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A Unified Model for Performance and Reliability
of Fault-Tolerant/Multi-Mode Systems*

V.G. Kulkarni, V.F. Nicola and
K.S. Trivedi

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Abstract

This paper unifies different models and relates different performance and reliability measures that have been proposed for the analysis of fault-tolerant computer systems. We model the changes in the structure of the system due to different events (such as degradation, failure or repair) as a continuous time Markov chain. In particular we consider the execution of a job on such a computer system where a service rate (or a reward rate) is associated with each structure-state. We allow different types of service-preemption interactions due to changes in the structure-state of the system. We derive the distribution of the completion time of a given job. Although the developed techniques are suitable for the analysis of complex systems, we demonstrate their use through a simple switching server example.

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1. *Introduction*

The increased reliability requirements have caused fault-tolerant and degradable systems to become more important. A separate performance or reliability analysis of such sytems is inadequate. It is of much interest to introduce measures that reflect both performance and reliability of the system. Several authors have proposed such measures and studied their analysis in specific cases [1-7,10-15]. Different models were often developed to study different performance and/or reliability measures. This paper is an attempt to unify different performance and reliability models into a single model that is useful for assessing the behaviour of degradable computer systems.

Such performance and reliability measures can be either system- oriented or job-oriented. A system-oriented reliability measure is the distribution of the time to failure [15], while a job-oriented reliability measure is the probability that the job completes before system failure [7,16]. System or job oriented performance measures such as throughput or response time are evaluated assuming no failures [15]. A system-oriented combined (performance and reliability) measure is the accumulated reward (or performance measure) up to time t [4,10] or up to system failure [2,15,16]. A combined job-oriented measure is the distribution of the job completion time in the presence of failure/repair preemptions [3,5,8,12]. The analysis of this measure and its relation to the accumulated reward [8] is a major concern of this paper.

In the model we develop, a computer system is described by a stochastic process that represents the structural state of the system which changes due to different events. Associated with each structure- state is a reward rate (e.g., computation capacity, throughput or response time). It is particularly useful for our unifying analysis to consider the execution of a particular job on the system described above, where the reward rate represents the service rate (e.g., the number of instructions executed per unit time). It is obvious that the completion time of the job is affected by the preemptions and the possible variations in the service rate due to changes in the structure-

state of the system (e.g., system degradation, failure or repair). We will show that, when the job service is resumed after preemptions, the completion time of a given job and the accumulated service (reward) until a given time are dual measures so that the distribution of one of them allows us to compute the distribution of the other. Furthermore we show how to derive different common performance and reliability measures from our model.

It is important to distinguish the type of a structure-state according to the service-preemption interaction associated with a transition to that structure-state. We will consider the following types of structure-states:

a) preemptive-resume (*prs*): upon preemption, the job's service is resumed at a (possibly) different service rate.

b) preemptive-repeat (*prt*): upon preemption, the job's service is restarted at (possibly) a different service rate. This can be further classified into the following two types

i) preemptive-repeat-identical (*pri*): the same (identical) job is restarted.

ii) preemptive-repeat-different (*prd*): a different independent job with the same distribution of service-requirement is restarted.

In [8] we considered the pure cases where all the structure-states in a given model are assumed to be of the same type (i.e., *prs*, *pri*, or *prd*). In that paper the analysis was based on conditional transform techniques. In the present paper, we allow a mixture of two different types of structure-states in the model (i.e., *prs* with *pri*, *prs* with *prd* and *pri* with *prd*). In the mixed cases the analysis is based on exact aggregation and conditional transform techniques.

In section 2, we introduce the mathematical model, basic assumptions, some definitions and notations. In section 3 we review the main results of the pure cases considered in [8]. Section 4 contains some preliminaries and introduces the main idea of the solution technique in the mixed cases. In sections 5, 6 and 7 we consider the

analysis of the mixed cases; namely, *prs* with *pri*, *prs* with *prd* and *pri* with *prd*, respectively. The techniques developed here are demonstrated by means of a simple and illustrative example - the switching server. We conclude our paper in section 8.

2. *The Basic Model*

Consider a particular job to be run on a particular computer system. The work requirement of a job is a random variable B, and is measured in work units (e.g., the number of instructions to be executed). It has the cumulative distribution function $G(x) = P(B \leq x)$ and *LST* (Laplace Stieltjes Transform) $G^{\sim}(s) = E(e^{-sB})$. It is assumed that $G(0+) = 0$. Now consider the execution of such a job on a computer system. The system changes its mode of operation in response to different sources of interruptions (e.g., failure/repair of a part or the whole system, increase/decrease of the degree of multiprogramming, system calls and I/O interruptions, etc.). The change of the mode of operation is reflected in the rate at which that particular job is serviced. The service rate is measured in number of work units (e.g., instructions) per unit time (we also refer to it as the work rate or reward rate).

The process that describes the mode changes in the system's operation in time is called the structure-state process. It is a stochastic process $\{Z(t), t \geq 0\}$, where $Z(t)$ is the state of the system (i.e., the mode of operation) at time t. This stochastic process is assumed to have piecewise constant paths and finite number of jumps in finite intervals of time. At any time, the system can be in one of the $n+1$ states numbered $0,1,2,...,n$. In state i the system serves the job at a rate $r_i \geq 0$, $1 \leq i \leq n$. The state 0 is an absorbing "failure" state, i.e., once the system enters state 0 it stays there and offers no more service. There may be other absorbing "non-failure" states among the states $1,2,...,n$, with service rates greater than zero, so that if the system enters such a state the job will eventually complete.

It is reasonable to assume that the structure-state process is independent of the work requirement of the job. A state i, $1 \le i \le n$, is of the type *prs* or *pri* or *prd* (as defined in section 1).

Now let us introduce a cumulative quantity that is very useful in our unifying analysis.

The cumulative measure, $W(t)$: defined to be the total reward gained since the last transition to a *prt* state, until (global) time t. W is the cumulative measure until system's failure. We note that $W(t)$ has the following properties:

i) $W(0)=0$

ii) $Z(t) = i \Rightarrow dW(t)/dt = r_i$

iii) If the structure-state process makes a transition at time t and $Z(t+)=i$ then
     $W(t+)=0$ if i is a *prt* state and $W(t+)=W(t-)$ if i is a *prs* state.

Typical sample paths of the structure-state process and the cumulative measure $W(t)$ are shown in figure 1, for the following case: The set of states is $\{0,1,2,3\}$; state 1 is *prs* with $r_1=1$, state 2 and 3 are *prt* with $r_2=2$ and $r_3=0$, state 0 is an absorbing failure state.

The following job-oriented performance measures relate to the cumulative measure $W(t)$.

The job completion time, $T(x)$: defined to be the time needed to complete a job that requires x units of work. T denotes the completion time of a job that requires a random amount of work, B (note that if there are any *prd* structure-states, the job that completes at time T need not have the same initial work requirement).

Since $W(t)$ represents the useful work done on the job upto time t, and since $W(t)$ has piecewise continuous paths with only downward jumps, it follows that

$$T(x) = \min\{t \ge 0: W(t)=x\}$$

and

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$$T = \min\{t \geq 0 : W(t) = B\} .$$

The probability of omission failure, $\eta(x)$: defined to be the probability that the system fails before the completion of a job that requires x units of work. Thus

$$\eta(x) = P(W(t) < x, \text{ for all } t \geq 0)$$
$$= P(T(x) = \infty) .$$

η is the probability that the system fails before the completion of a job with random work requirement. Hence

$$\eta = P(W(t) < B, \text{ for all } t \geq 0)$$
$$= P(T = \infty) .$$

The related notion of the dynamic failure probability in real-time systems with a hard deadline, d [7], is given by $\eta = P(T > d)$. Since our techniques allow us to compute the distribution function of T, the dynamic failure probability for a given deadline is readily obtained. Obviously, for systems with no absorbing failure states (repairable system) and with no hard deadlines, $\eta = 0$.

The cumulative measure $W(t)$ can be further specialized to the following two cumulative quantities that yield different performance measures.

i) *The preemptive-resume cumulative measure, $Y(t)$:* Let all structure-states be of the *prs* type. Then $Y(t)$ is the total reward gained in all structure-states until time t [4,10,14]. Y is the *prs* cumulative measure until system's failure [2,13,15,16].

The following system-oriented measures can be derived as special cases of the *prs* cumulative measure $Y(t)$.

The system reliability, $R(t)$: Let X be the time until system's failure, i.e., the time until the structure-state process enters the absorbing failure state 0. The reliability $R(t)$ of the system is defined to be the probability $P(X > t)$.

Clearly, if we set all $r_i = 1$, $1 \leq i \leq n$, then

$$R(t) = P(X > t)$$
$$= \lim_{\tau \to \infty} P(Y(\tau) > t) .$$

It should be noted that the analysis of the related measure of saftey [9] is the same as that of the system reliability, provided that all unsafe system states are grouped into an absorbing failure state.

The total "up" or "down" time until time t, $U(t)$ or $D(t)$: The system is said to be "up" if it is in a state i with $r_i > 0$, else it is said to be "down". $U(t)$ (or $D(t)$) is defined to be the total time the system spends in "up" (or "down") states until time $\min \{t, X\}$, where X is the system's life time. Clearly if we set all $r_i > 0$ to 1, then

$$P(U(t) \leq x) = P(Y(t) \leq x)$$
and, since

$$U(t) + D(t) = \min \{t, X\}$$
it follows that

$$P(D(t) \leq x) = P(Y(t) \geq \min \{t, X\} - x) .$$

ii) *The preemptive-repeat cumulative measure, $C(t)$:* Let all structure-states be of the *prt* type. Then $C(t)$ is the reward gained since the last transition of the structure-state process, until (global) time t. C is the *prt* cumulative measure until system's failure.

The following system-oriented measure can be related to the *prt* cumulative measure $C(t)$.

The instantaneous system availability at time t, $A(t)$: defined to be the probability that the system is in an "up" state at time t, i.e., in a state with $r_i > 0$. The steady-state availability, A, is the limit of $A(t)$ as $t \to \infty$; it is greater than zero only if there is no absorbing failure state. Clearly

$$A(t) = P(C(t) > 0)$$

and

$$A = \lim_{t \to \infty} P(C(t) > 0) .$$

From the foregoing discussion it is clear that the cumulative measures introduced are of central importance. The following theorem presents a useful dual relationship between the cumulative measure and the completion time of a given job.

Theorem 2.1. The probability distribution function of the cumulative measure, $\sup \{W(u): 0 \leq u \leq t\}$, is related to the probability distribution function of $T(x)$, as follows

$$P(\sup \{W(u): 0 \leq u \leq t\} < x) = 1 - P(T(x) \leq t)$$

and

$$P(\sup \{W(u): all \ u \geq 0\} < x) = 1 - P(T(x) < \infty) .$$

Proof: It is clear that

$$\{ \sup \{W(u): 0 \leq u \leq t\} < x \} \Leftrightarrow \{ T(x) > t \}$$

and

$$\{ \sup \{W(u): all \ u \geq 0\} < x \} \Leftrightarrow \{ T(x) = \infty \} .$$

Hence,

$$P(\sup \{W(u): 0 \leq u \leq t\} < x) = P(T(x) > t)$$

and

$$P(\sup \{W(u): all \ u \geq 0\} < x) = P(T(x) = \infty) . \quad Q.E.D.$$

Corollary 2.1. When all structure-states are *prs*, the probability distribution function of the *prs* cumulative measure, $Y(t)$, is related to the probability distribution function of $T(x)$, as follows

$$P(Y(t) < x) = 1 - P(T(x) \leq t)$$

and

$$P(Y < x) = 1 - P(T(x) < \infty) .$$

Proof: Follows directly from theorem 2.1, since when all structure-states are *prs*, it holds that

$$Y(t) = sup \{W(u): 0 \le u \le t\} . \quad Q.E.D.$$

As a result of the above corollary, knowing the distribution of $T(x)$ allows us to determine the distribution of $Y(t)$ which can be further specialized to determine different performance measures as discussed above.

The following sections are devoted to the analysis of the random variable $T = \min \{t \ge 0: W(t) = B\}$. In the remainder of this section we introduce some notations and relations that will be used later.

Define the distribution functions

$$F_i(t,x) = P(T(x) \le t \,|\, Z(0){=}i), \quad x \ge 0, \; 1 \le i \le n,$$

$$F(t,x) = P(T(x) \le t), \quad x \ge 0,$$

$$F_i(t) = P(T \le t \,|\, Z(0) = i), \quad 1 \le i \le n,$$

$$F(t) = P(T \le t)$$

and the *LST* (Laplace Stieltjes Transforms)

$$F_i^{\sim}(s,x) = E(e^{-sT(x)}|\, Z(0){=}i), \quad x \ge 0, \; 1 \le i \le n. \tag{2.1}$$

From the independence of $\{Z(t), t \ge 0\}$ and B it follows that

$$F^{\sim}(s,x) = E(e^{-sT(x)}) = \sum_{i=1}^{n} F_i^{\sim}(s,x) \, P(Z(0){=}i), \quad x \ge 0, \tag{2.2}$$

$$F_i^{\sim}(s) = E(e^{-sT}|\, Z(0){=}i) = \int_0^{\infty} F_i^{\sim}(s,x) \, dG(x), \quad 1 \le i \le n, \tag{2.3}$$

$$F^{\sim}(s) = E(e^{-sT}) = \sum_{i=1}^{n} F_i^{\sim}(s) \, P(Z(0){=}i) . \tag{2.4}$$

The omission failure probability η follows from

$$\eta = P(T = \infty) = 1 - \lim_{s \to 0} F^{\sim}(s) . \tag{2.5}$$

It is clear that the transforms $F_i^{\sim}(s,x)$ and $F_i^{\sim}(s)$ are the key quantities that need to be determined in the analysis of T. To proceed with the analysis we make the assumption that the structure-state process $\{Z(t), t \geq 0\}$ is a time homogeneous continuous time Markov chain (CTMC). Let q_{ij}, $1 \leq i \neq j \leq n$, be infinitesimal transition rate from state i to state j and q_{i0} be the absorbing failure rate from state i. Let $Q = [q_{ij}]$, $1 \leq i,j \leq n$, be the n by n generator matrix where

$$q_i = \sum_{\substack{j=0 \\ j \neq i}}^{n} q_{ij} = -q_{ii} .$$

Note that row sums of Q are ≤ 0.

Let H be the holding (sojourn) time in the initial state. Then

$$H = \min \{t \geq 0 : Z(t) \neq Z(0)\} .$$

From the properties of the CTMC, we have

$$P(H \leq x, Z(H+)=j \mid Z(0)=i) = \frac{q_{ij}}{q_i}(1-e^{-q_i x}), \quad i \neq j . \tag{2.6}$$

In the next section we review the pure cases where all states are of the same type.

3. *The Pure Cases: Review*

The cases where all structure-states of the process are of the same type (i.e., *prs*, *pri* or *prd*) have been analysed in detail in a recent paper [8]. In this section, we review the basic results. For detailed proofs the reader is refered to [8].

3.1 *The Preemptive-resume Case*

In this case we assume that the states $1,2,...,n$ are all preemptive-resume. Define the double transform $F_i^{\sim *}(s,u)$ as follows:

$$F_i^{\sim\circ}(s,u) = \int_0^\infty e^{-xu}\, F_i^{\sim}(s,x)\, dx, \quad 1 \le i \le n.$$

The main result is given in the following theorem.

Theorem 3.1. The double tansforms $F_i^{\sim\circ}(s,u)$, $1 \le i \le n$, satisfy the following equations

$$F_i^{\sim\circ}(s,u) = \frac{r_i}{s+q_i+r_i u} + \sum_{\substack{j=1 \\ j \ne i}}^{n} \frac{q_{ij}}{s+q_i+r_i u} F_j^{\sim\circ}(s,u), \quad 1 \le i \le n. \tag{3.1}$$

Equations (3.1) can be solved to obtain $F_i^{\sim\circ}(s,u)$, $1 \le i \le n$, which are rational functions in the Laplace transform variable u. Therefore the conditional *LST*s $F_i^{\sim}(s,x)$, $1 \le i \le n$, can be obtained using partial fraction and standard inversion techniques. $F^{\sim}(s)$ follows from equation (2.4) and hence the omission failure probability from equation (2.5).

The following corollary is used to determine the *LST* of the *prs* cumulative measure (or reward) for a given initial state.

Corollary 3.1. For a given $t \ge 0$, let $Y(t)$ be the *prs* cumulative measure upto time t. Then

$$E(e^{-uY(t)} \mid Z(0)=i) = L_s^{-1}\left[\frac{1 - u F_i^{\sim\circ}(s,u)}{s}\right] \tag{3.2}$$

where L_s^{-1} represents the Laplace inverse with respect to s, to yield a function of t.

In order to demonstrate the derivation of performance measures from the above theorem, let us consider the case where $r_i = 1$, $1 \le i \le n$. The system of equations (3.1) reduces to

$$F_i^{\sim\circ}(s,u) = \frac{1}{s+q_i+u} + \sum_{\substack{j=1 \\ j \ne i}}^{n} \frac{q_{ij}}{s+q_i+u} F_j^{\sim\circ}(s,u), \quad 1 \le i \le n. \tag{3.3}$$

Let $R_i(x)$ be the system reliability given that the initial structure-state is i, and denote by $R_i^\circ(u)$ its Laplace transform. In this special case, it is clear that $(1-R_i(x))$ is identical to $\eta_i(x)$, the omission failure probability, given that the initial structure-state is i.

Therefore

$$R_i(x) = 1 - \eta_i(x) = P(T(x) < \infty \mid Z(0) = i)$$

$$= F_i^{\sim}(0,x), \quad 1 \le i \le n.$$

It follows that

$$R_i^*(u) = F_i^{\sim*}(0,u), \quad 1 \le i \le n.$$

Therefore $R_i^*(u)$, $1 \le i \le n$, are obtained by solving the following system of equations

$$R_i^*(u) = \frac{1}{q_i + u} + \sum_{\substack{j=1 \\ j \ne i}}^{n} \frac{q_{ij}}{q_i + u} R_j^*(u), \quad 1 \le i \le n.$$

Similar results can be derived for the time to absorption in Markov chains [15].

Now let us determine the distribution of the completion time of a job that requires x units of work. From equations (3.3) we remark that

$$F_i^{\sim*}(s,u) = F_i^{\sim*}(0,s+u), \quad 1 \le i \le n$$

It follows that

$$F_i^{\sim}(s,x) = e^{-sx} L_u^{-1}[F_i^{\sim*}(0,u)]$$

$$= e^{-sx} R_i(x), \quad 1 \le i \le n$$

and finally

$$T(x) \mid Z(0)=i = \begin{cases} x, & \text{with probability } R_i(x) \\ \infty, & \text{with probability } (1 - R_i(x)) . \end{cases}$$

If $q_{i0} = 0$, $1 \le i \le n$, (i.e., there is no absorbing failure), then $T(x) = x$, which is clearly justified.

3.2 *The Preemptive-repeat-identical Case*

In this case we assume that the states $1, 2, ..., n$ are all preemptive-repeat-identical. The main result is given in the following theorem.

Theorem 3.2. The conditional *LST*s $F_i^{\sim}(s,x)$, $1 \le i \le n$, satisfy the following equations

$$F_i^{\sim}(s,x) = e^{-(s+q_i)x/\tau_i} + \sum_{\substack{j=1 \\ j \neq i}}^{n} \frac{q_{ij}}{(s+q_i)}(1-e^{-(s+q_i)x/\tau_i})F_j^{\sim}(s,x), \quad 1 \leq i \leq n. \quad (3.4)$$

Equations (3.4) can be solved to obtain $F_i^{\sim}(s,x)$, $1 \leq i \leq n$. $F^{\sim}(s)$ and the omission failure probability are obtained from equations (2.4) and (2.5), respectively.

3.3 *The Preemptive-repeat-different Case*

In this case the states $1,2,...,n$ are assumed to be preemptive-repeat-different. The main result is given in the following theorem.

Theorem 3.3. The *LST*s $F_i^{\sim}(s)$, $1 \leq i \leq n$, satisfy the following equations

$$F_i^{\sim}(s) = G^{\sim}((s+q_i)/\tau_i) + \sum_{\substack{j=1 \\ j \neq i}}^{n} \frac{q_{ij}}{(s+q_i)}[1-G^{\sim}((s+q_i)/\tau_i)]F_j^{\sim}(s), \quad 1 \leq i \leq n. \quad (3.5)$$

Where $G^{\sim}((s+q_i)/\tau_i) \to 0$ as $\tau_i \to 0$ (since $G(0+) = 0$). $F^{\sim}(s)$ and the omission failure probability are obtained from equations (2.4) and (2.5), respectively. Note that for a deterministic work requirement, say x, equations (3.5) reduce to equations (3.4).

In [8] the use of the above theorems is illustrated by means of examples.

4. *The Mixed Cases: Preliminaries*

In these cases we allow two different types of states in the same model. Let S and $\bar{S}$ be a partition of $\{1,2,...,n\}$ (i.e., $S \cap \bar{S} = 0$ and $S \cup \bar{S} = \{1,2,...,n\}$). All the states in the subset S are of a certain type and all the states in the subset $\bar{S}$ are of a different type. In the following sections, we consider all possible mixtures with two types of states (i.e. *prs* with *pri*, *prs* with *prd* and *pri* with *prd*). In this section, we introduce some basic definitions and notations that are useful and common to the analysis in the subsequent sections. The basic idea of the solution technique in the mixed cases is described below.

Let the structure-state process be initially in the set $\overline{S}$, in which the job starts being serviced. The job may be completed before a transition out of the set $\overline{S}$, otherwise the job service continues in the other set, S. Now, the job may be completed before a transition out of the set S, otherwise the job service continues in the other set, $\overline{S}$. We keep observing the time to transitions between the sets S and $\overline{S}$ until job completion (i.e., the service accumulated since the last transition to a *prt* state is equal to the job service requirement. Note that the service requirement of the job may be changed, if *prd* states are present). Thus we, effectively, aggregate the times spent in each of the sets, S and $\overline{S}$, until job completion.

We note that the structure-state process may jump into an absorbing state before the job completion. In such a case the job will eventually complete if the work (or reward) rate is greater than zero, otherwise the absorbing state is a failure state and the job will never complete.

In the following sections we show how these observations can be elegantly translated into equations leading to useful theorems for the computation of important quantities.

Let the structure-state process be initially in the set $\overline{S}$. We introduce some useful random variables. Let

$$U = \min \{t \geq 0: Z(t) \in S\} . \tag{4.1}$$

U represents the total time spent in the set $\overline{S}$ by the structure-state process until it hits the set S. Now consider a job with work requirement equal to x. This job starts being processed at time $t = 0$ in the set $\overline{S}$. Let $T(x)$ be its completion time; it is given by

$$T(x) = \min \{t \geq 0: W(t) = x\} . \tag{4.2}$$

The following two quantities are important to our analysis.

$$T'(x) = \begin{cases} T(x), & \text{if } T(x) \le U \\ \infty, & \text{if } T(x) > U \end{cases} \qquad (4.3)$$

and

$$T''(x) = \begin{cases} U, & \text{if } T(x) > U \\ \infty, & \text{if } T(x) \le U. \end{cases} \qquad (4.4)$$

$T'(x)$ represents the time needed to accomplish x units of work (i.e., to complete the job) before leaving the subset $\bar{S}$. If the structure-state process hits the subset S before completing x units of work then $T'(x) = \infty$.

$T''(x)$ represents the time spent by the structure-state process in the subset $\bar{S}$ until it hits the subset S and before completing x units of work (i.e., before completing the job). If x units of work are completed before leaving the subset $\bar{S}$ then $T''(x) = \infty$. Note that both $T'(x)$ and $T''(x)$ are defective random variables.

The following different transforms are associated with the above random variables. For $k \in \bar{S}$, define the double transform

$$M_k^{\sim \bullet}(s,w) = \int_0^{\infty} e^{-wx} M_k^{\sim}(s,x)\, dx \qquad (4.5)$$

where

$$M_k^{\sim}(s,x) = E(e^{-sT'(x)}|Z(0) = k) \qquad (4.6)$$

and the unconditional LST

$$M_k^{\sim}(s) = \int_0^{\infty} M_k^{\sim}(s,x)\, dG(x) \qquad (4.7)$$

where $G(x)$ is the probability distribution function of the work requirement of a job. For $k \in \bar{S}$ and $j \in S$, define the double transform

$$M_{kj}^{\sim \bullet}(s,w) = \int_0^{\infty} e^{-wx} M_{kj}^{\sim}(s,x)\, dx \qquad (4.8)$$

where

$$M_{kj}^{\sim}(s,x) = E(e^{-sT''(x)}; Z(U) = j | Z(0) = k) \qquad (4.9)$$

and the unconditional *LST*

$$M_{kj}^{\sim}(s) = \int_0^{\infty} M_{kj}^{\sim}(s,x)\, dG(x) \,. \tag{4.10}$$

The quantities defined above will be used in the analysis of subsequent sections.

5. *The Mixed Preemptive-resume with Preemptive-repeat-identical Case*

In this case we assume that all states in a subset, say $\bar{S}$, are of the *prs* type, and that all states in the complementary subset S are of the *pri* type. Let $k \in \bar{S}$ be the initial state. The following proposition gives a method of computing $M_k^{\sim*}(s,w)$ as defined by equation (4.5).

Proposition 5.1. The double transforms $M_k^{\sim*}(s,w)$, $k \in \bar{S}$, satisfy the following equations

$$M_k^{\sim*}(s,w) = \frac{\tau_k}{(s+q_k+\tau_k w)} + \sum_{i \in \bar{S}-\{k\}} \frac{q_{ki}}{(s+q_k+\tau_k w)} M_i^{\sim*}(s,w), \quad k \in \bar{S}. \tag{5.1}$$

Proof: Conditioning on the sojourn time in the initial state, H, we get

$$E(e^{-sT(x)} \mid H=h, Z(0)=k) = \begin{cases} e^{-sx/\tau_k}, & \text{if } h \geq x/\tau_k \\ e^{-sh} \sum_{i \in \bar{S}-\{k\}} \dfrac{q_{ki}}{q_k} M_i^{\sim}(s, x-\tau_k h), & \text{if } h < x/\tau_k \end{cases}$$

Unconditioning on H, yields

$$M_k^{\sim}(s,x) = \int_0^{\infty} E(e^{-sT(x)} \mid H=h, Z(0)=k)\, q_k\, e^{-q_k h}\, dh$$

$$= e^{-(s+q_k)x/\tau_k} + \sum_{i \in \bar{S}-\{k\}} q_{ki} \int_0^{x/\tau_k} e^{-(s+q_k)h}\, M_i^{\sim}(s, x-\tau_k h)\, dh$$

Multiplying both sides by e^{-wx} and integrating we get equation (5.1). *Q.E.D.*

The double transforms $M_k^{\sim*}(s,w)$, for $k \in \bar{S}$, can be obtained by solving equations (5.1). These are rational functions in the Laplace transform variable, w, and therefore

can be inverted to obtain the conditional *LSTs* $M_k^{\sim}(s,x)$, $k \in \overline{S}$.

For $k \in \overline{S}$ and $j \in S$, the following proposition gives a method of computing $M_{kj}^{\sim \circ}(s,w)$ as defined by equation (4.8).

Proposition 5.2. The double transforms $M_{kj}^{\sim \circ}(s,w)$, $k \in \overline{S}$, $j \in S$, satisfy the following equations:

$$M_{kj}^{\sim \circ}(s,w) = \frac{q_{kj}}{w \,(s+q_k+\tau_k w)} + \sum_{i \in \overline{S}-\{k\}} \frac{q_{ki}}{(s+q_k+\tau_k w)} M_{ij}^{\sim \circ}(s,w), \quad k \in \overline{S}, j \in S. \qquad (5.2)$$

Proof: Conditioning on the sojourn time in the initial state, H, we get

$$E(e^{-sT^{\circ}(s)}; Z(U)=j \mid H=h, Z(0)=k) = \begin{cases} e^{-sh}, & \text{if } h < x/\tau_k \text{ and } Z(h) = j \in S \\ e^{-sh} M_{ij}^{\sim}(s, x-\tau_k h), & \text{if } h < x/\tau_k \text{ and } Z(h) = i \in \overline{S}-\{k\} \end{cases}$$

$$= \frac{q_{kj}}{q_k} e^{-sh} + \sum_{i \in \overline{S}-\{k\}} \frac{q_{ki}}{q_k} e^{-sh} M_{ij}^{\sim}(s, x-\tau_k h), \quad h < x/\tau_k$$

Unconditioning on the holding time, H, we get

$$M_{kj}^{\sim} = q_{kj} \int_0^{x/\tau_k} e^{-(s+q_k)h} \, dh + \sum_{i \in \overline{S}-\{k\}} q_{ki} \int_0^{x/\tau_k} e^{-(s+q_k)h} M_{ij}^{\sim}(s, x-\tau_k h) \, dh$$

Multiplying both sides by e^{-wx} and integrating yields equation (5.2). *Q.E.D.*

The double transforms $M_{kj}^{\sim \circ}(s,w)$, $k \in \overline{S}$, $j \in S$, are obtained by solving equations (5.2). These transforms are rational functions in the Laplace transform variable, w, and therefore the corresponding conditional *LSTs* $M_{kj}^{\sim}(s,x)$, $k \in \overline{S}$, $j \in S$, can be determined.

The next two theorems show how the conditional *LSTs* $M_k^{\sim}(s,x)$ and $M_{kj}^{\sim}(s,x)$, $k \in \overline{S}$, $j \in S$, are used to compute the conditional *LSTs* $F_i^{\sim}(s,x)$, $1 \le i \le n$, as defined by equation (2.1). First, we define the following quantities:

$$h_i(s,x) = \begin{cases} r_i - \sum\limits_{k \in S} q_{ik} \int\limits_0^x e^{-(s+q_i)h/r_i}\, \tilde{M}_{ki}(s,x-h)\, dh, & \text{if } r_i > 0 \\ 1 - \sum\limits_{k \in S} \dfrac{q_{ik}}{(s+q_i)} \tilde{M}_{ki}(s,x), & \text{if } r_i = 0, \ i \in S, \end{cases} \tag{5.3}$$

$$h_{ij}(s,x) = \begin{cases} \dfrac{r_i q_{ij}}{(s+q_i)}(1-e^{-(s+q_i)x/r_i}) + \sum\limits_{k \in S} q_{ik} \int\limits_0^x e^{-(s+q_i)h/r_i}\, \tilde{M}_{kj}(s,x-h)\, dh, & \text{if } r_i > 0 \\ [q_{ij} + \sum\limits_{k \in S} q_{ik}\, \tilde{M}_{kj}(s,x)]/(s+q_i), & \text{if } r_i = 0, \ i,j \in S \end{cases} \tag{5.4}$$

and

$$g_i(s,x) = \begin{cases} r_i\, e^{-(s+q_i)x/r_i} + \sum\limits_{k \in S} q_{ik} \int\limits_0^x e^{-(s+q_i)h/r_i}\, \tilde{M}_k(s,x-h)\, dh, & \text{if } r_i > 0 \\ \sum\limits_{k \in S} \dfrac{q_{ik}}{(s+q_i)} \tilde{M}_k(s,x), & \text{if } r_i = 0, \ i \in S. \end{cases} \tag{5.5}$$

Theorem 5.1. The conditional *LSTs* $F_i^{\sim}(s,x)$, $i \in S$, satisfy the following equations

$$h_i(s,x)\, F_i^{\sim}(s,x) = g_i(s,x) + \sum_{j \in S-\{i\}} h_{ij}(s,x)\, F_j^{\sim}(s,x), \quad i \in S. \tag{5.6}$$

Proof: Conditioning on H, the sojourn time in the initial state, we get

$$E(e^{-sT(x)} | H=h, Z(0)=i) = \begin{cases} e^{-sx/r_i}, & \text{if } h \geq x/r_i \\ e^{-sh} \sum\limits_{j \in S-\{i\}} \dfrac{q_{ij}}{q_i} F_j^{\sim}(s,x) \end{cases}$$

$$+ e^{-sh} \sum_{k \in S} \frac{q_{ik}}{q_i} \tilde{M}_k(s, x-r_i h)$$

$$+ e^{-sh} \sum_{j \in S} \sum_{k \in S} \frac{q_{ik}}{q_i} \tilde{M}_{kj}(s, x-r_i h)\, F_j^{\sim}(s,x), \quad \text{if } h < x/r_i .$$

Obviously, if $h \geq x/r_i$, the job completes before the first transition, and $T(x)=x/r_i$. If $h < x/r_i$, the job is preempted before it completes. There are three possibilities:

i)   The structure-state process makes a transition to another state $j \in S$.

ii) The structure-state process makes a transition to a state $k \in \bar{S}$, and the job completes before a transition to a state in S.

iii) The structure-state process makes a transition to a state $k \in \bar{S}$, and it returns to a state $j \in S$ before the job is completed.

These three possibilities are represented by the three terms on the right handside for the case when $h < x/\tau_i$. Unconditioning and rearranging yields equation (5.6). *Q.E.D.*

Theorem 5.2. The conditional *LST*s $F_i^\sim(s,x)$, $i \in \bar{S}$, are given by the following relations

$$F_i^\sim(s,x) = M_i^\sim(s,x) + \sum_{j \in S} M_{ij}^\sim(s,x) \, F_j^\sim(s,x), \quad i \in \bar{S}. \tag{5.7}$$

Proof: Consider a job with work requirement equal to x. It starts being processed at the initial state $i \in \bar{S}$. Its completion time is given by

$$T(x)\big|_{Z(0)=i \in \bar{S}} = \begin{cases} \dfrac{T'(x)}{P(T(x) \le U)}, & \text{if } T(x) \le U \\[2ex] \dfrac{T''(x)}{P(T(x) > U)} + T(x)\big|_{Z(0)=Z(U) \in S}, & \text{if } T(x) > U \end{cases}$$

where U, $T(x)$, $T'(x)$ and $T''(x)$ are random variables as defined in equations (4.1)-(4.4). $Z(U)$ is a state $j \in S$ which is reached at time U. Since the random variables $T''(x)$ and $T(x)\big|_{Z(0)=Z(U)}$ are independent we can write the following

$$E(e^{-sT(x)}|Z(0)=i) = E(e^{-sT'(x)}|Z(0)=i)$$
$$+ \sum_{j \in S} E(e^{-sT''(x)}; Z(U)=j \,|\, Z(0)=i) \, E(e^{-sT(x)}|Z(0)=j)$$

Using the definitions in equations (4.6) and (4.9) yields equation (5.7). *Q.E.D.*

We conclude with the procedure to compute the *LST* $F^\sim(s)$ of the job completion time in the mixed *prs-pri* case in the following steps:

Procedure 5.1.

1. Compute $M_k^{\sim'}(s,w)$, $k \in \bar{S}$, by solving equations (5.1).

2. Compute $M_k^{\sim}(s,x)$, $k \in \overline{S}$, by inverting the Laplace transform $M_k^{\sim*}(s,w)$ with respect to w using partial fraction expansion.

3. Compute $M_{kj}^{\sim*}(s,w)$, $k \in \overline{S}$, $j \in S$, by solving equations (5.2).

4. Compute $M_{kj}^{\sim}(s,x)$, $k \in \overline{S}, j \in S$ by inverting the Laplace transform $M_{kj}^{\sim*}(s,w)$ with respect to w using partial fraction expansion.

5. Compute $F_i^{\sim}(s,x)$, $i \in S$, by solving equations (5.6).

6. Compute $F_i^{\sim}(s,x)$, $i \in \overline{S}$, by using equations (5.7).

7. Compute $F^{\sim}(s)$ by using equation (2.4).

We illustrate the procedure by applying it to a simple case of interest.

Example 5.1. The switching server

Consider a system that operates in two modes with different work rates, say r_1 and r_2, for modes "1" and "2", respectively. The system switches between the two modes according to a Poisson process at different rates, say λ and μ from modes "1" and "2", respectively. A total system failure may occur from any mode of operation; let λ_0 and μ_0 be the failure rates from modes "1" and "2", respectively. The CTMC representing the switching server is shown in figure 2.

The holding times in states 1 and 2 are exponential with parameters $\lambda' = \lambda+\lambda_0$ and $\mu' = \mu+\mu_0$, respectively. This system will be used as an example throughout the paper to illustrate the use of the techniques developed.

Let state 1 be of the *prs* type and state 2 be of the *pri* type. We follow procedure 5.1.

Step 1. Equation (5.1) yields

$$M_1^{\sim*}(s,w) = \frac{r_1}{s+\lambda'+r_1 w}$$

Step 2. Inverting the Laplace transform $M_1^{\sim*}(s,w)$ with respect to w yields

$$M_1^-(s,x) = e^{-(s+\lambda')x/\tau_1}$$

Step 3. Equation (5.2) yields

$$M_{12}^{\sim\circ}(s,w) = \frac{\lambda}{w\ (s+\lambda'+\tau_1 w)}$$

Step 4. Inverting the Laplace transform $M_{12}^{\sim\circ}(s,w)$ with respect to w yields

$$M_{12}^{\sim}(s,x) = \frac{\lambda}{s+\lambda'}(1-e^{-(s+\lambda')x/\tau_1})$$

Step 5. Equations (5.3)-(5.5) yield

$$h_2(s,x) = \tau_2\Big[\frac{\tau((s+\lambda')(s+\mu')-\lambda\mu) + \lambda\mu((s+\mu')\tau_1 e_1 - (s+\lambda')\tau_2 e_2)}{\tau(s+\lambda')(s+\mu')}\Big]$$

and

$$g_2 = \frac{\tau_2}{\tau}[(\tau-\mu\tau_1)e_2 + \mu\tau_1 e_1]$$

where

$$e_1 = e^{-(s+\lambda')x/\tau_1},$$

$$e_2 = e^{-(s+\mu')x/\tau_2}$$

and

$$\tau = \tau_1(s+\mu') - \tau_2(s+\lambda').$$

Equation (5.6) yields

$$F_2^{\sim}(s,x) = \frac{g_2(s,x)}{h_2(s,x)}$$

$$= \frac{(s+\lambda')(s+\mu')\ [(\tau-\mu\tau_1)e_2 + \mu\tau_1 e_1]}{\tau((s+\lambda')(s+\mu')-\lambda\mu) + \lambda\mu((s+\mu')\tau_1 e_1 - (s+\lambda')\tau_2 e_2)}.$$

Step 6. From equation (5.7) we have

$$F_1^{\sim}(s,x) = e_1 + \frac{\lambda}{(s+\lambda')}(1-e_1)\ F_2^{\sim}(s,x).$$

Step 7. From equation (2.4) we get

$$F^{\sim}(s) = P(Z(0)=1) \int_0^{\infty} F_1^{\sim}(s,x) \, dG(x) + P(Z(0)=2) \int_0^{\infty} F_2^{\sim}(s,x) \, dG(x) \; .$$

6. *The Mixed Preemptive-resume with Preemptive-repeat-different Case*

In this case we assume that all states in a subset, say $\overline{S}$, are of the *prs* type and that all states in the complementary subset S are of the *prd* type. The computation of the double tansforms $M_k^{\sim \circ}(s,w)$ and $M_{kj}^{\sim \circ}(s,w)$, $k \in \overline{S}$, $j \in S$, follow from equations (5.1) and (5.2) of propositions (5.1) and (5.2), respectively. The conditional *LST*s $M_k^{\sim}(s,x)$ and $M_{kj}^{\sim}(s,x)$ are obtained by using partial fraction expansion and standard inversion techniques.

The next two theorems give a method to compute the *LST*s $F_i^{\sim}(s)$, $1 \le i \le n$, as defined by equation (2.3), using the conditional *LST*s $M_k^{\sim}(s,x)$ and $M_{kj}^{\sim}(s,x)$, $k \in \overline{S}$, $j \in S$.

First, we give the following definitions:

$$h_i(s) = \int_0^{\infty} h_i(s,x) \, dG(x), \quad i \in S, \tag{6.1}$$

$$h_{ij}(s) = \int_C^{\infty} h_{ij}(s,x) \, dG(x), \quad i,j \in S \tag{6.2}$$

and

$$g_i(s) = \int_0^{\infty} g_i(s,x) \, dG(x), \quad i \in S \tag{6.3}$$

where $h_i(s,x)$, $h_{ij}(s,x)$ and $g_i(s,x)$ are as given in equations (5.3)-(5.5).

Theorem 6.1. The *LST*s $F_i^{\sim}(s)$, $i \in S$, satisfy the following equations

$$h_i(s) \, F_i^{\sim}(s) = g_i(s) + \sum_{j \in S-\{i\}} h_{ij}(s) \, F_j^{\sim}(s), \quad i \in S. \tag{6.4}$$

Proof: Conditioning on H, the sojourn time in the initial state, and on the initial work requirement, B, we get

$$E(e^{-sT(x)}|H=h,Z(0)=i) = \begin{cases} e^{-sx/\tau_i}, & \text{if } h \geq x/\tau_i \\ e^{-sh} \displaystyle\sum_{j \in S-\{i\}} \frac{q_{ij}}{q_i} F_j^{\sim}(s) \end{cases}$$

$$+ e^{-sh} \sum_{k \in S} \frac{q_{ik}}{q_i} M_k^{\sim}(s, x - \tau_i h)$$

$$+ e^{-sh} \sum_{j \in S} \sum_{k \in S} \frac{q_{ik}}{q_i} M_{kj}^{\sim}(s, x - \tau_i h) F_j^{\sim}(s), \text{ if } h < x/\tau_i .$$

Unconditioning on H, yields

$$\tau_i F_i^{\sim}(s,x) = \tau_i e^{-(s+q_i)x/\tau_i} + \sum_{j \in S-\{i\}} q_{ij} \int_0^x e^{-(s+q_i)h/\tau_i} F_j^{\sim}(s) \, dh$$

$$+ \sum_{k \in S} q_{ik} \int_0^x e^{-(s+q_i)h/\tau_i} M_k^{\sim}(s, x-h) \, dh$$

$$+ \sum_{j \in S} \sum_{k \in S} q_{ik} \int_0^x e^{-(s+q_i)h/\tau_i} M_{kj}(s, x-h) F_j^{\sim}(s) \, dh$$

Unconditioning on the initial work requirement, B, and rearranging, yields equation (6.4). *Q.E.D.*

Theorem 6.2. The *LSTs* $F_i^{\sim}(s)$, $i \in \overline{S}$, are given by the following equations

$$F_i^{\sim}(s) = M_i^{\sim}(s) + \sum_{j \in S} M_{ij}^{\sim}(s) F_j^{\sim}(s), \quad i \in \overline{S} \tag{6.5}$$

where $M_i^{\sim}(s)$ and $M_{ij}^{\sim}(s)$, $i \in \overline{S}$, $j \in S$, are given by equations (4.7) and (4.10), respectively.

Proof: Conditioning on the initial work requirement, B, and following similar arguments as in theorem 5.2, we have

$$F_i^{\sim}(s,x) = M_i^{\sim}(s,x) + \sum_{j \in S} M_{ij}^{\sim}(s,x) F_j^{\sim}(s)$$

Unconditioning on the initial work requirement, B, yields equation (6.5). *Q.E.D.*

We conclude with the procedure to compute the *LST* $F^{\sim}(s)$ of the job completion time in the mixed *prs-prd* case in the following steps:

Procedure 6.1.

1. Compute $M_k^{\sim\bullet}(s,w)$, $k \in \overline{S}$, by solving equations (5.1).

2. Compute $M_k^{\sim}(s,x)$, $k \in \overline{S}$, by inverting the Laplace transform $M_k^{\sim\bullet}(s,w)$ with respect to w, using partial fraction expansion, and get $M_k^{\sim}(s)$ from equation (4.7).

3. Compute $M_{kj}^{\sim\bullet}(s,w)$, $k \in \overline{S}$, $j \in S$, by solving equations (5.2).

4. Compute $M_{kj}^{\sim}(s,x)$, $k \in \overline{S}$, $j \in S$, by inverting the Laplace transform $M_{kj}^{\sim\bullet}(s,w)$ with respect to w, using partial fraction expansion, and get $M_{kj}^{\sim}(s)$ from equation (4.10).

5. Compute $F_i^{\sim}(s)$, $i \in S$, by solving equations (6.4).

6. Compute $F_i^{\sim}(s)$, $i \in \overline{S}$, by using equations (6.5).

7. Compute $F^{\sim}(s)$ by using equation (2.4).

We illustrate the procedure by means of an example.

Example 6.1. Consider the switching server of example 5.1. Let state 1 be of the *prs* type and state 2 be of the *prd* type. We follow procedure 6.1.

Steps 1 & 2. Equation (5.1) yields

$$M_1^{\sim\bullet}(s,w) = \frac{\tau_1}{s+\lambda'+\tau_1 w}$$

and hence

$$M_1^{\sim}(s,x) = e^{-(s+\lambda')x/\tau_1}$$

Unconditioning, we get

$$M_1^{\sim}(s) = G^{\sim}((s+\lambda')/\tau_1) .$$

Steps 3 & 4. Equation (5.2) yields

$$M_{12}^{\sim\bullet}(s,w) = \frac{\lambda}{w\,(s+\lambda'+r_1 w)}$$

and hence

$$M_{12}^{\sim}(s,x) = \frac{\lambda}{s+\lambda'}(1-e^{-(s+\lambda')x/r_1})$$

Unconditioning, we get

$$M_{12}^{\sim}(s) = \frac{\lambda}{s+\lambda'}[1-G^{\sim}((s+\lambda')/r_1)] \ .$$

Steps 5 & 6. Substituting from equations (5.3)-(5.5) in equations (6.1)-(6.3), we get

$$h_2(s) = r_2\Big[\frac{r((s+\lambda')(s+\mu')-\lambda\mu) + \lambda\mu((s+\mu')r_1 e_1^{\sim}-(s+\lambda')r_2 e_2^{\sim})}{r(s+\lambda')(s+\mu')}\Big]$$

and

$$g_2(s) = \frac{r_2}{r}[(r-\mu r_1)e_2^{\sim} + \mu r_1 e_1^{\sim}]$$

where

$$e_1^{\sim} = G^{\sim}((s+\lambda')/r_1) \ ,$$

$$e_2^{\sim} = G^{\sim}((s+\mu')/r_2)$$

and

$$r = r_1(s+\mu') - r_2(s+\lambda') \ .$$

Equation (6.4) yields

$$F_2^{\sim}(s) = \frac{g_2(s)}{h_2(s)}$$

$$= \frac{(s+\lambda')(s+\mu')[(r-\mu r_1)e_2^{\sim} + \mu r_1 e_1^{\sim}]}{r((s+\lambda')(s+\mu')-\lambda\mu) + \lambda\mu((s+\mu')r_1 e_1^{\sim}-(s+\lambda')r_2 e_2^{\sim})} \ .$$

From equation (6.5) we get

$$F_1^{\sim}(s) = e_1^{\sim} + \frac{\lambda}{s+\lambda'}(1-e_1^{\sim})\, F_2^{\sim}(s) \ .$$

Step 7. From equation (2.4) we get

$$F^{\sim}(s) = P(Z(0)=1)\, F_1^{\sim}(s) + P(Z(0)=2)\, F_2^{\sim}(s) \ .$$

Note that for a deterministic work requirement $(B=x)$, $F_1^\sim(s)$ and $F_2^\sim(s)$ in the mixed *prs-prd* case are identical to $F_1^\sim(s,x)$ and $F_2^\sim(s,x)$ in the mixed *prs-pri* case.

7. The Mixed Preemptive-repeat-identical with Preemptive-repeat-different Case

In this case we assume that all states in a subset, say $\bar{S}$, are of the *pri* type and that all states in the complementary subset S are of the *prd* type. The following proposition gives a method of computing $M_k^\sim(s,x), k \in \bar{S}$, as defined by equation (4.6).

Proposition 7.1. The conditional *LST*s $M_k^\sim(s,x)$, $k \in \bar{S}$, satisfy the following equations

$$M_k^\sim(s,x) = e^{-(s+q_k)x/\tau_k} + \frac{(1-e^{-(s+q_k)x/\tau_k})}{(s+q_k)} \sum_{i \in \bar{S}-\{k\}} q_{ki} \, M_i^\sim(s,x), \quad k \in \bar{S}. \qquad (7.1)$$

Proof: Conditioning on H, the sojourn time in the initial state, we have

$$E(e^{-sT(x)}|H=h, Z(0)=k) = \begin{cases} e^{-sx/\tau_k}, & \text{if } h \geq x/\tau_k \\ e^{-sh} \displaystyle\sum_{i \in \bar{S}-\{k\}} \frac{q_{ki}}{q_k} M_i(s,x), & \text{if } h < x/\tau_k \end{cases}$$

Unconditioning on H, yields equation (6.1). *Q.E.D.*

The next proposition gives a method of computing $M_{kj}^\sim(s,x)$, $k \in \bar{S}$, $j \in S$, as defined by equation (4.9).

Proposition 7.2. The conditional *LST*s $M_{kj}^\sim(s,x)$, $k \in \bar{S}$ and $j \in S$, satisfy the following equations

$$M_{kj}^\sim(s,x) = \frac{q_{kj}}{(s+q_k)}(1-e^{-(s+q_k)x/\tau_k})$$

$$+ \frac{(1-e^{-(s+q_k)x/\tau_k})}{(s+q_k)} \sum_{i \in \bar{S}-\{k\}} q_{ki} M_{ij}^\sim(s,x), \, k \in \bar{S}, \quad j \in S. \qquad (7.2)$$

Proof: Conditioning on H, the sojourn time in the initial state, we have

$$E(e^{-sT^{\sim}(x)}; Z(U) = j \mid H=h, Z(0)=k)$$

$$= \begin{cases} e^{-sh}, & \text{if } h < x/\tau_k \text{ and } Z(h)=j \in S \\ e^{-sh} M_{ij}^{\sim}(s,x), & \text{if } h < x/\tau_k \text{ and } Z(h)=i \in \bar{S}-\{k\} \end{cases}$$

$$= \frac{q_{kj}}{q_k} e^{-sh} + \sum_{i \in \bar{S}-\{k\}} \frac{q_{ki}}{q_k} e^{-sh} M_{ij}^{\sim}(s,x), \quad h < x/\tau_k$$

Unconditioning on the holding time, H, yields equation (7.2). *Q.E.D.*

The next two theorems give a method to compute the *LST*s $F_i^{\sim}(s)$, $1 \le i \le n$, as defined by equation (2.3), using the conditional *LST*s $M_k^{\sim}(s,x)$ and $M_{kj}^{\sim}(s,x)$, $k \in \bar{S}$, $j \in S$, as obtained from propositions 7.1 and 7.2. First, we give some definitions

$$h_i'(s,x) = \left[1 - \frac{(1-e^{-(s+q_i)x/\tau_i})}{(s+q_i)} \sum_{k \in S} q_{ik} M_{ki}^{\sim}(s,x) \right], \quad i \in S, \tag{7.3}$$

$$h_{ij}'(s,x) = \frac{(1-e^{-(s+q_i)x/\tau_i})}{(s+q_i)} \left[q_{ij} + \sum_{k \in S} q_{ik} M_{kj}^{\sim}(s,x) \right], \quad i,j \in S \tag{7.4}$$

and

$$g_i'(s,x) = e^{-(s+q_i)x/\tau_i} + \frac{(1-e^{-(s+q_i)x/\tau_i})}{(s+q_i)} \sum_{k \in S} q_{ik} M_k^{\sim}(s,x), \quad i \in S. \tag{7.5}$$

The corresponding unconditional quantities are given by

$$h_i'(s) = \int_0^{\infty} h_i'(s,x) \, dG(x), \quad i \in S, \tag{7.6}$$

$$h_{ij}'(s) = \int_0^{\infty} h_{ij}'(s,x) \, dG(x), \quad i,j \in S \tag{7.7}$$

and

$$g_i'(s) = \int_0^{\infty} g_i'(s,x) \, dG(x), \quad i \in S. \tag{7.8}$$

Theorem 7.1. The *LST*s $F_i^{\sim}(s)$, $i \in S$, satisfy the following equations

$$h_i'(s) F_i'(s) = g_i'(s) + \sum_{j \in S-\{i\}} h_{ij}'(s) F_j'(s), \quad i \in S. \tag{7.9}$$

Proof: Conditioning on H, the sojourn time in the initial state, and on the initial work

requirement, B, we get

$$E(e^{-sT(x)}|H=h,Z(0)=i) = \begin{cases} e^{-sx/\tau_i}, & \text{if } h \geq x/\tau_i \\ e^{-sh} \sum\limits_{j \in S-\{i\}} \dfrac{q_{ij}}{q_i} F_j^{\sim}(s) \end{cases}$$

$$+ e^{-sh} \sum_{k \in S} \frac{q_{ik}}{q_i} M_k^{\sim}(s,x)$$

$$+ e^{-sh} \sum_{j \in S} \sum_{k \in S} \frac{q_{ik}}{q_i} M_{kj}^{\sim}(s,x) F_j^{\sim}(s), \quad \text{if } h < x/\tau_i$$

Unconditioning on H, yields

$$F_i^{\sim}(s,x) = e^{-(s+q_i)x/\tau_i} + \frac{(1-e^{-(s+q_i)x/\tau_i})}{(s+q_i)} \sum_{j \in S-\{i\}} q_{ij} F_j^{\sim}(s)$$

$$+ \frac{(1-e^{-(s+q_i)x/\tau_i})}{(s+q_i)} \sum_{k \in S} q_{ik} M_k^{\sim}(s,x)$$

$$+ \frac{(1-e^{-(s+q_i)x/\tau_i})}{(s+q_i)} \sum_{j \in S} \sum_{k \in S} q_{ik} M_{kj}^{\sim}(s,x) F_j^{\sim}(s)$$

Unconditioning on the initial work requirement, B, and rearranging yields equation (7.9). Q.E.D.

Theorem 7.2. The LSTs $F_i^{\sim}(s)$, $i \in \overline{S}$, are given by the following equations

$$F_i^{\sim}(s) = M_i^{\sim}(s) + \sum_{j \in S} M_{ij}^{\sim} F_j^{\sim}(s), \quad i \in \overline{S} \tag{7.10}$$

where $M_i^{\sim}(s)$ and $M_{ij}^{\sim}(s)$ are obtained from equations (7.1), (7.2), (4.7) and (4.10).

Proof: Conditioning on the initial work requirement, B, and following similar arguments as in theorem 5.2, we have

$$F_i^{\sim}(s,x) = M_i^{\sim}(s,x) + \sum_{j \in S} M_{ij}^{\sim}(s,x) F_j^{\sim}(s)$$

Unconditioning on the initial work requirement, B, yields equation (7.10). Q.E.D.

We conclude with the procedure to compute the *LST* $F^{\sim}(s)$ of the job completion time in the mixed *pri-prd* case in the following steps:

Procedure 7.1.

1. Compute $M_k^{\sim}(s,x)$, $k \in \bar{S}$, by solving equations (7.1).

2. Compute $M_{kj}^{\sim}(s,x)$, $k \in \bar{S}$, $j \in S$, by solving equations (7.2).

3. Compute $F_i^{\sim}(s)$, $i \in S$, by solving equations (7.9).

4. Compute $F_i^{\sim}(s)$, $i \in \bar{S}$, from equations (7.10).

5. Compute $F^{\sim}(s)$ from equation (2.4).

We illustrate the use of the above technique by means of an example.

Example 7.1. Once again we consider the switching server of example 5.1. Let state 1 be of the *pri* type and state 2 be of the *prd* type. We follow procedure 7.1.

Step 1. From equation (7.1) we have

$$M_1^{\sim}(s,x) = e^{-(s+\lambda')x/r_1} \,.$$

Step 2. From equation (7.2) we have

$$M_{12}^{\sim}(s,x) = \frac{\lambda}{s+\lambda'}(1-e^{-(s+\lambda')x/r_1}) \,.$$

Step 3. From steps 1 and 2 and equations (7.3)-(7.5) we have

$$h_{2'}(s,x) = [1 - \frac{\mu}{s+\mu'}(1-e^{-(s+\mu')x/r_2}) \, M_{12}^{\sim}(s,x)]$$

$$= \frac{(s+\lambda')(s+\mu') - \lambda\mu(1-e^{-(s+\lambda')x/r_1})(1-e^{-(s+\mu')x/r_2})}{(s+\lambda')(s+\mu')}$$

and

$$g_{2'}(s,x) = e^{-(s+\mu')x/r_2} + \frac{\mu}{s+\mu'}(1-e^{-(s+\mu')x/r_2}) \, M_1^{\sim}(s,x)$$

$$= \frac{(s+\lambda')(s+\mu')e^{-(s+\mu')x/r_2} + \mu(s+\lambda')e^{-(s+\lambda')x/r_1}(1-e^{-(s+\mu')x/r_2})}{(s+\lambda')(s+\mu')} \,.$$

From equations (7.6)-(7.8) it follows that

$$h_{2'}(s) = \frac{(s+\lambda')(s+\mu') - \lambda\mu[1-e_1^{\sim}-e_2^{\sim}+e_{12}^{\sim}]}{(s+\lambda')(s+\mu')}$$

and

$$g_{2'}(s) = \frac{(s+\lambda')(s+\mu')e_2^{\sim} + \mu(s+\lambda')[e_1^{\sim}-e_{12}^{\sim}]}{(s+\lambda')(s+\mu')}$$

where

$$e_1^{\sim} = G^{\sim}((s+\lambda')/\tau_1),$$

$$e_2^{\sim} = G^{\sim}((s+\mu')/\tau_2)$$

and

$$e_{12}^{\sim} = G^{\sim}((s+\lambda')/\tau_1 + (s+\mu')/\tau_2).$$

Equation (7.9) yields

$$F_2^{\sim}(s) = \frac{g_{2'}(s)}{h_{2'}(s)}$$

$$= \frac{(s+\lambda')(s+\mu')e_2^{\sim} + \mu(s+\lambda')[e_1^{\sim}-e_{12}^{\sim}]}{(s+\lambda')(s+\mu') - \lambda\mu[1-e_1^{\sim}-e_2^{\sim}+e_{12}^{\sim}]}.$$

Step 4. From equation (7.10) we get

$$F_1^{\sim}(s) = e_1^{\sim} + \frac{\lambda}{s+\lambda'}[1-e_1^{\sim}] F_2^{\sim}(s)$$

$$= \frac{(s+\lambda')(s+\mu')e_1^{\sim} + \lambda(s+\mu')e_2^{\sim}[1-e_1^{\sim}] + \lambda\mu[e_1^{\sim}e_2^{\sim}-e_{12}^{\sim}]}{(s+\lambda')(s+\mu') - \lambda\mu[1-e_1^{\sim}-e_2^{\sim}+e_{12}^{\sim}]}.$$

Step 5. From equation (2.4) we have

$$F^{\sim}(s) = P(Z(0)=1) F_1^{\sim}(s) + P(Z(0)=2) F_2^{\sim}(s).$$

8. *Conclusions*

In this paper, we have presented a unified modeling approach to the combined evaluation of performance and reliability of fault-tolerant/multi-mode systems.

The structure-state process of the system is modelled as a CTMC, in which transitions occur in response to events such as failure/repair or system degradation. A reward rate (or performance measure) is associated with each structure-state. We have introduced different types of cumulative measures, and related them to different performance and/or reliability measures.

Our main concern was the analysis of the job completion time. We extended the results in [8] to allow the simultaneous presence of two different types of service-preemption interaction, due to changes in the structure-state of the system. We derived the distribution of the job completion time. Other job-oriented measures, such as the omission/dynamic failure probability are readily obtained from the distribution of the completion time.

It is shown that, when all structure-states are of the preemptive-resume type, the cumulative measure up to a given time is dual to the completion time of a given job. This is a useful relationship, since it enables us to specialize the analysis of the completion time to derive different system-oriented measures such as system reliability/saftey and up/down time.

It remains of interest to extend the present model to allow a semi-Markov structure-state process, and the simultaneous pesence of more types of service-preemption interaction. Needless to say, the evaluation of performance and/or reliability in existing systems could benefit from the results presented in this paper. Therefore, emphasis should be given to practical applications.

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