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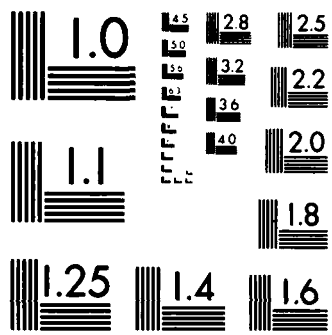

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Revised Version

BOUNDING THE STOCHASTIC PERFORMANCE

OF CONTINUUM STRUCTURE FUNCTIONS I*

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ABSTRACT

A continuum structure function γ is a nondecreasing mapping from the unit hypercube to the unit interval. Minimal path (cut) sets of upper (lower) simple continuum structure functions are introduced and are used to determine bounds on the distribution of $\gamma(\underset{\sim}{X})$ when $\underset{\sim}{X}$ is a vector of associated random variables and when γ is right (left)-continuous. It is shown that, if γ admits of a modular decomposition, improved bounds may be obtained.

KEYWORDS: Continuum structure function; weak coherency; minimal path set; minimal cut set; upper simple; lower simple; module; modular decomposition; associated random variable.

1. INTRODUCTION

Let $C = \{1,2,\ldots,n\}$ denote a set (of components) of finite cardinality n. A continuum structure function (CSF) is a nondecreasing mapping $\gamma\colon [0,1]^n \mapsto [0,1]$; we assume, without loss of generality, that $\gamma(\underset{\sim}{0}) = 0$ and $\gamma(\underset{\sim}{1}) = 1$ where $\underset{\sim}{\alpha}$ denotes $(\alpha,\ldots,\alpha) \in \Delta = [0,1]^n$. Such a function is said to be weakly coherent if $\sup_{\underset{\sim}{X}\in\Delta}[\gamma(1_i,\underset{\sim}{X}) - \gamma(0_i,\underset{\sim}{X})] > 0$ for each $i \in C$ where $(\delta_i,\underset{\sim}{X})$ denotes $(X_1,\ldots,X_{i-1},\delta,X_{i+1},\ldots,X_n)$. See Baxter (1984a,b) and Block and Savits (1984) for further details of CSFs.

In this paper, we show how minimal path (cut) sets of CSFs, subsets of C which are necessary and sufficient for a CSF to attain any value in its image when the state of every other component is zero (one), can be used to determine bounds on the distribution of $\gamma(\underset{\sim}{X})$ when the X_i's are associated random variables. We also continue our study of modules of CSFs (Baxter and Kim, 1984) by showing how the assumption of a modular decomposition yields uniformly improved bounds, thereby extending the well-known results for binary structure functions (Barlow and Proschan (1975), Chapter 2) to the continuum case.

We shall make use of the following notation:

$\underset{\sim}{Y} < \underset{\sim}{X}$ means that $Y_i \leq X_i$ for $i=1,2,\ldots,n$, but $\underset{\sim}{Y} \neq \underset{\sim}{X}$

$|A|$ denotes the cardinality of A

$\mathrm{Im}\,\gamma$ denotes the image of γ

$\underset{\sim}{X}^P$ denotes $\{X_i \in \underset{\sim}{X} \mid i \in P \subset C\}$

P_1 denotes $\{\underset{\sim}{X} \in \Delta \mid \gamma(\underset{\sim}{X}) = 1$ but $\gamma(\underset{\sim}{Y}) < 1$ for all $\underset{\sim}{Y} < \underset{\sim}{X}\}$

K_0 denotes $\{\underset{\sim}{X} \in \Delta \mid \gamma(\underset{\sim}{X}) = 0$ but $\gamma(\underset{\sim}{Y}) > 0$ for all $\underset{\sim}{Y} > \underset{\sim}{X}\}$

$\coprod_{i=1}^{n} X_i$ denotes $1 - \prod_{i=1}^{n}(1-X_i)$.

2. BOUNDS USING MINIMAL PATH AND CUT SETS

A <u>minimal vector</u> to level $\alpha \in \mathrm{Im}\,\gamma - \{0\}$ is a vector $\underset{\sim}{X} \in \Delta$ such that $\gamma(\underset{\sim}{X}) = \alpha$ whereas $\gamma(\underset{\sim}{Y}) < \alpha$ for all $\underset{\sim}{Y} < \underset{\sim}{X}$. The corresponding <u>path set</u> to level α is $T_\alpha = T_\alpha(\underset{\sim}{X}) = \{i \in C \mid X_i \neq 0\}$.

<u>Definition</u> (Baxter and Kim, 1984)

Let T be a nonempty subset of C. If T is a path set to level α for all $\alpha \in \mathrm{Im}\,\gamma - \{0\}$, then T is a <u>minimal path set</u> of γ.

Minimal path sets do not necessarily exist for an arbitrary CSF nor, if they do exist, do they necessarily exhibit the desirable properties of minimal path sets of binary structure functions. The following definition yields a large class of CSFs for which minimal path sets exist and are "well-behaved".

<u>Definition</u> (Baxter and Kim, 1984)

A CSF γ is <u>upper simple</u> if it satisfies the following four conditions:

C1 $P_1 \neq \emptyset$ and $P_1 \subset \{0,1\}^n - \{\underset{\sim}{0}\}$.

C2 $\bigcup_{i=1}^{r} T_{1i} = C$ where $T_{11},\ldots,T_{1r}$ are the r path sets of γ to level 1.

C3 If T is a path set to level $\alpha \in \mathrm{Im}\,\gamma - \{0\}$, then T is also a path set to level $\beta \in \mathrm{Im}\,\gamma - \{0\}$ for all $\beta \leq \alpha$.

C4 If T_α is a path set to level $\alpha < 1$, then $T_\alpha \subset T_{1i}$ for some path set T_{1i} to level 1.

Proposition 2.1

Let γ be an upper simple CSF. Then

(i) γ has at least one minimal path set

(ii) γ is weakly coherent

(iii) No proper subset of a minimal path set is itself a minimal path set

(iv) $\bigcup_{i=1}^{r} T_i = C$ where $T_1, \ldots, T_r$ denote the r minimal path sets of γ.

Proof. See Baxter and Kim (1984).

Similarly, we can define minimal cut sets and lower simple CSFs.

A maximal vector to level $\alpha \in \operatorname{Im} \gamma - \{1\}$ is a vector $\underset{\sim}{X} \in \Delta$ such that $\gamma(\underset{\sim}{X}) = \alpha$ whereas $\gamma(\underset{\sim}{Y}) > \alpha$ for all $\underset{\sim}{Y} > \underset{\sim}{X}$. The corresponding cut set to level α is $S_\alpha = S_\alpha(\underset{\sim}{X}) = \{i \in C \mid X_i \neq 1\}$.

Definition

Suppose that S is a nonempty subset of C. If S is a cut set to level α for all $\alpha \in \operatorname{Im} \gamma - \{1\}$, then S is a minimal cut set of γ.

Definition

A CSF γ is lower simple if it satisfies the following four conditions:

D1 $K_0 \neq \emptyset$ and $K_0 \subset \{0,1\}^n - \{\underset{\sim}{1}\}$.

D2 $\bigcup_{j=1}^{t} S_{0j} = C$ where $S_{01}, \ldots, S_{0t}$ are the t cut sets of γ to level 0.

D3 If S is a cut set to level $\alpha \in \operatorname{Im} \gamma - \{1\}$, then S is also a cut set to level $\beta \in \operatorname{Im} \gamma - \{1\}$ for all $\beta \geq \alpha$.

D4 If S_α is a cut set to level $\alpha > 0$, $S_\alpha \subset S_{0j}$ for some cut set S_{0j} to level 0.

An argument similar to the proof of Proposition 2.1 yields the following results.

Proposition 2.2

Let γ be a lower simple CSF. Then

(i) γ has at least one minimal cut set

(ii) γ is weakly coherent

(iii) No proper subset of a minimal cut set is itself a minimal cut set

(iv) $\bigcup_{j=1}^{t} S_j = C$ where $S_1,\ldots,S_t$ are the t minimal cut sets of γ.

Definition

If γ is both upper simple and lower simple, it is simple.

Examples of upper simple CSFs include the CSFs ζ and η studied by Baxter (1984a) and the CSF $\mu(\underset{\sim}{X}) = \frac{1}{n}\sum_{i=1}^{n} X_i$; examples of lower simple CSFs include ζ,μ and $\gamma(\underset{\sim}{X}) = \coprod_{i=1}^{n} X_i$.

Theorem 2.3 (Decomposition Theorem)

(i) Let γ be a right-continuous, upper simple CSF with minimal path sets $T_1,\ldots,T_r$. Then

$$\gamma(\underset{\sim}{X}) = \max_{1\leq i\leq r} \gamma(\underset{\sim}{X}^{T_i},\underset{\sim}{0}^{T_i^c}).$$

(ii) Let γ be a left-continuous, lower simple CSF with minimal cut sets $S_1,\ldots,S_t$. Then

$$\gamma(\underset{\sim}{X}) = \min_{1\le j\le t} \gamma(\underset{\sim}{X}^{S_j}, \underset{\sim}{1}^{S_j^c}).$$

(iii) Let γ be a continuous, simple CSF with minimal path (cut) sets $T_1,\ldots,T_r$ $(S_1,\ldots,S_t)$. Then

$$\gamma(\underset{\sim}{X}) = \max_{1\le i\le r} \gamma(\underset{\sim}{X}^{T_i}, \underset{\sim}{0}^{T_i^c}) = \min_{1\le j\le t} \gamma(\underset{\sim}{X}^{S_j}, \underset{\sim}{1}^{S_j^c}).$$

Proof: See Baxter and Kim (1984) for the proof of (i). The proof of (ii) is similar, and (iii) follows immediately from (i) and (ii). $\square$

This theorem extends formulae (3.2) and (3.4) of Barlow and Proschan (1975, p. 10) to the continuum case.

Suppose that the CSF γ is right-continuous. Then $\{\underset{\sim}{X} \in \Delta \mid \gamma(\underset{\sim}{X}) \ge \alpha\}$ is closed (Block and Savits, 1984) and is therefore a Borel set for all α. Thus γ is Borel-measurable and hence, if $X_1,\ldots,X_n$ are random variables defined on the same probability space $(\Omega,\mathcal{F},P)$, $\gamma(\underset{\sim}{X})$ is also a random variable on $(\Omega,\mathcal{F},P)$. The same is true if γ is left-continuous.

It is henceforth assumed that $X_1,\ldots,X_n$ are associated (Barlow and Proschan (1975), p. 29). The following property of associated random variables will subsequently prove useful.

Proposition 2.4

If $X_1,\ldots,X_n$ are associated random variables, then

(i) $P\{X_1 \ge x_1,\ldots,X_n \ge x_n\} \ge \prod_{i=1}^{n} P\{X_i \ge x_i\}$

(ii) $P\{X_1 < x_1, \ldots, X_n < x_n\} \geq \prod_{i=1}^{n} P\{X_i < x_i\}$

for any choice of $x_1, \ldots, x_n$.

Proof: Barlow and Proschan (1975), p. 33, prove a similar result.

We now derive bounds on the distribution of $\gamma(\underset{\sim}{X})$ using the minimal path and cut sets of γ. These bounds generalise Theorems 3.4 and 3.9 of Barlow and Proschan (1975, Chapter 2).

Theorem 2.5

(i) Let γ be a right-continuous, upper simple CSF with minimal path sets $T_1, \ldots, T_r$. Then, if $X_1, \ldots, X_n$ are associated random variables,

$$\max_{1 \leq i \leq r} P\{\gamma(\underset{\sim}{X}^{T_i}, \underset{\sim}{0}^{T_i^c}) \geq x\} \leq P\{\gamma(\underset{\sim}{X}) \geq x\} \leq \coprod_{i=1}^{r} P\{\gamma(\underset{\sim}{X}^{T_i}, \underset{\sim}{0}^{T_i^c}) \geq x\}$$

for all $x \in \mathbb{R}$.

(ii) Let γ be a left-continuous, lower simple CSF with minimal cut sets $S_1, \ldots, S_t$. Then, if $X_1, \ldots, X_n$ are associated random variables,

$$\prod_{j=1}^{t} P\{\gamma(\underset{\sim}{X}^{S_j}, \underset{\sim}{1}^{S_j^c}) > x\} \leq P\{\gamma(\underset{\sim}{X}) > x\} \leq \min_{1 \leq j \leq t} P\{\gamma(\underset{\sim}{X}^{S_j}, \underset{\sim}{1}^{S_j^c}) > x\}$$

for all $x \in \mathbb{R}$.

Proof: (i) By the Decomposition Theorem, $P\{\gamma(\underset{\sim}{X}) \geq x\} = P\{\max_{1 \leq i \leq r} \gamma(\underset{\sim}{X}^{T_i}, \underset{\sim}{0}^{T_i^c}) \geq x\}$

and so

$$(2.1)\quad P\{\gamma(\underset{\sim}{X}) \geq x\} = 1 - P\{\gamma(\underset{\sim}{X}^{T_1},\underset{\sim}{0}^{T_1^c}) < x,\ldots,\gamma(\underset{\sim}{X}^{T_r},\underset{\sim}{0}^{T_r^c}) < x\}.$$

Since γ is nondecreasing, and nondecreasing functions of associated random variables are themselves associated random variables, it follows that $\gamma(\underset{\sim}{X}^{T_1},\underset{\sim}{0}^{T_1^c}),\ldots,\gamma(\underset{\sim}{X}^{T_r},\underset{\sim}{0}^{T_r^c})$ are associated random variables. Thus, from (2.1) and Proposition 2.4,

$$P\{\gamma(\underset{\sim}{X}) \geq x\} \leq \coprod_{i=1}^{r} P\{\gamma(\underset{\sim}{X}^{T_i},\underset{\sim}{0}^{T_i^c}) \geq x\}.$$

To establish the lower bound, observe that

$$P\{\gamma(\underset{\sim}{X}^{T_1},\underset{\sim}{0}^{T_1^c}) < x,\ldots,\gamma(\underset{\sim}{X}^{T_r},\underset{\sim}{0}^{T_r^c}) < x\} \leq \min_{1\leq i\leq r} P\{\gamma(\underset{\sim}{X}^{T_i},\underset{\sim}{0}^{T_i^c}) < x\}$$

and hence, from (2.1),

$$P\{\gamma(\underset{\sim}{X}) \geq x\} \geq 1 - \min_{1\leq i\leq r} P\{\gamma(\underset{\sim}{X}^{T_i},\underset{\sim}{0}^{T_i^c}) < x\} = \max_{1\leq i\leq r} P\{\gamma(\underset{\sim}{X}^{T_i},\underset{\sim}{0}^{T_i^c}) \geq x\}.$$

(ii) The proof is similar and is omitted. $\square$

<u>Remark 2.6</u>

If γ is continuous and simple, both sets of bounds hold. Neither lower bound dominates the other. Consider, for example, the binary structure function $\gamma(\underset{\sim}{x}) = x_1 \vee (x_2 \wedge x_3 \wedge x_4)$, $\underset{\sim}{x} \in \{0,1\}^4$, which is clearly simple. Suppose that x_1, x_2, x_3 and x_4 are independent Bernoulli random variables with parameter $p \in (0,1)$. Then $\max_{i=1,2} P\{\gamma\{(\underset{\sim}{x}^{T_i},\underset{\sim}{0}^{T_i^c}) = 1\} = p$

whereas $\prod_{j=1}^{3} P\{\gamma(\underset{\sim}{x}^{S_j}, \underset{\sim}{1}^{S_j^c}) = 1\} = p^3(2-p)^3$. Thus, if $p = 0.1$, the lower bound based on minimal cut sets is majorised by the lower bound based on minimal path sets whereas, if $p = 0.9$, the order of majorisation is reversed.

Similarly, neither upper bound majorises the other.

3. IMPROVED BOUNDS USING A MODULAR DECOMPOSITION

Let A be a nonempty subset of C and suppose that γ is weakly coherent. Suppose, further, that there exists a weakly coherent CSF $\gamma_1: [0,1]^{|A|} \mapsto [0,1]$ and a CSF $\chi: [0,1]^{n-|A|+1} \mapsto [0,1]$ such that $\gamma(\underset{\sim}{X}) = \chi[\gamma_1(\underset{\sim}{X}^A), \underset{\sim}{X}^{A^c}]$ for all $\underset{\sim}{X} \in \Delta$. Then (A,γ_1) is a module of (C,γ).

This definition is due to Baxter and Kim (1984).

Definition

Let γ be a weakly coherent CSF. Suppose that $(A_1,\gamma_1),\ldots,(A_N,\gamma_N)$ are modules of (C,γ) where $\{A_1,\ldots,A_N\}$ is a partition of C, i.e. there exists a CSF $\chi: [0,1]^{n-|A|+1} \mapsto [0,1]$, the organising structure, such that $\gamma(\underset{\sim}{X}) = \chi[\gamma_1(\underset{\sim}{X}^{A_1}),\ldots,\gamma_N(\underset{\sim}{X}^{A_N})]$ for all $\underset{\sim}{X} \in \Delta$. Then $\{\chi,(A_1,\gamma_1),\ldots,(A_N,\gamma_N)\}$ is a modular decomposition of (C,γ).

In this section, we show that, if γ admits of a modular decomposition, improved bounds on the distribution of $\gamma(\underset{\sim}{X})$ may be obtained. It is first necessary to introduce some further notation.

Suppose that γ is upper (lower) simple with minimal path (cut) sets $T_1,\dots,T_r$ $(S_1,\dots,S_t)$ and that the organising structure χ is also upper (lower) simple with minimal path (cut) sets $\mu_1,\dots,\mu_p$ $(\nu_1,\dots,\nu_k)$. We write $\mu_i = \{A_{i1},\dots,A_{iM_i}\}$ for $i=1,2,\dots,p$ $(\nu_j = \{A_{j1},\dots,A_{jM_j}\}$ for $j=1,2,\dots,k)$ where each $A_{\ell h} \in \{A_1,\dots,A_N\}$. Further, we write $B_i = \bigcup_{\ell=1}^{M_i} A_{i\ell}$ $(D_j = \bigcup_{\ell=1}^{M_j} A_{j\ell})$; these are those elements of C which are contained in the elements of $\mu_i(\nu_j)$.

Proposition 3.1

(i) Suppose that γ admits of a modular decomposition $\{\chi,(A_1,\gamma_1),\dots,(A_N,\gamma_N)\}$ and that γ and χ are both upper simple and right-continuous. Then each B_i contains at least one of the minimal path sets of γ, and each of these minimal path sets is a subset of precisely one of $B_1,\dots,B_p$.

(ii) Suppose that γ admits of a modular decomposition $\{\chi,(A_1,\gamma_1),\dots,(A_N,\gamma_N)\}$ and that γ and χ are both lower simple and left-continuous. Then each D_j contains at least one of the minimal cut sets of γ, and each of these minimal cut sets is a subset of precisely one of $D_1,\dots,D_k$.

Proof: (i) Suppose, firstly, that there exists a minimal path set of γ, T say, such that $T \not\subset B_i$ for each $i=1,2,\dots,p$. Since T is a path set to level $\alpha \in \text{Im}\,\gamma - \{0\}$, there exists a minimal vector $\underset{\sim}{X}^T$ to level α. Let $Y_h = \gamma_h(\underset{\sim}{X}^{A_h \cap T}, 0^{A_h \cap T^c})$ for $h=1,2,\dots,N$.

Since $\mu_1,\dots,\mu_p$ are the minimal path sets of χ, it follows from the Decomposition Theorem that

$$\chi(\underset{\sim}{Y}) = \max_{1\le i\le p} \chi(\underset{\sim}{Y}^{\mu_i}, \underset{\sim}{0}^{\mu_i^c})$$

$$= \max_{1\le i\le p} \chi[\gamma_{i1}(\underset{\sim}{X}^{A_{i1}\cap T}, \underset{\sim}{0}^{A_{i1}\cap T^c}), \ldots, \gamma_{iM_i}(\underset{\sim}{X}^{A_{iM_i}\cap T}, \underset{\sim}{0}^{A_{iM_i}\cap T^c}), \underset{\sim}{0}^{\mu_i^c}]$$

where γ_{ij} is the CSF associated with A_{ij}

$$= \max_{1\le i\le p} \gamma(\underset{\sim}{X}^{B_i\cap T}, \underset{\sim}{0}^{B_i^c\cup T^c})$$

$$< \gamma(\underset{\sim}{X}^{T}, \underset{\sim}{0}^{T^c})$$

since, by hypothesis, $B_i \cap T$ is a proper subset of T for each i. This is a contradiction, and hence there exists a B_i containing T. Since T is arbitrary, it follows that each minimal path set of γ is a subset of some B_i.

Since, clearly, no B_i is empty, and thus contains at least one minimal path set of γ, it remains to show that no T can lie in more than one of the B_i's. Suppose, conversely, that $T \subset B_i$ and $T \subset B_j$ $(i \neq j)$. Suppose, further, that, for some $\underset{\sim}{Y} = (\gamma_1(\underset{\sim}{X}^{A_1}), \ldots, \gamma_N(\underset{\sim}{X}^{A_N}))$, $(\underset{\sim}{Y}^{\mu_i}, \underset{\sim}{0}^{\mu_i^c})$ is a minimal vector to level α, i.e.

$$\max_{1\le \ell\le p} \chi(\underset{\sim}{Y}^{\mu_\ell}, \underset{\sim}{0}^{\mu_\ell^c}) = \chi(\underset{\sim}{Y}^{\mu_i}, \underset{\sim}{0}^{\mu_i^c}) > \chi(\underset{\sim}{Y}^{\mu_h}, \underset{\sim}{0}^{\mu_h^c})$$

for all $h \neq i$. Now

$$\chi(\underset{\sim}{Y}^{\mu_i},\underset{\sim}{0}^{\mu_i^c}) = \chi[\gamma_{i1}(\underset{\sim}{X}^{A_{i1}\cap T},\underset{\sim}{0}^{A_{i1}\cap T^c}),\ldots,\gamma_{iM_i}(\underset{\sim}{X}^{A_{iM_i}\cap T},\underset{\sim}{0}^{A_{iM_i}\cap T^c}),\underset{\sim}{0}^{\mu_i^c}]$$

$$= \gamma(\underset{\sim}{X}^{B_i\cap T},\underset{\sim}{0}^{B_i^c\cup T^c})$$

$$= \gamma(\underset{\sim}{X}^{T},\underset{\sim}{0}^{T^c}) \text{ since } T \subset B_i$$

$$= \gamma(\underset{\sim}{X}^{B_j\cap T},\underset{\sim}{0}^{B_j^c\cup T^c}) \text{ since } T \subset B_j$$

$$= [\gamma_{j1}(\underset{\sim}{X}^{A_{j1}\cap T},\underset{\sim}{0}^{A_{j1}\cap T^c}),\ldots,\gamma_{jM_j}(\underset{\sim}{X}^{A_{jM_j}\cap T}),\underset{\sim}{0}^{A_{jM_j}\cap T^c}),\underset{\sim}{0}^{\mu_j^c}]$$

since μ_j is a minimal path set of χ

$$= \chi(\underset{\sim}{Y}^{\mu_j},\underset{\sim}{0}^{\mu_j^c}),$$

thereby contradicting the assumption that $\chi(\underset{\sim}{Y}^{\mu_i},\underset{\sim}{0}^{\mu_i^c}) > \chi(\underset{\sim}{Y}^{\mu_j},\underset{\sim}{0}^{\mu_j^c})$. It follows that T is a subset of precisely one of $B_1,\ldots,B_p$. $\square$

This completes the proof of part (i). The proof of part (ii) is similar, and is omitted.

<u>Theorem 3.2</u>

Suppose that γ is a CSF with modular decomposition $\{\chi,(A_1,\gamma_1),\ldots,(A_N,\gamma_N)\}$ and that $\underset{\sim}{X}$ is a vector of associated random variables. Let $Y_j = \gamma_j(\underset{\sim}{X}^{A_j})$ for $j=1,2,\ldots,N$.

(i) If γ and χ are both right-continuous and upper simple with minimal path sets $T_1,\ldots,T_r$ and $\mu_1,\ldots,\mu_p$ respectively,

$$\max_{1\le i\le r} P\{\gamma(\underset{\sim}{X}^{T_i},\underset{\sim}{0}^{T_i^c}) \ge x\} \le \max_{1\le i\le p} P\{\chi(\underset{\sim}{Y}^{\mu_i},\underset{\sim}{0}^{\mu_i^c}) \ge x\} \le P\{\gamma(\underset{\sim}{X}) \ge x\} \le$$

$$\coprod_{i=1}^{p} P\{\chi(\underset{\sim}{Y}^{\mu_i},\underset{\sim}{0}^{\mu_i^c}) \ge x\} \le \coprod_{i=1}^{r} P\{\gamma(\underset{\sim}{X}^{T_i},\underset{\sim}{0}^{T_i^c}) \ge x\}.$$

(ii) If γ and χ are both left-continuous and lower simple with minimal cut sets $S_1,\dots,S_t$ and $\nu_1,\dots,\nu_k$ respectively,

$$\prod_{j=1}^{t} P\{\gamma(\underset{\sim}{X}^{S_j},\underset{\sim}{1}^{S_j^c}) > x\} \le \prod_{j=1}^{k} P\{\chi(\underset{\sim}{Y}^{\nu_j},\underset{\sim}{1}^{\nu_j^c}) > x\} \le P\{\gamma(\underset{\sim}{X}) > x\} \le$$

$$\min_{1\le j\le k} P\{\chi(\underset{\sim}{Y}^{\nu_j},\underset{\sim}{1}^{\nu_j^c}) > x\} \le \min_{1\le j\le t} P\{\gamma(\underset{\sim}{X}^{S_j},\underset{\sim}{1}^{S_j^c}) > x\}.$$

<u>Proof</u>: (i) That $P\{\gamma(\underset{\sim}{X}) \ge x\} \ge \max\limits_{1\le i\le p} P\{\chi(\underset{\sim}{Y}^{\mu_i},\underset{\sim}{0}^{\mu_i^c}) \le x\}$ follows immediately from the modular decomposition of γ and from Theorem 2.5, so it is sufficient to verify that the assumption of a modular decomposition yields an improved bound.

Since $\chi(\underset{\sim}{Y}^{\mu_i},\underset{\sim}{0}^{\mu_i^c}) = \gamma(\underset{\sim}{X}^{B_i},\underset{\sim}{0}^{B_i^c})$ on Ω and, by Proposition 3.1, there exists at least one minimal path set of γ which is contained in B_i, it follows from the Decomposition Theorem that

$$\gamma(\underset{\sim}{X}^{B_i},\underset{\sim}{0}^{B_i^c}) = \max_{\{T \mid T\subset B_i\}} \gamma(\underset{\sim}{X}^{T},\underset{\sim}{0}^{T^c}) \text{ on } \Omega. \tag{3.1}$$

Thus $\gamma(\underset{\sim}{X}^{B_i},\underset{\sim}{0}^{B_i^c}) \ge \gamma(\underset{\sim}{X}^{T},\underset{\sim}{0}^{T^c})$ on Ω for each $T \subset B_i$ and so $P\{\chi(\underset{\sim}{Y}^{\mu_i},\underset{\sim}{0}^{\mu_i^c}) \ge x\} \ge P\{\gamma(\underset{\sim}{X}^{T},\underset{\sim}{0}^{T^c}) \ge x\}$ for each $T \subset B_i$. Hence,

$$P\{\chi(\underset{\sim}{Y}^{\mu_i},\underset{\sim}{0}^{\mu_i^c}) \ge x\} \ge \max_{\{T \mid T\subset B_i\}} P\{\gamma(\underset{\sim}{X}^{T},\underset{\sim}{0}^{T^c}) \ge x\}$$

for i=1,2,...,p, from which it follows that

$$\max_{1\le i\le p} P\{\chi(\underset{\sim}{Y}^{\mu_i},\underset{\sim}{0}^{\mu_i^c}) \ge x\} \ge \max_{1\le i\le r} P\{\gamma(\underset{\sim}{X}^{T_i},\underset{\sim}{0}^{T_i^c}) \ge x\}$$

as claimed.

It follows from the modular decomposition of γ and from Theorem 2.5 that $P\{\gamma(\underset{\sim}{X}) \ge x\} \ge \coprod_{i=1}^{p} P\{\chi(\underset{\sim}{Y}^{\mu_i},\underset{\sim}{0}^{\mu_i^c}) \ge x\}$ and so it is again sufficient to show that the assumption of a modular decomposition leads to a uniformly improved bound.

From (3.1), we see that

$$\chi(\underset{\sim}{Y}^{\mu_i},\underset{\sim}{0}^{\mu_i^c}) = \max_{\{T|T\subset B_i\}} \gamma(\underset{\sim}{X}^{T},\underset{\sim}{0}^{T^c}) \text{ on } \Omega$$

for i=1,2,...,p, and so

$$\coprod_{i=1}^{p} P\{\chi(\underset{\sim}{Y}^{\mu_i},\underset{\sim}{0}^{\mu_i^c}) \ge x\} = \coprod_{i=1}^{p} P\{\max_{\{T|T\subset B_i\}} \gamma(\underset{\sim}{X}^{T},\underset{\sim}{0}^{T^c}) \ge x\}$$

$$\le \coprod_{i=1}^{p} \coprod_{\{\ell|T_\ell\subset B_i\}} P\{\gamma(\underset{\sim}{X}^{T_\ell},\underset{\sim}{0}^{T_\ell^c}) \ge x\}$$

on appealing to Proposition 2.4 and performing some manipulations

$$= \coprod_{i=1}^{r} P\{\gamma(\underset{\sim}{X}^{T_i},\underset{\sim}{0}^{T_i^c}) \ge x\}$$

from Proposition 3.1.

(ii) This is proved by similar arguments. $\square$

Remark 3.3

If γ is simple and continuous, both sets of bounds hold. Except for the fact that the assumption of a modular decomposition leads to an improved bound, none of the upper (lower) bounds majorises any of the others; counterexamples are easily constructed.

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