AD-A148 620 MODULES OF CONTINUUM STRUCTURES REVISED(U) STATE UNIV OF NEW YORK AT STONY BROOK DEPT OF APPLIED MATHEMATICS AND STATISTICS L A BAXTER ET AL. JUN 84 1/1

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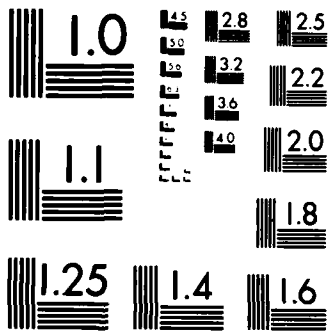

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REPORT DOCUMENTATION PAGE

1a. REPORT SECURITY CLASSIFICATION: UNCLASSIFIED	1b. RESTRICTIVE MARKINGS
2a. SECURITY CLASSIFICATION AUTHORITY	3. DISTRIBUTION/AVAILABILITY OF REPORT: Approved for public release; distribution unlimited.
2b. DECLASSIFICATION/DOWNGRADING SCHEDULE	
4. PERFORMING ORGANIZATION REPORT NUMBER(S)	5. MONITORING ORGANIZATION REPORT NUMBER(S): AFOSR-TR- 84-1083
6a. NAME OF PERFORMING ORGANIZATION: State University of New York at Stony Brook; 6b. OFFICE SYMBOL (If applicable)	7a. NAME OF MONITORING ORGANIZATION: Air Force Office of Scientific Research
6c. ADDRESS (City, State and ZIP Code): Department of Applied Mathematics and Statistics, Stony Brook NY 11794	7b. ADDRESS (City, State and ZIP Code): Directorate of Mathematical & Information Sciences, Bolling AFB DC 20332-6448
8a. NAME OF FUNDING/SPONSORING ORGANIZATION: AFOSR; 8b. OFFICE SYMBOL (If applicable): NM	9. PROCUREMENT INSTRUMENT IDENTIFICATION NUMBER: AFOSR-84-0243
8c. ADDRESS (City, State and ZIP Code): Bolling AFB DC 20332-6448	10. SOURCE OF FUNDING NOS. PROGRAM ELEMENT NO.: 61102F; PROJECT NO.: 2304; TASK NO.: A5; WORK UNIT NO.

11. TITLE (Include Security Classification): MODULES OF CONTINUUM STRUCTURES

12. PERSONAL AUTHOR(S): Laurence A. Baxter and Chul Kim

13a. TYPE OF REPORT	13b. TIME COVERED	14. DATE OF REPORT (Yr., Mo., Day)	15. PAGE COUNT
Technical	FROM ____ TO ____	JUN 84	19

16. SUPPLEMENTARY NOTATION

17. COSATI CODES			18. SUBJECT TERMS (Continue on reverse if necessary and identify by block number)
FIELD	GROUP	SUB. GR.	

DTIC ELECTE DEC 14 1984 E

19. ABSTRACT (Continue on reverse if necessary and identify by block number)

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20. DISTRIBUTION/AVAILABILITY OF ABSTRACT: UNCLASSIFIED/UNLIMITED ☒ SAME AS RPT. ☐ DTIC USERS ☐	21. ABSTRACT SECURITY CLASSIFICATION: UNCLASSIFIED
22a. NAME OF RESPONSIBLE INDIVIDUAL: MAJ Brian W. Woodruff	22b. TELEPHONE NUMBER (Include Area Code): (202) 767- 5027; 22c. OFFICE SYMBOL: NM

DD FORM 1473, 83 APR — EDITION OF 1 JAN 73 IS OBSOLETE.

UNCLASSIFIED
SECURITY CLASSIFICATION OF THIS PAGE

ITEM #19, ABSTRACT, CONTINUED:

χ such that $\gamma(\underset{\sim}{X}) = \chi[\gamma_1(\underset{\sim}{X}^A), \underset{\sim}{X}^{A^c}]$ for all $\underset{\sim}{X}$. A minimal path set of γ is, essentially, a subset of C which is necessary and sufficient for the CSF to attain any value in its image when every other component is in state zero.

Using these concepts, the main results of Birnbaum and Esary's theory of modules of binary structure functions, in particular the Three Modules Theorem, generalize to the continuum case.

AFOSR-TR- 84-1083

Revised Version

MODULES OF CONTINUUM STRUCTURES*

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Invited paper prepared for presentation at the
Conference on Reliability Theory and Quality Control
University of Missouri-Columbia, June 4-8, 1984.

*Research supported by the National Science Foundation under grant ECS-8306871 and, in part, by the Air Force Office of Scientific Research under grant AFOSR-84-0243.

ABSTRACT

A continuum structure function ~~(CSF) is a~~ nondecreasing mapping from the unit hypercube to the unit interval. Such a function, γ say, is said to be weakly coherent if $\sup[\gamma(1_i,\underset{\sim}{X})-\gamma(0_i,\underset{\sim}{X})]>0$ for each component $i \in C$. Suppose that γ is weakly coherent and that $A \subseteq C$ is nonempty. Then (A,γ_1) is a module of (C,γ) if γ_1 is weakly coherent and if there exists a CSF χ such that $\gamma(\underset{\sim}{X}) = \chi[\gamma_1(\underset{\sim}{X}^A),\underset{\sim}{X}^{A^c}]$ for all $\underset{\sim}{X}$. A minimal path set of γ is, essentially, a subset of C which is necessary and sufficient for the CSF to attain any value in its image when every other component is in state zero.

Using these concepts, the main results of Birnbaum and Esary's theory of modules of binary structure functions, in particular the Three Modules Theorem, generalize to the continuum case.

1. INTRODUCTION

A continuum structure function (CSF) on the unit hypercube is a nondecreasing mapping $\gamma: [0,1]^n \mapsto [0,1]$. The dimension of the domain space is $|C|$, the cardinality of $C = \{1,2,\ldots,n\}$, a set of components, and is assumed to be finite. It is supposed, without any loss of generality, that $\gamma(\underset{\sim}{0}) = 0$ and $\gamma(\underset{\sim}{1}) = 1$ where $\underset{\sim}{\alpha}$ denotes $(\alpha,\ldots,\alpha)$. Definitions of various types of component relevancy are given by Baxter (1984b); only the following will be required here.

Definition

A CSF γ is said to be weakly coherent if $\sup_{\underset{\sim}{X}\in\Delta}[\gamma(1_i,\underset{\sim}{X})-\gamma(0_i,\underset{\sim}{X})] > 0$ for each $i \in C$.

In the above, $(\delta_i,\underset{\sim}{X})$ denotes $(X_1,\ldots,X_{i-1},\delta,X_{i+1},\ldots,X_n) \in \Delta = [0,1]^n$. See Baxter (1984a,b) and Block and Savits (1984) for further details of CSFs.

The purpose of the present paper is to describe a theory of modules of CSFs analogous to that of modules of binary coherent structures (Birnbaum and Esary, 1965). Indeed, our main results are extensions of those of Birnbaum and Esary (1965), though our approach is rather different; in particular, a continuum version of ϕ-equivalence is not required.

It will be necessary to introduce the notion of minimal path sets of CSFs, subsets of C which are necessary and sufficient for a CSF to attain any value in its image when the state of every other component is zero. In order to do so, however, it is necessary to restrict the class of CSFs considered so that the minimal path sets have certain desirable properties. Hence, before proceeding to a definition of modules, it is convenient to define and study minimal path sets of upper simple CSFs.

2. MINIMAL PATH SETS OF UPPER SIMPLE CSFs

A minimal vector to level $\alpha \in \text{Im}\,\gamma-\{0\}$ is a vector $\underset{\sim}{X} \in \Delta$ such that $\gamma(\underset{\sim}{X}) = \alpha$ whereas $\gamma(\underset{\sim}{Y}) < \alpha$ for all $\underset{\sim}{Y} < \underset{\sim}{X}$. (Im γ denotes the image of γ and $\underset{\sim}{Y} < \underset{\sim}{X}$ means that $Y_i \leq X_i$ for $i=1,2,\ldots,n$ but $\underset{\sim}{Y} \neq \underset{\sim}{X}$.) Let P_α denote the set of all minimal vectors to level α. (Notice that our definition of P_α differs from that of Block and Savits (1984) as we require that $\alpha \in \text{Im}\,\gamma$.)

A path set to level $\alpha \in \text{Im}\,\gamma-\{0\}$ is a nonempty set $T_\alpha = T_\alpha(\underset{\sim}{X}) = \{i \in C \mid X_i \neq 0\}$ where $\underset{\sim}{X} \in P_\alpha$.

Definition

Let T be a nonempty subset of C. If T is a path set to level α for all $\alpha \in \text{Im}\,\gamma-\{0\}$, then T is a minimal path set of γ.

Remarks

1. Replacing Δ and γ by $\{0,1\}^n$ and ϕ, a binary coherent structure function, respectively, in this definition yields the minimal path sets of ϕ, and hence the above definition is a direct generalization of the definition of the minimal path sets of ϕ.

2. There is an analogous definition of minimal cut sets of CSFs, but this concept will not be needed here.

Minimal path sets do not necessarily exist for arbitrary CSFs and, if they do exist, they may exhibit undesirable properties. The following definition yields a large class of CSFs for which minimal path sets exist and are "well-behaved".

Notation

$A \subseteq B$ means that A is a subset of B; $A \subset B$ means that A is a proper subset of B; $\underset{\sim}{X}^P$ denotes $\{X_i | i \in P \subseteq C\}$.

Definition

A CSF γ is <u>upper simple</u> if it satisfies the following four conditions.

C1: $P_1 \neq \emptyset$ and $P_1 \subseteq \{0,1\}^n - \{\underset{\sim}{0}\}$

C2: $\bigcup_{i=1}^{r} T_{1i} = C$ where $T_{11}, \ldots, T_{1r}$ are the r path sets of γ to level 1.

C3: If T is a path set to level $\alpha \in \text{Im}\,\gamma - \{0\}$, then T is also a path set to level $\beta \in \text{Im}\,\gamma - \{0\}$ for all $\beta \leq \alpha$.

C4: If T_α is a path set to level $\alpha < 1$, then $T_\alpha \subseteq T_{1i}$ for some path set T_{1i} to level 1.

Condition C1 asserts that, if a component is necessary to ensure that the system is fully operational, that component must itself be fully operational. Conditions C2 and C3 state, respectively, that each component is required in at least one of the path sets to level 1 and that, if a set of components is sufficient to attain a given level, it must also be sufficient to attain any lower level. Condition C4 is, essentially, a regularity condition.

Examples of upper simple CSFs include the CSFs

$$\zeta(\underset{\sim}{X}) = \max_{1 \le r \le p} \min_{i \in P_r} X_i$$

and

$$\eta(\underset{\sim}{X}) = \max_{1 \le r \le p} \prod_{i \in P_r} X_i$$

(Baxter, 1984a), where $P_1, \ldots, P_p$ denote the minimal path sets of ϕ, a binary coherent structure function, and

$$\mu(\underset{\sim}{X}) = \frac{1}{n} \sum_{i=1}^{n} X_i .$$

Some properties of upper simple CSFs are given in the following proposition.

Proposition 2.1

Let γ be an upper simple CSF. Then

(i) γ has at least one minimal path set

(ii) γ is weakly coherent

(iii) No proper subset of a minimal path set is itself a minimal path set.

(iv) $\bigcup_{i=1}^{r} T_i = C$ where $T_1,\ldots,T_r$ denote the r minimal path sets of γ.

Proof: (i) This is immediate from C1 and C3.

(ii) By C1, there exist path sets to level 1, $T_{11},\ldots,T_{1r}$ say, such that $\gamma(\underset{\sim}{1}^{T_{1i}},\underset{\sim}{0}^{T_{1i}^c}) = 1$ whereas $\gamma(\underset{\sim}{Y}^{T_{1i}},\underset{\sim}{0}^{T_{1i}^c}) < 1$ for all $\underset{\sim}{Y}^{T_{1i}} < \underset{\sim}{1}^{T_{1i}}$ $(i=1,2,\ldots,r)$. Thus, for each $j \in T_{1i}$, $\gamma(1_j,(\underset{\sim}{1}^{T_{1i}-\{j\}},\underset{\sim}{0}^{T_{1i}^c})) = 1$ whereas $\gamma(0_j,(\underset{\sim}{1}^{T_{1i}-\{j\}},\underset{\sim}{0}^{T_{1i}^c})) < 1$. This holds for each $j \in C$ by C2, and hence γ is weakly coherent.

(iii) Suppose that T_1 and T_2 are minimal path sets of γ such that $T_1 \subset T_2$. Clearly, T_1 and T_2 are both path sets to level 1. By C1, $\gamma(\underset{\sim}{1}^{T_1},\underset{\sim}{0}^{T_1^c}) = 1$ whereas $\gamma(\underset{\sim}{Y}^{T_1},\underset{\sim}{0}^{T_1^c}) < 1$ for all $\underset{\sim}{Y}^{T_1} < \underset{\sim}{1}^{T_1}$, and $\gamma(\underset{\sim}{1}^{T_2},\underset{\sim}{0}^{T_2^c}) = 1$ whereas $\gamma(\underset{\sim}{Z}^{T_2},\underset{\sim}{0}^{T_2^c}) < 1$ for all $\underset{\sim}{Z}^{T_2} < \underset{\sim}{1}^{T_2}$. Since, however, $T_1 \subset T_2$, we have $(\underset{\sim}{1}^{T_1},\underset{\sim}{0}^{T_1^c}) < (\underset{\sim}{1}^{T_2},\underset{\sim}{0}^{T_2^c})$, thereby contradicting the minimality of T_2. Thus $T_1 \not\subset T_2$.

(iv) This is immediate from C2 and C3. □

The following decomposition will subsequently prove useful. It can be viewed as a generalization of formula (3.2) of Barlow and Proschan (1975, p. 10).

Theorem 2.2 (Decomposition Theorem)

Let γ be a right-continuous, upper simple CSF with minimal path sets $T_1,\ldots,T_r$. Then

$$\gamma(\underset{\sim}{X}) = \max_{1\leq i\leq r} \gamma(\underset{\sim}{X}^{T_i},\underset{\sim}{0}^{T_i^c}).$$

Proof: The result is trivially true for $r = 1$, so suppose that $r \geq 2$.

By the monotonicity of γ, $\gamma(\underset{\sim}{X}) \geq \gamma(\underset{\sim}{X}^{T_i},\underset{\sim}{0}^{T_i^c})$ for $i = 1,2,\ldots r$ and hence $\gamma(\underset{\sim}{X}) \geq \max_{1\leq i\leq r} \gamma(\underset{\sim}{X}^{T_i},\underset{\sim}{0}^{T_i^c})$. Suppose that

$\gamma(\underset{\sim}{X}) > \max_{1\leq i\leq r} \gamma(\underset{\sim}{X}^{T_i},\underset{\sim}{0}^{T_i^c})$ for some $\underset{\sim}{X} \in \Delta$.

Consider firstly the case where $\underset{\sim}{X} \in P_\alpha$ (observe that, by Proposition 2.1(i), $P_\alpha \neq \emptyset$ for all $\alpha \in \text{Im}\ \gamma$). If $X_j \neq 0$ for all $j \in T_i$ $(1\leq i\leq r)$, then, since $\underset{\sim}{X} > (\underset{\sim}{X}^{T_i},\underset{\sim}{0}^{T_i^c})$, we must have $X_k \neq 0$ for some $k \in T_i^c$ and hence there exists a path set, T_α say, to level α such that $T_\alpha \supset T_i$, in contradiction to C4. If $X_j = 0$ for some $j \in T_i$ $(1\leq i\leq r)$, then, again, $X_k \neq 0$ for some $k \in T_i^c$ and hence there exists a path set, T_α say, to level α such that $T_\alpha \not\subseteq T_i$ and $T_\alpha \not\supset T_i$, again a contradiction to C4. Thus equality holds for all minimal vectors.

Suppose, now, that $\gamma(\underset{\sim}{X}) = \alpha$, but that $\underset{\sim}{X} \notin P_\alpha$. Since γ is right-continuous, $\underset{\sim}{X} > \underset{\sim}{Y} \in P_\alpha$ (Block and Savits, 1984) and so

$$\gamma(\underset{\sim}{Y}) = \max_{1 \le i \le r} \gamma(\underset{\sim}{Y}^{T_i}, \underset{\sim}{0}^{T_i^c}) = \gamma(\underset{\sim}{Y}^{T_m}, \underset{\sim}{0}^{T_m^c}) \text{ (say) } > \gamma(\underset{\sim}{X}^{T_m}, \underset{\sim}{0}^{T_m^c}),$$

thereby contradicting the monotonicity of γ.

It follows that $\gamma(\underset{\sim}{X}) = \max_{1 \le i \le r} \gamma(\underset{\sim}{X}^{T_i}, \underset{\sim}{0}^{T_i^c})$ as claimed. □

3. MODULES AND THEIR MINIMAL PATH SETS

We start by defining modules and modular sets of CSFs.

Definition

Suppose that γ is a weakly coherent CSF and that $A \subseteq C$ is nonempty. Suppose, further, that there exists a weakly coherent CSF γ_1: $[0,1]^{|A|} \mapsto [0,1]$ and a CSF χ: $[0,1]^{n-|A|+1} \mapsto [0,1]$ such that $\gamma(\underset{\sim}{X}) = \chi[\gamma_1(\underset{\sim}{X}^A), \underset{\sim}{X}^{A^c}]$ for all $\underset{\sim}{X} \in \Delta$. Then (A,γ_1) is a <u>module</u> of (C,γ) and A is a <u>modular set</u> of (C,γ).

In this section, we present three of our main results. We prove that, if (A,γ_1) is a module of (C,γ), and if T is a minimal path set of the upper simple CSF γ, then $A \cap T$ is a minimal path set of γ_1, thereby generalizing Theorem 4.1 of Birnbaum and Esary (1965) to the continuum case. We also prove a partial converse to that theorem: a condition on the minimal path sets of γ under which $A \subseteq C$ is a modular set of (C,γ). The third result is that if (A,γ_1) is a module of (C,γ), and if γ and γ_1 are both upper simple, then $(A \cap T) \cup (A^c \cap T')$ is a minimal path set of γ whenever T and T' are minimal path sets of γ which intersect A; this generalizes part of the Birnbaum-Esary Test for Modularity.

Remark

It is easily seen that χ is weakly coherent.

Examples

1. For the CSF μ, any nonempty subset of C is a modular set.
2. For the CSF ζ, any modular set of (C,ϕ) is also a modular set of (C,ζ).
3. For the CSF η, any minimal path set of ϕ which does not intersect any of the other minimal path sets is a modular set.

Theorem 3.1

Let γ be an upper simple CSF with minimal path sets $T_1,\ldots,T_r$. Suppose that (A,γ_1) is a module of (C,γ) and that $A \cap T_j \neq \emptyset$ for $j = 1,2,\ldots,k$ whereas $A \cap T_j = \emptyset$ for $j = k+1,\ldots,r$. Suppose further that γ_1 is upper simple. Then the minimal path sets of γ_1 are $A \cap T_1,\ldots,A \cap T_k$.

<u>Proof</u>: If $k = 1$, the result is trivial, so suppose that $k \geq 2$.

Firstly, we show that $A \cap T_j$ $(1 \leq j \leq k)$ is a minimal path set of γ_1. Since γ_1 is upper simple, it is sufficient to verify that $A \cap T_j$ is a path set of γ_1 to level 1.

Suppose that $\gamma_1(\underset{\sim}{1}^{A\cap T_j}, \underset{\sim}{0}^{A\cap T_j^c}) = \beta < 1$. If $\beta = 0$, then, using the definitions of T_j and (A,γ_1), we see that $\gamma_1(\underset{\sim}{1}^{A\cap T_j}, \underset{\sim}{0}^{A\cap T_j^c}) = 0 = \gamma_1(\underset{\sim}{0}^A)$ and thus $\gamma(\underset{\sim}{1}^{T_j}, \underset{\sim}{0}^{T_j^c}) =$ $\chi[\gamma_1(\underset{\sim}{0}^A), \underset{\sim}{1}^{A^c\cap T_j}, \underset{\sim}{0}^{A^c\cap T_j^c}] = \gamma(\underset{\sim}{1}^{A^c\cap T_j}, \underset{\sim}{0}^{A\cup T_j^c}) = 1$, thereby contradicting the assumption that $(\underset{\sim}{1}^{T_j}, \underset{\sim}{0}^{T_j^c}) \in P_1$. Suppose, now, that $0 < \beta < 1$. Since γ_1 is upper simple, it possesses a minimal path set, W say. Let $(\underset{\sim}{V}^W, \underset{\sim}{0}^{A-W})$ be a minimal vector of γ_1 to level β. By the definitions of T_j and (A,γ_1), $\chi[\gamma_1(\underset{\sim}{V}^W, \underset{\sim}{0}^{A-W}), \underset{\sim}{1}^{A^c\cap T_j}, \underset{\sim}{0}^{A^c\cap T_j^c}] = 1$ whereas $\chi(y, \underset{\sim}{Y}^{A^c\cap T_j}, \underset{\sim}{0}^{A^c\cap T_j^c}) < 1$ for all $(y, \underset{\sim}{Y}^{A^c\cap T_j}, \underset{\sim}{0}^{A^c\cap T_j^c}) <$ $(\gamma_1(\underset{\sim}{V}^W, \underset{\sim}{0}^{A-W}), \underset{\sim}{1}^{A^c\cap T_j}, \underset{\sim}{0}^{A^c\cap T_j^c})$. Thus, $\gamma(\underset{\sim}{V}^W, \underset{\sim}{0}^{A-W}, \underset{\sim}{1}^{A^c\cap T_j}, \underset{\sim}{0}^{A^c\cap T_j^c}) = 1$ whereas $\gamma(\underset{\sim}{Z}^W, \underset{\sim}{0}^{A-W}, \underset{\sim}{Y}^{A^c\cap T_j}, \underset{\sim}{0}^{A^c\cap T_j^c}) < 1$ for all $(\underset{\sim}{Z}^W, \underset{\sim}{Y}^{A^c\cap T_j}) <$ $(\underset{\sim}{V}^W, \underset{\sim}{1}^{A^c\cap T_j})$, i.e. $(\underset{\sim}{V}^W, \underset{\sim}{0}^{A-W}, \underset{\sim}{1}^{A^c\cap T_j}, \underset{\sim}{0}^{A^c\cap T_j^c}) \in P_1$. It is clear, however, that $\underset{\sim}{0} < \underset{\sim}{V}^W < \underset{\sim}{1}^W$, and hence we have a contradiction to C1. It follows that $\gamma(\underset{\sim}{1}^{A\cap T_j}, \underset{\sim}{0}^{A\cap T_j^c}) = 1$.

Suppose that there exists a vector $\underset{\sim}{Y}^{A\cap T_j} < \underset{\sim}{1}^{A\cap T_j}$ such that $\gamma_1(\underset{\sim}{Y}^{A\cap T_j}, \underset{\sim}{0}^{A\cap T_j^c}) = 1$. Then, since (A,γ_1) is a module of (C,γ),

$$\gamma(\underset{\sim}{1}^{T_j}, \underset{\sim}{0}^{T_j^c}) = \chi[\gamma_1(\underset{\sim}{1}^{A\cap T_j}, \underset{\sim}{0}^{A\cap T_j^c}), \underset{\sim}{1}^{A^c\cap T_j}, \underset{\sim}{0}^{A^c\cap T_j^c}]$$

$$= \chi[\gamma_1(\underset{\sim}{Y}^{A\cap T_j}, \underset{\sim}{0}^{A\cap T_j^c}), \underset{\sim}{1}^{A^c\cap T_j}, \underset{\sim}{0}^{A^c\cap T_j^c}]$$

$$= \gamma((\underset{\sim}{Y}^{A\cap T_j}, \underset{\sim}{1}^{A^c\cap T_j}), \underset{\sim}{0}^{T_j^c})$$

$$= 1,$$

contradicting the assumption that $(\underset{\sim}{1}^{T_j}, \underset{\sim}{0}^{T_j^c}) \in P_1$.

It follows that $A\cap T_1, \ldots, A\cap T_k$ are minimal path sets of γ_1 as claimed.

Suppose, now, that there exists a minimal path set T of γ_1 such that $T \neq A\cap T_j$ for $j=1,2,\ldots,k$. An argument similar to that of the first part of the proof shows that $T \cup (A^c \cap T_j)$ is a minimal path set of γ for $j=1,2,\ldots,k$ and is hence equal to one of $T_1, \ldots, T_r$. Since, however, T intersects at least two of $A\cap T_1, \ldots, A\cap T_k$, $T_\ell \cap [T\cup(A^c\cap T_j)] = (T_\ell \cap T) \cup (T_\ell \cap A^c \cap T_j) \subset (T_\ell \cap A) \cup (T_\ell \cap A^c) = T_\ell$ for all $j=1,2,\ldots,k$, $\ell=1,2,\ldots,r$. This is a contradiction, and hence T must be of the form $A \cap T_j$ $(1 \leq j \leq k)$. □

Theorem 3.1 generalizes Theorem 4.1 of Birnbaum and Esary (1965). These authors do not prove a converse to their theorem, and a converse to our own result is not immediate. The following theorem gives a condition under which a nonempty subset of C is a modular set; the proof is a construction of the "natural" CSF induced by the restriction γ to that subset and by the minimal path sets of γ.

Theorem 3.2

Let γ be a right-continuous, upper simple CSF with minimal path sets $T_1,\ldots,T_r$. Suppose that A is a nonempty subset of C such that $T_j \subseteq A$ for $j=1,2,\ldots,k$ whereas $A\cap T_j = \emptyset$ for $j=k+1,\ldots,r$. Then there exists a weakly coherent CSF γ_A: $[0,1]^{|A|} \to [0,1]$ such that (A,γ_A) is a module of (C,γ).

Proof: By the Decomposition Theorem,

$$(3.1)\qquad \gamma(\underset{\sim}{X}) = \max\left\{\max_{1\le j\le k} \gamma(\underset{\sim}{X}^{T_j},\underset{\sim}{0}^{T_j^c}),\ \max_{k+1\le j\le r} \gamma(\underset{\sim}{X}^{T_j},\underset{\sim}{0}^{T_j^c})\right\}$$

for all $\underset{\sim}{X} \in \Delta$. Since $T_j = A\cap T_j$ and $T_j^c = (A\cap T_j^c)\cup A^c$, it follows that $\gamma(\underset{\sim}{X}^{T_j},\underset{\sim}{0}^{T_j^c}) = \gamma(\underset{\sim}{X}^{A\cap T_j},\underset{\sim}{0}^{A\cap T_j^c},\underset{\sim}{0}^{A^c})$ for $j=1,2,\ldots,k$.

Define the CSF γ_A: $[0,1]^{|A|} \to [0,1]$ by

$$(3.2)\qquad \gamma_A(\underset{\sim}{X}^A) = \gamma(\underset{\sim}{X}^A,\underset{\sim}{0}^{A^c}).$$

From (3.1) and (3.2),

$$(3.3)\qquad \gamma(\underset{\sim}{X}) = \max\left\{\max_{1\le j\le k} \gamma_A(\underset{\sim}{X}^{A\cap T_j},\underset{\sim}{0}^{A\cap T_j^c}),\ \max_{k+1\le j\le r} \gamma(\underset{\sim}{X}^{T_j},\underset{\sim}{0}^{T_j^c})\right\}.$$

Clearly, γ_A is right-continuous. We show that γ_A is upper simple and that its minimal path sets are $A\cap T_1,\ldots,A\cap T_k$.

Since T_j $(1\le j\le k)$ is a minimal path set of γ, $\gamma(\underset{\sim}{1}^{T_j},\underset{\sim}{0}^{T_j^c}) = \gamma_A(\underset{\sim}{1}^{A\cap T_j},\underset{\sim}{0}^{A\cap T_j^c}) = 1$ whereas $\gamma(\underset{\sim}{Y}^{T_j},\underset{\sim}{0}^{T_j^c}) = \gamma_A(\underset{\sim}{Y}^{A\cap T_j},\underset{\sim}{0}^{A\cap T_j^c}) < 1$ for all $\underset{\sim}{Y}^{T_j} < \underset{\sim}{1}^{T_j}$, i.e. for all $\underset{\sim}{Y}^{A\cap T_j} < \underset{\sim}{1}^{A\cap T_j}$. Thus $A\cap T_j$ is a path set of γ_A to level 1, and it is easily seen that there are no path sets of γ_A to level 1 other than $A\cap T_1,\ldots,A\cap T_k$. Further, it is obvious that γ_A satisfies C1 and, since $A\cap T_j = \emptyset$ for $j = k+1,\ldots,r$,

$$\bigcup_{j=1}^{k}(A\cap T_j) = \bigcup_{j=1}^{r}(A\cap T_j) = A\cap\bigcup_{j=1}^{r}T_j = A,$$

so that γ_A satisfies C2.

<u>Claim</u>: Let T be a nonempty subset of C such that $A\cap T \neq \emptyset$. Then T is a path set of γ to level $\alpha \leq 1$ if and only if $T \subseteq A$ and $A\cap T$ is a path set of γ_A to level α.

<u>Proof of Claim</u>: Suppose that T is a path set of γ to level $\alpha \leq 1$. We show that $T \subseteq A$ or, equivalently, that $A^c \cap T = \emptyset$. Suppose, conversely, that $A^c\cap T \neq \emptyset$; since T is a path set to level α, there exists an $\underset{\sim}{X} \in P_\alpha$ such that

$$\gamma(\underset{\sim}{X}^T,\underset{\sim}{0}^{T^c}) = \max_{1\leq j\leq r}\gamma(\underset{\sim}{X}^{T_j\cap T},\underset{\sim}{0}^{T_j^c\cup T^c}) = \gamma(\underset{\sim}{X}^{T_m\cap T},\underset{\sim}{0}^{T_m^c\cup T^c}),$$

say. Since $A \cap T$ and $A^c \cap T$ are both nonempty, $T_m \cap T \subset T$, thereby contradicting the minimality of $(\underset{\sim}{X}^T,\underset{\sim}{0}^{T^c})$. Thus $A^c \cap T = \emptyset$ as claimed, and it is obvious that $A \cap T$ is a path set of γ_A to level .

The proof of the converse is straightforward and is omitted. □

The claim enables us easily to verify that γ_A satisfies C3 and C4. Suppose that T is a path set of γ to level α and that $A \cap T \neq \emptyset$. Then, by the claim, $A \cap T$ is a path set of γ_A to level α. Since T is a path set of γ to level β for all $\beta \leq \alpha$, it follows that $A \cap T$ is a path set of γ_A to level β for all $\beta \leq \alpha$. Further, if T is a path set of γ to level α, then, by C4, $T \subseteq T_j$ for some $j=1,2,\ldots,k$. Thus, by the claim, $A \cap T$ is a path set of γ_A to level α and $A \cap T \subseteq A \cap T_j$.

In summary, γ_A is right-continuous and upper simple, and its minimal path sets are $A \cap T_1, \ldots, A \cap T_k$. Thus, by the Decomposition Theorem and (3.3),

$$\gamma(\underset{\sim}{X}) = \max\left\{\gamma_A(\underset{\sim}{X}^A),\ \max_{k+1 \leq j \leq r} \gamma(\underset{\sim}{X}^{T_j}, \underset{\sim}{0}^{T_j^c})\right\}.$$

Define the CSF χ: $[0,1]^{n-|A|+1} \to [0,1]$ by

$$\chi(y, \underset{\sim}{X}^{A^c}) = \max\left\{y,\ \max_{k+1 \leq j \leq r} \gamma(\underset{\sim}{X}^{T_j}, \underset{\sim}{0}^{T_j^c})\right\}. \tag{3.4}$$

The CSFs γ and γ_A are upper simple and hence weakly coherent. Thus, from (3.2) and (3.4), $\gamma(\underset{\sim}{X}) = \chi[\gamma_A(\underset{\sim}{X}^A), \underset{\sim}{X}^{A^c}]$ for all $\underset{\sim}{X} \in \Delta$ and so (A, γ_A) is a module of (C, γ). □

Theorem 3.3

Let γ be an upper simple CSF with minimal path sets $T_1, \ldots, T_r$. Suppose that (A, γ_1) is a module of (C, γ) and that $A \cap T_j \neq \emptyset$ for $j=1,2,\ldots,k$ whereas $A \cap T_j = \emptyset$ for $j=k+1,\ldots,r$. Suppose, further, that γ_1 is upper simple. Then $(A \cap T_j) \cup (A^c \cap T_\ell)$ is a minimal path set of γ for $j, \ell = 1,2,\ldots,k$.

Proof: Since γ is upper simple, it is sufficient to show that $(A \cap T_j) \cup (A^c \cap T_\ell)$ is a path set of γ to level 1, and this follows easily from the definitions. □

This theorem generalizes part of the Birnbaum-Esary Test for Modularity. Birnbaum and Esary (1965) also show that, if $(A \cap T_j) \cup (A^c \cap T_\ell)$ is a minimal path set of a binary coherent structure function, then A is a modular set of that structure. We have not yet been able to determine conditions on γ under which a converse to our theorem holds.

4. THE THREE MODULES THEOREM

The main result of Birnbaum and Esary (1965) is their Three Modules Theorem which asserts that, if $A_1 \cup A_2$ and $A_2 \cup A_3$ are modular sets of a binary coherent structure, then so are A_1, A_2, A_3 and $A_1 \cup A_2 \cup A_3$. The following theorem extends this result to upper simple CSFs. Since, as previously noted, a converse to Theorem 3.3 has not yet been established, the following additional condition will be assumed.

C5 Suppose that γ is upper simple with minimal path sets $T_1,\ldots,T_r$. Suppose, further, that A is a nonempty subset of C such that $A \cap T_j \neq \emptyset$ for $j=1,2,\ldots,k$ whereas $A \cap T_j = \emptyset$ for $j=k+1,\ldots,r$. If $(A \cap T_j) \cup (A^c \cap T_\ell)$ is a minimal path set of γ for all $j,\ell=1,2,\ldots,k$, then A is a modular set of (C,γ) and the associated CSF is upper simple.

Theorem 4.1 (Three Modules Theorem)

Let γ be an upper simple CSF which satisfies C5. Suppose that A_1, A_2 and A_3 are disjoint, nonempty subsets of C such that $A_1 \cup A_2$ and $A_2 \cup A_3$ are modular sets of (C,γ) and that the associated CSFs are upper simple. Then A_1, A_2, A_3, $A_1 \cup A_3$ and $A_1 \cup A_2 \cup A_3$ are all modular sets of (C,γ). Further, those minimal path sets of γ, $T_1,\ldots,T_r$ say, which intersect $A_1 \cup A_2 \cup A_3$ all intersect each of A_1, A_2 and A_3, or else they all intersect exactly one of these sets.

Proof: Define the following subcollections of $T_1,\ldots,T_r$:

$B_1 = \{T | A_1 \cap T_i \neq \emptyset,\ A_2 \cap T_i = \emptyset,\ A_3 \cap T_i = \emptyset \text{ for some } i\}$

$B_2 = \{T | A_1 \cap T_j = \emptyset,\ A_2 \cap T_j \neq \emptyset,\ A_3 \cap T_j = \emptyset \text{ for some } j\}$

$B_3 = \{T | A_1 \cap T_k = \emptyset,\ A_2 \cap T_k = \emptyset,\ A_3 \cap T_k \neq \emptyset \text{ for some } k\}$

$B_4 = \{T | A_1 \cap T_\ell \neq \emptyset,\ A_2 \cap T_\ell \neq \emptyset,\ A_3 \cap T_\ell = \emptyset \text{ for some } \ell\}$

$B_5 = \{T | A_1 \cap T_m \neq \emptyset,\ A_2 \cap T_m = \emptyset,\ A_3 \cap T_m \neq \emptyset \text{ for some } m\}$

$B_6 = \{T | A_1 \cap T_v = \emptyset,\ A_2 \cap T_v \neq \emptyset,\ A_3 \cap T_v \neq \emptyset \text{ for some } v\}$

$B_7 = \{T | A_1 \cap T_w \neq \emptyset,\ A_2 \cap T_w \neq \emptyset,\ A_3 \cap T_w \neq \emptyset \text{ for some } w\}$.

Clearly, $B_1,\ldots,B_7$ form a partition of $\bigcup_{h=1}^{r} T_h$.

The first stage of the proof is to show that B_4, B_5 and B_6 are all empty.

Suppose that B_4 is not empty. Then, since A_3 is, by hypothesis, nonempty, there exists a minimal path set, T say, such that $A_3 \cap T \neq \emptyset$, and so $B_3 \cup B_5 \cup B_6 \cup B_7 \neq \emptyset$. We show that this latter event, in conjunction with the event $B_4 \neq \emptyset$, leads to a contradiction.

Suppose, firstly, that $B_3 \neq \emptyset$. Then, since B_3 and B_4 are nonempty, there exist distinct minimal path sets $T_k \in B_3$ and $T_\ell \in B_4$ both of which intersect the modular set $A_2 \cup A_3$. Noting that $(A_2 \cup A_3) \cap T_\ell = A_2 \cap T_\ell$, it follows from Theorem 3.3 that $(A_2 \cap T_\ell) \cup ((A_2 \cap A_3)^c \cap T_k)$ is a minimal path set of γ. This assertion is not, however, true: consider, for example, the case $A_1 \cup A_2 \cup A_3 = C$. Then $(A_2 \cup A_3)^c \cap T_k = \emptyset$ and $A_2 \cap T_\ell$ is not a minimal path set of γ since $A_2 \cap T_\ell \subset T_\ell$.

By similar arguments, it can be shown that B_4 and B_5 cannot simultaneously be nonempty; neither can B_4 and B_6, nor B_4 and B_7. Thus, if $B_4 \neq \emptyset$, then $B_3 \cup B_5 \cup B_6 \cup B_7 = \emptyset$ and A_3 does not intersect any minimal path set of γ. This is clearly a contradiction, and hence we must have $B_4 = \emptyset$. By similar arguments, B_5 and B_6 are also empty.

The same arguments show that, if $B_1 \cup B_2 \cup B_3 \neq \emptyset$, then $B_7 = \emptyset$ and, conversely, that if $B_7 \neq \emptyset$, then $B_1 \cup B_2 \cup B_3 = \emptyset$.

<u>Case I</u>: $B_1 \cup B_2 \cup B_3 = \bigcup_{h=1}^{r} T_h$.

From the definitions of B_1, B_2 and B_3, it follows that each of $T_1, \ldots, T_r$ intersects exactly one of A_1, A_2 or A_3.

To show that A_1 is a modular set, observe that, since $A_1 \cup A_2$ is a modular set, it follows from Theorem 3.3 that $T = [(A_1 \cup A_2) \cap T_{i_1}] \cup [(A_1 \cup A_2)^c \cap T_{i_2}]$ is a minimal path set of γ where $T_{i_1}, T_{i_2} \in B_1$. Let $A_4 = C - (A_1 \cup A_2 \cup A_3)$. Then

$T = (A_1 \cap T_{i_1}) \cup (A_4 \cap T_{i_2}) = (A_1 \cap T_{i_1}) \cup (A_1^c \cap T_{i_2})$. Since, by assumption, γ satisfies C5, it follows that A_1 is a modular set of (C,γ).

Similarly, it can also be shown that A_2, A_3, $A_1 \cup A_3$ and $A_1 \cup A_2 \cup A_3$ are also modular sets of (C,γ).

<u>Case II</u>: $B_7 = \bigcup_{h=1}^{r} T_h$

From the definition of B_7, each of $T_1,\ldots,T_r$ intersects A_1, A_2 and A_3.

To show that A_1 is a modular set, observe that, since $A_1 \cup A_2$ is a modular set of (C,γ), it follows from Theorem 3.3 that $T = [(A_1 \cup A_2) \cap T_{w_1}] \cup [(A_1 \cup A_2)^c \cap T_{w_2}]$ is a minimal path set of γ where T_{w_1}, $T_{w_2} \in B_7$. Further, again from Theorem 3.3, since $A_2 \cup A_3$ is a modular set of (C,γ), $[(A_2 \cup A_3) \cap T_{w_2}] \cup [(A_2 \cup A_3)^c \cap T]$ is a minimal path set of γ. The latter term reduces to $(A_1 \cap T_{w_1}) \cup (A_1^c \cap T_{w_2})$, from which it follows (by C5) that A_1 is a modular set of (C,γ).

Similarly, A_2, A_3, $A_1 \cup A_3$ and $A_1 \cup A_2 \cup A_3$ are also modular sets of γ.

This completes the proof. □

Remarks

1. Birnbaum and Esary (1965), in the statemenet of their Three Modules Theorem, do not explicitly assume that A_1, A_2 and A_3 are mutually disjoint. That this is a necessary requirement is easily seen. Consider, for example, the binary structure function $\phi(\underset{\sim}{x}) = x_1 \wedge (x_2 \vee x_3) \wedge x_4$ where $\underset{\sim}{x} \in \{0,1\}^4$ and let $A_1 = \{1,2\}$, $A_2 = \{2,3\}$ and $A_3 = \{3,4\}$. Clearly, $A_1 \cup A_2$ and $A_2 \cup A_3$ are modular sets, but neither A_1 nor A_3 is a modular set of $(\{1,2,3,4\},\phi)$.

2. The Three Modules Theorem can be interpreted as follows: a sufficient condition for the union of two modular sets to be a modular set is that the two sets have a nonempty intersection, in which case the intersection, the union, the two differences and the symmetric difference of these sets are also modular sets.

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