio, in agreement with heuristic expectation.

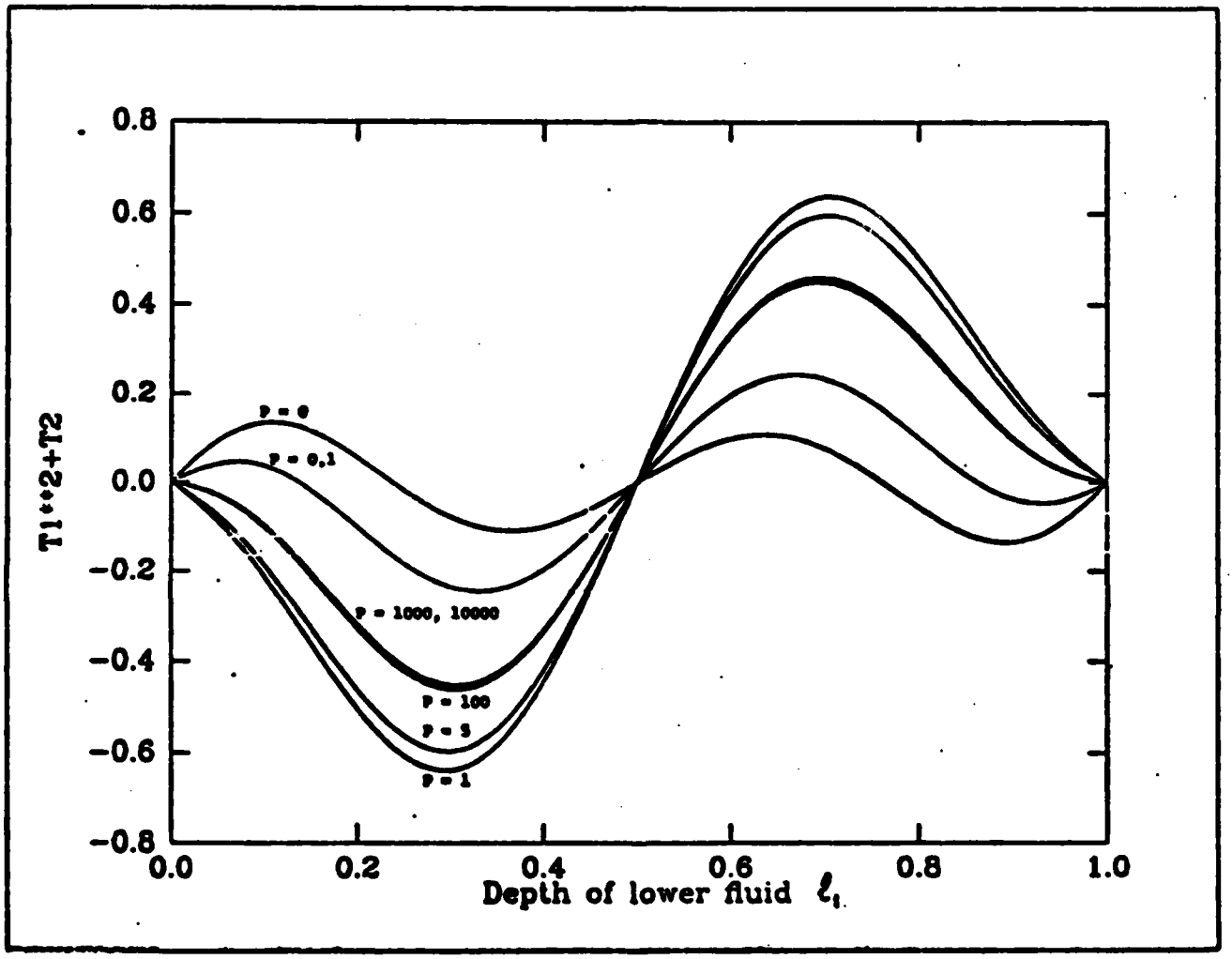
Case vi. $\bar{\gamma} \neq 0, \bar{m} = \bar{\zeta} = \bar{S} = \bar{r} = \bar{\beta} = 0$. The ratio γ of thermal diffusivities plays a similar role here to the ratio m of viscosities, which measures the rates of diffusion of momenta. As in Case v above, $\tau_1^2 = 0$ and the graph of $\tau_1^2 + \tau_2$ versus l_1 is shown in Fig.6. If Fluid 1 has the lesser coefficient of thermal diffusivity, then σ is stable. If Fluid 1 has

Fig. 3



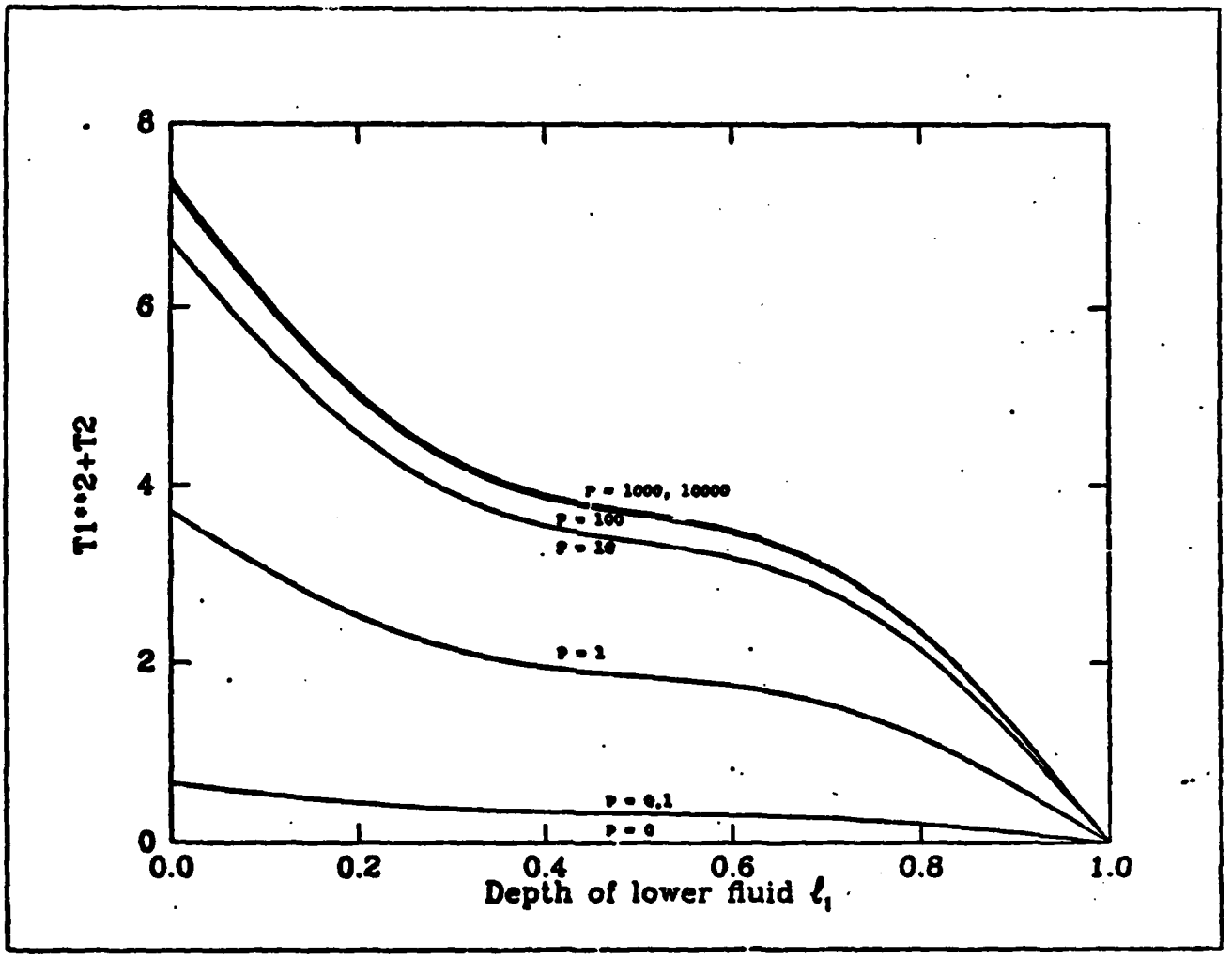
Graph of $r_1^2 + r_2$ versus l_1 for various Prandtl numbers; $\bar{\beta} = 1$, $\bar{S} = \bar{\xi} = \bar{r} = \bar{m} = \bar{\gamma} = 0$, $\bar{\alpha}_1 \Delta T^* = 0.001$.

Fig. 4



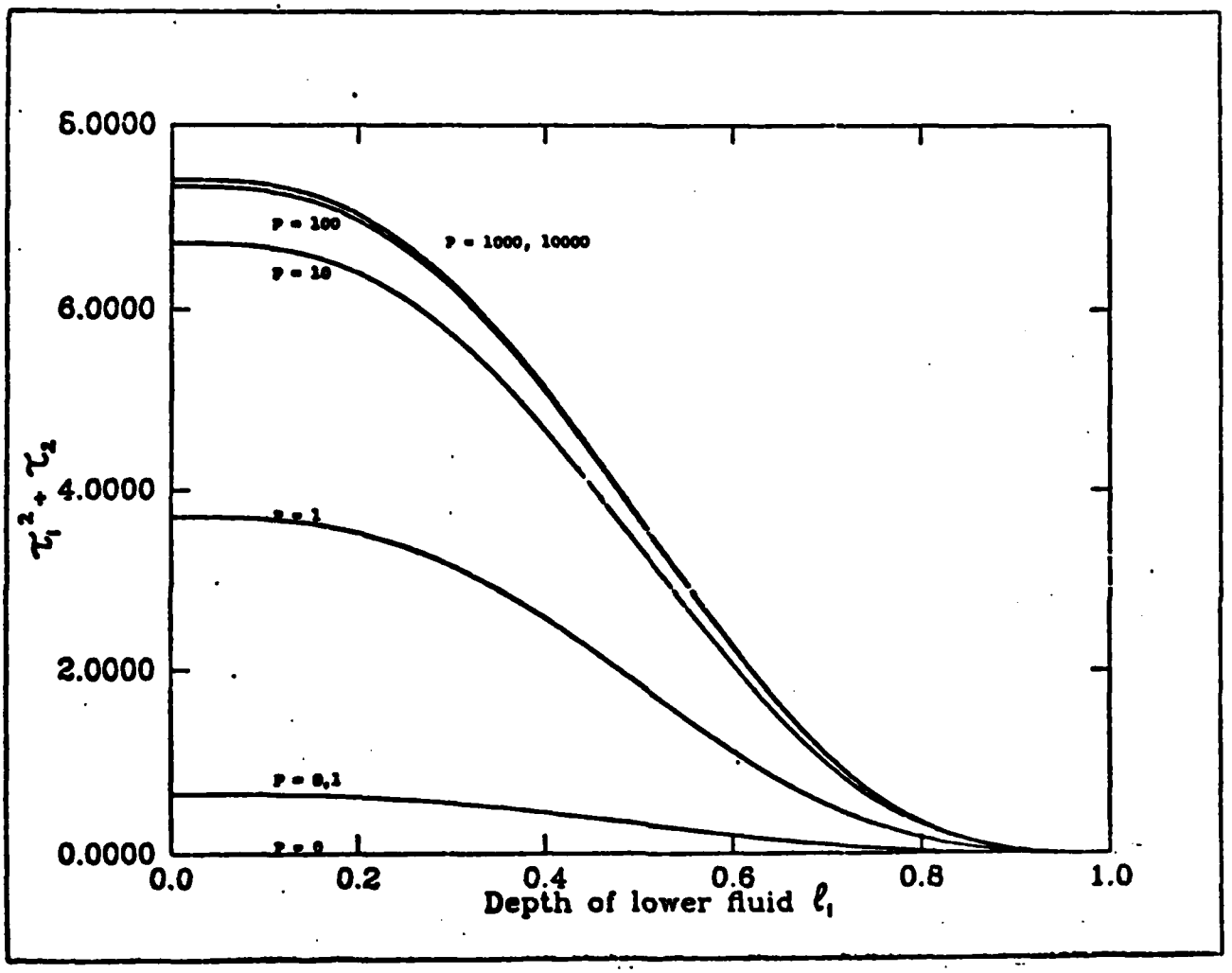
Graph of $\tau_1^2 + \tau_2$ versus l_1 for various Prandtl numbers; $\bar{\zeta} = 1$, $\bar{S} = \bar{r} = \bar{\beta} = \bar{m} = \bar{\gamma} = 0$, $\hat{\alpha}_1 \Delta T^* = 0.001$.

Fig. 5



Graph of $\tau_1^2 + \tau_2$ versus l_1 for various Prandtl numbers; $\bar{m} = 1$, $\bar{\zeta} = \bar{S} = \bar{\tau} = \bar{\beta} = \bar{\gamma} = 0$, $\hat{\alpha}_1 \Delta T^* = 0.001$.

Fig. 6



Graph of $\tau_1^2 + \tau_2$ versus l_1 for various Prandtl numbers; $\bar{\gamma} = 1$, $\bar{\zeta} = \bar{S} = \bar{r} = \bar{\beta} = \bar{m} = 0$, $\hat{\alpha}_1 \Delta T^* = 0.001$.

the greater coefficient of thermal diffusivity, then convective instability ensues, regardless of the depth ratio, as expected.

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APPENDIX A: The adjoint problem for ϵ=0

We denote the domains occupied by the two fluids by

$$\Omega_1 = \{0 \le x \le 2\pi/\alpha, 0 \le z \le l_1\}$$

and

$$\Omega_2 = \{0 \le x \le 2\pi/\alpha, l_1 \le z \le 1\}.$$

We denote the interface by I, and the lower and upper boundaries by Γ_1 and Γ_2, respectively. Let $X_1 = (\Theta, u, w, h)$ and $X_2 = (\Theta^*, u^*, w^*, h^*)$. Asterisks denote the adjoint. We have

$$\langle X_2, L_0 X_1 \rangle = \int_{\Omega_1} \bar{\Theta}^* \triangle \Theta + \bar{\Theta}^* w - \bar{u}^* \frac{\partial p}{\partial x} + P\bar{u}^* \triangle u - \bar{w}^* \frac{\partial p}{\partial z} + RP\bar{w}^* \Theta + P\bar{w}^* \triangle w$$
$$+ \int_{\Omega_2} \bar{\Theta}^* \triangle \Theta + \bar{\Theta}^* w - \bar{u}^* \frac{\partial p}{\partial x} + P\bar{u}^* \triangle u - \bar{w}^* \frac{\partial p}{\partial z} + RP\bar{w}^* \Theta + P\bar{w}^* \triangle w$$
$$+ \int_I \bar{h}^* w_1. \qquad (A1)$$

We integrate by parts and obtain, using the divergence condition div $\underline{u}$ = div $\underline{u}^*$ = 0,

$$\langle X_2, L_0 X_1 \rangle = \int_{\Omega_1} \Theta(\triangle \bar{\Theta}^* + RP\bar{w}^*) + u(P \triangle \bar{u}^* - \frac{\partial \bar{p}^*}{\partial x}) + w(P \triangle \bar{w}^* - \frac{\partial \bar{p}^*}{\partial z} + \bar{\Theta}^*)$$
$$+ \int_{\Omega_2} \Theta(\triangle \bar{\Theta}^* + RP\bar{w}^*) + u(P \triangle \bar{u}^* - \frac{\partial \bar{p}^*}{\partial x}) + w(P \triangle \bar{w}^* - \frac{\partial \bar{p}^*}{\partial z} + \bar{\Theta}^*)$$
$$+ \int_{\Gamma_1} -\bar{\Theta}^* \frac{\partial \Theta}{\partial z} + \Theta \frac{\partial \bar{\Theta}^*}{\partial z} - P\bar{u}^* \frac{\partial u}{\partial z} + Pu \frac{\partial \bar{u}^*}{\partial z} - P\bar{w}^* \frac{\partial w}{\partial z} + Pw \frac{\partial \bar{w}^*}{\partial z} + \bar{w}^* p - w\bar{p}^*$$
$$+ \int_{\Gamma_2} \bar{\Theta}^* \frac{\partial \Theta}{\partial z} - \Theta \frac{\partial \bar{\Theta}^*}{\partial z} + P\bar{u}^* \frac{\partial u}{\partial z} - Pu \frac{\partial \bar{u}^*}{\partial z} + P\bar{w}^* \frac{\partial w}{\partial z} - Pw \frac{\partial \bar{w}^*}{\partial z} - \bar{w}^* p + w\bar{p}^*$$
$$+ \int_I \bar{h}^* w_1 + [[\bar{\Theta}^* \frac{\partial \Theta}{\partial z} - \Theta \frac{\partial \bar{\Theta}^*}{\partial z} + P\bar{u}^* \frac{\partial u}{\partial z} - Pu \frac{\partial \bar{u}^*}{\partial z}$$
$$+ P\bar{w}^* \frac{\partial w}{\partial z} - Pw \frac{\partial \bar{w}^*}{\partial z} - \bar{w}^* p + w\bar{p}^*]]. \qquad (A2)$$

From this, we immediately read off the adjoint differential operator to be given by

$$L_0^* X_2 = \begin{pmatrix} \triangle \Theta^* + RPw^* \\ P \triangle u^* - \frac{\partial p^*}{\partial x} \\ P \triangle w^* - \frac{\partial p^*}{\partial z} + \Theta^* \\ 0 \end{pmatrix}. \qquad (A3)$$

Moreover, since X_1 satisfies the boundary conditions (6) on Γ_1 and Γ_2, the integrals over these boundaries vanish if

$$\Theta^* = w^* = \frac{\partial u^*}{\partial z} = 0. \qquad (A4)$$

Into the interface term in (A2), we add

$$-P\bar{w}^*(\frac{\partial u}{\partial x} + \frac{\partial w}{\partial z}) + Pw(\frac{\partial \bar{u}^*}{\partial x} + \frac{\partial \bar{w}^*}{\partial z})$$

which is zero. We integrate the x-derivative by parts and use periodicity. This yields

$$\int_I \bar{h}^* w_1 + [[\bar{\Theta}^* \frac{\partial \Theta}{\partial z} - \Theta \frac{\partial \bar{\Theta}^*}{\partial z} + P\bar{u}^*(\frac{\partial u}{\partial z} + \frac{\partial w}{\partial x}) - Pu(\frac{\partial \bar{u}^*}{\partial z} + \frac{\partial \bar{w}^*}{\partial x})$$

$$-\bar{w}^*(p - 2P\frac{\partial w}{\partial z}) + w(\bar{p}^* - 2P\frac{\partial \bar{w}^*}{\partial z})]]. \qquad (A5)$$

From this we find the adjoint interface conditions:

$$[[\Theta^*]] = 0,$$

$$[[\frac{\partial \Theta^*}{\partial z}]] = 0,$$

$$[[u^*]] = 0, \qquad (A6)$$

$$[[w^*]] = 0,$$

$$[[\frac{\partial u^*}{\partial z} + \frac{\partial w^*}{\partial x}]] = 0,$$

$$[[p^* - 2P\frac{\partial w^*}{\partial z}]] + h^* = 0.$$

APPENDIX B: Eigenfunctions of the unperturbed problem

If ϵ=0, the variable h does not occur in the right hand sides of (4), (5) ,(8), or in the interface conditions, and we have the eigenfunction

$$a_1 = e^{i\alpha x} \begin{pmatrix} 0 \\ 0 \\ 0 \\ 1 \end{pmatrix}. \tag{B1}$$

The equations (4)-(7) are precisely those characterizing the one-fluid Bénard problem. The eigenfunction for this problem [1] now yields a generalized eigenfunction:

$$a_2 = \frac{e^{i\alpha x}}{\sin \pi l_1} \begin{pmatrix} \frac{\sin \pi z}{\pi^2+\alpha^2} \\ \frac{i\pi}{\alpha} \cos \pi z \\ \sin \pi z \\ 0 \end{pmatrix}. \tag{B2}$$

The adjoint equations agree with the one-fluid Bénard problem if we set $h^* = 0$. Thus the eigenfunction of the one-fluid Bénard problem yields the eigenfunction for the adjoint:

$$b_1 = e^{i\alpha x} \begin{pmatrix} \frac{9}{2} P\pi^2 \sin \pi z \\ \frac{i\pi}{\alpha} \cos \pi z \\ \sin \pi z \\ 0 \end{pmatrix}. \tag{B3}$$

The generalized eigenvector b_2 of the adjoint satisfies

$$L_0^* b_2 = b_1,$$

$$B_0^* b_2 = 0.$$

This leads to the equations:

$$\begin{aligned} \triangle \Theta^* + RPw^* &= \frac{9}{2} P\pi^2 e^{i\alpha x} \sin \pi z, \\ P \triangle u^* - \frac{\partial p^*}{\partial x} &= \frac{i\pi}{\alpha} e^{i\alpha x} \cos \pi z, \end{aligned} \tag{B4}$$

$$P \triangle w^* + \Theta^* - \frac{\partial p^*}{\partial z} = e^{i\alpha x} \sin \pi z,$$

$$\frac{\partial u^*}{\partial x} + \frac{\partial w^*}{\partial z} = 0.$$

We set $w^* = w_0^* e^{i\alpha x}$ etc., and obtain by combining the equations:

$$(\frac{d^2}{dz^2} - \alpha^2)^3 w_0^* + \frac{27}{8}\pi^6 w_0^* = \frac{9}{4}\pi^4(1 + \frac{1}{P}) \sin \pi z. \qquad (B5)$$

The general solution of this equation is

$$w_0^* = c_1 \sinh Q_1 z + c_2 \cosh Q_1 z + c_3 \sinh Q_2 z + c_4 \cosh Q_2 z$$

$$+ c_5 \sin \pi z + c_6 \cos \pi z - \frac{1}{6\pi}(1 + \frac{1}{P}) z \cos \pi z \qquad (B6)$$

in fluid 1, and

$$w_0^* = d_1 \sinh Q_1(z-1) + d_2 \cosh Q_1(z-1) + d_3 \sinh Q_2(z-1) + d_4 \cosh Q_2(z-1)$$

$$+ d_5 \sin \pi(z-1) + d_6 \cos \pi(z-1) - \frac{1}{6\pi}(1 + \frac{1}{P}) z \cos \pi z \qquad (B7)$$

in fluid 2, where

$$Q_1 = \frac{\pi}{2}\sqrt[4]{52}\, e^{i\phi/2}, Q_2 = \frac{\pi}{2}\sqrt[4]{52}\, e^{-i\phi/2}, \qquad (B8)$$

and ϕ is determined by $\cos\phi = 5/\sqrt{52}$, $\sin\phi = 3\sqrt{3/52}$. The coefficients c_1 - c_6 and d_1 - d_6 must be determined such that the boundary conditions are satisfied. By using (B4), we can show that the conditions (A4) at the walls reduce to

$$w_0^* = \frac{d^2}{dz^2} w_0^* = \frac{d^4}{dz^4} w_0^* = 0. \qquad (B9)$$

At z=0, this yields

$$c_2 + c_4 + c_6 = 0,$$

$$Q_1^2 c_2 + Q_2^2 c_4 - \pi^2 c_6 = 0, \qquad (B10)$$

$$Q_1^4 c_2 + Q_2^4 c_4 + \pi^4 c_6 = 0.$$

From this, we obtain $c_2 = c_4 = c_6 = 0$. At z=1, we find

$$d_2 + d_4 + d_6 + \frac{1}{6\pi}(1 + \frac{1}{P}) = 0,$$

$$Q_1^2 d_2 + Q_2^2 d_4 - \pi^2 d_6 - \frac{\pi}{6}(1 + \frac{1}{P}) = 0, \qquad (B11)$$

$$Q_1^4 d_2 + Q_2^4 d_4 + \pi^4 d_6 + \frac{\pi^3}{6}(1 + \frac{1}{P}) = 0.$$

This yields $d_2 = d_4 = 0$, $d_6 = -\frac{1}{6\pi}(1 + \frac{1}{P})$. The first five of the conditions (A6) lead, after eliminating u^* and Θ^* from (B4), to the conditions

$$[[w_0^*]] = [[\frac{dw_0^*}{dz}]] = [[\frac{d^2 w_0^*}{dz^2}]] = [[\frac{d^4 w_0^*}{dz^4}]]$$

$$= [[\frac{d^5 w_0^*}{dz^5} - \pi^2 \frac{d^3 w_0^*}{dz^3}]] = 0. \qquad (B12)$$

We can set the coefficient d_5 to zero for the following reason. The coefficient d_5 multiplies $w = \sin \pi(z-1)$ in the generalized eigenvector. The eigenvector b_1 has $w = \sin \pi z$. Since any multiple of b_1, added to b_2, i.e. $b_2 + Cb_1$, is also a generalized eigenvector, we choose C to be d_5. This essentially gets rid of d_5 in Ω_2, replaces c_5 in Ω_1 by $c_5 + d_5$, and we rename $c_5 + d_5$ as c_5. We thus obtain the following system of equations.

$$c_1 \sinh Q_1 l_1 + c_3 \sinh Q_2 l_1 + c_5 \sin \pi l_1 = -d_1 \sinh Q_1 l_2 - d_3 \sinh Q_2 l_2 + d_6 \cos \pi l_2,$$

$$c_1 Q_1 \cosh Q_1 l_1 + c_3 Q_2 \cosh Q_2 l_1 + \pi c_5 \cos \pi l_1 = Q_1 d_1 \cosh Q_1 l_2$$

$$+ Q_2 d_3 \cosh Q_2 l_2 + \pi d_6 \sin \pi l_2,$$

$$c_1 Q_1^2 \sinh Q_1 l_1 + c_3 Q_2^2 \sinh Q_2 l_1 - \pi^2 c_5 \sin \pi l_1 = -Q_1^2 d_1 \sinh Q_1 l_2$$

$$-Q_2^2 d_3 \sinh Q_2 l_2 - \pi^2 d_6 cos \pi l_2,$$

$$c_1 Q_1^4 \sinh Q_1 l_1 + c_3 Q_2^4 \sinh Q_2 l_1 + \pi^4 c_5 \sin \pi l_1 = -Q_1^4 d_1 \sinh Q_1 l_2$$

$$-Q_2^4 d_3 \sinh Q_2 l_2 + \pi^4 d_6 \cos \pi l_2, \qquad (B13)$$

$$c_1 (Q_1^5 - \pi^2 Q_1^3) \cosh Q_1 l_1 + c_3 (Q_2^5 - \pi^2 Q_2^3) \cosh Q_2 l_1 + 2 c_5 \pi^5 \cos \pi l_1$$

$$= d_1 (Q_1^5 - \pi^2 Q_1^3) \cosh Q_1 l_2 + d_3 (Q_2^5 - \pi^2 Q_2^3) \cosh Q_2 l_2 + 2 d_6 \pi^5 \sin \pi l_2.$$

We eliminate c_1 from the third and fourth equations by using the first equation.

$$(Q_2^2 - Q_1^2) \sinh Q_2 l_1 c_3 - (\pi^2 - Q_1^2) \sin \pi l_1 c_5 + (Q_2^2 - Q_1^2) \sinh Q_2 l_2 = -(\pi^2 + Q_1^2) \cos \pi l_2 d_6,$$

$$(Q_2^3 (Q_2^2 - \pi^2) - Q_1^2 Q_2 (Q_1^2 - \pi^2)) \cosh Q_2 l_1 c_3 + (2\pi^5 - \pi Q_1^2 (Q_1^2 - \pi^2)) \cos \pi l_1 c_5$$

$$+ (-Q_2^3 (Q_2^2 - \pi^2) + Q_2 Q_1^2 (Q_1^2 - \pi^2)) \cosh Q_2 l_2 d_3 = (2\pi^5 - Q_1^2 \pi (Q_1^2 - \pi^2)) \sin \pi l_2 d_6, \quad (B14)$$

$$(Q_2^4 - Q_1^4) \sinh Q_2 l_1 c_3 + (\pi^4 - Q_1^4) \sin \pi l_1 c_5 + (Q_2^4 - Q_1^4) \sinh Q_2 l_2 d_3$$

$$= (\pi^4 - Q_1^4) \cos \pi l_2 d_6.$$

The first and third equations of (B14) yield

$$c_5 = d_6 \cot \pi l_2, \qquad (B15)$$

and

$$c_3 \sinh Q_2 l_1 + d_3 \sinh Q_2 l_2 = 0.$$

The second equation of (B14) yields

$$c_3 \cosh Q_2 l_1 - c_3 \cosh Q_2 l_2 = \frac{\pi d_6 (1 + \sqrt{3} i)}{2 Q_2 \sin \pi l_2}.$$

Hence,

$$c_3 = \frac{d_6\pi(1+\sqrt{3}i)\sinh Q_2 l_2}{2Q_2 \sin \pi l_2 \sinh Q_2}, \tag{B16}$$

and

$$d_3 = -\frac{d_6\pi(1+\sqrt{3}i)\sinh Q_2 l_1}{2Q_2 \sinh Q_2 \sin \pi l_2}. \tag{B17}$$

From the first and second equations of (B13), we find that c_1 is the complex conjugate of c_3 and that d_1 is the complex conjugate of d_3.

APPENDIX C: Evaluation of inner products

We first calculate the boundary integrals that arise in (20) and then the entries of the matrix Ψ_{ij} defined in (18). We denote $b_i = (\Theta^*, u^*, w^*, h^*)$ and $x_j^0 = (\Theta, u, w, h)$. In addition to the boundary terms arising in (20), we also integrate the term arising from $\tilde{p}$ in $\langle b_i, L_1 x_j^0 \rangle$ by parts. This yields another integral over the interface, which we combine with those from (20) into an expression Γ_{ij}. The form of Γ_{ij} can be read off from the calculation of the adjoint in Appendix A. The terms remaining in $\langle b_i, L_1 x_j^0 \rangle$ will be denoted by $\langle\langle b_i, L_1 x_j^0 \rangle\rangle$. We thus have

$$\langle b_i, L_0 x_j^1 \rangle + \langle b_i, L_1 x_j^0 \rangle = \langle L_0^* b_i, x_j^1 \rangle + \langle\langle b_i, L_1 x_j^0 \rangle\rangle + \Gamma_{ij}$$

where

$$\Gamma_{ij} = \int_I \bar{\Theta}^* \xi \frac{\partial \Theta}{\partial z} - \xi h \frac{\partial \bar{\Theta}^*}{\partial z} + P \bar{u}^* \bar{m} \left(\frac{\partial u}{\partial z} + \frac{\partial w}{\partial x} \right)$$
$$- \bar{w}^* \left(-2P\bar{m} \frac{\partial w}{\partial z} + h(\bar{M}_1 + \frac{\pi^2}{2} P \bar{S}) \right) dx \qquad (C1)$$

Here, the interval of integration I extends over one wavelength in x, at $z = l_1$. Hence, equations (19) become:

$$\langle b_1, x_1^1 \rangle = \Gamma_{11} + \langle\langle b_1, L_1 x_1^0 \rangle\rangle,$$

$$\langle b_1, x_2^1 \rangle = \Gamma_{12} + \langle\langle b_1, L_1 x_2^0 \rangle\rangle,$$

$$\langle b_2, x_1^1 \rangle = \Gamma_{21} + \langle\langle b_2, L_1 x_1^0 \rangle\rangle + \langle b_1, x_1^1 \rangle, \qquad (C2)$$

$$\langle b_2, x_2^1 \rangle = \Gamma_{22} + \langle\langle b_2, L_1 x_2^0 \rangle\rangle + \langle b_1, x_2^1 \rangle,$$

where

$$\Gamma_{11} = 2\sqrt{2} \left(9P \frac{\pi^3}{2} \xi \cos \pi l_1 + (\bar{M}_1 + \frac{\pi^2}{2} P \bar{S}) \sin \pi l_1 \right),$$

$$\Gamma_{12} = 2\sqrt{2}(P\pi\cos\pi l_1((-3+9\frac{\pi^2}{2})\bar{\zeta}-\bar{m})+(\bar{M}_1+\frac{\pi^2}{2}P\bar{S})\sin\pi l_1),$$

and

$$\Gamma_{21} = 2\frac{\pi}{\alpha}(\bar{\zeta}\frac{\partial\bar{\Theta}^*}{\partial z}+\bar{w}_1^*(\bar{M}_1+\frac{\pi^2}{2}P\bar{S}))_{z=l_1}$$

$$= 2\sqrt{2}(\frac{\pi}{4}(1-11P)\bar{\zeta}\cos\pi l_1+2P\frac{\bar{\zeta}}{\pi^2}(c_1(Q_1^2-\frac{\pi^2}{2})^2Q_1\cosh Q_1l_1+c_3(Q_2^2-\frac{\pi^2}{2})^2Q_2\cosh Q_2l_1)$$

$$+(1+P)\bar{\zeta}\frac{3}{4}\pi^2(\frac{cos^2\pi l_1}{sin\ \pi l_1}+l_1\sin\pi l_1)$$

$$+(\bar{M}_1+\frac{\pi^2}{2}P\bar{S})(c_1\sinh Q_1l_1+c_3\sinh Q_2l_1+(1+\frac{1}{P})\frac{l_2}{6\pi}\cos\pi l_1)).$$

We will see later that Γ_{22} is not required. Since $L_1x_1^0$ vanishes in both fluids, the first equation reduces to $\langle b_1, x_1^1\rangle = \Gamma_{11}$. Since $x_1^0 = -a_1$,

$$\Psi_{11}(\epsilon,\hat{\sigma}) = -(1+\hat{\sigma})\langle b_1,a_1\rangle+\epsilon\Gamma_{11}+O(\epsilon^2).$$

We find that $\langle b_1, a_1\rangle = 0$ so that

$$\Psi_{11}(\epsilon,\hat{\sigma}) = \epsilon\Gamma_{11}+O(\epsilon^2). \qquad (C3)$$

Since $x_2^0 = -a_1 - a_2$,

$$\Psi_{12}(\epsilon,\hat{\sigma}) = -(1+\hat{\sigma})\langle b_1,a_2\rangle+\epsilon(\Gamma_{12}+\langle\langle b_1, L_1x_2^0\rangle\rangle)+O(\epsilon^2)$$

We set

$$\hat{\sigma} = -1+\tau_1\epsilon^{1/2}+\tau_2\epsilon+O(\epsilon^{3/2})$$

so that

$$\Psi_{12}(\epsilon,\hat{\sigma}) = -\sqrt{\epsilon}\tau_1\langle b_1,a_2\rangle+\epsilon(-\tau_2\langle b_1,a_2\rangle+\Gamma_{12}+\langle\langle b_1,L_1x_2^0\rangle\rangle)+O(\epsilon^{3/2}). \qquad (C4)$$

We have

$$\langle b_1, a_2 \rangle = \frac{1}{\sin \pi l_1} \frac{2\pi}{\alpha} \int_0^1 \frac{9P\pi^2 \sin^2 \pi z}{2(\pi^2 + \alpha^2)} + \frac{\pi^2 \cos^2 \pi z}{\alpha^2} + \sin^2 \pi z dz \qquad (C5)$$

$$= \frac{3\sqrt{2}(1+P)}{\sin \pi l_1},$$

and

$$\langle\langle b_1, L_1 x_2^0 \rangle\rangle = \frac{2\pi}{\alpha \sin \pi l_1} (\bar{\zeta} l_2 \int_0^{l_1} \frac{9}{2} P\pi^2 (-\sin^2 \pi z) dz$$

$$+ \int_{l_1}^1 \sin^2 \pi z (\frac{9}{2} P\pi^2 \bar{\zeta} l_1 + \frac{9}{2} P\pi^2 \bar{\gamma} + \frac{3}{2} P\pi^2 \bar{m} - \frac{9}{2} P\pi^2 (\bar{r} + \bar{\beta})) + 3P\bar{m}\pi^2 \cos^2 \pi z dz$$

$$= \frac{2\sqrt{2}}{\sin \pi l_1} (\frac{9}{4} P\pi^2 l_2 (\bar{\gamma} + \bar{m} - \bar{r} - \bar{\beta}) + \frac{9}{8} P\pi \sin 2\pi l_1 (\bar{\zeta} + \bar{\gamma} - \bar{r} - \bar{\beta} - \frac{\bar{m}}{3})). \qquad (C6)$$

Next,

$$\Psi_{21}(\epsilon, \hat{\sigma}) = -(1 + \hat{\sigma}) \langle b_2, a_1 \rangle + \epsilon (\Gamma_{21} + \langle b_1, x_1^1 \rangle) + O(\epsilon^2).$$

We note that

$$\langle b_2, a_1 \rangle = \langle b_2, L_0 a_2 \rangle = \langle L_0^* b_2, a_2 \rangle = \langle b_1, a_2 \rangle. \qquad (C7)$$

Hence,

$$\Psi_{21}(\epsilon, \hat{\sigma}) = -\sqrt{\epsilon} \tau_1 \langle b_1, a_2 \rangle + \epsilon (\Gamma_{21} + \langle b_1, x_1^1 \rangle) + O(\epsilon^{3/2}). \qquad (C8)$$

We find

$$\Psi_{22}(\epsilon, \hat{\sigma}) = \langle b_2, x_2^0 \rangle - (-1 + \sqrt{\epsilon} \tau_1) \langle b_2, a_2 \rangle + O(\epsilon)$$

$$= -\langle b_2, a_1 \rangle - \sqrt{\epsilon} \tau_1 \langle b_2, a_2 \rangle + O(\epsilon). \qquad (C9)$$

Collecting the $O(\epsilon)$-terms from the equation $det \Psi_{ij} = 0$, we find

$$\tau_1^2 = -\frac{\Gamma_{11}}{\langle b_2, a_1 \rangle}. \qquad (C10)$$

Collecting the $O(\epsilon^{3/2})$-terms, we find

$$\tau_2 = \frac{\Gamma_{12} + \Gamma_{21} + \langle\langle b_1, L_1 x_2^0 \rangle\rangle}{2\langle b_1, a_2 \rangle} + \frac{\tau_1^2}{2}(-1 + \frac{\langle b_2, a_2 \rangle}{\langle b_1, a_2 \rangle}). \qquad (C11)$$

We now need to calculate $\langle b_2, a_2 \rangle$ given by

$$\langle b_2, a_2 \rangle = \frac{2\pi}{\alpha \sin \pi l_1}\{\int_0^{l_1} + \int_{l_1}^1\} \frac{\bar{\Theta}^* \sin \pi z}{(\pi^2 + \alpha^2)} + \frac{i\pi \cos \pi z \bar{u}^*}{\alpha} + \bar{w}^* \sin \pi z \, dz,$$

where $b_2 = (\Theta^*, u^*, w^*, h^*)$ and a_2 is given by (B2). We express the integrand in terms of w^* by using equations (B4): $i\bar{u}^* = \frac{\bar{w}_x^*}{\alpha}$ and $\bar{\Theta}^* = 3 \sin \pi z + \frac{2P}{\pi^2} \triangle^2 \bar{w}^*$. Hence,

$$\langle b_2, a_2 \rangle = \frac{2\sqrt{2}}{\sin \pi l_1}\{\frac{1}{\pi^2} + (\int_0^{l_1} + \int_{l_1}^1) \frac{4P}{3\pi^4} \triangle^2 \bar{w}^* \sin \pi z + 3\bar{w}^* \sin \pi z dz\}.$$

Integration by parts simplifies the first integral as follows:

$$\int \triangle^2 \bar{w}^* \sin \pi z dz = \sin \pi l_1 [[\frac{\partial^3 \bar{w}^*}{\partial z^3}]] + \frac{9\pi^4}{4} \int \bar{w}^* \sin \pi z dz,$$

so that we are essentially left with having to evaluate $[[\frac{\partial^3 \bar{w}^*}{\partial z^3}]]$ and $\int \bar{w}^* \sin \pi z dz$ and substituting them into

$$\langle b_2, a_2 \rangle = \frac{2\sqrt{2}}{\sin \pi l_1}\{\frac{1}{\pi^2} + \frac{4P \sin \pi l_1}{3\pi^4} [[\frac{\partial^3 \bar{w}^*}{\partial z^3}]] + 3(1 + P) \int \bar{w}^* \sin \pi z dz\}. \qquad (C12)$$

The former is facilitated by multiplying equation (B5) by $\sin \pi z$ and integrating by parts. This leads to

$$[[\frac{\partial^3 \bar{w}^*}{\partial z^3}]] = -\frac{3\pi^2(1 + \frac{1}{P})}{4 \sin \pi l_1}. \qquad (C13)$$

From Appendix B, we have

$$\bar{w}^* = c_1 \sinh Q_1 z + c_3 \sinh Q_2 z + c_5 \sin \pi z - \frac{(1 + \frac{1}{P})}{6\pi} z \cos \pi z$$

in fluid 1 and

$$\bar{w}^* = d_1 \sinh Q_1(z-1) + d_3 \sinh Q_2(z-1) + \frac{1}{6\pi}(1+\frac{1}{P})(1-z)\cos \pi z$$

in fluid 2, with the coefficients given by (B15) - (B17). After some algebra, we find

$$\int_0^1 \bar{w}^* \sin \pi z dz = -\frac{(1+\frac{1}{P})}{12\pi}(l_1 \cot \pi l_2 + \frac{1}{2\pi}). \qquad (C14)$$

Using (C12) - (C14),

$$\langle b_2, a_2 \rangle = -\frac{2\sqrt{2}}{\pi \sin \pi l_1}\{\frac{P}{\pi} + \frac{(1+P)^2}{4P}(l_1 \cot \pi l_2 + \frac{1}{2\pi})\}. \qquad (C15)$$

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In a recent paper, Y. Renardy and D. D. Joseph[7] study the Bénard problem for two layers of different fluids lying on top of each other and bounded by walls. Their study shows that, in contrast to the Bénard problem for one fluid, the onset of instability can be oscillatory. The number of parameters involved in the problem is large, and there is yet no comprehensive picture of when the instability is oscillatory and when it is not. The study of limiting cases, accessible by perturbation methods, may be helpful in this respect. In this paper, an analysis is given for the case when the properties of the two fluids are nearly equal and the fluids are allowed to slip at the boundaries.

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