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On Modeling the Performance and Reliability of Multi-Mode Computer Systems

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On Modeling the Performance and Reliability of Multi-Mode Computer Systems

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Abstract

We present an effective technique for the combined performance and reliability analysis of multi-mode computer systems. A reward rate (or a performance level) is associated with each mode of operation. The switching between different modes is characterized by a continuous time Markov chain. Different types of service-interruption interactions (as a result of mode switching) are considered. We consider the execution time of a given job on such a system and derive the distribution of its completion time. A useful dual relationship, between the completion time of a given job and the accumulated reward up to a given time, is noted. We demonstrate the use of our technique by means of a simple example.
1. Introduction

We consider a model for the combined evaluation of performance and reliability of a multi-mode computer system. Performance (e.g., throughput, response time, instruction execution rate) changes from mode to mode and a mode change occurs in response to an event such as a failure or a repair. The stochastic process representing the modes (structure-states) and mode changes can be thought of as a reward process by associating a reward (performance index) with each mode [4,10]. We can then study the distribution of the accumulated reward until time \( t \) by time domain methods [10] or by transform techniques [4,14].

The authors who have taken such a system-oriented view do not consider the effect of a fault occurring during the execution of a program. A task (job or program) oriented view of such a system recognizes the fact that it is possible for a system failure to occur before the completion of a task [7] and that even if the task is completed, its completion time is likely to be different from its execution time in a given mode [3,5,12]. The job in service is interrupted with each mode change and the type of service - interruption interaction depends upon the mode just entered. For example, the occurrence of a fault during the execution of a job preempts the job and a later system recovery may allow the job to resume from the point of interruption (the preemptive-resume (prs) discipline) or the job may have to be repeated from the beginning. In the latter case, the repeated job may have the identical work requirement as the original preempted job (the preemptive-repeat-identical (prid) discipline) or a different work requirement sampled from the same distribution (the preemptive-repeat-different (prdd) discipline).

The purpose of this paper is to develop a model that unifies and extends the efforts of these two groups of researchers. In particular, we show that if all interruptions are of the preemptive-resume type then the completion time of a given task and the accumulated reward until a given time are dual measures, so that the distribution
of one of them allows us to compute the distribution of the other. In fact, our model is even more general - in that both acyclic (closed or non-repairable) and cyclic (open or repairable) systems are modeled.

Our model provides an exact analysis of the completion time distribution of a program (job) executing in a multimode system. It is also possible to incorporate the effect of queueing in our model. If the time spent in each structure-state is large compared with the interarrival and processing times of jobs, then we can use steady-state performance measures as reward rates for each structure-state. Such approximate decomposition methods have been considered by several authors [4,7,10,15]. If the assumption of a wide separation between the structure-state holding times and job processing times does not hold, then a more complex analysis is required [1,5,12].

We develop the basic model in the next section. In sections 3, 4 and 5, we consider the individual cases where all structure-states are of the same type, that is, preemptive-resume, preemptive-repeat-identical, or preemptive-repeat-different, respectively.

2. The Basic Model

Consider a single server (e.g., a computer) serving a single job (e.g., a program). The job is characterized by its work requirement, $B$. For example, the work requirement of a computer program can be measured in terms of the number of instructions to be executed. We assume that $B$ is a random variable with cumulative distribution function $G(x) = P(B \leq x)$ and LST $L^T(s) = E(e^{-sB})$. To avoid trivialities we assume $G(0+) = 0$.

The rate at which the server performs work is assumed to change with time according to the following model: At any time the server is in one (and only one) of the $n+1$ states (modes) numbered $0,1,2,...,n$. In state $i$ the server performs work at rate $r_i > 0, 1 \leq i \leq n$, work units per unit time (e.g., the instruction execution rate). The state
0 is an absorbing "failure" state, i.e., once the server is in state 0, it stays there forever and the work rate in this state is zero \( r_0 = 0 \). We allow absorbing non-failure states among the states 1, ..., \( n \) with reward rates greater than zero so that if the server enters such a state, the job will eventually complete. Let \( Z(t) \) be the state of the server at time \( t \). \( \{Z(t), t \geq 0\} \) is called the structure-state process. We shall assume that the structure-state process is a stochastic process with piecewise constant paths with finite number of jumps in finite intervals of time. Furthermore, the structure-state process is assumed to be independent of the work requirement \( B \) of the job.

The states \( i = 1, 2, ..., n \) are classified as (i) \( prs \): preemptive-resume, (ii) \( pri \): preemptive-repeat-identical or (iii) \( prd \): preemptive-repeat-different.

The following quantities have been analyzed before in the literature for some special \( \{Z(t), t \geq 0\} \) processes:

I. The job completion time \( (T(z)) \): defined to be the total time the server takes to complete a job that requires \( z \) units of work. \( T \) denotes the unconditional completion time of a job that requires a random amount of work, say \( B \) units. Gaver\[5]\] studied the distribution of the r.v. \( T \) for a system subject to one type of failure and repair, in which the operating state is Markovian and the failure state is semi-Markovian. Nicola\[12]\] extended Gaver's model to allow for mixed types of failures and repairs. Castillo and Siewiorek \[3\] considered a system with two types of failures in which the preemptive-repeat type failure could occur during the repair-time of the preemptive-resume type failure.

II. The probability of dynamic failure \( (\eta) \): defined to be the probability that the system fails before the job is completed, i.e. the server enters state 0 before completing \( B \) units of work[7].

III. The cumulative reward upto time \( t \) \( (Y(t)) \): defined to be the total amount of work done by the system up to time \( t \). \( Y \) is the total accumulated work during the system's
lifetime; it is the limit of $Y(t)$ as $t \to \infty$. The r.v. $Y(t)$ was first studied by Puri [14] for Markovian $Z(t)$ processes. Meyer[10] and Donatiello and Iyer[4] studied the distribution of $Y(t)$ for an acyclic Markovian $Z(t)$ process. Beaudry[2] studied the r.v. $Y$ for a Markovian $Z(t)$ process, while Osaki and Nishio [13] studied the r.v. $Y$ for a semi-Markovian $Z(t)$ process.

To present a unifying view of the quantities defined above, we introduce the cumulative measure, $W(t)$, defined as follows: Suppose that at time $t = 0$ the server starts processing a job with infinite work requirement. $W(t)$ is the amount of useful work completed by the server until time $t$ (thus, excluding the work done prior to the last visit to a pri or a prd state). The following properties of the cumulative measure, $W(t)$, are immediately obvious:

(i) $W(0) = 0$.

(ii) $Z(t) = i \Rightarrow dW(t)/dt = r_i$.

(iii) If there is a transition in the structure-state process at time $t$ and $Z(t+) = i$, then $W(t+) = 0$ if $i$ is a pri or a prd state and $W(t+) = W(t-)$ if $i$ is a prs state.

Typical sample paths of the structure-state process and the cumulative measure, $W(t)$, are shown in figure 1, for the following case: Set of states = {0,1,2,3}, state 1 is prs with $r_1 = 1$, states 2 and 3 are pri or prd with $r_2 = 2$ and $r_3 = 0$, state 0 is the absorbing failure state.

The following theorem shows how the quantities $T$, $\eta$, $Y(t)$ and $Y$ can be related to each other via the cumulative measure, $W(t)$.

**Theorem 1.**

(i) $T = \min\{t \geq 0: W(t) = B\}$.

(ii) The dynamic failure probability, $\eta = P(T = \infty)$.

(iii) If all states are prs, then
\[ P(Y(t) \leq z) = 1 - P(T(x) < t) \]

and

\[ P(Y \leq x) = 1 - P(T(x) < \infty). \]

**Proof:** (i) Let \( T \) be the job completion time. It is clear that

\[ \{ T > t \} \iff \{ W(u) < B, \text{ for all } 0 \leq u \leq t \}, \]

since \( W(u) \) represents the useful work done up to time \( u \). As \( W(t) \) has piecewise continuous paths with only downward jumps, \( T \) is given by (i).

(ii) It is clear that

\[ \{ \text{Dynamic Failure} \} \iff \{ \text{system fails before job completion} \} \iff \{ W(t) < B \text{ for all } t \geq 0 \} \iff \{ T = \infty \}. \]

Hence \( \eta = P(T = \infty) \).

(iii) Let \( T(x) = \min \{ t \geq 0 : W(t) = x \} \). If all states are pros, then

\[ \{ Y(t) > x \} \iff \{ W(t) > x \} \iff \{ T(x) < t \}. \]

Hence

\[ P(Y(t) > x) = P(T(x) < t). \quad Q.E.D. \]

It is apparent from the above theorem that

\[ T = \min \{ t \geq 0 : W(t) = B \} \quad (2.1) \]

is the unifying random variable. This paper is devoted to the study of this random variable. Define the following distribution functions:

\[ F_i(t, x) = P(T \leq t \mid B = x, Z(0) = i), \quad x \geq 0, \quad 1 \leq i \leq n, \]

\[ F(t, x) = P(T \leq t \mid B = x), \quad x \geq 0, \]

\[ F_i(t) = P(T \leq t \mid Z(0) = i), \quad 1 \leq i \leq n. \]
\[ F(t) = P(T \leq t) \]

and the corresponding LSTs (Laplace Stieltjes Transforms).

\[ \begin{align*}
F_i^+(s,x) &= E(e^{-sT} | B = x, Z(0) = i), \quad x \geq 0, \quad 1 \leq i \leq n, \\
F^+(s,x) &= E(e^{-sT} | B = x), \quad x \geq 0, \\
F_i^-(s) &= E(e^{-sT} | Z(0) = i), \quad 1 \leq i \leq n, \\
F^-(s) &= E(e^{-sT}).
\end{align*} \]

(2.2) - (2.5)

From the independence of \( \{Z(t), t > 0\} \) and \( B \) it follows that

\[ F^+(s,x) = \sum_{i=1}^{n} F_i^+(s,x)P(Z(0) = i), \quad x \geq 0. \]

(2.6)

\[ F_i^-(s) = \int_0^s F_i^+(s,x)dG(x), \quad 1 \leq i \leq n. \]

(2.7)

\[ F^-(s) = \sum_{i=1}^{n} F_i^-(s)P(Z(0) = i). \]

(2.8)

From equations (2.6) - (2.8) it is clear that the conditional LSTs \( F_i^-(s,x) \) are of central importance to the analysis of \( T \). In order to obtain explicit formulae for \( F_i^-(s,x) \) it is necessary to make some further assumptions about the structure-state process. In the remaining paper we make the assumption that \( \{Z(t), t \geq 0\} \) is a time homogeneous continuous time Markov chain (CTMC). The results derived here can be extended in a straightforward manner to the case when the structure-state process is assumed to be semi-Markov. Let \( q_{ij}, 1 \leq i \neq j \leq n, \) be infinitesimal transition rate from state \( i \) to \( j \) and \( q_0 \) be the absorbing failure rate from state \( i \). Let \( Q = [q_{ij}], 1 \leq i, j \leq n, \) be the \( n \) by \( n \) generator matrix where \( q_i = \sum_{j \neq i} q_{ij} = -q_0. \) Note that row sums of \( Q \) are \( \leq 0. \) We mention one property of the CTMC for future reference. Define

\[ H = \min\{t \geq 0: Z(t) \neq Z(0)\} \]

(2.9)

as the holding (or sojourn) time in the initial state. Then we have

\(^{(*)}\) denotes LST, i.e., the Laplace transform of a probability density function.
\[ P(H \leq x, Z(H^+) = j | Z(0) = i) = \frac{q_j}{q_i} (1 - e^{-q_i x}) , \quad (i \neq j). \]  

(2.10)

In the next section we treat the case where all states \( i = 1, 2, \ldots, n \) are preemptive-resume (prs) and in sections 4 and 5 we consider the case where all states are preemptive-repeat (pri and prd, respectively). The mixed cases where some states are prs and some are pri or prd have been studied in [8].

3. The Preemptive-resume Case

In this section we assume that the states 1, 2, \ldots, n are all preemptive-resume states. Note that state 0 does not have to be classified since it is a failure state. Theorem 2 below gives a method of computing the conditional LSTs defined by equation (2.2). First, some notation:

\[ F_i^-(s, u) = \int_0^\infty e^{-sx} F_i^-(s, x) dx , \quad 1 \leq i \leq n. \]  

(3.1)

\[ E^-(s, u) = [F_1^-(s, u), F_2^-(s, u), \ldots, F_n^-(s, u)]^T, \]  

(3.2)

\[ R = \text{diag}[r_1, r_2, \ldots, r_n], \]  

(3.3)

\[ \pi = [r_1, r_2, \ldots, r_n]^T, \]  

(3.4)

where the superscript \( T \) denotes transpose.

**Theorem 2.** \( F_i^-(s, u) \), for \( 1 \leq i \leq n \), is given by

\[ F_i^-(s, u) = \frac{r_i}{s + r_i u + q_i} + \sum_{j \neq i} \frac{q_j}{s + r_j u + q_i} F_j^-(s, u), \quad 1 \leq i \leq n. \]  

(3.5)

**Proof:** Conditioning on the sojourn time \( H \) in the initial state we get

\[ E(e^{-sT} | H = h, B = x, Z(0) = i) = \begin{cases} e^{-s/h}, & \text{if } h \leq x / r_i \\ e^{-s/h} \sum_{j \neq i} \frac{q_j}{q_i} F_j^-(s, x - r_i h), & \text{if } h > x / r_i \end{cases} \]  

(*) denotes the Laplace transform of a function
Unconditioning yields

\[ F_i^{-}(s,x) = \int_0^\infty E(x_t \mid H = h, B = x, Z(0) = i) q_i e^{-\eta h} dh \]

\[ = e^{-(s+\eta)h_{t_i}} + \sum_{j \neq i} q_j \int_0^{x_t} e^{-(s+\eta)h} F_j^{-}(s,x - \eta h) dh \]

Multiplying both sides by \( e^{-\eta h} \) and integrating we get equation (3.5). Q.E.D.

Equation (3.5) can be put in a matrix form as follows:

\[ [sI + uR - Q] E^{-}(s,u) = x, \]

where \( I \) is the identity matrix. As it is well known that \([sI + uR - Q]\) is invertible, we get

\[ E^{-}(s,u) = [sI + uR - Q]^{-1} x. \] (3.5a)

A direct inversion with respect to \( s \) yields

\[ d_t E^{-}(t,u) = e^{(Q - uR)t} x. \]

After integration and some manipulations, we get

\[ E^{-}(t,u) = \frac{1}{u} [I - e^{(Q - uR)t}] x. \] (3.6)

We now describe how we can use the above theorem to compute \( F_i^{-}(s,x) \). Using Cramer's rule we can write

\[ F_i^{-}(s,u) = A_i(s,u) / C(s,u) \]

where \( C(s,u) = \det [sI + uR - Q] \) and \( A_i(s,u) \) are appropriate \( n \) by \( n \) subdeterminants of the augmented matrix \([sI + uR - Q; x]\). It is obvious that both \( A_i(s,u) \) and \( C(s,u) \) are polynomials in \( s \) and \( u \). Hence one can use partial fractions to invert \( F_i^{-}(s,u) \) with respect to \( u \). Let \( d = \{ t_i: t_i > 0 \} \), i.e. \( d \) is the number of states in which work rate is positive. Then \( C(s,u) \) is a \( d \)-degree polynomial in \( u \) for a fixed value of \( s \). Let \(-u_1(s), \ldots, -u_d(s)\) be the roots of \( C(s,u) \). In the special case when these roots are dis-
tinct, we can write
\[
F_i^{(s,u)} = \sum_{j=1}^{d} \frac{A_j(s)}{u + u_j(s)}, \quad 1 \leq i \leq n.
\]

where
\[
A_j(s) = \lim_{u \to u_j(s)} \frac{A(s,u)}{C(s,u)}(u + u_j(s)), \quad 1 \leq j \leq d.
\]

Inverting with respect to \( u \), we get
\[
F_i^{(s,u)} = \sum_{j=1}^{d} A_j(s) e^{-u_j(s)x}, \quad 1 \leq i \leq n.
\]

Hence from equation (2.4)
\[
F_i^{(s)} = \sum_{j=1}^{d} A_j(s) G(u_j(s)), \quad 1 \leq i \leq n.
\]

(recall that \( G(s) = \int_0^s e^{-x} dG(x) \), and
\[
F_i^{(s)} = \sum_{j=1}^{d} \left[ \sum_{i=1}^{n} \pi_i A_j(s) \right] G(u_j(s)).
\]

where \( \pi_i = P(Z(0) = i), 1 \leq i \leq n. \)

It is interesting to note that the LST of \( T \) for a given \( s \) is simply a linear combination of the LST of \( B \) evaluated at \( u_1(s), ..., u_d(s) \).

Now, assuming that state 0 is reachable from every other state, the probability of dynamic failure can be computed easily from Theorem 1 as
\[
\eta = P(T = \infty) = 1 - \lim_{s \to 0} F_i^{(s)}.
\]

The following corollary indicates how the LST of the cumulative reward \( Y(t) \), for a given \( t \), can be obtained from the \( F_i^{(s,u)} \) functions.

**Corollary 1.** For a given \( t \geq 0 \), let \( Y(t) \) be the cumulative reward up to time \( t \). Let
\[
Y_i(x,t) = P(Y(t) \leq x \mid Z(0) = i),
\]
\[ Y_t^*(u,t) = E(e^{-uY(t)} \mid Z(0) = i) \]

and

\[ Y_t^*(u,s) = \int_0^s e^{-st} Y_t(u,t) \, dt \]

Then

\[ Y_t^*(u,s) = \frac{1}{s} \left( 1 - u F_t^*(s,u) \right), \quad 1 \leq i \leq n. \quad (3.13) \]

**Proof:** Part (iii) of Theorem 1 implies that

\[ P(Y(t) < x \mid Z(0) = i) = P(T(x) > t \mid Z(0) = i). \]

Now,

\[ Y_t^*(u,s) = \int_0^s e^{-st} E(e^{-uY(t)} \mid Z(0) = i) \, dt \]

\[ = \int_0^s e^{-st} \int_{x=0} e^{-ux} \, dx \, P(Y(t) \leq x \mid Z(0) = i) \, dt \]

\[ = \int_0^s e^{-ux} \int_{x=0} e^{-st} \, dx \, P(Y(t) \leq x \mid Z(0) = i) \, dt \]

\[ = \int_{x=0} e^{-ux} \, dx \left[ \int_{t=0} e^{-st} [1 - P(T(x) \leq t \mid Z(0) = i)] \, dt \right] \]

\[ = - \int_{x=0} e^{-ux} \, dx \frac{F_t(s,x)}{s} = \left[ 1 - u F_t^*(s,u) \right] / s. \quad Q.E.D. \]

Using equation (3.5a), we can write in a matrix form

\[ Y^*(u,s) = [sI + uR - Q]^{-1} \mathbf{g} \quad (3.13a) \]

with

\[ Y^*(u,s) = [Y_1^*(u,s), Y_2^*(u,s), \ldots, Y_n^*(u,s)]^T. \]

A direct inversion with respect to \( s \) yields

\[ Y^*(u,t) = e^{(Q - uR)t} \mathbf{g}. \quad (3.14) \]
We end this section with a simple example.

Example 3.1. The switching server

Consider a system that operates in two modes each with a different work rate, say \( r_1 \) and \( r_2 \) for modes "1" and "2", respectively. The system switches between the two modes according to a Poisson process at different rates, say \( \lambda \) and \( \mu \) from modes "1" and "2", respectively. A total system failure may occur at any mode of operation at different rates, say \( \lambda_0 \) and \( \mu_0 \) for modes "1" and "2", respectively. The CTMC representing the switching server is shown in figure 2. In the case where a total system failure may not occur, i.e. \( \lambda_0 = \mu_0 = 0 \) and if \( r_2 = 0 \) then the switching server model reduces to the completion time model of job execution in a system subject to breakdowns and repairs considered by Gaver [5].

In this example we consider the case in which both states 1 and 2 are of the preemptive-resume type. We note that if we set \( \mu = 0 \) in this example we obtain the reward model of a two processor system considered by Meyer [10]. In our example the \( Q \) matrix is

\[
Q = \begin{bmatrix}
-\lambda' & \lambda \\
\mu & -\mu'
\end{bmatrix}
\]

where \( \lambda' = \lambda + \lambda_0 \) and \( \mu' = \mu + \mu_0 \). Then from Equation (3.5a)

\[
\begin{bmatrix}
F_{1r}(s,u) \\
F_{2r}(s,u)
\end{bmatrix} = \begin{bmatrix}
s + r_1 u + \lambda' & -\lambda \\
-\mu & s + r_2 u + \mu'
\end{bmatrix}^{-1} \begin{bmatrix}
r_1 \\
r_2
\end{bmatrix}
\]

Solving for \( F_{1r}(s,u) \) and \( F_{2r}(s,u) \) we get

\[
F_{1r}(s,u) = \frac{r_1 r_2 u + r_1 (s + \mu') + r_2 \lambda}{(s + \lambda') (s + \mu' + r_2 u) - \lambda \mu}
\]

\[
F_{2r}(s,u) = \frac{r_1 r_2 u + r_2 (s + \lambda') + r_1 \mu}{(s + \lambda') (s + \mu' + r_2 u) - \lambda \mu}
\]

Hence, using eq. (3.9) we get

\[
F_1(s,x) = A_{11}(s) \exp(-u_1(s)x) + A_{12}(s) \exp(-u_2(s)x),
\]
$F^*(s,x) = A_{81}(s) \exp(-u_1(s)x) + A_{82}(s) \exp(-u_2(s)x)$

where

$u_1(s) = \left[ \frac{r_1(s + \mu') + r_2(s + \lambda')}{\sqrt{(r_1(s + \mu') - r_2(s + \lambda'))^2 + 4\lambda\mu r_2}} \right] / (2r_1 r_2)$

$u_2(s) = \left[ \frac{r_1(s + \mu') + r_2(s + \lambda')}{\sqrt{(r_1(s + \mu') - r_2(s + \lambda'))^2 + 4\lambda\mu r_2}} \right] / (2r_1 r_2)$

$A_{11}(s) = \left[ \frac{r_1(s + \mu') + r_2\lambda - r_1 r_2 u_1(s)}{(u_2(s) - u_1(s)) r_1 r_2} \right]$

$A_{12}(s) = \left[ \frac{r_1(s + \mu') + r_2\lambda - r_1 r_2 u_2(s)}{(u_1(s) - u_2(s)) r_1 r_2} \right]$

$A_{21}(s) = \left[ \frac{r_2(s + \lambda') + r_1\mu - r_1 r_2 u_1(s)}{(u_2(s) - u_1(s)) r_1 r_2} \right]$

$A_{22}(s) = \left[ \frac{r_2(s + \lambda') + r_1\mu - r_1 r_2 u_2(s)}{(u_1(s) - u_2(s)) r_1 r_2} \right]$

Then

$F^*(s) = [\pi_1 A_{11}(s) + \pi_2 A_{21}(s)] G^*(u_1(s)) + [\pi_1 A_{12}(s) + \pi_2 A_{22}(s)] G^*(u_2(s))$

And $\eta$, the probability of dynamic failure, is given by $1 - F^*(0)$.

From corollary 1, we have

$Y_{1-\alpha}^{s}(u,s) = \frac{1}{s} \left[ 1 - u F_{1-\alpha}^{s}(s,u) \right]$

$= \frac{(s + \lambda')(s + \mu') + r_2 u(s + \lambda) - \lambda\mu}{(s + \lambda' + r_1 u)(s + \mu' + r_2 u) - \lambda\mu}$

$= \frac{B_{10}(u)}{s} + \frac{B_{11}(u)}{s + s_1(u)} + \frac{B_{12}(u)}{s + s_2(u)}$

where

$s_1(u) = \frac{n_1}{2} \left[ (\lambda' + r_1 u + \mu' + r_2 u) + \sqrt{(\lambda' + r_1 u + \mu' + r_2 u)^2 + 4\lambda\mu} \right]$

$s_2(u) = \frac{n_1}{2} \left[ (\lambda' + r_1 u + \mu' + r_2 u) - \sqrt{(\lambda' + r_1 u + \mu' + r_2 u)^2 + 4\lambda\mu} \right]$

$B_{10}(u) = \frac{\lambda' + r_2 u - \lambda\mu}{s_2(u) s_1(u)}$

$B_{11}(u) = \frac{(\lambda' - s_1(u)) (\mu' - s_1(u))^2 + r_2 u (\lambda' - s_1(u)) - \lambda\mu}{s_1(u) [s_1(u) - s_2(u)]}$

$B_{12}(u) = \frac{(\lambda' - s_2(u)) (\mu' - s_2(u))^2 + r_2 u (\lambda' - s_2(u)) - \lambda\mu}{s_2(u) [s_2(u) - s_1(u)]}$
Inverseing with respect to $s$, yields

$$Y_1(u,t) = B_{10}(u) + B_{11}(u)e^{-s_1(u)t} + B_{12}(u)e^{-s_2(u)t}.$$ 

In a similar manner we can compute $Y_2(u,t)$. We note that the above LSTs can be inverted in this case to obtain the distribution function of $Y(t)$ as an infinite sum of Bessel functions owing to the occurrences of radicals in the expressions of $s_1(u)$ and $s_2(u)$. However, in the case that $\mu=0$ (as considered by Meyer), the radicals disappear and the inversion is relatively easy (as derived in [4] for arbitrary number of processors).

4. The Preemptive-repeat-identical Case

In this section we assume that all states are preemptive repeat-identical. The main result is given in the following:

**Theorem 3.** The conditional LSTs $F_i^- (s,x)$, 1 ≤ $i$ ≤ n as defined in equation (2.2) satisfy the following simultaneous equations:

$$F_i^-(s,x) = e^{-s_1(x)} + \sum_{j \neq i}^{n} \frac{\gamma_j}{s_1(x) + s_j} (1-e^{-s_1(x)/\gamma_j} F_j^-(s,x)), 1 \leq i \leq n. \quad (4.1)$$

**Proof.** Conditioning on the holding time $H$ in the initial state we have

$$E(s^{-x}|H=h, B=x, Z(0)=i) = \begin{cases} e^{-x/\gamma_i}, & \text{if } h \geq x/\gamma_i \\ e^{-x} \sum_{j \neq i}^{n} \frac{\gamma_j}{\gamma_i} F_j^-(s,x), & \text{if } h < x/\gamma_i \end{cases}$$

Unconditioning yields equation (4.1). Q.E.D.

Solving equations (4.1) we get $F_i^-(s,x)$, for 1 ≤ $i$ ≤ n. Then equations (2.7) and (2.8) can be used to compute $F^-(s)$. Finally, $\eta = 1 - F^-(0)$.

**Example 4.1.** Consider the switching server of example 3.1, except now we assume that
states 1 and 2 are preemptive-repeat-identical.

Equations (4.1) become:

\[(s+\lambda')F^{-}_{1}(s,x) = (s+\lambda)e^{-\left[(s+\lambda)x/r_{1}\right]} + \lambda\left(1-e^{-\left[(s+\lambda)x/r_{1}\right]}\right)F^{-}_{2}(s,x)\]

\[(s+\mu')F^{-}_{2}(s,x) = (s+\mu)e^{-\left[(s+\mu)x/r_{2}\right]} + \mu\left(1-e^{-\left[(s+\mu)x/r_{2}\right]}\right)F^{-}_{1}(s,x).\]

Solving the above equations we get

\[F^{-}_{1}(s,x) = (s+\mu)[s+\lambda']a + \lambda(1-a)b]/\Delta\]

\[F^{-}_{2}(s,x) = (s+\lambda)[s+\mu']b + \mu(1-b)a]/\Delta\]

where \(a = \exp\left(-\left[(s+\lambda)x/r_{1}\right]\right)\), \(b = \exp\left(-\left[(s+\mu)x/r_{2}\right]\right)\) and

\[\Delta = (s+\lambda')(s+\mu') - \lambda\mu(1-a)(1-b).\]

\(F^{-}_{i}(s)\), for \(i = 1,2\) and \(F^{-}(s)\) can be obtained from equations (2.7) and (2.8).

5. The Preemptive-repeat-different Case

Here we consider the case, where all structure-states of the process are preemptive-repeat-different (prd).

The following theorem holds

**Theorem 4.** The LSTs \(F^{-}_{i}(s)\), for \(1 \leq i \leq n\), as defined in equation (2.4) satisfy the following simultaneous equations

\[F^{-}_{i}(s) = G^{-}\left((s+q_{i})/r_{i}\right) + \sum_{j=1}^{n} \frac{q_{j}}{(s+q_{i})}\left[1-G^{-}\left((s+q_{j})/r_{i}\right)\right]F^{-}_{j}(s), \ 1 \leq i \leq n. \quad (5.1)\]

Note that when \(r_{i} \to 0\), \(G^{-}\left((s+q_{i})/r_{i}\right) \to 0\), since \(G(0+) = 0\) and hence \(\lim_{s \to \infty}G^{-}(s) \to 0\).

**Proof:** Conditioning on the work requirement \(B\) of the job to be executed and on the holding time \(H\) in the initial state we get
Note that if a structure state transition occurs before the job is completed then a different job with independent and identical distribution is restarted.

Now, unconditioning on \( B \) (the job's work requirement) yields

\[
E(e^{-rT} | B=Z, Z(0)=t) = \begin{cases} 
    e^{-h/r_i} & \text{if } h \geq z/r_i \\
    e^{-h} \sum_{j=1}^{r_i} q_j f_j^{-1}(z) & \text{if } h < z/r_i 
\end{cases}
\]

Unconditioning on \( H \) (the holding time in the initial state), yields equation (5.1).

Q.E.D.

Solving equations (5.1) we get \( F_i^{-}(s) \), for \( 1 \leq i \leq n \). Equation (2.8) can be used to get \( F^{-}(s) \). The dynamic failure probability \( \eta \) follows immediately

\[
\eta = P(T=\infty) = 1 - F^{-}(0).
\]

Note that the preemptive-repeat-different case with a constant (or deterministic) work requirement of a job \( (B=x) \) corresponds to the preemptive-repeat-identical case.

**Example 5.1.** Again we consider the switching server of example 3.1 with the states 1 and 2 being preemptive-repeat-different. From equations (5.1) we have

\[
F_i^{-}(s) = G^{-}((s+\lambda')/r_i) + \frac{\lambda}{(s+\lambda)}[1 - G^{-}((s+\lambda)/r_i)]F_i^{-}(s)
\]

\[
F_2^{-}(s) = G^{-}((s+\mu')/r_2) + \frac{\mu}{(s+\mu')}[1 - G^{-}((s+\mu')/r_2)]F_1^{-}(s)
\]

It follows that

\[
F_1^{-}(s) = \frac{G^{-}((s+\lambda)/r_i) + (\frac{\lambda}{s+\lambda})(1 - G^{-}((s+\lambda)/r_i))G^{-}((s+\mu')/r_2)}{[1 - (\frac{\lambda}{s+\lambda})(\frac{\mu}{s+\mu'})(1 - G^{-}((s+\lambda)/r_i))(1 - G^{-}((s+\mu')/r_2))]} \]
\[ F_g(s) = \frac{G((s+\mu')/r) + \left(\frac{\lambda}{s+\lambda'}\right) (1-G((s+\mu')/r)) G((s+\lambda')/r)}{[1 - \left(\frac{\lambda}{s+\lambda'}\right) (1-G((s+\lambda')/r)) (1-G((s+\mu')/r))]} \]

\( F^-(s) \) can be obtained from equation (2.8).

6. Conclusions and Extensions

We have developed a unified model for the combined evaluation of performance and reliability of multi-mode computer systems. This allows us to compute both system-oriented measures (such as the accumulated reward) and task-oriented measures (such as the completion time and the dynamic failure probability) from a single model. We model preemptive-resume and preemptive-repeat interactions between task execution and mode change (failure/repair) events. It is clearly of interest to allow mixed preemptive-resume and preemptive-repeat interactions in the same model. This and other extensions have been studied and reported recently [8]. The techniques developed in this paper can be extended to the case where the structure-state process is a semi-Markov process.

References


Typical Sample Paths of $Z(t)$ and $W(t)$

Figure 1
Figure 2

The Switching Server
A direct inversion with respect to $s$ yields

$$\mathcal{Y}(u,t) = e^{(q-uR)t} \mathcal{Q}.$$  \hspace{1cm} (3.14)
Hence, using eq. (3.9) we get

\[ F_1^+(s,x) = A_{11}(s)e^{u_1(s)x} + A_{12}(s)e^{u_2(s)x}, \]
\[
B_{11}(u) = \frac{(\lambda - s_1(u))(\mu - s_1(u)) + r_0 u (\lambda_0 - s_1(u)) - \lambda \mu}{s_1(u)[s_1(u) - s_2(u)]}
\]

\[
B_{12}(u) = \frac{(\lambda - s_2(u))(\mu - s_2(u)) + r_0 u (\lambda_0 - s_2(u)) - \lambda \mu}{s_2(u)[s_2(u) - s_1(u)]}
\]
Solving equations (4.1) we get $F_i^*(s,x)$, for $1 \leq i \leq n$. Then equations (2.7) and (2.8) can be used to compute $F^*(s)$. Finally, $\eta = 1 - F^*(0)$.

Example 4.1. Consider the switching server of example 3.1, except now we assume that...
holding time \( H \) in the initial state we get