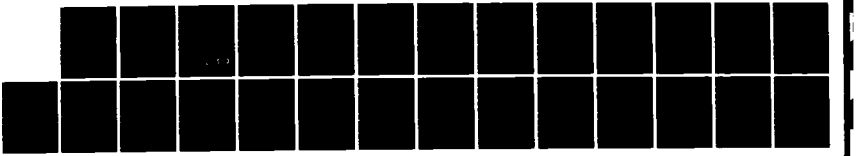
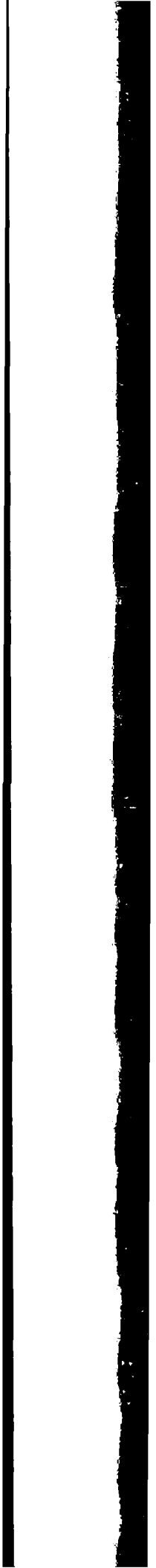
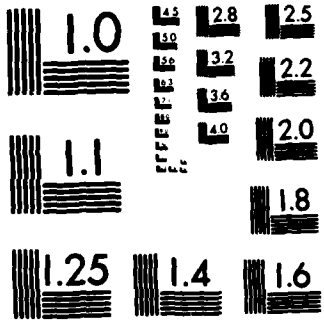
AD-A149 050 A CONFORMAL MAPPING SUITABLE FOR PROBLEMS INVOLVING 1/1
INTERACTION BETWEEN G..(U) AERONAUTICAL RESEARCH LABS
MELBOURNE (AUSTRALIA) T TRAN-CONG SEP 84
UNCLASSIFIED ARL-AERO-TM-367 F/G 12/1 NL



END
FILMED
DTIC





MICROCOPY RESOLUTION TEST CHART
NATIONAL BUREAU OF STANDARDS-1963-A

UNCLASSIFIED

(12)

ARL-AERO-TM-367

AR-003-960

DEPARTMENT OF DEFENCE

DEFENCE SCIENCE AND TECHNOLOGY ORGANISATION

AERONAUTICAL RESEARCH LABORATORIES

MELBOURNE, VICTORIA

AD-A149 050

Aerodynamics Technical Memorandum 367

A CONFORMAL MAPPING SUITABLE FOR PROBLEMS INVOLVING INTERACTION BETWEEN GIVEN GEOMETRIES AND KNOWN FAR FIELDS

by

TON TRAN-CONG

Approved for Public Release

COPY No

SEPTEMBER 1984

UNCLASSIFIED

AR-003-960

DEPARTMENT OF DEFENCE

DEFENCE SCIENCE AND TECHNOLOGY ORGANISATION

AERONAUTICAL RESEARCH LABORATORIES

Aerodynamics Technical Memorandum 367

A CONFORMAL MAPPING SUITABLE FOR PROBLEMS INVOLVING INTERACTION BETWEEN GIVEN GEOMETRIES AND KNOWN FAR FIELDS

By

TON TRAN-CONG

Summary

A conformal transformation formula using Riemann-Stieltjes integrals is derived for use with problems involving the interaction between a given finite-sized geometry and a known far field. The derivative of this transformation is non-singular in the domain considered and tends to one at infinity. A formula is derived for transformation from the unit circle to the exterior of an arbitrarily given continuous curve with bounded variation. A special case of the transformation is very similar to that of Schwarz-Christoffel. Application to the generation of aerofoils gives some fairly flexible formulae of the finite product type.

POSTAL ADDRESS: Director, Aeronautical Research Laboratories,
P.O. Box 4331, Melbourne, Victoria, 3001, Australia.

CONTENTS

1. INTRODUCTION

A number of problems involving the interaction between a given finite-sized geometry and a far field such as potential flows about aerofoil sections [1], [2], [3], scattering of two dimensional long waves [4], [5] and stress concentration around two dimensional holes [6], [7] can all be neatly solved using external conformal mappings. Their solutions are derived from those for the simple basic circular geometry. Therefore the common problem is to find the conformal transformation from the unit circle $|\zeta| = 1$ to a given closed curve in the z plane [8]. This is equivalent to determining all the equipotentials and force lines external to the given closed curve. Solution of the potential problem using the greatest lower bound of superharmonic functions [9], [10] is not a practically attractive method unless the field is discretized and handled in a Finite Difference scheme. This leaves conformal mapping as the only direct method. However the required conformal transformation must satisfy the following conditions.

(a) The mapping must be one-to-one onto between the regions under consideration. (1a)

(b) The mapping must be continuous and conformal between the two regions. (1b)

(c) The difference $(z - \zeta)$ must vanish to order $\zeta^{-\nu}$ at infinity, where ν is a positive number. (1c)

These conditions are illustrated by examining the following familiar transformations:

The Joukowski transformation

$$z = (\zeta - m)(\zeta - n)(\zeta - (m + n)/2)^{-1} \tag{2}$$

where $m = 1$, $|n| < 1$ satisfies all three requirements and is widely used in theoretical aerodynamics. Likewise a slight change in the roots m, n turns the curve in the z plane into an ellipse which is also widely used in elastostatic and elastodynamics.

Finite trailing edge aerofoils and lenticular shapes bounded by two circular arcs are generated by the Kármán - Trefftz formula ([11], page 78)

$$\frac{z + \nu}{z - \nu} = \left(\frac{\zeta + 1}{\zeta - 1}\right)^{\nu} \tag{3}$$

where ν is a positive number $0<\nu\leq 2$. Again the three requirements are

satisfied. Two cuts are required, between $\zeta = -1$, $\zeta = +1$ and between $z = -\nu$, $z = +\nu$, for the ζ plane and z plane respectively. The sucess of these two transformations is in part owing to the containment of all singularities of $dz/d\zeta$ inside the circles in the ζ plane.

The Schwarz - Christoffel transformation $S(\sigma)$, ([10], page 236) defined by

$$\frac{dS}{d\sigma} = M \prod_{k=1}^{n} (\sigma - \sigma_k)^{-2\beta_k}, \quad \sum_{k=1}^{n} \beta_k = 1, \text{ M a complex constant,} \tag{4}$$

is general enough to generate a number of polygonal curves but it does not satisfy the condition (1c), and therefore is not appropriate to the present problems. It should be noted that if the σ_k's are placed at the vertices of a regular n-sided polygon, all β_k are equal to 1/n, then we get a transformation from the circle passing through n points σ_k's in the σ plane to a regular n-sided polygon in the z plane. As the number n tends to infinity the Schwarz - Christoffel transformation becomes

$$\frac{dS}{d\sigma} = \frac{M}{\sigma^2} ,$$

which is a mapping from the outside of one circle to the inside of another. A modification of this transformation will be considered in Section 5.

Another transformation method uses the infinite series

$$z(\zeta) = \zeta + a_1\zeta^{-1} + a_2\zeta^{-2} + \ldots \tag{5}$$

and can generate any shape as the series is a Laurent expansion for the complex function $z(\zeta)$ which is analytic outside the unit circle $|\zeta| = 1$. However it requires quite a number of terms even for the simplest aerofoil sections and yields no information whatsoever about the singularities or one-to-one onto property. It is noted that Glauert [11] has applied this method to the problems of aerofoils. A well developed theory on the determination of these coefficients for aerofoils is due to Theodorsen and can be found in [12].

Here we propose a transformation of the kind given below by equation (6) or (10) subjected to the condition (7). This transformation will be shown to be indeed the general form for any conformal mapping satisfying the three conditions (1). A formula

is then derived for transformation from the unit circle to an arbitrary given shape. A test case is given for transformations of the finite product type. One special case of this transformation is very similar to that of Schwarz and Christoffel. Application to the problem of aerofoil sections gives some fairly flexible formulae of the finite product type.

2. ANALYSIS OF THE NEW TRANSFORMATION

Let t be a real variable ($0 \le t \le 1$), $\alpha(t)$ be a periodic complex valued function of t with bounded variation and $\phi(t)$ ($|\phi(t)| \le 1$) be a periodic continuous complex-valued function of t. We have the function

$$z(\zeta) = \int_0^1 \phi(t)\,d\alpha(t) + \int_0^1 \phi(t)\,dt + \exp\left[\int_0^1 \log(\zeta-\phi(t))\,d\alpha(t) + \int_0^1 \log(\zeta-\phi(t))\,dt\right] \tag{6}$$

defined for all $|\zeta| > 1$ as all the conditions for the Riemann - Stieltjes integrals are satisfied. The function $\log(\zeta-\phi(t))$ is chosen such that it is continuous for $0 \le t \le 1$ with each given value of ζ. Furthermore, if the images of the circles $|\zeta| = 1 + \epsilon$, $\epsilon > 0$, ϵ very small, are non-self-intersecting closed paths and also if it is assumed that

$$\int_0^1 \frac{1}{\zeta-\phi(t)}\,d\,(\alpha(t) + t) \neq 0 \text{ for all finite } \zeta,\ |\zeta| > 1 \tag{7}$$

then all the requirements (1) are satisfied. This significance of the assumption (7) is discussed in detail below, in Section 4.

First $z(\zeta)$ is continuous, its derivative is

$$\frac{dz}{d\zeta} = \exp\left[\int_0^1 \log(\zeta-\phi(t))\,d(\alpha(t) + t)\right] \int_0^1 \frac{1}{\zeta-\phi(t)}\,d(\alpha(t) + t), \tag{8}$$

and $z(\zeta)$ is thus an analytic function. The exponential factor is always non-zero and the last integral has been assumed in equation (7) to be non-zero. Hence condition (1b) is satisfied.

For large value of ζ we have the expansion

$$z - \int_0^1 \phi(t)\,d(\alpha(t) + t) = \zeta \exp\left\{\int_0^1 \log\left(1 - \frac{\phi(t)}{\zeta}\right) d(\alpha(t) + t)\right\}$$

$$= \zeta \exp\left\{-\sum_{k=1}^{\infty} \frac{1}{k\zeta^k} \int_0^1 \phi^k(t)\,d(\alpha(t) + t)\right\}, \tag{9}$$

using Arzelà's theorem. After an expansion for the exponential function we get the condition (1c) satisfied. It is then easy to show that for large enough value of $|\zeta|$ the mapping $z(\zeta)$ is one-to-one.

The remaining part of the proof is to prove that the mapping $z(\zeta)$ is one-to-one onto between the exterior of the unit circle in the ζ plane and the exterior of its image in the z plane.

Now let ζ describes a circle of very large radius centered on the origin of the ζ plane. The image $z(\zeta)$ describes a closed path in the z plane which is very close to this large circle if we superimpose the two planes. Let the large circle in the ζ plane expand continuously as $|\zeta|$ tends to infinity. The closed path in the z plane then also expands continuously and sweeps through every point exterior to its initial state. Hence outside some circle $|\zeta| = M$, the mapping $z(\zeta)$ is one-to-one onto.

Since the mapping is conformal with $dz/d\zeta$ non-zero everywhere we can unambiguously define the "interior" and "exterior" of each closed path which is the image of some circle centred on the origin in the ζ plane. Now draw a nest of concentric circles in the ζ plane between $|\zeta| = 1 + \varepsilon$ and $|\zeta| = M$. The image of this nest is another nest of closed paths in the ζ plane. Each closed path of this nest is wholly on the interior or exterior of every other of the nest, and it can have closed paths as close to it as we like on both its exterior or interior. We now draw a circle $|\zeta| = M$ in the z plane and let it shrink continuously to $|\zeta| = 1 + \varepsilon$. Its image closed path also shrinks continuously from its initial state to its final state. No region on the z plane bounded by these two closed curves can be left outside this sweep by consideration of the properties of the nest of image closed paths in the z plane.

The mapping $z(\zeta)$ is also one-to-one in the range $M \geq |\zeta| \geq 1 + \varepsilon$. For points of different radii in the ζ plane we have different closed paths in the z plane. The image of each circle centered on the origin of the ζ plane can not fold back on itself either, as $dz/d\zeta$ is everywhere non-zero. Therefore no two points in the ζ plane between the circles $|\zeta| = 1 + \varepsilon$ and $|\zeta| = M$ can have the same image.

This completes the proof.

The function $z(\zeta)$ for $|\zeta| = 1$ is defined to be the limit as $|\zeta| \to 1$, $|\zeta| > 1$ of $z(\zeta)$. Note that due to the one-to-one property of the mapping, $|z(\zeta)|$ is bounded as $|\zeta|$ tends to unity.

When $\phi(t)$ assumes the value $\exp(i2\pi t)$, we have a special case of equation (6), which is

$$z(\zeta) = \int_0^1 e^{i2\pi t}\, d\alpha(t) + \exp\left[\int_0^1 \log(\zeta - e^{i2\pi t})[d\alpha(t) + dt]\right]. \qquad (10)$$

With an appropriate cut in the ζ plane, we can show that equation (10) is indeed the general form for any conformal transformation satisfying conditions (1), as in the following:

Let $W(\zeta)$ be a given conformal transformation satisfying conditions (1) and $W(e^{i2\pi t})$ is a function of t with bounded variation

Choose a point W_c strictly <u>interior</u> to the image of the unit circle $|\zeta| = 1$. Consider the contour integral

$$\oint_\gamma \log\,(W(\zeta) - W_c)\frac{d\sigma}{\sigma - \zeta} = 0$$

with the cut on the ζ plane following the unit circle ω then radiating from the point +1 to $+\infty$. The curve γ is along the unit circle, the radial cut, the infinite circle, and back to the unit circle along the radial cut. In this way the function $\log\,(W(\zeta) - W_o)$ is defined and single-valued for all points ζ not on the cut. The components of this contour integral give

$$-2\pi i \log\,(W(\zeta) - W_c) = \oint_\omega \log\,[W(\sigma) - W_c]\,\frac{d\sigma}{\sigma - \zeta} + \int_1^R 2\pi i\,\frac{dx}{x - \zeta}$$

$$- \oint_{\Omega_R} \log\,[W(\sigma) - W_c]\,\frac{d\sigma}{\sigma - \zeta} \quad ,$$

where Ω_R is a circle of radius R centered on the origin, R tending to infinity.

The first integral on the right hand side is then integrated by parts. The third integral has its integrand expanded using the condition (1c). The result is

$$\log\,(W(\zeta) - W_c) = \int_{t=0}^{1} \log\,(\zeta - e^{i2\pi t})d\left[\frac{\log\,(W(e^{i2\pi t}) - W_c)}{2\pi i}\right]$$

The square bracket is a continuous complex-valued function of t. Due to our assumption that $W(e^{i2\pi t})$ is of bounded variation the square bracket in the last integral also has bounded variation. Therefore

$$W(\zeta) - W_c = \exp\left(\int_{t=0}^{1} \log\,(\zeta - e^{i2\pi t})d(\beta(t) + t)\right)$$

where $\beta(t)$ is a continuous periodic, complex valued function of t, having bounded variation. (The function $\beta(t)$ may not have bounded variation if W_c is on the image of the unit circle).

Routine integration shows that

$$\int_{t=0}^{1} e^{i2\pi t} \, d\left[\frac{\log\,(W(e^{i2\pi t}) - W_c}{i2\pi}\right] = W_c \,,$$

satisfying the requirement

$$W_c = \int_0^1 e^{i2\pi t} d\beta(t)$$

of equation (6).

Hence formula (10) is indeed the general form for any conformal transformation $z(\zeta)$ satisfying the conditions (1) subjected to the conditions (7) and that $W(e^{i2\pi t})$ is of bounded variation.

Section 3 will show that equation (6) is only a special case of equation (10), both subjected to the condition (7). Henceforth we take (10) as the standard form for our transformation.

The form of equation (10) can be simplified even further when the imaginary part of the function $\alpha(t)$ is continuous. Let us denote $\mathrm{Im}(\alpha(t))$ by $b(e^{i2\pi t})$ and consider the integral

$$\int_{t=0}^{1} \log(\zeta - e^{i2\pi t}) db(e^{i2\pi t}) = - \int_{t=0}^{1} b(e^{i2\pi t}) d \log(\zeta - e^{i2\pi t})$$

which is the imaginary part of the second integral in equation (10) Since $b(e^{i2\pi t})$ is continuous, the internal boundary value problem for the unit circle has a solution $b(\xi + i\eta)$ which is harmonic in the (ξ, η) plane and takes the given boundary values on the unit circle. Therefore

$$\oint_\gamma b(\sigma) d\tau(\sigma) - \oint_\gamma i\tau(\sigma) \frac{\partial b}{\partial n} ds \quad \text{where} \quad b(\xi, \eta) \text{ is harmonic}$$

and $\tau(\xi + i\eta)$ is an analytic function, is identically zero for any closed loop γ which can be shrunk to a single point without crossing any pole of $b(\xi, \eta)$ and $(\xi + i\eta)$ (Proof at the end of this section). The unit circle can be shrunk to the origin without passing through any pole of $b(\xi, \eta)$ or $\tau(\xi + i\eta)$ and is one such closed path. Therefore we have

$$\int_{t=0}^{1} b(e^{i2\pi t})d \log(\zeta - e^{i2\pi t}) = i\int_{t=0}^{1} \log(\zeta - e^{i2\pi t}) \frac{\partial b}{\partial n} ds.$$

In a similar manner, we have

$$\int_{t=0}^{1} e^{i2\pi t} db(e^{i2\pi t}) = - \int_{t=0}^{1} b(e^{i2\pi t}) d(e^{i2\pi t})$$

$$= - \int_{t=0}^{1} e^{i2\pi t} \frac{\partial b}{\partial n} ds$$

Consequently if the periodic function $\alpha(t)$ has its real part of bounded variation and its imaginary part continuous formula (10) becomes

$$z(\zeta) - \int_0^1 e^{i2\pi t} d\,\alpha(t) = z(\zeta) - \int_0^1 e^{i2\pi t} d[\mathrm{Re}(\alpha(t)) + 2\pi \frac{\partial b}{\partial n}\, t]$$

$$= \exp \int_{t=0}^{1} \log(\zeta - e^{i2\pi t}) d[\mathrm{Re}(\alpha(t)) + (1 + 2\pi \frac{\partial b}{\partial n})\, t] \qquad (11)$$

where $b(\zeta, \eta)$ is a harmonic function defined on the disc $|\zeta| \leq 1$ and equal to $\mathrm{Im}\,(\alpha(\zeta))$ for $|\zeta| = 1$.

Putting $\psi = 1/\bar{\zeta}$, $W(\psi) - W_o = \bar{z}(1/\psi) - \bar{z}_o$ we have a formula corresponding to (10) for the internal problem.

$$\log(W(\psi) - W_o) = - \int_0^1 \log(\frac{1}{\psi} - e^{-i2\pi t}) d[\gamma(t) + t] \qquad (12)$$

where $\gamma(t)$ is a complex valued periodic function of t with bounded variation.

Notes to Section 2

Let an integral I be defined by

$$I = \oint_\gamma b(\sigma) d\tau(\sigma) = [b(\sigma)\tau(\sigma)]_\gamma - \oint_\gamma \tau(\sigma) db(\sigma)$$

where γ is a closed path containing no pole of either $b(\sigma)$ or $\tau(\sigma)$, $b(\sigma)$ being a real-valued harmonic function of $(\xi, \eta) = (\mathrm{Re}(\sigma), \mathrm{Im}(\sigma))$ and $\tau(\sigma)$ being a (complex valued) analytic function of σ inside the curve γ. The integrals are defined when one of the functions is continuous and the other is of bounded variation on γ.

We have

$$I = \oint_{\gamma} b(\xi, \eta)\left(\frac{\partial \tau_r(\xi,\eta)}{\partial s} + i\frac{\partial \tau_i(\xi,\eta)}{\partial s}\right) ds.$$

By the analyticity of $\tau(\sigma) = \tau_r + i\tau_i$ we can write

$$I = i\oint_{\gamma} b(\xi,\eta)\left(\frac{\partial \tau_r(\xi,\eta)}{\partial \eta} + \frac{\partial \tau_i(\xi,\eta)}{\partial \eta}\right) ds \quad ,$$

The real and imaginary part of the above integral is of the form $B\int_{\gamma}\frac{\partial C}{\partial n} ds$ where B and C are harmonic and n is the normal for the curve γ. Noting that Green's identity is

$$\int_{\partial A} (B\nabla C - C\nabla B)\cdot n ds = \int_{A} (B\nabla^2 C - C\nabla^2 B) da = 0,$$

where A is the part of the plane (ξ, η) enclosed by the curve γ, we have

$$\oint_{\gamma} b(\xi,\eta) d\tau(\xi + i\eta) - i\oint_{\gamma} \tau(\xi + i\eta)\frac{\partial b}{\partial n} ds = 0 \qquad (13)$$

for any closed path γ not containing any pole of $b(\xi, \eta)$ or $\tau(\sigma)$, and s is the arc length of the curve γ.

As an example we choose $b(\xi, \eta)$ such that it vanishes on the unit circle. The familiar real valued Green's function becomes.

$$b(\sigma) = \log\left|\frac{1 - 1/\bar{a}}{1 - a}\right| + \log\left|\frac{\sigma - a}{\sigma - 1/\bar{a}}\right| \quad .$$

Its poles are at a ($|a| < 1$) and $1/\bar{a}$ which is outside the unit circle. Choosing the curve γ encircling the point a, then along a radial line, the unit circle and back to the small circle around a we have

$$\oint_{\gamma_a} b(\sigma) d\tau(\sigma) - \oint_{\omega} b(\sigma) d\tau(\sigma) = i\oint_{\gamma_a} \tau(\sigma)\frac{\partial b}{\partial n} ds - i\oint_{\omega} \tau(\sigma)\frac{\partial b}{\partial n} ds.$$

The second integral is zero as $b(\sigma)$ is identically zero on ω. The first integral vanishes as the radius γ_a reduces to zero

as $\tau(\sigma)$ is analytic at the point a. The left hand side is thus zero. The right hand side gives

$$i2\pi\iota(a) = i\oint_{\omega} \tau(\sigma)\left[\frac{1}{|\sigma-a|}\cos\left(\arg\frac{\sigma-a}{\sigma}\right) - \frac{1}{(\sigma-\frac{1}{\bar{a}})}\cos\left(\arg\frac{\sigma-\frac{1}{\bar{a}}}{\sigma}\right)\right] ds$$

$$= \oint_{\omega} \tau(\sigma)\,\frac{\sigma}{(\sigma-a)(\sigma-\frac{1}{\bar{a}})}\,(a-\tfrac{1}{\bar{a}})\,\frac{d\sigma}{\sigma}$$

which reduces to a standard Poisson formula upon simplification (see [13] and its notes).

3. DETERMINATION OF THE FUNCTION $\alpha(t)$

Let us consider initially the problem of determining $\alpha(t)$ of equation (10) for a given function $z(\zeta)$ of the form

$$z(\zeta) = z_c + \prod_{k=1}^{n} (\zeta - \phi_k)^{\Gamma_k}, \quad \sum_{k=1}^{n} \Gamma_k = 1 , \tag{14}$$

where Γ_k's may be complex and each ϕ_k is inside the unit circle ($|\phi_k| \leq 1$) unless its corresponding Γ_k has a positive integer value (1, 2, 3,...). In this way, the function $z(\zeta)$ is defined unambiguously outside the unit circle $|\zeta| = 1$. Its derivative is

$$\frac{dz}{d\zeta} = \prod_{k=1}^{2n-1} (\zeta - \phi_k)^{\Delta_k}, \quad \sum_{k=1}^{2n-1} \Delta_k = 0, \tag{15}$$

with all the ϕ_k assumed to be contained in the unit disk unless its corresponding Δ_k is equal to zero. The (n-1) values of ϕ_k for $k = n + 1, 2n - 1$, are the roots of a polynominal resulting from the differentiation of $z(\zeta)$. When two conditions of (14) and (15) are satisfied we have all our requirements (1) satisfied with z_c standing for

$$z_c = \int_0^1 e^{i2\pi t}\, d\alpha(t). \tag{16}$$

Now let the function $\alpha(t)$ be the value of a complex function $\eta(\zeta)$ on the unit circle $\zeta = e^{i2\pi t}$. The problem is to determine a function $\eta(\zeta)$ such that equation (10) is satisfied on the unit circle of the ζ plane.

Since the contour integral

$$\oint_{\omega} \left(\prod_{k=1}^{2n-1} (\sigma - \emptyset_k)^{(\Delta_k - \Gamma_k)} \right) d\sigma$$

has the purely imaginary value of $2\pi i$ we can define a single-valu analytic function $\eta(\zeta)$

$$\eta(\zeta) = \frac{1}{2\pi i} \left\{ \int_{\sigma=1}^{\zeta} \left(\prod_{k=1}^{2n-1} (\sigma - \emptyset_k)^{(\Delta_k - \Gamma_k)} \right) d\sigma - \log \zeta \right\} ,$$

where the integral path is outside the unit disc. On the unit ci we have

$$\alpha(t) = \frac{1}{2\pi i} \left\{ \int_{\sigma=1}^{e^{i2\pi t}} \left(\prod_{k=1}^{2n-1} (\sigma - \emptyset_k)^{(\Delta_k - \Gamma_k)} \right) d\sigma \right\} - t$$

Substitution of this function $\alpha(t)$ into equation (10) noting that

$$\prod_{k=1}^{2n-1} (\sigma - \emptyset_k)^{(\Delta_k - \Gamma_k)} = \frac{d}{d\sigma} \left(\log \prod_{k=1}^{n} (\sigma - \emptyset_k)^{\Gamma_k} \right),$$

does give equation (14). Also substitution of this function $\alpha($ into equation (8) gives equation (15). Therefore the functio $\alpha(t)$ is the required function.

As an application of the above calculations we can wr down the expression for $(\zeta - \emptyset)$ where $|\emptyset| < 1$ as

$$\zeta - \emptyset = \exp\left\{ \int_{t=0}^{1} \log (\zeta - e^{i2\pi t}) \frac{1}{2\pi i} d [\log (e^{i2\pi t} - \emptyset)] \right\}$$

The familiar transformation from a circle to an ellip or a Joukowski aerofoil therefore can be written as

$$z(\zeta) - \frac{m+n}{2} = (\zeta - m)(\zeta - n)(\zeta - \frac{m+n}{2})^{-1}, \quad |m|, |n| \leq 1$$

$$= \exp \left\{ \int_{t=0}^{1} \log (\zeta - e^{i2\pi t}) \frac{1}{2\pi i} d[\log (\frac{e^{i2\pi t} - m}{e^{i2\pi t}}$$

If we employ the method of deriving equation (11) then the quantities inside the square brackets in the last two integrals can be replaced by two purely real ones. However it is simpler in practice to use finite factorization rather than the formulae just derived.

The next case we consider is very practical. It is the problem of determining the transformation (10) when the shape of a non-self-intersecting, continuous, closed curve C in the z-plane is given in parametric form

$$w(u) \varepsilon C,\ w(0) = w(1),\ 0 \leq u \leq 1 \tag{19}$$

(An example is the ellipse $w = \cos(2\pi u^2) + 2i\sin(2\pi u^2)$ given in the z-plane).

The problem is to determine a function $z(\zeta)$ such that

$$z(\zeta) = \zeta + \sum_{n=0}^{\infty} a_n \zeta^{-n} \tag{20a}$$

and

$$w(u) = kz(\zeta) \text{ for } \zeta = e^{i2\pi t}, \tag{20b}$$

where k is a positive scaling constant and t is a monotonic continuous function of u with

$$t(0) = 0,\ t(1) = 1. \tag{20c}$$

For the moment we assume that such a function t(u) and the function $z(\zeta)$ has been determined.

Take a point z_c <u>interior</u> to the image of the unit circle $|\zeta| = 1$ and examine the function $(z(\zeta) - z_c)/\zeta$. This function is single-valued and analytic for all values of ζ outside the unit circle. The image of any continuous curve external to $|\zeta| = 1$ in the ζ plane can be continuously shrunk to a single point +1 in the $(z(\zeta) - z_c)/\zeta$ plane. Therefore a single-valued function $\log((z(\zeta) - z_c)/\zeta)$ can be defined for all points $|\zeta| \geq 1$ and has a Laurent expansion

$$\log\left(\frac{z(\zeta) - \zeta_c}{\zeta}\right) = \sum_{n=0}^{\infty} b_n \zeta^{-n}$$

with b_o equal to zero due to the condition (20a).

By taking a contour integral around the unit circle $|\zeta| = 1$ we have

$$\int_{u=0}^{1} (\log(z(\zeta) - z_c) - i2\pi t) e^{i2\pi mt} dt(u) = 0 \text{ for } m = 0,1,2,3.. \quad (21)$$

Especially, for $m = 0$

$$\int_{u=0}^{1} (\log(w(u) - w_c) - i2\pi t) dt(u) = \log k.$$

Thus

$$\int_{0}^{1} \log|w(u) - w_c| \, dt(u) = \log k \quad (22)$$

for all points w_c interior to the curve C.

The function t(u) determined by this equation (21) is the required function if it is monotonic. It is not difficult to see that dt/du is the charge distribution (with respect to u) on the boundary C of a two dimensional conductor in an external potential problem.

Once the function t(u) is solved from (22), the required function $\alpha(t) = \log (z(e^{i2\pi t}) - z_c)$ is determined. There is only one function t(u) satisfying equations (20). Its uniqueness is proved by showing that any function $z(\zeta)$ satisfying all conditions (1) and

$$|z(\zeta)| = 1 \text{ for } |\zeta| = 1$$

must be identically equal to ζ. The proof uses Laurent series expansion and the Maximum Modulus theorem.

4. THE RESTRICTION POSED BY EQUATION (7)

The restriction (7) has to be observed for the proposed transformation (10) to work. The physical meaning of this restriction is now examined. Equation (11) has shown that for any continuous $\alpha(t)$ we can find another real function $[\mathrm{Re}(\alpha(t)) + (1 + 2\pi\partial b/\partial n)t]$ of equivalent effect therefore we shall content ourselves with the case of only real $\alpha(t)$ in this section.

Consider the real part of the last integral on the right hand side of equation (10). It is the potential generated by a charge distribution proportional to $d\alpha/dt$ on the periphery of the unit circle. The restriction (7) then requires that its corresponding electric field should not vanish anywhere outside the unit circle. This vanishing can precipitate the formation of kinked potential lines (such as those apple shaped curves associated with $\log z = (1 + k) \log \zeta - k \log (\zeta - a)$) which leads to a finite area in the z plane being the image of a line segment in the ζ plane. In other words the area of uniform potential in the z plane is more than the area interior to the curve C, hence there may exist some area of uniform potential adjacent to some two-dimensional conductors which is not totally enclosed by the conductors. There are analogous results for three dimensional problems. One advantageous use of such results is with the operation of Van der Graaff static generators.

There is more than one way to satisfy the condition (7) The easiest way to satisfy this condition is to have $[\alpha(t) + t]$ monotonically increasing.

The existence of those "regions of degeneration" has diverse physical meanings besides that already cited: In problems of potential flows we have regions of separation, in wave scattering problems we require points of complex mass attached to the boundary of the basic circular hole to generate all patterns of far-field scattering.

5. RELATIONSHIP TO THE SCHWARZ-CHRISTOFFEL TRANSFORMATION

Remember that a Schwarz-Christoffel transformation is of the form given by equation (4). This makes the point $|\zeta| = \infty$ a singular point. We now consider a modified formula

$$\frac{dz}{d\zeta} = \frac{1}{\zeta^2} \prod_{k=1}^{n} (\zeta - \phi_k)^{2\gamma_k}, \quad \gamma_k\text{'s real, all } |\phi_k| = 1, \quad \sum_{k=1}^{n} \gamma_k = 1 \qquad (23)$$

It is obvious that $z(\zeta)$ describes a polygonal curve when ζ describes the unit circle just as with the Schwarz-Christoffel transformation. However the transformation (23) can satisfy condition (1c) whereas the other can not.

By writing

$$\zeta = \lim_{m\to\infty} \prod_{k=1}^{m} (\zeta - e^{i2\pi k/m})^{\frac{1}{m}} \qquad (24)$$

we can readily put equation (23) into the form (15) which has been proved to be a particular result of the formula (10).

Using σ to denote $1/\zeta$ we can write (23) as $dz/d\sigma = \prod_{k=1}^{n} (1/\sigma - \phi_k)^{2\gamma_k}$, which gives

$$\left\{ \prod_{k=1}^{n} [(\frac{1}{\sigma} - \phi_k)(\sigma - \bar{\phi}_k)] ^{-2\gamma_k} \right\} \frac{dz}{d\sigma} = \prod_{k=1}^{n} (\sigma - \bar{\phi}_k)^{-2\gamma_k} \text{ on the circle } |\sigma| = 1 \qquad (25)$$

This last expression is very similar to the Schwarz-Christoffel formula (4). We can therefore consider the latter transformation as a composite one from the $\sigma = 1/\zeta$ plane to the final S plane with the ratio dS/dz real on the circle $|\sigma| = 1$. This consideration explains its peculiar mapping from the interior to the exterior of the unit circle in the σ-plane as n tends to infinity.

Just as with the Schwarz-Christoffel transformation there is a linear form for formula (10). This is obtained by applying the inverse Mobius transformation $\tau = i(1 + \zeta)/(1 - \zeta)$ to the ζ plane. The combined transformation $z(\zeta(\tau))$ is then from the half-plane $Im(\tau) < 0$ onto the exterior of a closed continuous curve C.

6. APPLICATION OF FORMULA (6) TO THE GENERATION OF AEROFOIL SECTIONS

In generating aerofoil sections it is preferable to employ formula (6). The method is to have one zero of order (1 - b) for

$dz/d\zeta$ on the unit circle $|\zeta| = 1$ while keeping all its other singular points inside the unit circle. The real value b is the internal angle of the corner (trailing edge) of the aerofoil section.

The Joukowski sections are thus the case with

$$z(\zeta) - z_c = (\zeta - m)(\zeta - n)(\zeta - \frac{m+n}{2})^{-1} , \tag{26}$$

$$\frac{dz}{d\zeta} = \frac{(\zeta - \frac{m+n}{2} + i\frac{m-n}{2})(\zeta - \frac{m+n}{2} - i\frac{m-n}{2})}{(\zeta - \frac{m+n}{2})^2} \tag{27}$$

and the complex constants m,n are chosen such that one of the values $(\frac{m+n}{2} \pm i\frac{m-n}{2})$ is on the unit circle and the other is inside. The pole of $dz/d\zeta$ is the center of these two points and is therefore put inside the unit circle.

The Karman-Trefftz sections are given by

$$q(\sigma) = \nu \frac{(\frac{\sigma + 1}{\sigma - 1})^{\nu} + 1}{(\frac{\sigma + 1}{\sigma - 1})^{\nu} - 1} , \quad 1 < \nu \leq 2 \tag{28}$$

and

$$\frac{dq}{d\sigma} = \frac{4\nu^2}{(\sigma + 1)(\sigma - 1)} \frac{(\frac{\sigma + 1}{\sigma - 1})^{\nu}}{\left[(\frac{\sigma + 1}{\sigma - 1})^{\nu} - 1\right]^2} \tag{29}$$

which require cuts between $q = -\nu$, $q = +\nu$ and $\sigma = -1$, $\sigma = +1$ in the q and σ planes respectively (see [14], page 64).

Employing the intermediate transformations $w(q)$ and $u(\sigma)$ we have

$$\frac{q + \nu}{q - \nu} = w(q) = u(\sigma) = (\frac{\sigma + 1}{\sigma - 1})^{\nu} .$$

This shows clearly that there is no singularity of $dq/d\sigma$ outside the circle $|\sigma| \leq 1$ save for a possible singularity at $|\sigma| = \infty$. This latter point is shown by to be not a singular one. Two new variables $z(q)$ and $\zeta(\sigma)$ are then introduced as the same linear function of q and σ respectively. The composite transformation $z(\zeta)$ then gives a Karman-Trefftz section as the image of a unit circle $|\zeta| = 1$.

Note that the image $\sigma(\zeta)$ of $|\zeta| = 1$ in the σ plane must not cross the cut between $\sigma = -1$ and $\sigma = +1$. By examining the intermediate variables $u(\sigma)$ and $w(q)$ we see that the last condition automatically ensures that no singularity for $z(\zeta)$ can exist outside the disc $|\zeta| \leq 1$.

Following the previous two methods of generating aerofoil sections we can have quite a range of transformations in the form

$$z(\zeta) - z_c = (\zeta + m_1)^{\alpha_1}(\zeta + m_2)^{\alpha_2}\ldots(\zeta + m_k)^{\alpha_k}, \quad \sum_{i=1}^{k} \alpha_i = 1, \tag{30}$$

where α_i's are all real. For $z(\zeta)$ to be defined uniquely every m_i must be contained inside the unit disc $|\zeta| \leq 1$ unless its corresponding power index α_i is a positive integer (1, 2, 3 ...).

The derivative $dz/d\zeta$ is given by

$$\frac{dz}{d\zeta} = (z(\zeta) - z_c)\left[\frac{\alpha_1}{\zeta + m_1} + \frac{\alpha_2}{\zeta + m_2} + \ldots + \frac{\alpha_k}{\zeta + m_k}\right] \tag{31}$$

and all singularities (zeros and poles) of $dz/d\zeta$ are contained within the disc $|\zeta| \leq 1$. This last condition in turn requires that $|m_i|$ is inside the unit disc $|\zeta| \leq 1$ unless its corresponding power index α_i is equal to one. (The Joukowski aerofoil given by (6.1) has one zero of order one of $z(\zeta)$ outside the unit circle).

For examples, a few aerofoil shapes have been produced using the following formulae

$$z(\zeta) - z_c = (\zeta + 1)^{1.9}(\zeta - 0.1 - 0.2i)^{0.1}(\zeta + 0.1 - 0.2i)^{-1}$$

$$z(\zeta) - z_c = (\zeta + 1)^{1.9}(\zeta - 0.1 - 0.2i)^{0.1}(\zeta + 0.1i)^{-1}$$

$$z(\zeta) - z_c = (\zeta + 1)^{1.9}(\zeta)^{-0.45}(\zeta + 0.1)^{-0.45},$$

or we can even use complex powers (in this case we require that $|m_i| \leq 1$ if $(\alpha_i - 1)$ is real and non-zero, $|m_i| < 1$ if α_i is complex) to have

$$z(\zeta) - z_c = (\zeta + 1)^{1.9}(\zeta)^{0.1-0.3i}(\zeta + 0.1)^{-1+0.3i}.$$

Not all choices of m_i and α_i give acceptable aerofoil shapes. For the above four functions $z(\zeta)$ a computer program has been used to ensure that all singularities of $dz/d\zeta$ are contained in the unit disc $|\zeta| \leq 1$ except for one at $\zeta = 1$ to generate a finite acute angle at the trailing edge. The resulting aerofoil sections have been plotted by a computer and the effect of m_i's and α_i's on thickness, camber etc.. was examined (The computer programs are written by Mr. C. A. Martin, the author acknowledges his help to substantiate the proposed method).

7. CONCLUSIONS

The transformation (10) subjected to condition (7) is a continuous form of that given by equation (23). Our representation for $z(\zeta)$ in equation (6) or (10) has some similarities to Wierstrass' theorem on entire functions (see [10], page 195).

The shapes not covered by the transformations (6) or (10) due to the restriction (7) may have regions of finite area which are not in the range of the conformal mapping. For these cases it is mathematically satisfactory to modify the boundary to exclude those regions.

It is noted that the conformal mapping here has some relationship to the more general problem of the existence of a conformal mapping from the interior of a unit circle to an arbitrary simply connected region (see [10], page 230). The latter problem is associated with such famous mathematicians as Riemann and Wierstrass (see [15], page 186). The difference between the mapping considered here and that of the general problem is our requirement (1c). It ought to be mentioned here that the numerical method which combines numerical solution to the external potential field for a given conductor shape with a Fourier series analysis to determine the series expansion (5) for the corresponding conformal mapping can be considered a logical outflow of Riemann's idea for the existential problem.

REFERENCES

1. CURLE N. and DAVIES, H.J., "Modern Fluid Dynamics Vol. 1, p. 73, D. Van Nostrand Co. Ltd., New Jersey, 1968.

2. HOUGHTON E.L., and BROCK, A.E., "Aerodynamics for Engineering Students", Chapter 11, Edward Arnold Ltd, London, 1970.

3. VALLENTINE, H.R., "Applied Hydrodynamics", Chapters 5-6 Butterworth and Co. Ltd., London, 1969.

4. BUCHWALD, V.T. "The diffraction of elastic waves by small cylindrical cavities, J. Austral. Math, Soc. 20 B (1978), 495 - 507.

5. BUCHWALD, V.T. and TRAN-CONG, T. The diffraction of long elastic waves by elliptic cylindrical cavities, J. Austral. Math. Soc. (1984).

6. MUSKHELISHVILI, N.I., "Some basic problems of the mathematical theory of elasticity", transulated by J.R.M. Radok, Nordhoff, 1953.

7. TIMOSHENKO, S.P. and GOODIER, J.N., "Thoery of elasticity", Chapter 6, McGraw-Hill, Kogakusha Ltd., Tokyo, 1970.

8. CHURCHILL, R.V., BROWN, J.W. and VERHEY, R.F., "Complex variables and applications", 3rd ed., McGraw-Hill Int. Book Co., Tokyo, 1974.

9. COPSON, E.T., "partial differential equations", Cabridge Univ. Press, Cambridge, 1975.

10. AHLFORS, L.V., "Complex analysis", 3rd ed., McGraw-Hill Kogakusha Ltd., Tokyo, 1979.

11. GLAUERT, H., "The elements of aerofoils and airscrews theory", 2nd ed., Cambridge Univ. Press, Cambridge, 1959.

12. ABBOTT, I.H. and DONEHOFF, A.E., "Theory of wing sections", Dover Publichations Inc., New York, 1959.

13. DERRICK, W.R., "Introductory complex analysis and applications", Academic Press, New York, 1972.

14. PHILLIPS, E.G., "Functions of a complex variable", 8th ed., Oliver and Boyd Ltd., Edinburgh, 1968.

15. COPSON, E.T. "An introduction to the theory of functions of a complex variable", Oxford Univ. Press, Oxford, 1955.

DISTRIBUTION

AUSTRALIA

Department of Defence

Central Office

Chief Defence Scientist)
Deputy Chief Defence Scientist)
Superintendent, Science and Program Administration) (1 copy)
Controller, External Relations, Projects and Analytical Studies)
Defence Science Advisor (U.K.) (Doc Data sheet only)
Counsellor, Defence Science (U.S.A.) (Doc Data sheet only)
Defence Science Representative (Bangkok)
Defence Central Library
Document Exchange Centre, D.I.S.B. (18 copies)
Joint Intelligence Organisation
Librarian H Block, Victoria Barracks, Melbourne
Director General - Army Development (NSO) (4 copies)

Aeronautical Research Laboratories

Director
Library
Superintendent - Aerodynamics
Divisional File - Aerodynamics
Author: T. Tran Cong
D. A. Secomb
C. A. Martin
M. Cooper
J. S. Dobik
R. A. Feik
L. F. Rose

Materials Research Laboratories

Director/Library

Defence Research Centre

Library

RAN Research Laboratory

Library

Navy Office

Navy Scientific Advisor

..2/..

DISTRIBUTION (CONT'D)

Army Office

Scientific Adviser - Army

Air Force Office

Air Force Scientific Adviser

Central Studies Establishment

Information Centre

Department of Defence Support

Government Aircraft Factories

Manager

Universities and Colleges

Sydney N.S.W.	Dr. G. P. Steven, Dept. of Aeronautical Engineering
	Professor V. T. Buchwald, School of Mathematics
	Dean, Faculty of Science

SPARES (20 copies)

TOTAL (69 copies)

Department of Defence

DOCUMENT CONTROL DATA

1. a. AR No	1. b. Establishment No	2. Document Date	3. Task No
AR-003-960	ARL-AERO-TM-367	SEPTEMBER 1984	DST 84/037

4. Title	5. Security a. document	6. No Pages
A CONFORMAL MAPPING SUITABLE FOR PROBLEMS INVOLVING INTERACTION BETWEEN GIVEN GEOMETRIES AND KNOWN FAR FIELDS.	UNCLASSIFIED b. title c. abstract U U	19 7. No Refs 15

8. Author(s)	9. Downgrading Instructions
TON TRAN-CONG	

10. Corporate Author and Address	11. Authority (as appropriate)
Aeronautical Research Laboratories, P.O. Box 4331, Melbourne Vic. 3001	a.Sponsor b.Security c.Downgrading d.Approval

12. Secondary Distribution (of this document)

Approved for public release

Overseas enquiries outside stated limitations should be referred through ASDIS, Defence Information Services Branch, Department of Defence, Campbell Park, CANBERRA ACT 2601

13 a. This document may be ANNOUNCED in catalogues and awareness services available to ..

No limitations

13 b. Citation for other purposes (ie casual announcement) may be (select) unrestricted (or) as for 13 a.

14. Descriptors	15. COSATI Group
Conformal mapping Air foils Far fields	12010 01010

16. Abstract

A conformal transformation formula using Riemann-Stieltjes integrals is derived for use with problems involving the interaction between a given finite-sized geometry and a known far field. The derivative of this transformation is non-singular in the domain considered and tends to one at infinity. A formula is derived for transformation from the unit circle to the exterior of an arbitrarily given continuous curve with bounded variation. A special case of the transformation is very similar to that of Schwarz-Christoffel. Application to the generation of aerofoils gives some fairly flexible formulae of the finite product type.

This page is to be used to record information which is required by the Establishment for its own use but which will not be added to the DISTIS data base unless specifically requested.

16 Abstract *(Contd)*

17. Imprint

Aeronautical Research Laboratories, Melbourne

18. Document Series and Number	19. Cost Code	20. Type of Report and Period Covered
Aerodynamics Technical Memorandum 367	527740	

21. Computer Programs Used

22. Establishment File Ref(s)

END

FILMED

2-85

DTIC