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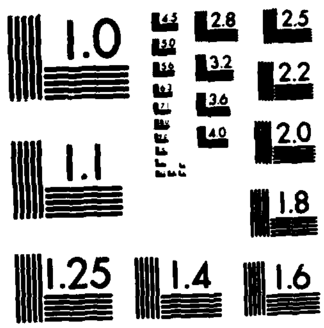

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FOREIGN TECHNOLOGY DIVISION

ON THE CLASSIFICATION AND NONLINEAR SOLUTION OF ONE TYPE OF STRATIFIED SHEAR FLOW

by

Liu Shida, Chen Jiayi, et al.

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On The Classification and Nonlinear Solution of One Type of Stratified Shear Flow /121

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ABSTRACT

Starting from the nonlinear internal gravity wave equation for stratified shear fluid, this paper considered a type of progressive wave motions and derived a set of two-variable (perturbation velocity and perturbation density) ordinary differential equations for the self-governing dynamic system of this type of fluid motion. Based on qualitative theories in differential equation, this paper also conducted a qualitative analysis on the geometric topological structure of the integration curve in the vicinity of the origin of a phase plane which used the perturbation velocity and perturbation density as coordinates. Based on the difference in the Richardson number on the phase diagram, the integration curve was separated into several areas with different properties. When Ri<0, the singular point was an unstable saddle point regardless of whether the velocity shear $d\bar{u}/dz$ was positive or negative. When Ri>0, the singular point was unstable in an area where the velocity shear $d\bar{u}/dz$ was positive, while the singular point was stable where the velocity shear $d\bar{u}/dz$ was negative (when 0<Ri<1/4, it was a stable nodal point; when Ri>1/4 it was a stable focal point, so the periodic solution of amplitude decay existed; when $Ri\to\infty$ it was a stable central point so there existed the periodic solution).

The second order system was obtained by expanding the nonlinear terms and then retaining the second order terms. Our analysis indicated that the topological structure was completely consistent with the first order system. The central point of the first order system was still the central point of the second order system. The periodic solution of the second order system

satisfied the famous K_dV equation, and it was an elliptical cosine wave. Under other conditions the solutions of the second order system (which was the velocity which varied only with time) were different from those of the first order system. This further reflected the actual situations associated with many atmospheric and oceanic phenomena (including turbulence).

I. INTRODUCTION

The density of the atmosphere or the ocean varies with height (this is being called "layered" or "stratified"), and the velocity also varies sharply with height (this is called the "vertical shear"). The most fundamental wave within the stratified shear flow is the internal gravity wave. The medium to small scale atmospheric systems such as typhoon, scowl line, and cumulus cloud; as well as temperature jump and turbulence in the ocean are all closely related to the internal gravity wave. In the past the discussion concerning the internal gravity waves was mainly concentrated in linear theories. Many important conclusions were obtained this way, such as the identification of the Brunt-Väisälä frequency N as the upper bound of the frequency of internal wave ω, and that the internal wave is stable when $Ri = N^2/(d\bar{u}/dz)^2$ is greater than 1/4[1] (here $\bar{u}(z)$ is the fundamental shear flow field). It was also established that Ri = 1/4 is the boundary between laminar flow and turbulent flow in stably stratified atmosphere ($N^2 > 0$). In recent years observations have revealed large amplitude fluctuations of the boundary layer at night[2] and blocked structures[3]. Intermittent turbulent flow[2] and solitary waves have also been observed. None of these phenomena can be explained by the linear theory. What are the characteristics of nonlinear stratified shear flow? What kinds of relations do they have with the Ri number? Can we find analytical solutions? All of these are topics that we are concerned about. Some of the relatively early studies concerning the theoretical examination of nonlinear internal waves include the work of Benney (1966)[5] and Benjamin

(1966)[6] (1967)[7]. Recently Maslowe and Redekopp[8] have conducted a systematic research into the nonlinear stratified shear flows. All of these studies applied the multiple scale perturbation method. This paper analyzed the set of equations which describes the flow, expanded the nonlinear terms of this set of equations by asymptotic expansion[9] and conducted an analysis on the phase plane. In addition, we have also obtained the analytical solution of the second order system which retained the second order terms. We obtained an elliptical cosine wave /122
where there was no shear ($d\bar{u}/dz = 0$). This paper carefully examined the effects of the shear $d\bar{u}/dz$ and Ri number on the stability of the flow within the stratified shear flows. We pointed out that the property of stability was different with a difference in the sign of the shear $d\bar{u}/dz$. Ri = 1/4 was not a line separating stability from instability.

II. BASIC SET OF EQUATIONS

By using $\bar{\rho}(z)$ to represent density stratification and $\bar{u}(z)$ to represent velocity shear, the two dimensional (x,z) motion of a Boussinesq fluid can be considered. The nonlinear equation of motion, adiabatic equation and continuity equation which describe the stratified shear flow can be represented as

$$\left.\begin{aligned}
&\frac{\partial u}{\partial t}+(\bar{u}+u)\frac{\partial u}{\partial x}+w\frac{d\bar{u}}{dz}=-\frac{1}{\bar{\rho}}\frac{\partial p}{\partial x}\\
&\frac{\partial w}{\partial t}+(\bar{u}+u)\frac{\partial w}{\partial x}=-\frac{1}{\bar{\rho}}\frac{\partial p}{\partial z}-g\frac{\rho}{\bar{\rho}}\\
&\frac{\left(\partial\frac{\rho}{\bar{\rho}}\right)}{\partial t}+(\bar{u}+u)\frac{\partial\left(\frac{\rho}{\bar{\rho}}\right)}{\partial x}-\frac{N^2}{g}w=0\\
&\frac{\partial u}{\partial x}+\frac{\partial w}{\partial z}=0
\end{aligned}\right\}\qquad(1)$$

Here u,w are the x and z components of velocity, respectively. p is pressure, ρ is density, g is gravitational acceleration,

$$N^2 = -\frac{g}{\bar\rho}\frac{d\bar\rho}{dz}$$

is a stratification parameter, $d\bar{u}/dz$ is a velocity shear parameter.

Due to the existence of the nonlinear advective terms in equation (1),

$$u\frac{\partial u}{\partial x},\quad u\frac{\partial w}{\partial x},\quad u\frac{\partial\left(\frac{\rho}{\bar\rho}\right)}{\partial x}$$

it is generally difficult to obtain an analytical solution which can explain the motion phenomenon. Here we will consider a type of progressive wave motion described below. We will assume that the wave solution has the following form:

$$u = U(\xi) \quad w = W(\xi) \quad p = P(\xi) \quad \frac{\rho}{\bar\rho} = \Pi(\xi) \qquad \xi = kx + nz - \omega t \tag{2}$$

Here k, n are wave numbers along the x,z direction, respectively, ω=kc is the angular frequency, c is the wave speed. None of the parameters ω, k, c will vary with x,z, or t.

Substitute (2) into (1) we have

$$\left.\begin{aligned} &(-\omega + kU + k\bar{u})U' + \frac{d\bar{u}}{dz}W = -\frac{1}{\bar\rho}kP' \\ &(-\omega + kU + k\bar{u})W' = -\frac{1}{\bar\rho}nP' - \Pi g \\ &(-\omega + kU + k\bar{u})\Pi' - \frac{N^2}{g}W = 0 \\ &kU' + nW' = 0 \end{aligned}\right\} \tag{3}$$

Here the symbol "'" is used to represent derivative taken against /123
ξ.

By eliminating P and W using the first and the second equations in (3), we can obtain two ordinary differential equations in U and ¶.

$$\left.\begin{aligned} &(n^2 + k^2)(-\omega + kU + k\bar{u})U' = nkg\Pi + nk\frac{d\bar{u}}{dz}U \\ &(-\omega + kU + k\bar{u})\Pi' = -\frac{N^2k}{gn}U \end{aligned}\right\} \tag{4}$$

If $(-\omega + kU + k\bar{u}) \neq 0$, (4) can be transformed into

$$U' = \frac{nkg\Pi + nk\frac{d\bar{u}}{dz}U}{(n^2+k^2)(-\omega+kU+k\bar{u})} = F(U,\Pi) \qquad (5)$$
$$\Pi' = \frac{-\frac{N^2k}{gn}U}{(-\omega+kU+k\bar{u})} = G(U)$$

Here $F(U, \Pi)$, $G(U)$ are nonlinear functions.

The right hand side of equation (5) does not contain ξ, explicitly so it is a set of ordinary differential equations for the self-governing system. Based on qualitative analysis of ordinary differential equations[10], it is easy to see that the origins $\Pi= 0$, $U = 0$ is the singular point on the (U,Π) phase plane. At this point U=0 and ρ=0 so this point represents the undisturbed static condition based on physical consideration.

In order to analyze (5), we will conduct Taylor expansion on the nonlinear F and G terms on the right hand side of equation (5) in the vicinity of the origin of the Π, U plane. We can thus obtain

$$\left.\begin{aligned} U' &= \frac{n\frac{d\bar{u}}{dz}}{(n^2+k^2)(-c+\bar{u})}U + \frac{ng}{(n^2+k^2)(-c+\bar{u})}\Pi - \frac{n\frac{d\bar{u}}{dz}}{(n^2+k^2)(-c+\bar{u})^2}U^2 \\ &\quad - \frac{ng}{(n^2+k^2)(-c+\bar{u})}\Pi U + \cdots\cdots \\ \Pi' &= -\frac{N^2}{gn(-c+\bar{u})}U + \frac{N^2}{gn(-c+\bar{u})^2}U^2 + \cdots\cdots \end{aligned}\right\} \qquad (6)$$

We will now explain that equation (6) is an equation which can be used for the discussion of internal gravity wave. We shall place our emphasis on the commonly seen practical situation where $C > \bar{u} > 0$. In order to simplify our description, we will first discuss the case where $k < n$, and we will keep the right hand side of equation (6) up to the second order terms. Equation (6) can thus be transformed into

$$\left.\begin{aligned} U' &= -\frac{\frac{d\bar{u}}{dz}}{nc}U - \frac{g}{nc}\Pi - \frac{\frac{d\bar{u}}{dz}}{nc^2}U^2 - \frac{g}{nc^2}\Pi U \\ &= -\frac{1}{nc^2}(U+c)\left(\frac{d\bar{u}}{dz}U + g\Pi\right) \\ \Pi' &= \frac{N^2}{gnc}U + \frac{N^2}{gnc^2}U^2 = \frac{N^2}{gnc^2}(U+c)U \end{aligned}\right\} \qquad (7)$$

Equation (7) is the second order self-governing system which will be used to analyze the nonlinear internal gravity waves from now on.

III. RESULTS OF LINEAR WAVES

We will first discuss the condition of linear internal gravity waves by using equations (6) and (7). If we were to take only the linear terms on the right hand side of equation (6), we can use these terms to describe the linear gravity waves at an infinitesimal distance away from the equilibrium point (0,0). At this time equation (6) becomes /124

$$\left.\begin{aligned} U' &= \frac{n\frac{d\bar{u}}{dz}}{(n^2+k^2)(-c+\bar{u})}U + \frac{ng}{(n^2+k^2)(-c+\bar{u})}\Pi \\ \Pi' &= \frac{-N^2}{gn(-c+\bar{u})}U \end{aligned}\right\} \tag{8}$$

If there is only density stratification $\rho(z)$ and no velocity shear $d\bar{u}/dz = 0$ (the same as $Ri \to \infty$), and we also assume N^2 is a constant, we can then obtain the following from (8)

$$U'' + \frac{N^2}{(n^2+k^2)(-c+\bar{u})^2}U = 0 \tag{9}$$

If the variation of U with ξ is periodic, we can obtain the dispersion relationship from (9):

$$(-c+\bar{u})^2 = \frac{N^2}{n^2+k^2} \tag{10}$$

Equation (10) is exactly the well known result of stratified flow[1], and we can also come to the conclusion that $\omega < N$.

If we now include the velocity shear in the discussion ($d\bar{u}/dz=0$) and assume that $d\bar{u}/dz$ varies with z, we can obtain the following equation from (8)

$$U'' = \frac{\left(\frac{n}{n^2+k^2} - \frac{1}{n}\right)\frac{d\bar{u}}{dz}}{(-c+\bar{u})}U' + \frac{1}{n^2+k^2}\left[\frac{\frac{d^2\bar{u}}{dz^2}}{\bar{u}-c} - \frac{\left(\frac{d\bar{u}}{dz}\right)^2}{(\bar{u}-c)^2}Ri\right]U \tag{11}$$

By setting $k \ll n$, (11) can be simplified as

$$U'' = \frac{1}{n^2}\left[\frac{\frac{d^2\bar{u}}{dz^2}}{\bar{u}-c} - \frac{\left(\frac{d\bar{u}}{dz}\right)^2 Ri}{(\bar{u}-c)^2}\right]U \qquad (12)$$

Here $Ri = \frac{N^2}{(du/dz)^2}$ is the Richardson number.

Equation 12 is consistent with the results of Taylor and Goldstein[11]. Furthermore, if $d\bar{u}/dz=0$ equation (12) become (9).

If we were to take only the linear terms in equation (7), it can be shown below that it is necessary for $Ri > 1/4$ in order to obtain a stable wave solution. By collecting only the linear terms in equation (7), it becomes

$$\left.\begin{aligned} U' &= -\frac{\frac{d\bar{u}}{dz}}{nc}U - \frac{g}{nc}\Pi \\ \Pi' &= \frac{N^2}{gnc}U \end{aligned}\right\} \qquad (13)$$

By assuming $d\bar{u}/dz$ and N^2 to be constants, we can obtain the following equation from (13)

$$U'' + \frac{\frac{d\bar{u}}{dz}}{nc}U' + \frac{N^2}{n^2c^2}U = 0 \qquad (14)$$

We will set the U in equation (14) to be /125

$$U = Ve^{-\frac{1}{2}\frac{\frac{d\bar{u}}{dz}}{nc}z} \qquad (15)$$

We can then eliminate the U' in equation (14) and obtain

$$V'' - \frac{1}{4}\frac{\left(\frac{d\bar{u}}{dz}\right)^2}{c^2}(1-4Ri)V = 0 \qquad (16)$$

It is obvious that it is necessary for the coefficient in front of V to be positive in order for equation (16) to have a wave solution. This is to say that

$$Ri > \frac{1}{4} \qquad (17)$$

All of the above are consistent with the existing results of stratified flow theories. This suggests that the basic equation sets (6) and (7) accurately reflect the results of linear internal gravity waves. If we keep the second order terms on the right hand side of either equation (6) or (7), we can then use them to discuss the finite amplitude nonlinear internal gravity waves.

IV. THE TOPOLOGICAL STRUCTURE OF STRATIFIED SHEAR FLOW

From (13) we can obtain the integration curve in the vicinity of the equilibrium point (0,0) on the phase plane (Π ,U). It is determined by the following equation

$$\frac{dU}{d\Pi} = \frac{-\frac{d\bar{u}}{dz}U - g\Pi}{\frac{N^2}{g}U} \tag{18}$$

The property of the phase path is determined by the characteristic root λ of the characteristic equation

$$\begin{vmatrix} 0-\lambda & \frac{N^2}{g} \\ -g & -\frac{d\bar{u}}{dz}-\lambda \end{vmatrix} = 0 \tag{19}$$

This is to say that

$$\lambda^2 + \frac{d\bar{u}}{dz}\lambda + N^2 = 0 \tag{20}$$

The characteristic root obtained from (20) is

$$\lambda = \frac{-\frac{d\bar{u}}{dz} \pm \frac{d\bar{u}}{dz}\sqrt{1-4Ri}}{2} \tag{21}$$

This is why the phase paths (Π ,U) in the vicinity of the origin (0,0) of the parameter ($d\bar{u}/dz, N^2$) are divided into several different areas as shown in Figure 1. The black dot ● in this figure represents the equilibrium point which is stable, the white dot o represents the unstable equilibrium point, and the direction of the row indicates the direction in which time is increasing. Since $\xi = kx + nz - \omega t$, so for a certain given point (X,Z) the direction in which time is increasing is the direction in which ξ is decreasing. The equation for the dashed line in this figure is

$$(N^2) = \frac{1}{4}\left(\frac{d\bar{u}}{dz}\right)^2, \qquad \text{(c)}$$

It is a parabola.

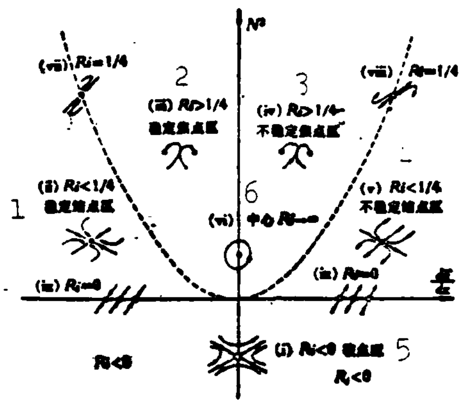

Figure 1

1. stable nodal point area
2. stable focal point area
3. unstable focal point area
4. unstable nodal point area
5. saddle point area
6. unstable

It can be seen from this figure that the parameter plane $(d\bar{u}/dz, N^2)$ divides the properties of the solution of the stratified shear flow (the phase diagram which used ¶ as abscissa and U as ordinate) into five areas (i), (ii), (iii), (iv), (v) and four boundaries (vi), (vii), (viii), and (ix):

(i) Ri< 0 unstable saddle point area; $\lambda_1 > 0$, $\lambda_2 < 0$

(ii) Ri< 1/4, $d\bar{u}/dz < 0$ stable nodal point area; $\lambda_1 > \lambda_2 > 0$

(iii) Ri> 1/4, $d\bar{u}/dz < 0$ stable focal point area; λ_1 and λ_2 are complex conjugates with real parts being positive.

(iv) Ri> 1/4, $d\bar{u}/dz > 0$ unstable focal point area; λ_1 and λ_2 are complex conjugates with real parts being negative.

(v) Ri $<1/4$, $d\bar{u}/dz > 0$ unstable nodal point area; $\lambda_1 < \lambda_2 < 0$

(vi) $Ri \to \infty$, $d\bar{u}/dz = 0$ stable central point; λ_1 and λ_2 are pure imaginary roots.

(vii) Ri = 1/4, $d\bar{u}/dz < 0$; $\lambda_1 = \lambda_2 > 0$

(viii) Ri = 1/4, $d\bar{u}/dz > 0$; $\lambda_1 = \lambda_2 < 0$

(ix) Ri = 0; λ_1 is either positive or negative, $\lambda_2 = 0$

It should be pointed out that the above analysis was done under the conditions of $C > \bar{u}$, $k << n$.

For the condition where $c < \bar{u}$ the above analysis can still be applied.
All we have to do is to switch the $d\bar{u}/dz > 0$ (or $d\bar{u}/dz < 0$) with $d\bar{u}/dz < 0$ (or $d\bar{u}/dz > 0$). This is to say that the first quadrant of the parameter plane $(d\bar{u}/dz, N^2)$ is a stable area while the second quadrant is an unstable area.

For the condition where k is not smaller than n, all we have to do is to replace the parameter Ri with $\frac{n^2 + k^2}{n^2}$ Ri in the above analysis. This is to say that the stable area is still related to k and n.

In this way, we can classify the flows according to their Ri numbers.

Up to now we have only analyzed the first order system (13), now we will analyze the second order system (7). According to the theory of Poincaré-Bendixson[13], since the nonlinear terms in (7) are separately

$$X(\Pi, U) = \frac{N^2}{gnc^2} U^2$$
$$Y(\Pi, U) = -\frac{\frac{d\bar{u}}{dz}}{nc^2} U^2 - \frac{g}{nc^2} \Pi U \quad (22)$$

so we will have /127

$$X(0,0) = Y(0,0) = \frac{\partial X(0,0)}{\partial U} = \frac{\partial Y(0,0)}{\partial U} = \frac{\partial X(0,0)}{\partial \Pi} = \frac{\partial Y(0,0)}{\partial \Pi} = 0 \quad (23)$$

In this way the qualitative properties of the second order system (7) in the vicinity of the origin (0,0) under the conditions of (i), (ii), (iii), (iv), (v), (vi), (vii), and (viii) are completely the same as the analysis done on the first order system (13).

At the same time, relative to the central point (Ri→∞, $d\bar{u}/dz = 0$) of the first order system, the second order system (7) becomes

$$U' = -\frac{g}{nc}\Pi - \frac{g}{nc^2}\Pi U = B(\Pi, U)$$

$$\Pi' = \frac{N^2}{gnc}U + \frac{N^2}{gnc^2}U^2 = A(\Pi, U) \qquad (24)$$

According to the symmetry principle[13], we have

$$A(\Pi, U) = A(-\Pi, U) \qquad B(\Pi, U) = -B(-\Pi, U) \qquad (25)$$

We can be sure that the central point of the first order system is still the central point of the second order system. Even under condition (vi), the qualitative property of the second order system is the same as that of the first order system.

This is why the qualitative structure of the solution of the first order system (13) is completely the same as that of the second order system (7) in the vicinity of the origin. The only difference is that the rate of variation of U and Π with t (or ξ) is different on the phase diagram.

From the complete phase diagram of the stratified shear flow we can see that when $C > \bar{u}$, we will have:

(1) $N^2 < 0$ (the same as $d\rho/dz > 0$) is always a factor for instability. As for whether the shear is a factor for stability or a factor for instability depends on whether $d\bar{u}/dz < 0$ or $d\bar{u}/dz > 0$. In the past, the discussion of a Kevin-Helmholz wave always treated the shear as a factor for instability.

(2) Miles (1961) and Howard (1961) gave the necessary condition for a stable internal gravity wave as Ri> 1/4, and they did not include the condition where 0< Ri< 1/4. It can be seen from the phase diagram that as long as $d\bar{u}/dz < 0$, the wave is still stable when 0 <Ri<1/4.

(3) In the past, people always believed that the wave is always stable when the Ri number is very large (at this time the shear $d\bar{u}/dz$ is very small). This is why the production of turbulence under the condition of stable stratification ($N^2 > 0$, $d\bar{\rho}/dz < 0$) could not be explained. According to our analysis,

instability can still be reached under very high Ri as long as $d\bar{u}/dz > 0$.

(4) When $1/4 < Ri < \infty$, there exist periodic solutions with either amplitude growth (under the condition of $d\bar{u}/dz > 0$) or amplitude decay (under the condition of $d\bar{u}/dz < 0$). When $Ri \to \infty$, there is a periodic solution and when $Ri \leq 1/4$ there is no periodic solution. During the night there is a phenomenon of frequent appearance or vanishing of large amplitude waves in the boundary layer[2]. We believe that this is exactly the condition of the stratified shear flow within the focal point area where $1/4 < Ri < \infty$.

V. THE NONLINEAR SOLUTION OF THE STRATIFIED SHEAR FLOW /128

In order to compare the rates of variation of U and Π with ξ (or t) between the nonlinear system (second order system) and the linear system (first order system), we will now derive the analytical solutions $U(\xi)$ and $\Pi(\xi)$ for the second order system (7).

By taking the derivative against ξ, the first equation in (7) becomes

$$U'' = -\frac{\frac{d\bar{u}}{dz}}{nc}U' - \frac{g}{nc}\Pi' - \frac{\frac{d\bar{u}}{dz}}{nc^2}\cdot 2UU' - \frac{g}{nc^2}(\Pi'U + U'\Pi) \qquad (26)$$

We can now obtain the second order nonlinear equation of U by substituting the Π' in the second equation of (7) and the Π' in the first equation (7) into the formulation.

$$U'' + \frac{\frac{d\bar{u}}{dz}}{nc}\left(1+\frac{U}{c}\right)U' - \frac{1}{c}\frac{U'^2}{1+\frac{U}{c}} + \frac{N^2}{n^2c^2}U\left(1+\frac{U}{c}\right)^2 = 0 \qquad (27)$$

We will now derive the analytical solution of the nonlinear equation (27) as follows. To serve as a contrast, we will first consider a relatively simple condition. This is to say that we will first eliminate the nonlinear terms from equation (27) and then try to solve the linearized equation:

$$U'' + \frac{\frac{d\bar{u}}{dz}}{nc}U' + \frac{N^2}{n^2c^2}U = 0 \qquad (28)$$

Equation (28) is actually equation (14), so its corresponding characteristic equation is

$$r^2 + \frac{\frac{d\bar{u}}{dz}}{nc} r + \frac{N^2}{n^2c^2} = 0 \qquad (29)$$

The characteristic root of (29) is

$$r = \frac{-\frac{\frac{d\bar{u}}{dz}}{nc} \pm \frac{\frac{d\bar{u}}{dz}}{nc}\sqrt{1 - 4Ri}}{2} = \frac{\lambda}{nc} \qquad (30)$$

Here λ is the representative equation (21).

The form of the solution of equation (28) $U(\xi)$ can thus be determined based on whether the characteristic root r (or λ) is a real root, an imaginary root or a complex conjugate root. We can then obtain $\Pi(\xi)$ from the second equation of (13). These special solutions are listed as follows:

(1) When λ is a real root, the $U(\xi)$ and $\Pi(\xi)$ corresponding to the two straight lines which pass through the saddle point on ncdal point along the phase path are

$$\begin{aligned} U(\xi) &= e^{\frac{\lambda}{nc}\xi} = e^{\frac{\lambda}{nc}(kx+nz-\omega t)} \\ \Pi(\xi) &= \frac{N^2}{g\lambda} e^{\frac{\lambda}{nc}\xi} = \frac{N^2}{g\lambda} e^{\frac{\lambda}{nc}(kx+nz-\omega t)} \end{aligned} \qquad (31)$$

(2) When λ are complex conjugate roots, the solutions of $U(\xi)$ and $\Pi(\xi)$ corresponding to the spiral line in the focal point area on the phase diagram are:

$$\begin{aligned} U(\xi) &= e^{\frac{\alpha}{nc}\xi} \cos\frac{\beta}{nc}\xi = e^{\frac{\alpha}{nc}(kx+nz-\omega t)} \cos\frac{\beta}{nc}(kx + nz - \omega t) \\ \Pi(\xi) &= \frac{N}{g} e^{\frac{\alpha}{nc}\xi} \cos\left(\frac{\beta}{nc}\xi + A\right) = \frac{N}{g} e^{\frac{\alpha}{nc}(kx+nz-\omega t)} \cos\left[\frac{\beta}{nc}(kx + nz - \omega t) + A\right] \end{aligned} \qquad (32)$$

Here α and β are the real and imaginary part of λ, respectively; /129
and $tgA = -\sqrt{4Ri-1}$.

(3) When λ is a pure imaginary root, the $U(\xi)$ and $\Pi(\xi)$ corresponding to the closed trajectory around the central point of the phase diagram are

$$U(\xi) = \cos\frac{N}{nc}\xi = \cos\frac{N}{nc}(kx + nz - \omega t)$$

$$\Pi(\xi) = -\frac{N}{g}\sin\frac{N}{nc}\xi = -\frac{N}{g}\sin\frac{N}{nc}(kx + nz - \omega t) \tag{33}$$

We will now solve for the nonlinear solutions corresponding to equations (31) and (33) as follows. Since the relationship between U and Π is given by the phase diagram, we will not try to find the solution by directly using equation (27). Instead, we will use the original set of equations (7).

For the two straight lines passing through the saddle point or the nodal point on the phase diagram, we can derive the following equation from (31)

$$U = \frac{N^2}{g\lambda}\Pi \tag{34}$$

By substituting (34) into (7) we can obtain

$$\frac{dU}{d\xi} = \frac{\lambda}{nc}U(U + c) \tag{35}$$

By integrating (35) we can get

$$\frac{U}{U + c} = e^{\frac{\lambda}{nc}\xi} \tag{36}$$

or

$$\frac{c + 2U}{c} = \mathrm{th}\left(-\frac{\lambda}{2nc}\xi\right) \tag{37}$$

So we set

$$U = -\frac{c}{2}\left(1 + \mathrm{th}\frac{\lambda}{2nc}\xi\right) = -\frac{c}{2}\left[1 + \mathrm{th}\frac{\lambda}{2nc}(kx + nz - \omega t)\right] \tag{38}$$

By substituting (38) into the second equation of (7) we can get

$$\Pi = -\frac{N^2c}{2g\lambda}\left(1 + \mathrm{th}\frac{\lambda}{2nc}\xi\right) = -\frac{N^2c}{2g\lambda}\left[1 + \mathrm{th}\frac{\lambda}{2nc}(kx + nz - \omega t)\right] \tag{39}$$

By comparing (38) and (39) with the linear solution (31), we can see that the velocities at which they approach (or get away from) the origin (the equilibrium point) are different. The solution obtained by considering only the linear terms has the form of an exponential function. The solution obtained by considering the nonlinear terms has the form of a hyperbolic tangent function.

What's more meaningful is that (38) and (39) are very similar to the progressive wave solution of the nonlinear dispersion equation (which is the Burgers equation)

$$\frac{\partial u}{\partial t} + u\frac{\partial u}{\partial x} = \mu\frac{\partial^2 u}{\partial x^2} \quad (\mu \text{ is the dispersion coefficient}) \qquad (40)$$

The progressive wave solution with $u = U(\xi)$ and $\xi = kx - \omega t$ is

$$U = c\left(1 - \mathrm{th}\frac{c}{2k\mu}\xi\right) \qquad (41)$$

In equations (38) and (39)

$$\frac{1}{|\lambda|} \text{ is equivalent to } \mu \qquad (42)$$

So the $|\lambda|$ value increases as $d\bar{u}/dz$ increases, corresponding to decreasing μ value. In this way, U will vary sharply at the location where ξ changes sign. This explains that for the nonlinear solutions of (38) and (39) there is a phenomenon of /130
sudden variations in U and Π when $d\bar{u}/dz$ is very large. This phenomenon will not occur if we only consider the linear system. The conclusion of this paper can be used to explain a lot of the intermittent phenomena in the atmosphere on the ocean.

Now we will derive the nonlinear solution corresponding to (33), at the time the origin of the phase plane is the central point. From (18) or (33) or (24) we can obtain

$$\frac{dU}{d\Pi} = -\frac{g\Pi}{\frac{N^2}{g}U} \qquad (43)$$

So the closed phase path around the central point is

$$\Pi^2 + \frac{N^2}{g^2}U^2 = D \qquad (44)$$

In this equation D is an integration constant. (44) is an ellipse.

By taking the derivative against ξ for the first equation of (24), then substituting the second equation of (24) and equation (44) into the formulation, and retaining only up to the second order terms, we can get

$$U'' = -\frac{3N^2}{\pi^2 c^3} U^2 - \frac{N^2}{\pi^2 c^3} U + \frac{\varepsilon^2}{\pi^2 c^3} D \qquad (45)$$

By taking the derivative of (45) with respect to ξ we can obtain the famous K_dV equation

$$U''' + \frac{6N^2}{\pi^2 c^3} UU' + \frac{N^2}{\pi^2 c^3} U' = 0 \qquad (46)$$

We have already obtained the solution of (45) and (46), it is an elliptical cosine function[14] cn

$$U(\xi) = U_2 + (U_1 - U_2) cn^2 \sqrt{\frac{N^2}{2\pi^2 c^3}(U_1 - U_3)}\,(kx + nz - \omega t) \qquad (47)$$

Here $U_1 > 0$, $U_2 < 0$, $U_3 < U_2 < 0$ is a third order equation

$$H(U) = U^3 + \frac{\varepsilon}{2} U^2 - \frac{\varepsilon^2}{N^2} DU + B \qquad (48)$$

These are independent single real roots, and B is a constant.

It can be seen from the comparison between (47) and the corresponding linear solution (33) that the linear wave has the form of a cosine (or sine) function while the solution obtained by taking into consideration the nonlinear effects is an elliptical cosine function. The periodicity and wavelength are also different. Furthermore, the amplitude of a nonlinear wave is related to its wave speed[14].

Now we have obtained nonlinear solutions for the two straight lines passing through the saddle point or the nodal point of the phase diagram as well as the closed curve around the central point of the phase diagram (π,U).

VI. CONCLUSION

We have obtained the structure of geometric topology for the phase diagram of the stratified shear flow in the vicinity of the origin of the (π,U) phase plane. Its property is determined by the stratification parameter N^2 and the shear parameter $d\bar{u}/dz$. Our analysis indicates that the stability of the flow depends not only on the sign of stratification N^2 (same as $\partial\bar{\rho}/\partial z$) but also on the sign of the shear ($d\bar{u}/dz$). This is why only the first quadrant ($c<\bar{u}$) and second quadrant ($c>\bar{u}$) on the parameter plane ($d\bar{u}/dz$, N^2) are stable. Ri = 1/4 is not the boundary between stable and unstable. It is actually the boundary between periodic and nonperiodic solutions.

The qualitative properties of a linear system are completely the same as a nonlinear system, but the rates of variations are different. For the nonlinear system there is a phenomenon of rapid variation when $|d\bar{u}/dz|$ is very large. The amplitude of a nonlinear periodic wave is related to its wave speed.

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REFERENCES

[1] Yih, C.S., Stratified Flow, Academic Press, New York (1980).

[2] Chimonas, G., Workshop on the Planetary Boundary Layer, 67-106 (1980)

[3] Zhou Ming Yu et al, The Black Structure of the Turbulence Field in the Atmospheric Boundary Layer, Academia Sinica 5 (1981), 614.

[4] Apel, J.R., et al, J.G.R. 80 (1975), 865-881.

[5] Bennery, D.J., J. Math. Phys., DE (1966), 52-63.

[6] Benjamin, T.B., J. Fluid. Mech., 29 (1966), 241-270.

[7] Benjamin, T.B., J. Fluid.. Mech., 29 (1967), 559-592.

[8] Maslowe, S.A., Redekopp, L.G., Fluid Mech., 101 (1980), 321-348.

[9] Liu Shida and Liu Shishi, The Nonlinear Elliptical Cosine Wave and Solitary Wave in the Atmosphere, Academia Sinica, 4 (1982).

[10] Jordan, J.W., Nonlinear Ordinary Differential Equation (1977).

[11] Turner, J.S., Buoyancy Effects in Fluid (1973).

[12] Bhatnagar, P.L., Nonlinear Waves in One-Dimensional Dispersion Systems (1980).

[13] Qin Yuan Xun, The Integration Curve Defined by the Differential Equation.

[14] Liu Shida and Liu Shishi, The Nonlinear Internal Wave in Stratified Fluids, Journal of Oceanography, 5 (1982).

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