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NONPARAMETRIC MAXIMUM PENALIZED LIKELIHOOD ESTIMATION  
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DEPT OF MATHEMATICS AND STATISTI. A M LUBECKE ET AL.

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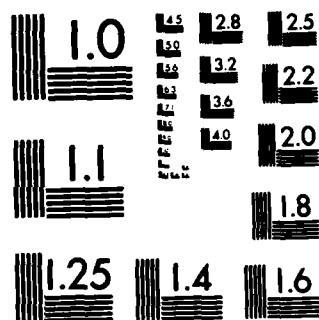
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NONPARAMETRIC MAXIMUM PENALIZED LIKELIHOOD  
ESTIMATION OF A DENSITY FROM ARBITRARILY  
RIGHT-CENSORED OBSERVATIONS \*

by

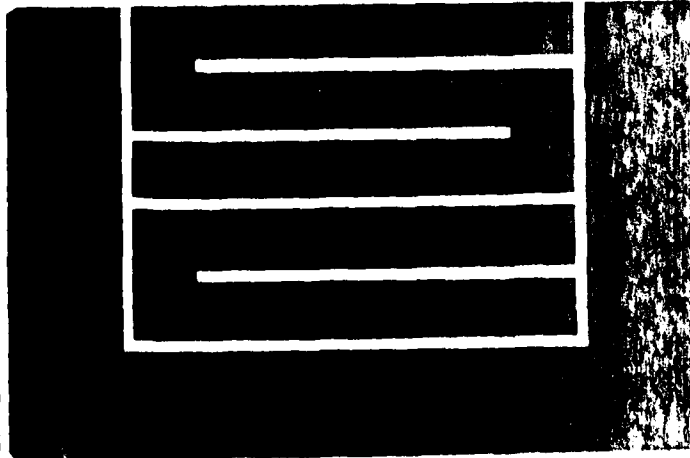
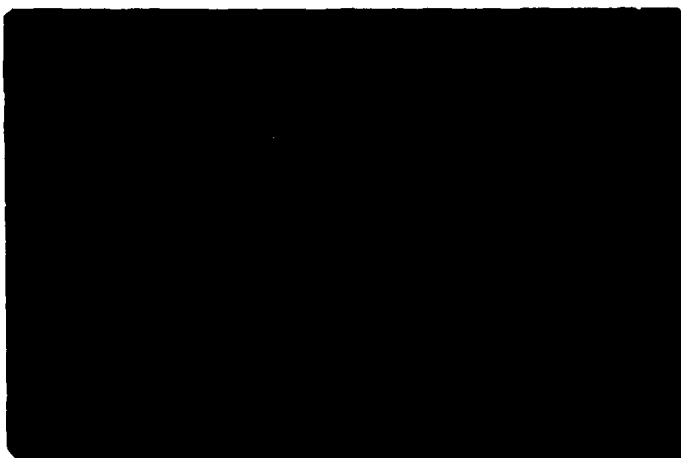
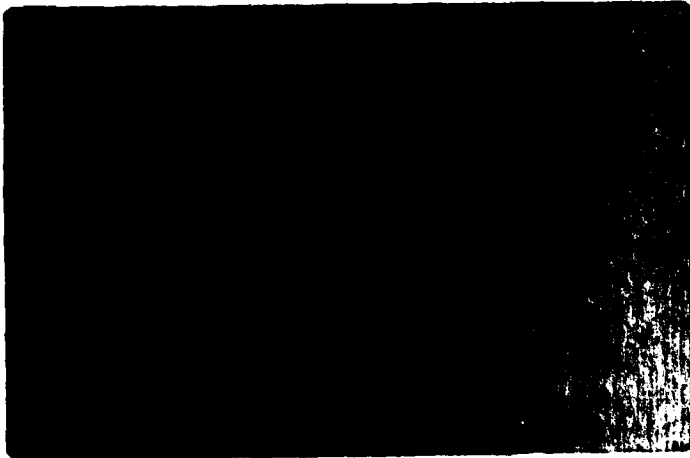
André M. Lubecke and W. J. Padgett

University of South Carolina  
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ABSTRACT

Based on arbitrarily right-censored observations from a probability density function  $f^0$ , the existence and uniqueness of the maximum penalized likelihood estimator (MPLE) of  $f^0$  is proven. In particular, the "first MPLE of Good and Gaskins" of a density defined on  $[0, \infty)$  is shown to exist and to be unique under arbitrary right-censorship. Furthermore, the MPLE is shown to be in the form of the solution to a linear integral equation whose forcing function is an exponential spline with knots at the observed censored and uncensored data points.

1. INTRODUCTION

The problem of nonparametric probability density estimation has been studied for many years. Summaries of results for com-

plete (uncensored) random samples have been listed by Tapia and Thompson (1978), Wertz and Schneider (1979), and Bean and Tsokos (1980), for example. Also, a review of results for censored samples has been given by Padgett and McNichols (1984). In addition to its importance in theoretical statistics, nonparametric density estimation has been used in hazard analysis, life testing, and reliability, as well as in the areas of nonparametric discrimination and high energy physics (Good and Gaskins, 1971).

One approach to estimating a density function nonparametrically is that of maximum likelihood. Nonparametric maximum likelihood estimates of a probability density function do not exist in general. That is, the likelihood function for a complete sample is unbounded over the class of all possible densities. However, by suitably restricting the class of densities, a nonparametric maximum likelihood estimator (MLE) may be found within the restricted class. For complete samples, the maximum likelihood estimator of a density  $g$  was given by Barlow, Bartholomew, Bremner and Brunk (1972) if  $g$  was assumed to be either decreasing (nonincreasing) or unimodal with known mode. Wegman (1970a,b) assumed unimodality with unknown mode and found the MLE of the density and studied its properties for complete samples. McNichols and Padgett (1982) studied the nonparametric MLE of monotonic or unimodal densities based on arbitrarily right-censored observations. Even within the class of decreasing (or unimodal) density functions, however, when the largest observation was censored, McNichols and Padgett (1982) had to restrict their estimator to a finite interval  $[0, T]$  where  $T$  was an arbitrarily large positive number, greater than the largest observation.

Another approach to the problem of nonparametric maximum likelihood estimation of a density from complete samples was proposed by Good and Gaskins (1971). This method allowed any smooth integrable function on the interval of interest  $(a, b)$  (which may be finite or infinite) as a possible estimator, but added a "penalty function" to the likelihood. The penalty function penal-

ized a density for its lack of smoothness, so that a very "rough" density would have a smaller likelihood than a "smooth" density, and hence, would not be admissible. De Montricher, Tapia, and Thompson (1975) showed that the natural mathematical setting for the solution of the maximum penalized likelihood estimation (MPLE) problem of Good and Gaskins (1971) was provided by the Sobolev subspaces of the Hilbert space  $L_2(R)$ , the square-integrable functions on the real line  $R$ . They proved existence and uniqueness results for the MPLE. Later, Klonias (1982) obtained the strong consistency of the MPLE of the density function in appropriate norms. He also derived the "first MPLE of Good and Gaskins" for the case that the density  $g$  has support only on the half line, essentially by reflecting  $g$  around zero and using results for  $g$  having support  $R$ .

In this paper we obtain existence and uniqueness results for the nonparametric MPLE of a density  $g$  based on arbitrarily right-censored observations from  $g$ . General results are first obtained for densities with support  $\Omega \subset R$  and penalty function  $\phi$  and then the problem of "Good and Gaskins' first MPLE" is considered for arbitrarily right-censored data observed on  $R$ . The existence and uniqueness results are then obtained for densities  $g$  with only positive support by using a symmetry argument, reflecting  $g$  about zero, and then utilizing the general results for support  $R$ . It is also shown that the MPLE is in the form of a solution to a linear integral equation whose forcing function is an exponential spline with knots at the data points.

## 2. NOTATION AND BASIC DEFINITIONS

Let  $\Omega \subset R$  be a finite or infinite interval and let  $f^0$  denote a probability density function with support in  $\Omega$ . Let  $X_1^0, \dots, X_n^0$  be  $n$  independent identically distributed random variables with common density  $f^0$ . Later,  $X_i^0$ ,  $i=1, \dots, n$ , will be interpreted as the true survival times of  $n$  items or indi-



viduals under observation, where  $f^0$  will have support in  $[0, \infty)$ . Suppose that  $U_1, U_2, \dots, U_n$  is a sequence of constants or random variables which "censor"  $X_i^0$ ,  $i=1, \dots, n$ , on the right. In survival analysis or reliability studies, the  $U_i$ 's represent possible "loss" times of items or individuals from the test.

The observed data are denoted by the pairs  $(X_i, \Delta_i)$ ,  $i=1, \dots, n$ , where

$$X_i = \min\{X_i^0, U_i\}, \quad \Delta_i = \begin{cases} 1 & \text{if } X_i^0 \leq U_i \\ 0 & \text{if } X_i^0 > U_i \end{cases}.$$

It is desired to obtain the MPLE of  $f^0$  based on these observations.

In reliability or survival analysis, where  $f^0$  has support in  $[0, \infty)$ , the nature of the censoring depends on the  $U_i$ 's.

(i) If  $U_1, \dots, U_n$  are fixed constants, the observations are time truncated. If all  $U_i$ 's are equal to the same constant, then the case of Type I censoring results. (ii) If all  $U_i = X_{(r)}^0$ , the  $r$ th order statistic of  $X_1^0, \dots, X_n^0$ , then the situation is that of Type II censoring. (iii) If  $U_1, \dots, U_n$  constitute a random sample from a distribution  $H$  (usually unknown) and are independent of  $X_1^0, \dots, X_n^0$ , then  $(X_i, \Delta_i)$ ,  $i=1, \dots, n$ , is called a randomly censored sample. See Gill (1980, Ch. 3 and Ex. 4.1.1) for further discussion. An observed value of  $(X_i, \Delta_i)$  will be denoted by  $(x_i, d_i)$ .

By  $L^p(\Omega)$  we will mean the space of functions  $v$  such that  $\int_{\Omega} |v(t)|^p dt < \infty$  with norm  $\|v\|_p = [\int_{\Omega} |v(t)|^p dt]^{1/p}$  for  $p \geq 1$ . Let  $H(\Omega)$  be a manifold in  $L^1(\Omega)$ .

Following notation similar to that of De Montricher, Tapia, and Thompson (1975), let  $\phi$  denote a functional  $\phi: H(\Omega) \rightarrow \mathbb{R}$ . Given the arbitrarily right-censored sample  $(x_i, d_i)$ ,  $i=1, 2, \dots, n$ , the  $\phi$ -penalized likelihood of  $v \in H(\Omega)$  is defined by

$$L(v) = \prod_{i=1}^n [v(x_i)]^{d_i} [1-v(x_i)]^{1-d_i} \exp(-\phi(v)),$$

where  $V(x_1) = \int_{-\infty}^{x_1} v(t)dt$  denotes the cumulative distribution function with density  $v$  and  $\phi$  is the penalty function. By the maximum penalized likelihood estimator (MPLE) of  $f^0$  corresponding to manifold  $H(\Omega)$  and penalty function  $\phi$ , we will mean any solution to the problem:

$$\begin{aligned} &\text{maximize } L(v) \text{ subject to} \\ &v \in H(\Omega), \int_{\Omega} v(t)dt = 1, \text{ and} \\ &v(t) \geq 0 \text{ for all } t \in \Omega. \end{aligned} \quad (2.1)$$

The function  $L(v)$  is the censored form of the penalized likelihood of Good and Gaskins (1971).

When  $H(\Omega)$  is a Hilbert space, a natural penalty function to use is  $\phi(v) = ||v||^2$ , where  $||\cdot||$  is the norm on  $H(\Omega)$ . If no reference is given to  $\phi$  when we are considering the MPLE corresponding to a Hilbert space  $H(\Omega)$ , it is assumed that  $\phi$  is the square of the norm on  $H(\Omega)$ . A Hilbert space inner product will be denoted by  $\langle \cdot, \cdot \rangle$  so that  $\langle v, v \rangle = ||v||^2$ . When  $H(\Omega)$  is a Hilbert space, it is a reproducing kernel Hilbert space (RKHS) if point evaluation is a continuous operation, that is,  $v_n \rightarrow v$  in  $H(\Omega)$  implies that  $v_n(t) \rightarrow v(t)$  for all  $t \in \Omega$ . See Goffman and Pedrick (1965) for further details.

### 3. EXISTENCE AND UNIQUENESS OF AN MPLE

In this section we establish the existence and uniqueness of a solution to problem (2.1) when  $H(\Omega)$  is a RKHS. The inner product on  $H(\Omega)$  is defined by  $\langle u, v \rangle = \int_{\Omega} u(t)v(t)dt$  for  $u, v \in H(\Omega)$ .

Theorem 3.1. Assume that  $H(\Omega)$  is a RKHS, integration over  $\Omega$  is a continuous functional, and  $D$  is a closed convex subset of  $\{v \in H(\Omega) : v(x_i) \geq 0, i=1, \dots, n\}$  with the property that  $D$  contains at least one function which is positive at the data points  $x_1, \dots, x_n$ . Then the MPLE of  $f^0$  corresponding to penalty function  $\phi(v) = ||v||^2$  in (2.1) exists in  $D$  and is unique, where  $||\cdot||$  denotes the norm on  $H(\Omega)$ .

The proof of Theorem 3.1 is omitted. It is analogous to the proof of Proposition 2.1 of De Montricher, Tapia, and Thompson (1975), using the inequality  $L(v) \leq \|v\|^k \exp(-\|v\|^2) (\prod_{i=1}^n K_i)$ , where  $k = \sum_{i=1}^n d_i$  is the number of uncensored observations and  $K_i$  is such that  $|v(x_i)| \leq K_i \|v\|$  for each  $i=1,2,\dots,n$ . Also, the first and second Fréchet derivatives of  $J(v) = \ln L(v)$  are given by (Tapia, 1971)

$$J'(v)(\eta) = \sum_{i=1}^n \frac{d_i \eta(x_i)}{v(x_i)} - \sum_{i=1}^n \frac{(1-d_i)}{1-v(x_i)} \int_{-\infty}^{x_i} \eta(t) dt - 2 \langle v, \eta \rangle$$

and

$$J''(v)(\eta, \eta) = - \sum_{i=1}^n \frac{d_i \eta^2(x_i)}{v^2(x_i)} - \sum_{i=1}^n \frac{(1-d_i) [\int_{-\infty}^{x_i} \eta(t) dt]^2}{[1-v(x_i)]^2} - 2 \langle \eta, \eta \rangle.$$

We note that the constraints in (2.1) define a closed convex subset of  $\{v \in H(\Omega) : v(x_i) \geq 0, i=1, \dots, n\}$ . Also, let  $(a,b)$  be a finite interval. For each integer  $s \geq 1$ , let  $H_0^s(a,b)$  denote the Sobolev space of functions on  $[a,b]$  whose  $s-1$  derivatives are absolutely continuous and vanish at  $a$  and  $b$  and whose  $s$ th derivative is in  $L^2(a,b)$ . The inner product on  $H_0^s(a,b)$  is defined by

$$\langle u, v \rangle = \int_a^b u^{(s)}(t) v^{(s)}(t) dt,$$

where  $u^{(s)}$  denotes the  $s$ th derivative. It is well known that  $H_0^s(a,b)$  is a RKHS with the above inner product and integration over  $(a,b)$  is a continuous operation (Lemma 2.1 of De Montricher, Tapia, and Thompson, 1975).

**Corollary 3.1.** The MPLE corresponding to  $H_0^s(a,b)$  with  $\phi(v) = \langle v, v \rangle = \|v\|^2$  exists and is unique.

As a special case of Corollary 3.1, we can consider the MPLE of a lifetime density  $f^0$  over a finite interval  $[0,T]$  for very large  $T > 0$  based on an arbitrarily right-censored sample from  $f^0$ . The MPLE exists and is unique in  $H_0^s(0,T)$  with penalty function  $\phi(v) = \int_0^T [v^{(s)}(t)]^2 dt$ . The extension to

$[0, \infty)$  is considered in the next section.

#### 4. THE FIRST ESTIMATOR OF GOOD AND GASKINS UNDER CENSORING

For complete samples, Good and Gaskins (1971) considered the penalty function

$$\phi(v) = \alpha \int_{-\infty}^{\infty} \frac{[v'(t)]^2}{v(t)} dt,$$

for  $\alpha > 0$ , which is equivalent to

$$\phi(v) = 4\alpha \int_{-\infty}^{\infty} \left[ \frac{d(v(t))^{1/2}}{dt} \right]^2 dt.$$

De Montricher, Tapia, and Thompson (1975) indicated that the underlying manifold for the MPLE with this penalty function should be  $v^{1/2} \in H^1(-\infty, \infty)$ , where  $H^1(-\infty, \infty)$  is the Sobolev space of functions  $f: \mathbb{R} \rightarrow \mathbb{R}$  such that the first derivative  $f'$  exists almost everywhere and  $f, f' \in L^2(-\infty, \infty)$  with inner product

$$\langle f, g \rangle = \int_{-\infty}^{\infty} f(t)g(t)dt + \int_{-\infty}^{\infty} f'(t)g'(t)dt.$$

Letting  $u = v^{1/2}$ , we have the penalty function

$$\phi(u^2) = 4\alpha \int_{-\infty}^{\infty} [u'(t)]^2 dt, \quad u \in H^1(-\infty, \infty).$$

This substitution avoids the nonnegativity constraint in problem (2.1).

For the data  $(x_i, d_i)$ ,  $i=1, \dots, n$ , described in Section 2, we now would like to maximize

$$L(u) = \prod_{i=1}^n [u^2(x_i)]^{d_i} \left[ \int_{x_i}^{\infty} u^2(t) dt \right]^{1-d_i} \exp(-4\alpha \|u'\|_2^2).$$

Since  $L(u) \geq 0$ , maximizing  $L(u)$  is equivalent to maximizing  $\hat{L}(u) = [L(u)]^{1/2}$ . Thus, we have the problem:

$$\text{Maximize } \hat{L}(u) = \prod_{i=1}^n [u(x_i)]^{d_i} \left[ \int_{x_i}^{\infty} u^2(t) dt \right]^{(1-d_i)/2} \exp(-2\alpha \|u'\|_2^2)$$

$$\text{subject to } \int_{-\infty}^{\infty} u^2(t) dt = 1. \quad (4.1)$$

Letting  $J(u) = \ln \hat{L}(u)$ , problem (4.1) is equivalent to:

$$\begin{aligned}
\text{Maximize } J(u) &= \sum_{i=1}^n d_i \ln u(x_i) \\
&+ \sum_{i=1}^n \frac{1}{2}(1-d_i) \ln[1 - \int_{-\infty}^{x_i} u^2(t) dt] - 2\alpha \int_{-\infty}^{\infty} [u'(t)]^2 dt \\
\text{subject to } &\int_{-\infty}^{\infty} u^2(t) dt = 1.
\end{aligned} \tag{4.2}$$

**Theorem 4.1.** Problem (4.2) has a unique solution in the set

$$S = \{u \in H^1(-\infty, \infty) : \int_{-\infty}^{\infty} u^2(t) dt = 1\}.$$

**Proof:** The first part of the proof is similar to arguments in the proof of Proposition 3.3 of De Montricher, Tapia, and Thompson (1975), but the details are somewhat different. The Fréchet derivatives of  $J(u)$  are

$$\begin{aligned}
J'(u)(\eta) &= \sum_{i=1}^n \frac{d_i \eta(x_i)}{u(x_i)} - \sum_{i=1}^n \frac{(1-d_i)}{U_2(x_i)} \int_{-\infty}^{x_i} u(t) \eta(t) dt \\
&- 4\alpha \int_{-\infty}^{\infty} u'(t) \eta'(t) dt,
\end{aligned}$$

where  $U_2(x_i) = \int_{x_i}^{\infty} u^2(t) dt$ , and

$$\begin{aligned}
J''(u)(\eta, \eta) &= - \sum_{i=1}^n \frac{d_i \eta^2(x_i)}{u^2(x_i)} - \sum_{i=1}^n \frac{(1-d_i)}{(U_2(x_i))^2} \{U_2(x_i) \int_{-\infty}^{x_i} \eta^2(t) dt \\
&+ 2[\int_{-\infty}^{x_i} u(t) \eta(t) dt]^2\} - 4\alpha \int_{-\infty}^{\infty} \eta'(t)^2 dt.
\end{aligned}$$

Since  $J''(u)$  is negative definite,  $J$  is strictly concave, and by Theorem 2, page 160, of Tapia and Thompson (1978),  $J(u)$  has at most one maximizer in the set

$$S' = \{u \in H^1(-\infty, \infty) : \int_{-\infty}^{\infty} u^2(t) dt \leq 1\}.$$

If  $J(u)$  is continuous on  $S'$ , by Theorem 4 on page 162 of Tapia and Thompson (1978),  $J$  will have at least one maximizer in  $S'$ . To show this continuity, we note that by properties of a RKHS, if  $u_m \rightarrow u$  as  $m \rightarrow \infty$  in  $H^1(-\infty, \infty)$ , then

$u_m(x_i) \rightarrow u(x_i)$  for each  $i=1, \dots, n$ . Also,  $\|u_m - u\| \rightarrow 0$  as  $m \rightarrow \infty$  implies, by definition of the norm in  $H^1(-\infty, \infty)$ , that  $\|u_m - u\|_2 \rightarrow 0$  and  $\|u'_m - u'\|_2 \rightarrow 0$  as  $m \rightarrow \infty$ . Furthermore, for any fixed constant  $c$ ,  $\int_c^\infty u_m^2(t) dt \rightarrow \int_c^\infty u^2(t) dt$  as  $m \rightarrow \infty$ . Hence,  $J: S' \rightarrow R$  is continuous. Therefore,  $J(u)$  has a unique maximizer  $u_*$  in  $S'$ .

Next, suppose that  $\int_{-\infty}^\infty u_*^2(t) dt < 1$ . Since  $u_*^2(t) \rightarrow 0$  as  $t \rightarrow \infty$ , then  $u_*(t)$  and  $u'_*(t)$  both converge to zero as  $t \rightarrow \infty$ . Thus, there exists a number  $M$  such that  $u'_*(t) < 1$  for  $t > M$ . Consider a function  $v_*(t)$  defined so that (i)  $v_*(t) = u_*(t)$  for  $t \leq M$ , (ii)  $v_*(t) > u_*(t)$  for  $t > M$  and  $\int_{-\infty}^\infty v_*^2(t) dt = 1$ , and (iii)  $[v'_*(t)]^2 \leq [u'_*(t)]^2$  for  $t > M$ . Then by (i) and (iii)  $\int_M^M [v'_*(t)]^2 dt = \int_M^M [u'_*(t)]^2 dt$  and  $\int_M^\infty [u'_*(t)]^2 dt \geq \int_M^\infty [v'_*(t)]^2 dt$ . Also, by (ii), for each  $x_i$ ,  $i=1, \dots, n$ ,  $\int_{x_i}^\infty v_*^2(t) dt > \int_{x_i}^\infty u_*^2(t) dt$ . These results imply that  $J(u_*) < J(v_*)$ , a contradiction, since  $u_*$  is the unique maximizer of  $J$  in  $S'$ . Therefore  $\int_{-\infty}^\infty u_*^2(t) dt = 1$ , completing the proof.

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Now, we assume that  $f^0$  is a lifetime density on the half-line  $R_+ = (0, \infty)$  and use a symmetry argument about zero to obtain the results for  $f^0$ . Thus, assume that the censored sample  $(X_i, \Delta_i)$ ,  $i=1, \dots, n$ , is such that  $X_i > 0$  with probability one. Then the problem (4.1) becomes:

$$\begin{aligned} \text{Maximize } \hat{L}(u) = & \prod_{i=1}^n [u(x_i)]^{d_i} [1 - \int_{-\infty}^{x_i} u^2(t) dt]^{\frac{1}{2}(1-d_i)} \\ & \times \exp[-2\alpha \int_0^\infty (u'(t))^2 dt], \end{aligned} \quad (4.3)$$

where  $x_i > 0$ ,  $i=1, \dots, n$ ,  $\int_0^\infty u^2(t) dt = 1$ , and  $u(t) \geq 0$ ,  $t > 0$ .

Let  $X_{-1} = X_1$  and  $d_{-1} = d_1$ ,  $i=1, \dots, n$ , and define  $\bar{u}(x) = u(|x|)$  for  $x \in R \setminus \{0\}$  and  $\bar{u}(0) = \lim_{x \rightarrow 0^+} u(x)$ . Then define the following problem:

$$\begin{aligned} \text{Maximize } L(\bar{u}) = & \prod_{|i|=1}^n [\bar{u}(x_i)]^{d_i} [2 - \int_{-\infty}^{x_i} \bar{u}^2(t) dt]^{\frac{1}{2}(1-d_i)} \\ & \times \exp[-2\alpha \int_{-\infty}^{\infty} (\bar{u}'(t))^2 dt], \end{aligned} \quad (4.4)$$

where  $\int_{-\infty}^{\infty} \bar{u}^2(t) dt = 2$  and  $\bar{u} \in H_S \equiv \{g \in H^1(-\infty, \infty) : g(x) = g(-x)\}$ .

Notice that  $L(\bar{u}) = [\hat{L}(u)]^2$ . Also,  $H_S$  is equivalent to the Sobolev space  $H^1(0, \infty)$ .

**Proposition 4.2.** If  $u^*$  solves (4.4), then  $u_+^*(t) = u^*(t)$ ,  $t \geq 0$ , and  $u_+^*(t) = 0$ ,  $t < 0$ , solves (4.3).

**Proof:** Suppose  $u^*$  solves (4.4). Since  $\hat{L}(u) = [L(\bar{u})]^{\frac{1}{2}}$  and  $u^*$  is symmetric about zero implies that  $\int_0^{\infty} [u^*(t)]^2 dt = 1$ ,  $u_+^*$  solves (4.3). ///

From Proposition 4.2, the "first MPLE of Good and Gaskins" under arbitrary right-censorship will be given by  $(u_+^*)^2(t)$ . We next show that this solution exists and is unique.

**Theorem 4.3.** Problem (4.3) has a unique solution.

**Proof:**  $H_S$  defines a closed convex subset of  $H^1(-\infty, \infty)$ . Thus, by a proof similar to that of Theorem 4.1, problem (4.4) has a unique solution. By Proposition 4.2,  $u_+^*$  is the unique solution to problem (4.3). ///

To discover the general form of the unique solution  $\bar{u}^*$  of problem (4.4), we consider the following problem:

For given  $\lambda > 0$  and  $\alpha$  in (4.1), let

$$\phi_{\lambda}(\bar{u}) = 2\alpha \int_{-\infty}^{\infty} [\bar{u}'(t)]^2 dt + \lambda \int_{-\infty}^{\infty} \bar{u}^2(t) dt.$$

$$\begin{aligned} \text{Maximize } L_{\lambda}(\bar{u}) = & \prod_{|i|=1}^n [\bar{u}(x_i)]^{d_i} [2 - \int_{-\infty}^{x_i} \bar{u}^2(t) dt]^{\frac{1}{2}(1-d_i)} \\ & \times \exp[-\phi_{\lambda}(\bar{u})], \end{aligned} \quad (4.5)$$

subject to  $\bar{u} \in H_S$  and  $\int_{-\infty}^{\infty} \bar{u}^2(t) dt = 2$ .

The inner product  $\langle u, v \rangle = 2\alpha \int_{-\infty}^{\infty} u'(t)v'(t) dt + \lambda \int_{-\infty}^{\infty} u(t)v(t) dt$  defines a norm  $\|\bar{u}\|_{\lambda}^2 = \phi_{\lambda}(\bar{u})$  equivalent to the original norm on  $H^1(-\infty, \infty)$ . Let  $v_i$  denote the representer in the  $\phi_{\lambda}$ -inner

product of the continuous linear functional given by point evaluation at  $x_i$ , that is  $\langle v_i, \eta \rangle_\lambda = \eta(x_i)$  for all  $\eta \in H^1(-\infty, \infty)$ .

Let  $S = \{v \in H_S : v(x_i) \geq 0\}$ . Then  $S$  is closed and convex. Letting  $J_\lambda = \ln L_\lambda$ , we have the first and second Fréchet derivatives,

$$J'_\lambda(\bar{u})(\eta) = \sum_{|i|=1}^n \frac{d_i \eta(x_i)}{\bar{u}(x_i)} - \sum_{|i|=1}^n \frac{(1-d_i)}{U_2(x_i)} \int_{-\infty}^{x_i} \bar{u}(t) \eta(t) dt - 2\langle \bar{u}, \eta \rangle_\lambda,$$

where  $U_2(x_i) \equiv \int_{x_i}^{\infty} \bar{u}^2(t) dt$ , and

$$J''_\lambda(\bar{u})(\eta, \eta) = - \sum_{|i|=1}^n \frac{d_i \eta^2(x_i)}{\bar{u}^2(x_i)} - \sum_{|i|=1}^n \frac{(1-d_i)}{[U_2(x_i)]^2} \{U_2(x_i) \int_{-\infty}^{x_i} \eta^2(t) dt + 2 \left[ \int_{-\infty}^{x_i} \bar{u}(t) \eta(t) dt \right]^2\} - 2\langle \eta, \eta \rangle_\lambda.$$

Thus  $-J''_\lambda$  is uniformly positive definite relative to  $S$ . This implies that  $-J_\lambda$  is uniformly convex on  $S$ . Therefore, if we can show that  $J_\lambda$  is continuous on  $S$ , by Theorem 6, page 162 of Tapia and Thompson (1978),  $J_\lambda$  will have a unique maximizer in  $S$ . By an argument similar to that in the proof of Theorem 4.1,  $J_\lambda$  is continuous on  $S$ , and has a unique maximizer  $\bar{u}_\lambda$  in  $S$ .

Now, at the solution  $\bar{u}_\lambda$ , we must have the gradient of  $J_\lambda$  vanish. Let  $g_i$  be the element of  $H^1(-\infty, \infty)$  such that  $\langle g_i, \eta \rangle_\lambda = \int_{-\infty}^{x_i} \bar{u}_\lambda(t) \eta(t) dt$ . Then  $\nabla J_\lambda(\bar{u}_\lambda) = \sum_{|i|=1}^n \frac{d_i v_i}{\bar{u}_\lambda(x_i)} - \sum_{|i|=1}^n \frac{(1-d_i) g_i}{U_{2\lambda}(x_i)} - 2 \bar{u}_\lambda = 0$ , where  $U_{2\lambda}(x_i) = \int_{x_i}^{\infty} \bar{u}_\lambda^2(t) dt$ . Hence

$$\bar{u}_\lambda = \frac{1}{2} \left[ \sum_{|i|=1}^n \frac{d_i v_i}{\bar{u}_\lambda(x_i)} - \sum_{|i|=1}^n (1-d_i) \frac{g_i}{U_{2\lambda}(x_i)} \right]. \quad (4.6)$$

In order to obtain the form of  $v_i$  in (4.6), from the



definition of the  $\lambda$ -inner product, we have

$$2\alpha \int_{-\infty}^{\infty} v_1'(t) \eta'(t) dt + \lambda \int_{-\infty}^{\infty} v_1(t) \eta(t) dt = \eta(x_1). \quad (4.7)$$

Integrating the left-hand side of (4.7) by parts (in the distribution sense) gives

$$\int_{-\infty}^{\infty} [\lambda v_1(t) - 2\alpha v_1''(t)] \eta(t) dt = \int_{-\infty}^{\infty} \delta_1(t) \eta(t) dt, \quad (4.8)$$

where  $\delta_1(t) = \delta_0(t-x_1)$  and  $\delta_0$  denotes the Dirac delta function. Equation (4.8) is equivalent to the differential equation

$$\lambda v_1(t) - 2\alpha v_1''(t) = \delta_1(t) \quad (4.9)$$

which, for  $i=0$ , has the solution

$$v_0(t) = (2\alpha\lambda)^{-1/2} \exp[-(\frac{\lambda}{2\alpha})^{1/2} |t|], \quad t \neq 0.$$

Hence,  $v_1(t) = v_0(t-x_1) + v_0(t+x_1)$  solves (4.9).

Next, to determine the form of  $g_1$  in (4.6), replacing the right-hand side of (4.7) with  $\int_{-\infty}^{x_1} \bar{u}_\lambda(t) \eta(t) dt$  yields the non-homogeneous differential equation

$$g_1''(t) - (\lambda/2\alpha) g_1(t) = - (2\alpha)^{-1} \bar{u}_\lambda(t) I_{(-\infty, x_1]}(t), \quad (4.10)$$

where  $I_A(t)$  denotes the indicator function of the set  $A$ . Using the theory of Green's functions, the solution to equation (4.10) is

$$\begin{aligned} g_1(t) = & \exp \left[ - \left( \frac{\lambda}{2\alpha} \right)^{1/2} t \right] + \exp \left[ \left( \frac{\lambda}{2\alpha} \right)^{1/2} t \right] \\ & - (2\alpha\lambda)^{-1/2} \int_0^t \sinh \left[ \left( \frac{\lambda}{2\alpha} \right)^{1/2} (t-\tau) \right] I_{(-\infty, x_1]}(|\tau|) \bar{u}_\lambda(\tau) d\tau. \end{aligned}$$

Substitution of  $v_1(t)$  and  $g_1(t)$  into equation (4.6) gives the result that  $\bar{u}_\lambda$  solves the linear integral equation

$$\begin{aligned} \bar{u}_\lambda(t) = & C(t; \underline{x}, \alpha, \lambda) + (8\alpha\lambda)^{-1/2} \int_0^t \left[ \sum_{|i|=1}^n \frac{(1-d_1)}{v_{2\lambda}(\underline{x}_1)} I_{(-\infty, x_1]}(|\tau|) \right] \\ & \times \sinh \left[ \left( \frac{\lambda}{2\alpha} \right)^{1/2} (t-\tau) \right] \bar{u}_\lambda(\tau) d\tau, \end{aligned} \quad (4.11)$$

where the forcing function is defined by

$$C(t; \underline{x}, \alpha, \lambda) \equiv \frac{1}{2} \left\{ \sum_{|i|=1}^n \frac{d_i (2\alpha\lambda)^{-\frac{1}{2}}}{\bar{u}_\lambda(x_i)} [\exp(-(\lambda/2\alpha)^{\frac{1}{2}} |t-x_i|) + \exp(-(\lambda/2\alpha)^{\frac{1}{2}} |t+x_i|)] - \sum_{|i|=1}^n \frac{(1-d_i)}{U_{2\lambda}(x_i)} [\exp(-(\lambda/2\alpha)t) + \exp((\lambda/2\alpha)t)] \right\}.$$

Note that over the constraints of problem (4.4), problems (4.4) and (4.5) have the same solution for any  $\lambda > 0$  since  $\int_{-\infty}^{\infty} \bar{u}^2(t) dt$  is constant. By an argument similar to that in Tapia and Thompson (1978, p. 113), the unique solution  $\bar{u}^*$  to (4.4) also must solve (4.5) with a positive  $\lambda$ , and thus, has the same form as the solution to (4.11). The unique solution to problem (4.3) is then  $\bar{u}_+^*(t) = \bar{u}^*(t)$ ,  $t > 0$ , from (4.11), and hence, the "first MPLE of Good and Gaskins" is  $(\bar{u}_+^*)^2$ .

## 5. CONCLUSION

In this paper we have shown the existence and uniqueness of the MPLE of a density function in an appropriate general mathematical setting, based on arbitrarily right-censored observations from that density. For the first penalty function of Good and Gaskins (1971), the existence and uniqueness of the MPLE of the density function on  $(0, \infty)$  was also shown for this type of data. This "first MPLE of Good and Gaskins" under arbitrary right-censoring was shown to be in the form of a solution to a linear integral equation. These results are analogous to the complete sample case, except that the form of the penalized likelihood, and therefore, the MPLE, is complicated by the terms involving the survival function.

Statistical properties of the MPLE under censoring have not been considered here. The consistency, which seems to be quite difficult to prove, and other statistical properties need to be

investigated. The problem of computation of the MPLE from the general form (4.11), at least approximately, is currently under study by the authors.

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