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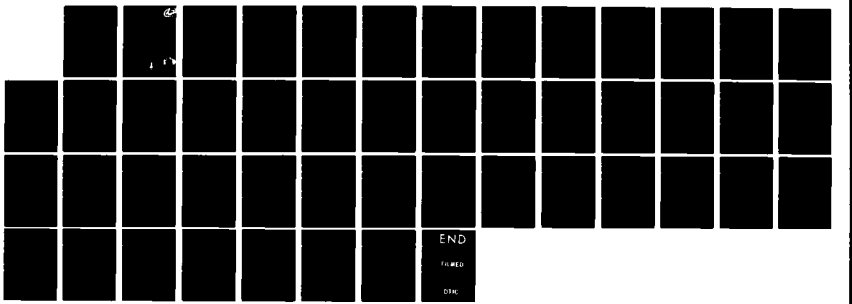

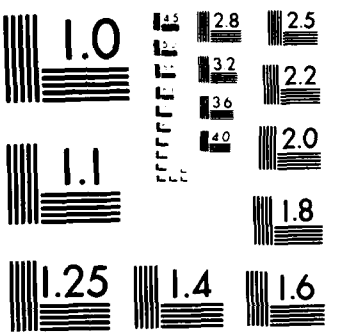

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# INEQUALITIES FOR THE M/G/∞ QUEUE AND RELATED SHOT NOISE PROCESSES

BY

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# Inequalities for the M/G/∞ Queue and Related Shot Noise Processes

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## 1. Introduction and Main Results.

This report deals with shot noise processes $X(t)$ which are formed by the superposition of pulses having random durations. The parameter $t$ will be either a point on the line or on the circle which are denoted by $L$ and $C$ respectively. The simplest such process is the M/G/∞ queue which is formally defined by

$$X = \sum_{i=-\infty}^{\infty} I_{[T_i, T_i+Z_i]}$$

where $\{T_i: -\infty < i < \infty\}$ are the ordered arrival times of a Poisson process having rate $\lambda$, $\{Z_i: -\infty < i < \infty\}$ are nonnegative random variables with finite expectation which are independent and identically distributed according to the distribution $F$, and $I_{[a,b]}$ denotes the indicator function of the closed interval $[a,b]$. Thus $X(t)$ is the number of busy servers at time $t$ in an M/G/∞ system where customers arrive at rate $\lambda$ and service times have the distribution $F$. $X$ is a stationary process with $EX(t) = \lambda(EZ_0)$ for all $t$. We use the notation $X \sim L(\lambda, F)$ to refer to the process $X$ determined by $\lambda$ and $F$.

We now define an analogous process $X$ on a circle having circumference $P$. Let $\{Z_i: 1 \leq i \leq n\}$ be independent random variables satisfying $0 \leq Z_i < P$ for all $i$. $F_i$ will denote the distribution of $Z_i$. Place

n arcs with lengths $Z_1, Z_2, \ldots, Z_n$ uniformly and independently on the circle. For any point t on the circle, let X(t) be the number of these arcs which cover t. It is more convenient notationally to think of X as a periodic step function (with period P) defined by

$$X = \sum_{i=1}^{n} J_{[T_i, T_i+Z_i]}$$

where $\{T_i : 1 \leq i \leq n\}$ are independent random variables distributed uniformly on the interval $[0,P)$ and $J_{[a,b]}$ is a periodic indicator function defined when $0 \leq b-a < P$ by

$$J_{[a,b]} = \sum_{k=-\infty}^{\infty} I_{[a+kP, b+kP]} .$$

We use the notation $X \sim C_p(F_1, F_2, \ldots, F_n)$ to refer to the process X on the circle (or the equivalent periodic process on the line) which is determined by the distributions $F_1, F_2, \ldots, F_n$.

We shall present results which indicate how the processes $L(\lambda, F)$ and $C_p(F_1, F_2, \ldots, F_n)$ change as the distributions F or $F_i$ are altered to increase their variability. The precise notion of "increasing variability" we shall use is contained in the following definition.

Definition: Suppose Y and Z are random variables with $E|Y| < \infty$ and $E|Z| < \infty$ having distributions F and G respectively. If $E\phi(Y) \leq E\phi(Z)$ for all convex functions $\phi: \mathbf{R} \to \mathbf{R}$, then we say that $F \prec G$ or equivalently $Y \prec Z$.

Note that $Y \prec Z$ implies $EY = EZ$ and $\text{Var } Y \leq \text{Var } Z$.

The variability ordering $\prec$ is well known and has been extensively used to derive inequalities in queueing theory. See Whitt [14] for an example of this use and Stoyan [11,12] for a survey of some of the literature. Rolski [6] and Stoyan [12] are good sources on partial orderings of probability distributions. Ross [7] gives an elementary treatment of variability orderings.

Our results concern random variables which may be written as functionals $H(X)$ of the process $X$. We assume that $H(f)$ is defined for all $f$ in some class of functions $F$. The relevant classes of functions are now given. Let $G$ be the set of upper semicontinuous step functions which take on only nonnegative integer values, have no jumps of magnitude greater than one, and have only finitely many jumps in any bounded interval. Clearly $P\{X \in G\} = 1$ for any of the processes $X \sim L(\lambda,F)$. Let $G_{n,P}$ be the set of periodic step functions (with period P) which can be written in the form

$$f = \sum_{i=1}^{n} J_{[a_i,b_i]}$$

for some values of $a_i$ and $b_i$.

The principal conditions we shall impose on the functionals are the following.

Definitions:

If $H(f \vee g) + H(f \wedge g) \geq H(f) + H(g)$ whenever $f,g,f \vee g,f \wedge g$ all belong to $F$, we say that $H$ is L-superadditive. If both $H$ and $-H$ are L-superadditive, we say that $H$ is L-additive.

A function $h$ with $k$ arguments is said to be L-superadditive if $h(x \vee y) + h(x \wedge y) \geq h(x) + h(y)$ for all $x$ and $y$ in $\mathbb{R}^k$. This condition was introduced by Lorentz [2]. L-superadditive functions have been used to obtain inequalities in a variety of settings. See Marshall and Olkin [3] for a survey of some of these uses.

Here are some other conditions we shall use. All functions below belong to the relevant domain $\mathcal{F}$. $H$ is <u>bounded</u> if there exists a constant $b$ such that $|H(f)| < b$ for all $f$. $H$ is <u>increasing</u> if $H(f) \leq H(g)$ whenever $f \leq g$. $H$ is <u>local</u> if there exists a bounded interval $[c,d]$ such that $H(f) = H(g)$ whenever $f(t) = g(t)$ for all $t$ in $[c,d]$.

We now state some special cases of our main results. Consider the following situations.

(1.1) Let $X \sim L(\lambda,F)$ and $X^* \sim L(\lambda^*,F^*)$ with $\lambda = \lambda^*$ and $F \prec F^*$. Suppose $H$ is a local functional defined for all $f$ in $\mathcal{G}$. Assume that $H$ is bounded, or alternatively, assume that $H$ is increasing with both $E|H(X)| < \infty$ and $E|H(X^*)| < \infty$.

(1.2) Let $X \sim C_p(F_1,F_2,\ldots,F_n)$ and $X^* \sim C_p(F_1^*,F_2^*,\ldots,F_n^*)$ with $F_i \prec F_i^*$ for all $i$. Assume that $H$ is bounded on $\mathcal{G}_{n,p}$.

(1.3) Theorem: The statements (a) and (b) given below hold true for both situations (1.1) and (1.2);

(a) If $H$ is L-superadditive, then $EH(X) \leq EH(X^*)$.

(b) If $H$ is increasing and L-additive, then $H(X) \prec H(X^*)$.

## 2. Examples.

Let $X$ and $X^*$ be as in (1.1). To obtain consequences of Theorem (1.3) we need to exhibit local functionals $H$ which are L-superadditive or increasing and L-additive.

(2.1) Example: Suppose $h$ is a k-dimensional L-superadditive function, $h(x \vee y) + h(x \wedge y) \geq h(x) + h(y)$ for all $x,y \in \mathbb{R}^k$. When $h$ has continuous partial derivatives $\partial^2 h(x)/\partial x_i \partial x_j$, the condition of L-superadditivity is equivalent to the requirement that $\partial^2 h(x)/\partial x_i \partial x_j \geq 0$ for all $x$ and all $i \neq j$. Let $t_1, t_2, \ldots, t_k$ be fixed times. Define the functional $H$ by $H(f) = h(f(t_1), f(t_2), \ldots, f(t_k))$. $H$ is easily seen to be local and L-superadditive. If $h(x_1, x_2, \ldots, x_k)$ is bounded (or increasing) when $x_i \geq 0$ for all $i$, then $H$ will be bounded (or increasing) and thus $EH(X) \leq EH(X^*)$. A simple special case is $h(x,y) = xy$ which gives $EX(s)X(t) \leq EX^*(s)X^*(t)$ for all $s$ and $t$. Since $EX(s) = EX^*(t)$ for all $s$ and $t$, this says that $X^*$ is "more correlated" than $X$.

(2.2) Example: Let $A$ be a bounded set and $\Psi$ be any function satisfying $\Psi(t) = 0$ when $t \notin A$. Define

$$H(f) = \begin{cases} 1 & \text{if } f(t) \geq \Psi(t) \text{ for all } t, \\ 0 & \text{otherwise .} \end{cases}$$

$H$ is clearly bounded and local and it is L-superadditive because $H(f \wedge g) = H(f) \wedge H(g)$, $H(f \vee g) \geq H(f) \vee H(g)$ and $H(f) \wedge H(g) + H(f) \vee H(g) = H(f) + H(g)$. Therefore $EH(X) = \text{Prob}\{X \geq \Psi\} \leq \text{Prob}\{X^* \geq \Psi\} = EH(X^*)$.

Suppose now that $\Psi$ satisfies $\Psi(t) = \infty$ for $t \notin A$ and define

$$H(f) = \begin{cases} 1 & \text{if } f(t) \leq \Psi(t) \text{ for all } t, \\ 0 & \text{otherwise .} \end{cases}$$

Again, $H$ is bounded, local and L-superadditive so that (1.3a) yields $\text{Prob}\{X \leq \Psi\} \leq \text{Prob}\{X^* \leq \Psi\}$.

By considering functions $\Psi$ of the form

$$\Psi(t) = \begin{cases} c & \text{for } t \in A \\ d & \text{for } t \notin A \end{cases}$$

where $d = 0$ or $\infty$ and applying the above facts we obtain the stochastic orderings

$$\sup\{X(t): t\epsilon A\} \geq_{st} \sup\{X^*(t): t\epsilon A\}, \text{ and}$$
$$\inf\{X(t): t\epsilon A\} \leq_{st} \inf\{X^*(t): t\epsilon A\}.$$

Here $\geq_{st}$ means "stochastically greater than".

(2.3) Example: Let $\pi$ be any measure on the strip $[a,b] \times [0,\infty)$. Use $(t,x)$ to denote a point in $[a,b] \times [0,\infty)$. For any measurable function $f$ define the set $D_f = \{(t,x): f(t) \geq x\}$. Now define $H$ by $H(f) = \pi(D_f)$. Since $D_{f \vee g} = D_f \cup D_g$ and $D_{f \wedge g} = D_f \cap D_g$ we have $H(f \vee g) + H(f \wedge g) = \pi(D_f \cup D_g) + \pi(D_f \cap D_g) = \pi(D_f) + \pi(D_g) = H(f) + H(g)$.

Thus $H$ is increasing, local and L-additive. If $\pi$ is chosen so that $EH(X) < \infty$ and $EH(X^*) < \infty$, then (1.3b) gives $H(X) \to H(X^*)$.

As special cases of this class of increasing L-additive functionals we give the following. In each case it is easy to describe the measure $\pi$ which yields the functional $H$.

(a) Choose any values $t_1, t_2, \ldots, t_k$ and $y_1, y_2, \ldots, y_k$ and define $H$ by

$$H(f) = \sum_{i=1}^{k} I_{\{f(t_i) \geq y_i\}} .$$

(b) Choose $T > 0$ and let $\Psi$ be any measurable function. A continuous version of (a) is

$$H(f) = \int_0^T I_{\{f(t) \geq \Psi(t)\}} dt .$$

(c) We may similarly define

$$H(f) = \int_0^T (f(t) - \Psi(t))_+ dt$$

where $(Z)_+$ denotes the positive part, $(Z)_+ = \max(Z,0)$.

(2.4) Example: We now present a functional $H$ which is <u>not</u> increasing but still yields $H(X) \to H(X^*)$. This functional does satisfy a weaker condition given in section (4). We shall describe the functional $H$ in terms of the $M/G/\infty$ queue. Choose an integer $k$ and a duration $L$. Let $H(X)$ be

the number of customers who arrive during the interval $[0,L]$ and find at least $k$ customers already being served. Using the notation in section (1) we may write

$$H(X) = \sum_{i=-\infty}^{\infty} I_{\{0 \le T_i \le L, X(T_i) \ge k+1\}} .$$

This functional depends explicitly on the values of the arrival times $T_i$ and cannot comfortably be regarded as a functional of just the sample path $X$. Alternatively, we can take $H(X)$ to be the number of customers who arrive during $[0,L]$ and find at <u>most</u> $k$ customers already being served. This definition also yields $H(X) \nearrow H(X^*)$.

(2.5) Example: Now take $X$ and $X^*$ to be the periodic processes of (1.2). All of the previous examples may be restated in terms of these periodic processes. We then obtain various inequalities concerning coverage problems on the circle. For example, let $H_k(X)$ be the indicator of the event that every point on the circumference is covered at least $k$ times, $H_k(X) = I_{\{X(t) \ge k \text{ for all } t\}}$. $H_k$ is bounded and L-superadditive (as in example (2.2)) so that $EH_k(X) \le EH_k(X^*)$. Taking $k = 1$ yields an inequality which implies the truth of a conjecture made by Siegel [8] concerning coverage probabilities. (In our notation Siegel's conjecture was that $EH_1(X) \le EH_1(X^*)$ when $F_1 = F_2 = \ldots = F_n$ and $F_1^* = F_2^* = \cdots = F_n^*$ with $F_1$ and $F_1^*$ obeying a condition somewhat stronger than $F_1 \nearrow F_1^*$.)

Siegel [8,9] also considered the distribution of the total length of the uncovered portion of the circumference. A more general quantity is the total length of that part of the circumference covered at most $k$ times, denoted by $V_k$. More formally,

$V_k(X) = \int_0^P I_{\{X(t) \leq k\}} dt$. As in example (2.3b), it is easily shown that $-V_k$ is bounded, increasing and L-additive and thus $V_k(X) \to V_k(X^*)$.

Siegel and Holst [10] give results concerning the distribution of the number of uncovered gaps on the circumference which we shall denote by G. Using the notation of section 1 we may write

$$G(X) = \sum_{i=1}^{n} I_{\{X(T_i)=1\}} .$$

As in example (2.4), the functional $-G$ does not quite satisfy the conditions in (1.3b) but does satisfy the weaker conditions given in section 4 so that $G(X) \to G(X^*)$.

The results in this example concerning the functionals $H_k$ and $V_k$ were first given by the present author in [1]. The methods and results of [1] are similar to but less general than those of this paper.

3. Corollaries.

Theorem (1.3) sometimes allows us to compare the processes $X$ and $X^*$ when the arrival rates differ, $\lambda \neq \lambda^*$. Using this we can prove a result concerning the behavior of $X$ when the time axis is rescaled.

Notation: Let $\delta_y$ denote the distribution which places all of its mass at $y$, $\delta_y(x) = 0$ for $x < y$ and $\delta_y(x) = 1$ for $x \geq y$.

(3.1) Corollary: Suppose that $X \sim L(\lambda,F)$ and $X^* \sim L(\lambda^*,F^*)$ and there exists a constant $\beta$ such that $0 < \beta \leq 1$, $\lambda^* = \beta\lambda$ and $F \prec (1-\beta)\delta_0+\beta F^*$. Let $A$ be the countable set of arrival times of a Poisson process with arbitrary rate $\alpha$. Let $H$ be a functional satisfying the conditions in (1.1) and assume also that for any $f \in G$, $H(f) = H(f+I_A)$ almost surely. Under these conditions (1.3a) and (1.3b) hold true.

In this corollary, $X^*$ is composed of pulses which tend to be longer than the pulses in $X$ but which arrive at a slower rate. These differences are balanced so that we still have $EX(t) = EX^*(t)$.

Proof of (3.1): Let $X^\# \sim L(\lambda^\#,F^\#)$ with $\lambda^\# = \lambda$ and $F^\# = (1-\beta)\delta_0+\beta F^*$. Since $F \prec F^\#$, Theorem (1.3) applies directly to $X$ and $X^\#$. Now note that $H(X^*)$ and $H(X^\#)$ have the same distribution because $X^*$ may be obtained from $X^\#$ by eliminating those pulses in $X^\#$ which have zero duration. More precisely, $X^\#$ has the same distribution as $X^* + I_A$ where $A$ is the set of arrival times of an independent Poisson process with rate of arrivals equal to $(1-\beta)\lambda$. By our assumption on $H$ we know that $H(X^*)$ and $H(X^\#)$ have the same distribution and this completes the proof.

The simplest application of (3.1) is to the case where $F = \delta_{\beta y}$ and $F^* = \delta_y$. The verification that $\delta_{\beta y} \prec (1-\beta)\delta_0+\beta\delta_y$ when $0 < \beta \leq 1$ is

immediate. This example may be generalized. Let $F \circ \beta$ denote the distribution defined by $(F \circ \beta)(x) = F(\beta x)$. We now show that $F \prec (1-\beta)\delta_0 + \beta(F \circ \beta)$ when $0 < \beta \leq 1$. Let $R$ be a random variable taking on the values 0 and 1 with probabilities $1-\beta$ and $\beta$ respectively. Let $V$ have the distribution $F$ and be independent of $R$. Define $W = RV/\beta$. The distribution of $W$ is $(1-\beta)\delta_0 + \beta(F \circ \beta)$. Since $E(W|V) = V$, Jensen's inequality for conditional expectations gives $E\phi(V) \leq E\phi(W)$ for all convex functions $\phi$ and thus $V \prec W$ as desired. This justifies the application of (3.1) when $X \sim L(\lambda, F)$ and $X^* \sim L(\lambda^*, F^*)$ with $\lambda^* = \beta\lambda$, $F^*(x) = F(\beta x)$ for all $x$ and $0 < \beta \leq 1$.

(3.2) Corollary: Let $X \sim L(\lambda, F)$ and $H$ be a bounded and local functional which also obeys the condition in (3.1). For $\beta \geq 0$ define $X \circ \beta$ by $(X \circ \beta)(t) = X(\beta t)$ for all $t$.

(a) If $H$ is L-superadditive, then $EH(X \circ \beta)$ is a decreasing function of $\beta$ for $\beta \geq 0$.

(b) If $H$ is increasing and L-additive, then $H(X \circ \beta) \prec H(X \circ \gamma)$ whenever $\beta \geq \gamma \geq 0$.

Proof of (3.2): Assume first that $\beta \neq 0$. To verify (3.2a) it suffices to show that $EH(X \circ \beta) \geq EH(X)$ when $\beta \leq 1$. Similarly, to verify (3.2b) it suffices to show that $H(X) \prec H(X \circ \beta)$ for $\beta \leq 1$. But these follow from (3.1) and the previous discussion because $X \circ \beta \sim L(\beta\lambda, F \circ \beta)$. The case $\beta = 0$ is handled by taking limits. Since $H$ is local and $X(t)$ is constant in some neighborhood of $t = 0$ almost surely, $H(X \circ \beta) \to H(X \circ 0)$ almost surely as $\beta \downarrow 0$. Thus, for example, $E\phi(H(X \circ \beta)) \uparrow E\phi(H(X \circ 0))$ for any convex $\phi$ as $\beta \downarrow 0$.

The condition in (3.2) that $H$ be bounded is just a convenient way to ensure that all expectations are finite. This assumption can easily be weakened.

For an application of (3.2a) we go back to example (2.1). Choose any constants $a_1, a_2, \ldots, a_k$ and define

$$h(x_1, x_2, \ldots, x_k) = \begin{cases} 1 & \text{if } x_i \geq a_i \text{ for all } i, \\ 0 & \text{otherwise .} \end{cases}$$

$h$ is an L-superadditive function on $\mathbb{R}^k$. By (3.2a), $Eh(X(\beta t_1), X(\beta t_2), \ldots, X(\beta t_k)) = \text{Prob}\{X(\beta t_i) \geq a_i \text{ for all } i\}$ is decreasing in $\beta$ for any values of $t_1, t_2, \ldots, t_k$.

For an application of (3.2b) we go back to example (2.3). Choose a constant $c > 0$ and define

$$Y_\beta = \frac{1}{\beta} \int_0^\beta I_{\{X(t) \geq c\}} dt .$$

(3.2b) implies that $Y_\beta \to Y_\gamma$ when $\beta \geq \gamma$.

## 4. Proofs of the Main Results.

In section 1 we stated a theorem concerning processes formed by the superposition of rectangular pulses. Now we shall consider pulses of a somewhat more general shape; the pulses will increase monotonically to unit amplitude, remain at unit amplitude for a random duration, and then decrease monotonically back to zero. The pulses form a parametric family of functions denoted $\{g_\theta : \theta \geq d\}$ where the parameter $\theta$ indicates the total duration of a pulse and $d$ is the duration of the briefest pulse in the family. The complete definition is given below.

Choose $b \geq 0$ and $c \geq 0$. Let $\xi$ be any increasing function on $[0,b]$ with $\xi(0) \geq 0$ and $\xi(b) = 1$. Let $\psi$ be any decreasing function on $[0,c]$ with $\psi(0) = 1$ and $\psi(c) \geq 0$. For $\theta \geq b+c$ define

$$g_\theta(t) = \begin{cases} \xi(t) & \text{for } 0 \leq t \leq b\ , \\ 1 & \text{for } b \leq t \leq \theta-c\ , \\ \psi(t-\theta+c) & \text{for } \theta-c \leq t \leq \theta, \\ 0 & \text{otherwise}\ . \end{cases}$$

In this parametric family the minimum duration is $d = b+c$.

The family $\{g_\theta : \theta \geq d\}$ has two properties we shall need. First note that

$$g_\theta(t) = 0 \quad \text{unless} \quad 0 \leq t \leq \theta\ . \tag{4.1}$$

The second and more important property is

(4.2) $$g_{\theta+\varepsilon+\delta} = g_{\theta+\varepsilon} \vee (S_\varepsilon g_{\theta+\delta}) \quad \text{and}$$

$$S_\varepsilon g_\theta = g_{\theta+\varepsilon} \wedge (S_\varepsilon g_{\theta+\delta})$$

for all $\theta \geq d$, $\varepsilon \geq 0$, $\delta \geq 0$ .

Here we have used $S_\varepsilon$ to represent a shift operator; $(S_\varepsilon f)(t) = f(t-\varepsilon)$ for all functions $f$ and all $t$.

We now define a shot noise process on the line ($L$) as in section 1. Let $\{T_i: -\infty < i < \infty\}$ be the ordered arrival times of a Poisson process having rate $\lambda$. Let $\{Z_i: -\infty < i < \infty\}$ be independent with the distribution $F$ and satisfy $\text{Prob}\{Z_i \geq d\} = 1$ and $EZ_i < \infty$. Define $X \sim L(\lambda, F)$ by

$$X(t) = \sum_{i=-\infty}^{\infty} g_{Z_i}(t-T_i) .$$

Next we define the analogous periodic process with period $P$ which is equivalent to a process on a circle with circumference $P$. For $d \leq \theta < P$ define the periodic pulse $g'_\theta$ by

$$g'_\theta(t) = \sum_{k=-\infty}^{\infty} g_\theta(t-kP) .$$

Let $Z_1, Z_2, \ldots, Z_n$ be independent random variables with distributions $F_1, F_2, \ldots, F_n$ and satisfying $\text{Prob}\{d_i \leq Z_i < P\} = 1$ for all $i$. Let $T_1, T_2, \ldots, T_n$ be independent random variables distributed uniformly on the interval $[0,P)$. Define $X \sim C_P(F_1, F_2, \ldots, F_n)$ by

$$X(t) = \sum_{i=1}^{n} g'_{Z_i}(t-T_i) .$$

We shall first prove a theorem concerning $C_p(F_1,F_2,\ldots,F_n)$. From this we obtain a corresponding result for $L(\lambda,F)$. The results stated in section 1 will follow as corollaries.

Reusing the notation of section 1, we take $G_{n,P}$ to be the class of all functions $f$ which can be written in the form

$$f = \sum_{i=1}^{n} S_{t_i} g'_{\theta_i}$$

for some values of $t_i$ and $\theta_i$ with $d \leq \theta_i < P$ for $1 \leq i \leq n$. In the discussion below, $H$ will always be a functional defined on some class of functions which includes $G_{n,P}$.

Lemma: If $H$ is L-superadditive, then

$$(4.3) \qquad H(f+S_t g'_{\theta+\varepsilon+\delta}) + H(f+S_{t+\varepsilon} g'_{\theta}) \geq H(f+S_t g'_{\theta+\varepsilon}) + H(f+S_{t+\varepsilon} g'_{\theta+\delta})$$

for all $f \in G_{n-1,P}$, all $t$ and all $\theta \geq d$, $\varepsilon \geq 0$, $\delta \geq 0$ satisfying $\theta+\varepsilon+\delta < P$.

Proof of Lemma: Using (4.1) and (4.2) we obtain

$$g'_{\theta+\varepsilon+\delta} = g'_{\theta+\varepsilon} \vee (S_\varepsilon g'_{\theta+\delta}) \quad \text{and}$$

$$S_\varepsilon g'_\theta = g'_{\theta+\varepsilon} \wedge (S_\varepsilon g'_{\theta+\delta})$$

for all $\theta \geq d$, $\varepsilon \geq 0$, $\delta \geq 0$ satisfying $\theta+\varepsilon+\delta < P$.

Applying $S_t$ and adding $f$ to both sides of these equations leads to

$$f+S_t g'_{\theta+\varepsilon+\delta} = (f+S_t g'_{\theta+\varepsilon}) \vee (f+S_{t+\varepsilon} g'_{\theta+\delta})$$

and

$$f+S_{t+\varepsilon} g'_{\theta} = (f+S_t g'_{\theta+\varepsilon}) \wedge (f+S_{t+\varepsilon} g'_{\theta+\delta}) .$$

Now L-superadditivity yields (4.3).

(4.4) Theorem:

Let $X \sim C_P(F_1,F_2,\ldots,F_n)$ and $X^* \sim C_P(F_1^*,F_2^*,\ldots,F_n^*)$ with $F_i \precsim F_i^*$ for all $i$. If $H$ satisfies (4.3) and $H(f) \geq 0$ for all $f \in G_{n,P}$, then $EH(X) \leq EH(X^*)$.

Proof of Theorem: The condition $H \geq 0$ ensures that $EH(X)$ and $EH(X^*)$ are well defined. It suffices to prove (4.4) in the special case where $F_i = F_i^*$ for $1 \leq i \leq n-1$ and $F_n \precsim F_n^*$. The general result is then obtained by repeated applications of this special case. Assuming $F_i = F_i^*$ for $1 \leq i \leq n-1$ allows us to define $X$ and $X^*$ on the same probability space as follows

$$X = W+S_{T_n} g'_{Z_n} \quad \text{and} \quad X^* = W+S_{T_n} g'_{Z_n^*}$$

where $W = \Sigma_{i=1}^{n-1} S_{T_i} g'_{Z_i}$, $Z_i \sim F_i$ for all $i$ and $Z_n^* \sim F_n^*$.

Let $A$ be the $\sigma$-field generated by $T_1,Z_1,T_2,Z_2,\ldots,T_{n-1},Z_{n-1}$. Conditional on $A$, we may regard $W$ as a fixed (nonrandom) function. The next lemma shows that $E(H(X)|A) \leq E(H(X^*)|A)$. Taking unconditional expectations completes the proof. (Note: If either $EH(X)$ or $EH(X^*)$ is infinite, the same basic argument works. However, instead of the convenient notation of conditional expectation, we must now write the expectations as multiple integrals and then use Fubini's theorem.)

(4.5) Lemma: Choose any $f \in G_{n-1,P}$. Suppose that $H \geq 0$ on $G_{n,P}$ and that $H$ satisfies (4.3). For $d \leq \theta < P$ define

$$\Phi(\theta) = \int_0^P H(f+S_t g'_\theta)dt \ .$$

Let $Y_1$ and $Y_2$ be random variables taking values in $[d,P)$. If $Y_1 \prec Y_2$, then $E\Phi(Y_1) \leq E\Phi(Y_2)$.

Proof of Lemma: Integrating with respect to $t$ in (4.3) and using the periodicity of $g'_\theta$ yields

$$\Phi(\theta+\varepsilon+\delta) + \Phi(\theta) \geq \Phi(\theta+\varepsilon) + \Phi(\theta+\delta)$$

for all $\theta \geq d$, $\varepsilon \geq 0$, $\delta \geq 0$ satisfying $\theta+\varepsilon+\delta < P$. This says that $\Phi$ is convex in the interval $[d,P)$. If $\Phi$ can be extended to be a convex function on the entire real line, then $Y_1 \prec Y_2$ implies $E\Phi(Y_1) \leq E\Phi(Y_2)$. Even when $\Phi$ cannot be so extended, we still have $E\Phi(Y_1) \leq E\Phi(Y_2)$. For this argument and further details see section 7.

Theorem (4.4) implies the truth of (1.3a) in situation (1.2). To obtain a result which implies (1.3b) we need the following lemmas.

Lemma: If $H$ is increasing and L-additive, then

$$(4.6) \qquad H(f+S_t g'_{\theta+\varepsilon+\delta}) + H(f+S_{t+\varepsilon} g'_\theta)$$

$$= H(f+S_t g'_{\theta+\varepsilon}) + H(f+S_{t+\varepsilon} g'_{\theta+\delta}),$$

$$H(f+S_t g'_\theta) \le H(f+S_t g'_{\theta+\varepsilon}), \quad \text{and}$$

$$H(f+S_{t+\varepsilon} g'_\theta) \le H(f+S_t g'_{\theta+\varepsilon})$$

for all $f \in G_{n-1,P}$, all $t$ and all $\theta \ge d$, $\varepsilon \ge 0$, $\delta \ge 0$ satisfying $\theta+\varepsilon+\delta < P$.

This lemma is trivial and we state it only because there are functionals which satisfy (4.6) but are not both increasing and L-additive. The functional $-G$ of (2.5) is such a functional.

Proof of Lemma: The lemma follows from noting that $g'_\theta \le g'_{\theta+\varepsilon}$, $S_\varepsilon g'_\theta \le g'_{\theta+\varepsilon}$ and both $H$ and $-H$ satisfy (4.3).

(4.7) Lemma: Let $\phi$ be any convex function. $\phi \circ H$ will denote the composition of $\phi$ and $H$; $(\phi \circ H)(f) = \phi(H(f))$.

(a) If $H$ is increasing and L-additive, then $\phi \circ H$ is L-superadditive.

(b) If $H$ satisfies (4.6), then $\phi \circ H$ satisfies (4.3).

Proof of Lemma: To prove (a) we must show that $(\phi oH)(f \vee g)+(\phi oH)(f \wedge g) \geq (\phi oH)(f)+(\phi oH)(g)$. Let $w = H(f)$, $x = H(g)$, $y = H(f \wedge g)$ and $z = H(f \vee g)$. By the assumptions on $H$ we have $y \leq w \wedge x$, $z \geq w \vee x$ and $y+z = w+x$. Thus by convexity $\phi(y)+\phi(z) \geq \phi(w)+\phi(x)$ as desired. The proof of (b) is similar.

(4.8) Corollary: Let $X$ and $X^*$ be as in (4.4). If $H$ is bounded on $G_{n,P}$ and satisfies (4.6), then $H(X) \prec H(X^*)$.

Proof of Corollary: Let $\phi$ be any convex function. $\phi oH$ is bounded on $G_{n,P}$ so that without loss of generality we may assume that $(\phi oH)(f) \geq 0$ for all $f \in G_{n,P}$. $\phi oH$ satisfies (4.3) by the preceding lemma. Now theorem (4.4) applies to yield $E\phi(H(X)) \leq E\phi(H(X^*))$ thus completing the proof.

To obtain results for $L(\lambda,F)$ we first extend (4.4) and (4.8) to allow for a random number of pulses. Let $M$ be a Poisson random variable with mean $\lambda P$. Let $T_1,T_2,T_3,\ldots$ be a sequence of independent random variables uniformly distributed on $[0,P)$. Let $Z_1,Z_2,Z_3,\ldots$ be i.i.d. according to $F$ and satisfy $P\{d \leq Z_i < P\} = 1$. The random variables $M$, $\{T_i\}$ and $\{Z_i\}$ are jointly independent. Define

$$X = \sum_{i=1}^{M} S_{T_i} g'_{Z_i} .$$

To refer to this periodic process we use the notation $X \sim C_p(\lambda,F)$.

(4.9) Corollary: Let $X \sim C_p(\lambda,F)$ and $X^* \sim C_p(\lambda^*,F^*)$ with $\lambda = \lambda^*$ and $F \prec F^*$. $H$ is a functional defined on some domain which

includes $\bigcup_{n=0}^{\infty} G_{n,P}$ where $G_{0,P}$ consists of a single function which is identically zero.

(a) If $H \geq 0$ and $H$ is L-superadditive, then $EH(X) \leq EH(X^*)$.

(b) If $H$ is bounded, increasing and L-additive, then $H(X) \prec H(X^*)$.

Remark: In (a) we may replace L-superadditivity by a condition like (4.3). In (b) we may replace the conditions increasing and L-additive by a condition like (4.6). The details are omitted.

Proof of Corollary: Conditional on the event $\{M=n\}$, the process $C_P(\lambda,F)$ has the same distribution as $C_P(F_1,F_2,\ldots,F_n)$ with $F_i = F$ for all $i$. Thus we may condition on the value of $M$ and use (4.4) and (4.8) to obtain (a) and (b) respectively.

Let $\mathcal{F}$ be the collection of all functions $f$ which can be written in the form

$$f(t) = \sum_{i=1}^{k} g_{\theta_i}(t-\tau_i)$$

where $0 \leq k \leq \infty$ and $\theta_i \geq d$ for all $i$. When $k = 0$, $f$ is identically zero. When $k = \infty$, we must also require that $f(t) < \infty$ for all $t$ and that no bounded set contains $\tau_i$ for infinitely many values of $i$. For $X \sim L(\lambda,F)$, it is clear that $\text{Prob}\{X \in \mathcal{F}\} = 1$.

We now state the basic result for $L(\lambda,F)$.

(4.10) Theorem: Let $X \sim L(\lambda,F)$ and $X^* \sim L(\lambda^*,F^*)$ with $\lambda = \lambda^*$ and $F \prec F^*$. Let $H$ be a local functional with domain $\mathcal{F}$. Assume that

$F$ and $F^*$ have bounded support, $F(B) = F^*(B) = 1$ for some $B < \infty$. Under these conditions statements (a) and (b) of (4.9) are true. We also have:

(c) If $H(f) \geq 0$ and $H(f+S_t g_{\theta+\varepsilon+\delta}) + H(f+S_{t+\varepsilon} g_\theta)$

$$\geq H(f+S_t g_{\theta+\varepsilon}) + H(f+S_{t+\varepsilon} g_{\theta+\delta})$$

for all $f \in F$, all $t$ and all $\theta \geq d$, $\varepsilon \geq 0$, $\delta \geq 0$, then $EH(X) \leq EH(X^*)$.

(d) If $H$ is bounded and $H(f+S_t g_{\theta+\varepsilon+\delta}) + H(f+S_{t+\varepsilon} g_\theta)$

$$= H(f+S_t g_{\theta+\varepsilon}) + H(f+S_{t+\varepsilon} g_{\theta+\delta}),$$

$$H(f+S_t g_\theta) \leq H(f+S_t g_{\theta+\varepsilon}) \quad \text{and}$$

$$H(f+S_{t+\varepsilon} g_\theta) \leq H(f+S_t g_{\theta+\varepsilon})$$

for all $f \in F$, all $t$ and all $\theta \geq d$, $\varepsilon \geq 0$ and $\delta \geq 0$, then $H(X) \prec H(X^*)$.

The slightly weaker conditions in (c) and (d) are necessary to handle examples like (2.4).

Proof of Theorem: Since $H$ is local, we may assume without loss of generality that $H(f)$ depends only on the values $f(t)$ for $t$ belonging to the interval $[0,L]$. Choose a value of $P$ such that $P > L+B$. Let $Y \sim C_p(\lambda,F)$ and $Y^* \sim C_p(\lambda^*,F^*)$. The periodic pulse $S_t g'_\theta$ is made up of translated copies of $g_\theta$ which are separated by intervals of length at least $L$ when $\theta \leq B$. Thus at most one of these copies can "intersect" the interval $[0,L]$. In the definition of $C_p(\lambda,F)$ we used a Poisson number of uniform random variables. Equivalently, we could have used the

arrival times of a Poisson process with rate $\lambda$ on the interval $[0,P)$. These remarks make it clear that $X$ and $Y$ can be defined on the same probability space in such a way that $X(t) = Y(t)$ for $0 \le t \le L$. To do this we take

$$Y(t) = \sum_{i=-\infty}^{\infty} g'_{Z_i}(t-T_i) I_{\{-B \le T_i < P-B\}}$$

where $Z_i$ and $T_i$ are the same random variables used in defining $X$. Thus $H(X)$ and $H(Y)$ have the same distribution. Now the result follows by applying (4.9). Statements (c) and (d) follow from the remark after (4.9). The details are omitted.

Many of the conditions in (4.10) can be weakened. First, we can eliminate the requirement that $F$ and $F^*$ have bounded support. To do this we need the following lemma which will be proved in Section 7. We shall also weaken the requirement in (4.9b) that $H$ be bounded.

Definitions: Given the random variables $V$ and $\{V_n : 1 \le n < \infty\}$ we say that $V_n \Rightarrow V$ if $\text{Prob}\{V_n = V \text{ for all sufficiently large } n\} = 1$. The symbol $\uparrow$ is used to indicate that a sequence is increasing.

(4.11) Lemma: Let $V$ and $W$ be nonnegative random variables with $EV < \infty$ and $EW < \infty$. If $V \prec W$, there exist sequences $\{V_n\}$ and $\{W_n\}$ of <u>bounded</u> nonnegative random variables satisfying $V_n \uparrow$, $W_n \uparrow$, $V_n \Rightarrow V$, $W_n \Rightarrow W$ and $V_n \prec W_n$ for all $n$.

Using lemma (4.11) we obtain the next fact.

(4.12) Lemma: Let $X$ and $X^*$ be as in (4.10) except that now $F$ and $F^*$ are no longer required to have bounded support. Let $H$ be any local functional. There exist sequences of processes $\{X_n\}$ and $\{X_n^*\}$ with the properties given below.

(a) $X_n \sim L(\lambda, F_n)$, $X_n^* \sim L(\lambda^*, F_n^*)$ and $F_n \Rightarrow F_n^*$ for all $n$.

(b) For all $n$, the distributions $F_n$ and $F_n^*$ have bounded support.

(c) $X_n \uparrow$ and $X_n^* \uparrow$.

(d) Choose any bounded interval $[a,b]$ and define $A_n$ to be the event that $X_n(t) = X(t)$ for all $t \in [a,b]$. Then $\text{Prob}\{A_n$ occurs for all sufficiently large $n\} = 1$. A similar property holds for $X_n^*$ and $X^*$.

(e) $H(X_n) \Rightarrow H(X)$ and $H(X_n^*) \Rightarrow H(X^*)$.

Proof of Lemma: Let $Z_\infty$ and $Z_\infty^*$ be random variables with distributions $F$ and $F^*$ respectively. Since $Z_\infty \prec\!\Rightarrow Z_\infty^*$ there are sequences $\{Z_n\}$ and $\{Z_n^*\}$ with the properties given in (4.11). Let $F_n$ and $F_n^*$ denote the distributions of $Z_n$ and $Z_n^*$ respectively so that $F_n \prec F_n^*$ for all $n$. We now construct the sequence $\{X_n\}$. The argument for $\{X_n^*\}$ is the same. Construct independent copies of the sequence $\{Z_n : 1 \le n \le \infty\}$ indexed by the letter $i$; $\{Z_{i,n} : 1 \le n \le \infty\} \overset{\text{dist.}}{=} \{Z_n : 1 \le n \le \infty\}$ for $-\infty < i < \infty$. Let $\{T_i : -\infty < i < \infty\}$ be as in the definition of $L(\lambda, F)$. For $1 \le n \le \infty$ and all $t$ define

$$X_n(t) = \sum_{i=-\infty}^{\infty} g_{Z_{i,n}}(t-T_i)$$

and take $X = X_\infty$. Clearly $X \sim L(\lambda,F)$ and $X_n \sim L(\lambda,F_n)$ so that (a) is true. Properties (b) and (c) follow immediately from the properties of $\{Z_n\}$ given in (4.11). Because $EZ_\infty < \infty$, the number of pulses in $X_\infty$ which "intersect" the interval $[a,b]$ is almost surely finite. (It is easily shown that the number of values of $i$ for which $[T_i,T_i+Z_{i,\infty}]$ intersects $[a,b]$ has a Poisson distribution with mean $\lambda(b-a+EZ_\infty)$.) Combining this observation with the fact that $Z_{i,n} \Rightarrow Z_{i,\infty}$ for all $i$ shows that (d) is true. Finally, (e) is an immediate consequence of (d).

We can now give a more widely applicable result for $L(\lambda,F)$.

(4.13) Theorem: Let $X \sim L(\lambda,F)$ and $X^* \sim L(\lambda^*,F^*)$ with $\lambda = \lambda^*$ and $F \lesssim F^*$. Let $H$ be a local functional with domain $\mathcal{F}$.

(a) If $H \geq 0$, $H$ is L-superadditive and $H$ satisfies either condition (i) or (ii) below, then $EH(X) \leq EH(X^*)$.

(i) $H$ is increasing.

(ii) There exists an increasing functional $Q$ such that $H(f) \leq Q(f)$ for all $f \in \mathcal{F}$, $EQ(X) < \infty$ and $EQ(X^*) < \infty$.

(b) If $H$ is increasing and L-additive, $E|H(X)| < \infty$ and $E|H(X^*)| < \infty$, then $H(X) \rightarrow H(X^*)$.

Remark: Results (a) and (b) above can be modified in the manner of statements (c) and (d) in (4.10).

Proof of (a): Let $\{X_n\}$ and $\{X_n^*\}$ be the sequences in (4.12). Using (4.10) we have $EH(X_n) \le EH(X_n^*)$ for all $n$ and by (4.12e) we have $H(X_n) \Rightarrow H(X)$ and $H(X_n^*) \Rightarrow H(X^*)$. To obtain $EH(X) \le EH(X^*)$ we need only show that $EH(X) = \lim_{n\to\infty} EH(X_n)$ and $EH(X^*) = \lim_{n\to\infty} EH(X_n^*)$. If condition (i) holds, then $H(X_n)\uparrow$ and $H(X_n^*)\uparrow$, so that the monotone convergence theorem completes the proof. If condition (ii) holds, then $H(X_n) \le Q(X_n) \le Q(X)$ and $H(X_n^*) \le Q(X_n^*) \le Q(X^*)$ for all $n$, so the result follows from the dominated convergence theorem.

Proof of (b): First note that $H$ is bounded below. $H$ is local and there is positive probability that no pulses "intersect" the interval which affects $H$. Thus $E|H(X)| < \infty$ implies $|H(\underset{\sim}{0})| < \infty$ where $\underset{\sim}{0}$ denotes the function which is identically zero. This shows that $H$ is bounded below by $H(\underset{\sim}{0})$.

Let $\phi$ be any <u>increasing</u> convex function. Clearly $\phi \circ H$ is increasing and bounded below and by Lemma (4.7) it is also L-superadditive. Thus we may apply the result of part (a) to conclude $E\phi(H(X)) \le E\phi(H(X^*))$. Taking $\phi(x) = x$ yields $EH(X) \le EH(X^*)$.

Now take $\phi$ to be any convex function and let $u = H(\underset{\sim}{0})$. Since the graph of $\phi$ has a supporting line at the point $(u,\phi(u))$, we can write $\phi$ as $\phi(x) = \phi(u) + a(x-u) + \gamma(x)$ where $a$ is a constant and $\gamma$ is a function which is convex and increases on the interval $[u,\infty)$. From above, we know that $E\gamma(H(X)) \le E\gamma(H(X^*))$. Therefore, to prove that $E\phi(H(X)) \le E\phi(H(X^*))$ it suffices to show that $EH(X) = EH(X^*)$.

For all $b > 0$ define $\phi_b$ by $\phi_b(x) = -(x \wedge b)$. $\phi_b$ is convex and bounded below. Therefore $\phi_b \circ H$ is bounded below and L-superadditive

(using lemma (4.7)). This allows us to use (4.10) to conclude $E\phi_b(H(X_n)) \leq E\phi_b(H(X_n^*))$ for all $n$ where $\{X_n\}$ and $\{X_n^*\}$ are as in (4.12). Equivalently $E(H(X_n) \wedge b) \geq E(H(X_n^*) \wedge b)$. Letting $b \to \infty$ and $n \to \infty$ gives $EH(X) \geq EH(X^*)$ and completes the proof.

The results in (4.13) contain those of (1.3) for situation (1.1) as special cases.

5. Various Extensions.

We can use (4.13) to obtain results for many functionals $H$ which are not local. To do this we need a sequence $\{H_n\}$ of local functionals which satisfy the conditions in (a) or (b) and converge to $H$ in a sufficiently strong sense. Two examples of this are now given. The details are omitted.

Our first example generalizes (2.2). Let $\psi$ be an arbitrary function. For $1 \leq n \leq \infty$ define $H_n$ by

$$H_n(f) = \begin{cases} 1 & \text{if } f(t) \leq \psi(t) \text{ for all } t\varepsilon(-n,n), \\ 0 & \text{otherwise .} \end{cases}$$

$H_\infty(f) = \lim_{n\to\infty} H_n(f)$ for all $f$. For $n < \infty$ the functionals $H_n$ satisfy the conditions of part (a) so that $EH_n(X) \leq EH_n(X^*)$. Letting $n \to \infty$ yields $P\{X \leq \psi\} \leq P\{X^* \leq \psi\}$. To avoid the triviality $0 \leq 0$ we must demand that $\psi(t) \to \infty$ sufficiently fast as $t \to \pm\infty$.

The second example resembles those of (2.3). Let $\xi$ be any nonnegative function satisfying $\int_{-\infty}^{\infty} \xi(t)dt < \infty$. Let $\psi$ be any increasing nonnegative function which satisfies $E\psi(X(t)) < \infty$. For $1 \leq n \leq \infty$ define $H_n$ by

$$H_n(f) = \int_{-n}^{n} \xi(t)\psi(f(t))dt .$$

$H_\infty(f) = \lim_{n\to\infty} H_n(f)$ for all $f$. For $n < \infty$ the functionals $H_n$ satisfy the conditions of part (b) so that $H_n(X) \prec H_n(X^*)$. Letting $n \to \infty$ yields

$H_\infty(X) \prec\prec H_\infty(X^*)$. We briefly justify this last step. Clearly $EH_\infty(X) < \infty$. For $n < \infty$ we have $EH_n(X) = EH_n(X^*)$ so that monotone convergence yields $EH_\infty(X) = EH_\infty(X^*)$. Therefore, as in the proof of (b), to verify $H_\infty(X) \prec H_\infty(X^*)$ it suffices to show that $E\phi(H_\infty(X)) \leq E\phi(H_\infty(X^*))$ for all <u>increasing</u> convex functions $\phi$. Since $E\phi(H_n(X)) \leq E\phi(H_n(X^*))$ for all $n < \infty$, this follows by monotone convergence.

Various conditions concerning the pulses may be relaxed without changing the character of the results in (4.13). We have so far considered only pulses which satisfy (4.1); the pulses are zero outside of a bounded interval. Using (4.13) and taking limits, we can deal with pulses satisfying the weaker condition

$$\int_{-\infty}^{\infty} g_\theta(t)dt < \infty \quad \text{for all} \quad \theta \ .$$

This integral must be finite to ensure that $P\{X(t) < \infty \text{ for all } t\} = 1$.

We now sketch some of the development. Let $\xi$ be an increasing function on $(-\infty,0]$ satisfying $\int_{-\infty}^{0}|\xi(t)|dt < \infty$ and $\xi(0) = 1$. Let $\psi$ be a decreasing function on $[0,\infty)$ satisfying $\int_0^\infty |\psi(t)|dt < \infty$ and $\psi(0) = 1$. For $1 \leq n \leq \infty$ and $\theta \geq 0$ let

$$g_\theta^n(t) = \begin{cases} \xi(t) & \text{for } -n \leq t \leq 0, \\ 1 & \text{for } 0 \leq t \leq \theta, \\ \psi(t-\theta) & \text{for } \theta \leq t \leq n+\theta \ , \\ 0 & \text{otherwise .} \end{cases}$$

and define

$$X^n(t) = \sum_{i=-\infty}^{\infty} g^n_{Z_i}(t-T_i)$$

with $Z_i$ and $T_i$ as in the definition of $L(\lambda,F)$. Clearly $X^n(t) \uparrow X^\infty(t)$ almost surely as $n \to \infty$. In any bounded interval this convergence will be uniform (almost surely). For $n < \infty$, the processes $X^n$ satisfy the requirements of (4.13). If $H$ is sufficiently regular so that $EH(X^\infty) = \lim_{n\to\infty} EH(X^n)$ or more strongly $E\phi(H(X^\infty)) = \lim_{n\to\infty} E\phi(H(X^n))$ for all convex $\phi$, then the results of (4.13) will extend to $H(X^\infty)$. For example, if $H$ is an increasing local functional satisfying $H(X^\infty) = \lim_{n\to\infty} H(X^n)$ almost surely, then all of the results in (4.13) extend by just using the monotone convergence theorem.

In sections 1 and 4 the amplitude of the pulses was arbitrarily chosen to be one. The results proved in section 4 also hold when the amplitude and shape of the pulses are allowed to vary randomly. Let $\{g_{\theta,\alpha}: \theta \geq d(\alpha), \alpha \in A\}$ be a parametric family of pulses with the parameter $\alpha$ taking values in a set $A$. Assume that for any fixed value $\alpha_0 \in A$, the family $\{g_{\theta,\alpha_0}: \theta \geq d(\alpha_0)\}$ satisfies (4.1) and (4.2).

For example, suppose $\{g^{(i)}_\theta: \theta \geq d_i\}$ for $1 \leq i \leq k$ are $k$ different parametric families each satisfying (4.1) and (4.2). Take $A = \{1,2,\ldots,k\}$ and define $g_{\theta,\alpha} = g^{(\alpha)}_\theta$ and $d(\alpha) = d_\alpha$ for $\alpha \in A$. For another example, let $\{g_\theta: \theta \geq d\}$ satisfy (4.1) and (4.2) and define $g_{\theta,\alpha} = \alpha g_\theta$, $A = \{\alpha: \alpha > 0\}$ and $d(\alpha) = d$. In this example, the parameter $\alpha$ determines the amplitude of the pulse and $\theta$ determines the duration.

With periodic pulses $\{g'_{\theta,\alpha}: d(\alpha) \leq \theta < P, \alpha \in A\}$ defined by

$$g'_{\theta,\alpha}(t) = \sum_{i=-\infty}^{\infty} g_{\theta,\alpha}(t-iP) ,$$

we may define a periodic process

$$X(t) = \sum_{i=1}^{n} g'_{Z_i,W_i}(t-T_i)$$

where $T_1,T_2,\ldots,T_n$ are i.i.d. uniform on $[0,P)$ and $(Z_1,W_1)$, $(Z_2,W_2),\ldots,(Z_n,W_n)$ are indpendent random vectors. The probability measure of $(Z_i,W_i)$ is denoted $\mu_i$ and we assume that $\text{Prob}\{d(W_i) \le Z_i < P, W_i \in A\} = 1$ for all $i$. The notation $X \sim C_p(\mu_1,\mu_2,\ldots,\mu_n)$ is used to refer to this process. The analog of (4.4) for this process is now given.

(5.1) Theorem: Let $X \sim C_p(\mu_1,\mu_2,\ldots,\mu_n)$ and $X^* \sim C_p(\mu_1^*,\mu_2^*,\ldots,\mu_n^*)$. Assume for all $i$ that $(Z_i,W_i) \sim \mu_i$ and $(Z_i^*,W_i^*) \sim \mu_i^*$ satisfy $W_i \overset{\text{dist.}}{=} W_i^*$ and $E(\phi(Z_i)|W_i=\alpha) \le E(\phi(Z_i^*)|W_i^*=\alpha)$ for all $\alpha \in A$ and all convex functions $\phi$. If $H \ge 0$ and $H$ is L-superadditive, then $EH(X) \le EH(X^*)$.

The proof is basically the same as that of (4.4). Assume that $\mu_i = \mu_i^*$ for $1 \le i \le n-1$. Without loss of generality, we can then take $T_i = T_i^*$, $W_i = W_i^*$ and $Z_i = Z_i^*$ for $1 \le i \le n-1$. Since $W_n \overset{\text{dist.}}{=} W_n^*$ we can also assume $W_n = W_n^*$. Now lemma (4.5) shows that the theorem holds conditionally given the values of $T_1,T_2,\ldots,T_{n-1}$, $Z_1,Z_2,\ldots,Z_{n-1},W_1,W_2,\ldots,W_n$.

We could parallel the development of section 4 and prove a more general version of (4.13) based on (5.1). Instead, we shall just state a special case. Let $T_i$, $Z_i$ and $\{g_\theta:\theta \ge d\}$ be as in the definition of

$L(\lambda,F)$. Let $\{Y_j:-\infty < j < \infty\}$ be nonnegative random variables which are i.i.d. according to a distribution $G$. Define the process $X \sim L(\lambda,F,G)$ by

$$X(t) = \sum_{i=-\infty}^{\infty} Y_i g_{Z_i}(t-T_i) .$$

Thus $X$ is a superposition of pulses having both random amplitudes and durations.

(5.2) Theorem: Let $X \sim L(\lambda,F,G)$ and $X^* \sim L(\lambda^*,F^*,G^*)$ with $\lambda = \lambda^*$, $G = G^*$ and $F \lesssim F^*$. If $H$ is local, nonnegative and L-superadditive, then $EH(X) \leq EH(X^*)$.

The proof is omitted.

## 6. Weaker Results for More General Pulse Shapes.

Weaker conclusions may be obtained under weaker assumptions on the form and duration of the pulses. Let $\{g_\theta : \theta \geq d\}$ be a family of pulses which satisfies (4.1) and the condition given below.

$$
\begin{aligned}
& g_{\theta+\varepsilon+\delta} \geq g_{\theta+\varepsilon} \vee (S_\varepsilon g_{\theta+\delta}) \quad \text{and} \\
& S_\varepsilon g_\theta \geq g_{\theta+\varepsilon} \wedge (S_\varepsilon g_{\theta+\delta}) \quad \text{for all} \\
& \theta \geq d,\ \varepsilon \geq 0,\ \delta \geq 0 \ .
\end{aligned} \tag{6.1}
$$

This condition is the same as (4.2) except that "=" has been replaced by "$\geq$".

Two examples of parametric families of pulses which satisfy (6.1) with $d=0$ will now be given. Let $\psi$ be any increasing function on $[0,\infty)$ with $\psi(0) \geq 0$. For our first example, for all $\theta \geq 0$ define

$$
g_\theta(t) = \begin{cases} \psi(\theta - t) & \text{for } 0 \leq t \leq \theta, \\ 0 & \text{otherwise .} \end{cases}
$$

The second example is a family of symmetric pulses defined for $\theta \geq 0$ by

$$
g_\theta(t) = \begin{cases} \psi(t) & \text{for } 0 \leq t \leq \theta/2 \\ \psi(\theta - t) & \text{for } \theta/2 \leq t \leq \theta, \\ 0 & \text{otherwise .} \end{cases}
$$

The verification of (6.1) is easy for both examples.

We also use a weaker version of the variability ordering defined as follows. Let $Y$ and $Z$ have distributions $F$ and $G$ respectively. Assume $EY > -\infty$ and $EZ > -\infty$. If $E\phi(Y) \leq E\phi(Z)$ for all functions $\phi : \mathbf{R} \to \mathbf{R}$ which are <u>increasing</u> and convex, then we say that $F \prec\uparrow G$ or equivalently $Y \precsim\uparrow Z$. This version of the variability ordering is frequently used. See the references given in section 1.

Using the family $\{g_\theta : \theta \geq d\}$ satisfying (4.1) and (6.1), we define the process $X \sim L(\lambda, F)$ as in section 4. With only minor changes in the proof we can obtain the following modification of theorem 4.10.

(6.2) Theorem: Let $X \sim L(\lambda, F)$ and $X^* \sim L(\lambda^*, F^*)$ with $\lambda = \lambda^*$ and $F \precsim\uparrow F^*$. Suppose that $H$ is a local functional defined on a sufficiently large class of functions. Assume that $F$ and $F^*$ have bounded support, $F(B) = F^*(B) = 1$ for some $B < \infty$. If $H$ is nonnegative, increasing and L-superadditive, then $H(X) \precsim\uparrow H(X^*)$ which further implies that $EH(X) \leq EH(X^*)$.

Proof: Let $\{g'_\theta : d \leq \theta < P\}$ be periodic pulses with period $P$ defined as in section 4. Property (4.3) holds as stated because replacing (4.2) by the weaker condition (6.1) is compensated for by imposing the additional condition that $H$ be increasing. The proof of (4.4) proceeds as before except that the crucial lemma 4.5 must be slightly modified as follows.

(6.3) Lemma: Let $H$ satisfy the conditions in (6.2). For $d \leq \theta < P$ and any function $f$ define

$$\Phi(\theta) = \int_0^P H(f + S_t g'_\theta)\,dt \ .$$

Let $Y_1$ and $Y_2$ be random variables taking values in $[d,P)$. If $Y_1 \precsim\uparrow Y_2$, then $E\Phi(Y_1) \leq E\Phi(Y_2)$.

To prove this lemma just show that $\Phi$ is increasing and convex on $[d,P)$ by following the pattern of (4.5).

We have now shown that when $X$ and $X^*$ are periodic processes like those in (4.4), $EH(X) \leq EH(X^*)$ for all $H$ satisfying the conditions of (6.2). By the next lemma, this implies $H(X) \precsim\uparrow H(X^*)$.

(6.4) Lemma: If $\phi$ is increasing and convex and the functional $H$ is increasing and L-superadditive, then the composition $\phi \circ H$ is increasing and L-superadditive.

This property is given by Topkis in [13]. The proof is similar to lemma 4.7.

Transforming the result for the periodic processes into theorem 6.2 is accomplished by the same arguments which took us from (4.4) to (4.9) and then to (4.10).

7. Notes on the Variability Ordering.

The next three lemmas are needed to complete the proof of lemma 4.5.

Let $\Phi$: $J \to [0,\infty]$ where $J$ is any convex subset of the real numbers. We assume that $\Phi(w) + \Phi(z) \geq \Phi(x) + \Phi(y)$ whenever $w,x,y,z \in J$, $w < x < y < z$ and $w+z = x+y$. Let $D$ denote the effective domain of $\Phi$, $D = \{x:\Phi(x) < \infty\}$.

(7.1) Lemma: The set $D$ is convex and $\Phi$ is a convex function on $D$.

Proof: If $D$ is empty or consists of a single point, the lemma is trivial. Suppose $\Phi(a) < \infty$, $\Phi(b) < \infty$ and $a < x < b$. Then $\Phi(a) + \Phi(b) \geq \Phi(x) + \Phi(a+b-x) \geq \Phi(x)$ since $\Phi \geq 0$. Thus $\Phi$ is bounded above by $\Phi(a) + \Phi(b)$ on the interval $[a,b]$. The function $\Phi$ is midconvex; $\Phi(w) + \Phi(z) \geq 2\Phi((w+z)/2)$ for all $w,z \in J$. A result due to Jensen (see section 72 of Roberts and Varberg [4]) shows that a midconvex function bounded above on $[a,b]$ is convex on $[a,b]$. This completes the proof since $a$ and $b$ are arbitrary members of $D$.

(7.2) Lemma: Suppose that $X \prec Y$. If $\text{Prob}\{Y \geq c\} = 1$, then $\text{Prob}\{X \geq c\} = 1$ and $\text{Prob}\{X = c\} \leq \text{Prob}\{Y = c\}$. If $\text{Prob}\{Y \leq d\} = 1$, then $\text{Prob}\{X \leq d\} = 1$ and $\text{Prob}\{X = d\} \leq \text{Prob}\{Y = d\}$.

Proof: $(z)_+$ will denote the positive part of $z$, $(z)_+ = \max(z,0)$. Define $f(x) = (x-d)_+$. $f$ is convex and finite on the entire real line so that $Ef(X) \leq Ef(Y)$. $\text{Prob}\{Y \leq d\} = 1$ implies $Ef(Y) = 0$ so that $Ef(X) = 0$ and $\text{Prob}\{X \leq d\} = 1$. Now define $f_n(x) = n(x-d+1/n)_+$. For $x \leq d$ we have $|f_n(x)| \leq 1$ for all $n$ and

$$\lim_{n\to\infty} f_n(x) = \begin{cases} 1 & \text{if } x = d, \\ 0 & \text{if } x < d. \end{cases}$$

$f_n$ is convex and finite so that $Ef_n(X) \leq Ef_n(Y)$ for all $n$. Assuming $\text{Prob}\{Y \leq d\} = 1$ and using the bounded convergence theorem yields $\text{Prob}\{X = d\} \leq \text{Prob}\{Y = d\}$. The proof of the other statement is similar.

A convex function $\psi$ is called <u>proper</u> if $\psi(x) < \infty$ for at least one $x$ and $\psi(x) > -\infty$ for all $x$.

(7.3) Lemma: Let $\psi$ by any proper convex function. If $X \prec Y$, then $E\psi(X) \leq E\psi(Y)$.

Proof: We assume first that $\psi$ is <u>closed</u>. This means that $\{x:\psi(x) \leq \alpha\}$ is a closed set for all $\alpha$. Define $\psi^*(y) = \sup_x\{xy-\psi(x)\}$ and $\psi^{**}(x) = \sup_y\{xy-\psi^*(y)\}$. $\psi^*$ is a proper closed convex function and $\psi^{**} = \psi$ (see section 12 of Rockafellar [5]). Choose $y_0$ such that $\psi^*(y_0) < \infty$. Define $\psi_n(x) = \sup\{xy-\psi^*(y):y_0-n < y < y_0+n\}$. It is easily seen that $\psi_n$ is convex for all $n$. The following argument shows that $\psi_n(x) < \infty$ for all $n$ and $x$. Choose $x_0$ such that $\psi(x_0) < \infty$. Clearly $\psi^*(y) \geq x_0y - \psi(x_0)$ for all $y$. Thus $\psi_n(x) \leq \sup\{y(x-x_0) + \psi(x_0):y_0 - n < y < y_0 + n\} \leq p|x-x_0| + \psi(x_0)$ where $p = \max(|y_0-n|, |y_0+n|)$. Therefore $E\psi_n(X) \leq E\psi_n(Y)$ for all $n$. Since $\psi^{**} = \psi$ we have $\psi_n(x) \uparrow \psi(x)$ as $n \to \infty$ for all $x$. Also note that $\psi_n(x) \geq xy_0-\psi^*(y_0)$ for all $x$ and $n$. Thus $\psi_n(X)$ and $\psi_n(Y)$ are bounded below by random variables having finite means and we may apply the monotone convergence theorem to conclude $E\psi(X) \leq E\psi(Y)$.

Now let $\psi$ be any proper convex function. Let $B$ be the effective domain of $\psi$, $B = \{x:\psi(x) < \infty\}$. There exists a proper <u>closed</u> convex function $f$ which agrees with $\psi$ except perhaps on the boundary of $B$ (denoted $\partial B$). More precisely, $\psi(x) = f(x)$ for $x \notin \partial B$ and $\psi(x) \geq f(x)$ for $x \in \partial B$ (see section 7 of Rockafellar [5]). Our conclusion now follows from lemma 7.2 and the previous paragraph. For example, suppose $B = [c,d]$. If $\text{Prob}\{c \leq Y \leq d\} < 1$, then $E\psi(Y) = \infty$ and therefore $E\psi(X) \leq E\psi(Y)$ as desired. Now assume $\text{Prob}\{c \leq Y \leq d\} = 1$. Lemma 7.2 implies $\text{Prob}\{c \leq X \leq d\} = 1$, $\text{Prob}\{X = c\} \leq \text{Prob}\{Y = c\}$ and $\text{Prob}\{X = d\} \leq \text{Prob}\{Y = d\}$. Let $\alpha = \psi(c)-f(c)$ and $\beta = \psi(d)-f(d)$. We can write

$$E\psi(X) = Ef(X) + \alpha\,\text{Prob}\{X=c\} + \beta\,\text{Prob}\{X=d\}$$

and similarly for $E\psi(Y)$. Since $\alpha \geq 0$, $\beta \geq 0$ and $Ef(X) \leq Ef(Y)$, we immediately conclude $E\psi(X) \leq E\psi(Y)$.

Extending the definition of $\Phi$ by taking $\Phi(x) = \infty$ for $x \notin J$ makes $\Phi$ into a proper convex function. Thus $X \precsim Y$ implies $E\Phi(X) \leq E\Phi(Y)$ and the proof of (4.5) is complete.

We now give a proof of lemma 4.11. The following facts will be needed.

(7.4) Let $V$ and $W$ be nonnegative random variables with $EV < \infty$ and $EW < \infty$ having distributions $F$ and $G$ respectively.

(a) $EV = EW$ if and only if $\int_0^\infty (G(x) - F(x))dx = 0$ .

(b) If $EV = EW$, then $V \precsim W$ if and only if $\int_0^t (G(x)-F(x))dx \geq 0$ for all $t$.

Fact (a) is elementary. Fact (b) is a minor variant of results in sections 1.3 and 1.4 of Stoyan [12] or section 8.5 of Ross [7].

Proof of (4.11): Suppose $V \lesssim W$ with $V$ and $W$ as in (7.4). Define $\alpha = \sup\{x:F(x) < 1\}$ and $\beta = \sup\{x:G(x) < 1\}$. $F \lesssim G$ implies $\alpha \leq \beta$ by lemma 7.2. Note that $E(V \wedge t)$ is continuous and strictly increasing in $t$ for $t < \alpha$ and $E(W \wedge t)$ is continuous and strictly increasing in $t$ for $t < \beta$. Since $\phi(x) = x \wedge t$ is a concave function, $E(V \wedge t) \geq E(W \wedge t)$ for all $t$. If $\beta < \infty$, then $\alpha < \infty$ and the lemma is trivial, just take $V_n = V$ and $W_n = W$ for all $n$. So assume $\beta = \infty$. $\alpha$ may be finite or infinite. The proof proceeds in two cases.

First case, assume $\text{Prob}\{V=\alpha\} = 0$. Choose a nonnegative sequence $\{\alpha_n\}$ satisfying $\alpha_n \uparrow \alpha$ and $\alpha_n < \alpha$ for all $n$. Define $V_n = V \wedge \alpha_n$ for all $n$. Clearly $V_n \uparrow$ and $V_n \Rightarrow V$. Since $E(W \wedge \alpha_n) \leq E(V \wedge \alpha_n) < EV = EW$, there exists a unique $\beta_n$ such that $\beta_n \geq \alpha_n$ and $E(W \wedge \beta_n) = EV_n$. Define $W_n = W \wedge \beta_n$ for all $n$. Since $EV_n \uparrow EV$, we have $\beta_n \uparrow \infty$ and thus $W_n \uparrow$ and $W_n \Rightarrow W$. Let $F_n$ and $G_n$ be the distributions of $V_n$ and $W_n$ respectively. Define $\psi_n(t) = \int_0^t (G_n(x)-F_n(x))dx$. $\psi_n(t) \geq 0$ for $t \leq \alpha_n$ because $G_n(x) - F_n(x) = G(x) - F(x)$ for $x < \alpha_n$. $\psi_n(t)$ is decreasing when $\alpha_n \leq t < \beta_n$ because $F_n(x) = 1$ for $x \geq \alpha_n$. Finally, $\psi_n(t) = 0$ for $t \geq \beta_n$ because $EV_n = EW_n$. Thus $\psi_n(t) \geq 0$ for all $t$ so that $F_n \lesssim G_n$ by (7.4b).

Second case, assume $\alpha < \infty$ and $\text{Prob}\{V=\alpha\} > 0$. $E(W \wedge \alpha) < EV$ so there exist $c$ and $d$ such that $c < \alpha < d$ and $E(V \wedge c) = E(W \wedge d)$.

As in the first case we have $(V \wedge c) \precsim (W \wedge d)$. Let $U$ be a random variable which is uniformly distributed on the interval $(0,1)$ and independent of $V$ and $W$. Define

$$V_n = \begin{cases} V \wedge c & \text{if } U < \frac{1}{n}, \\ V & \text{if } U \geq \frac{1}{n}. \end{cases}$$

Clearly $V_n \uparrow$ and $V_n \Rightarrow V$. Since $E(W \wedge d) \leq EV_n < EV = EW$, there exists $\beta_n \geq d$ such that $E(W \wedge \beta_n) = EV_n$. Define $W_n = W \wedge \beta_n$. $EV_n \uparrow EV$ implies $\beta_n \uparrow \infty$ so that $W_n \uparrow$ and $W_n \Rightarrow W$. Now we use $V \precsim W$, $(V \wedge c) \precsim (W \wedge d)$ and (7.4) to show this construction gives $V_n \precsim W_n$ as desired. Let $V_n$, $W_n$, $V \wedge c$, $W \wedge d$ have distributions denoted by $F_n$, $G_n$, $F_{(c)}$, $G_{(d)}$ respectively. Note that $F_n = \frac{1}{n} F_{(c)} + (1 - \frac{1}{n})F$. The ordering $\precsim$ is preserved by mixtures so that $F_n \precsim \frac{1}{n} G_{(d)} + (1 - \frac{1}{n})G$. Again define $\psi_n(t) = \int_0^t (G_n(x)-F_n(x))dx$. Since $\frac{1}{n} G_{(d)}(x) + (1-\frac{1}{n})G(x) = G(x) = G_n(x)$ for $x < d$, $\psi_n(t) \geq 0$ for $t \leq d$. $\psi_n(t)$ is decreasing when $d \leq t < \beta_n$ because $d > \alpha$ and $F_n(x) = 1$ for $x \geq \alpha$. $\psi_n(t) = 0$ for $t \geq \beta_n$ because $EV_n = EW_n$. Thus $\psi_n(t) \geq 0$ for all $t$ and the proof is complete.

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## 8. An Attempt to Use Shot Noise in Models for Neuron Firing.

This report deals with a shot noise process $X(t)$ which is formed by the superposition of random pulses (or shot effects) which arrive at random times according to a Poisson process. One area in which shot noise processes arise is in the construction of models in the neural sciences. This is because the cell membrane potential in neurons is formed by the superposition of distinct contributions (pulses) arising from the many hundreds (or thousands) of synapses which impinge on each neuron. A contribution from an excitatory (inhibitory) synapse is represented by a pulse having a positive (negative) amplitude. In the simplest models the neuron fires (produces an action potential) whenever the membrane potential at the axon hillock exceeds a certain threshold value (except for complications introduced by the presence of absolute and relative refractory periods).

For a brief description of the functioning of neurons and a catalog of mathematical models which have been developed to explain the spike trains of single neurons, see Sampath and Srinivasan [B]. Many of the models contained therein share some of the features of shot noise (for example, see model 7.6 on page 100). We note also that Bevan, Kullberg and Rice [A] have used a shot noise process with rectangular pulses to model the acetylcholine induced membrane noise which occurs at the neuromuscular junction.

This report was initially motivated by a desire to prove results concerning the spike trains of single neurons. In particular, let $U(X)$ denote the number of upcrossings of the threshold value made by the shot

noise process $X$ during the time period $(0,T)$. In a simple model of neuron firing, $U(X)$ will be the random number of spikes produced during a period of length $T$. In more complicated models, the number of spikes can be written as a more complicated functional of the process $X$. However, I was unable to obtain strong results for these functionals. The functional $U$ does not satisfy the conditions for applying Theorem (1.3). Attempts were made to weaken these conditions. The conditions used in statements (c) and (d) of Theorem (4.10) are weaker than those in (1.3). The functional $U$ satisfies the conditions in (c) but the conclusion $EU(X) \leq EU(X^*)$ is quite weak since in fact $EU(X) = EU(X^*)$. The conclusion of (d) is stronger, but $U$ fails to satisfy all the conditions in (d).

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Suppose that pulses arrive according to a Poisson process of rate $\lambda$ with the duration of each pulse independently chosen from a distribution F having finite mean. Let X(t) be the shot noise process formed by the superposition of these pulses. We consider functionals H(X) of the sample path of X(t). H is said to be L-superadditive if $H(f\vee g)+H(f\wedge g)\geq H(f)+H(g)$ for all functions f and g. For any distribution F for the pulse durations, we define H(F) = EH(X). We prove that if H is L-superadditive and $\int\phi(u)dF(u)\leq\int\phi(u)dG(u)$ for all convex functions $\phi$, then $H(F)\leq H(G)$. Various consequences of this result are explored.

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