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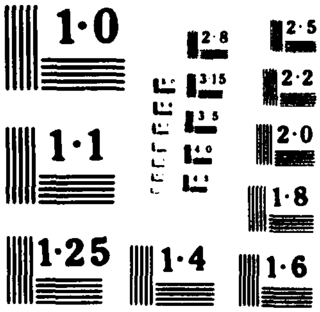
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Minimax Multiple t-tests for Comparing k Normal Populations with a Control*

by

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Population, with a Control

theta sub i >= theta sub 0

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sigma squared

theta

ABSTRACT

Let $\pi_1, \ldots, \pi_k$ be k normal populations with unknown means $\theta_1, \ldots, \theta_k$, and a common unknown variance $\sigma^2 > 0$. Based on independent samples of sizes $n_1, \ldots, n_k$, the populations are to be partitioned into two sets, where the first one contains all π_i with $\theta_i \geq \theta_0$, and where the other one contains the rest. At first it is assumed that θ_0 is known. Under an additive "a_i-b_i" loss function a minimax procedure is derived which is of a simple natural form. The proof of minimaxity makes use of the Bayes approach and involves a sequence of nonsymmetric priors, which play a similar role as a least favorable prior in simpler problems. Analogous results are presented for the case that θ_0 is not known. In this case, a control normal population is assumed to exist from which an additional sample of size n_0 can be drawn.

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1. Introduction.

Let $\pi_1 = N(\theta_1,\sigma^2),\ldots,\pi_k = N(\theta_k,\sigma^2)$ be k normal populations with unknown means $\theta_1,\ldots,\theta_k$ and a common unknown variance σ^2. A population π_i is considered to be "good" if $\theta_i \geq \theta_0$, and to be "bad" if $\theta_i < \theta_0$, $i = 1,\ldots,k$. The control value θ_0 may either be known or unknown, where in the latter case a control population $\pi_0 = N(\theta_0,\sigma^2)$ is assumed to be also available. The purpose of this paper is to derive statistical procedures which partition the k populations into "good" and "bad" ones, respectively, under the minimax criterion.

Let $X_i = (X_{i1},\ldots,X_{in_i})$ be a random sample from π_i, $i = 1,\ldots,k$. If θ_0 is unknown, let $X_0 = (X_{01},\ldots,X_{0n_0})$ be an additional sample from the control population π_0. All samples are assumed to be mutually independent. For notational convenience, let $X = (X_1,\ldots,X_k)$ if θ_0 is known, and let $X = (X_0,X_1,\ldots,X_k)$ if θ_0 is unknown. In either case, a multiple decision rule can be represented in the form $\delta = (\delta_1,\ldots,\delta_k)$, where, after having observed $X = x$, $\delta_i(x)$ denotes the probability of deciding that "π_i is good", $i = 1,\ldots,k$. Let $\mathcal{D}$ denote the class of all such rules which are Borel-measurable.

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For a decision theoretic treatment of the problem a loss function has to be specified. Assume that in each ith component problem a nonnegative loss $a_i(b_i)$ occurs if π_i is "bad" ("good"), but wrongly classified as "good" ("bad"), and that no loss occurs if the classification is correct, $i = 1,\ldots,k$. The overall loss is then assumed to be the total sum of these k losses. Formally, the loss function is thus of the form

$$(1) \qquad L(\theta,d) = \sum_{\substack{i=1\\ d_i=1}}^{k} a_i I_{(-\infty,\theta_0)}(\theta_i) + \sum_{\substack{i=1\\ d_i=0}}^{k} b_i I_{[\theta_0,\infty)}(\theta_i),$$

where $\theta \in \mathbb{R}^{k+1}$, $d \in \{0,1\}^k$, and $d_i = 0(1)$ stands for the decision that π_i is "bad" ("good"), $i = 1,\ldots,k$.

For the case of θ_0 known, let δ^* be the following rule.

$$(2) \qquad \delta_i^*(X) = 0(1) \text{ iff } n_i^{1/2}(\bar{X}_i-\theta_0)/S < (>)\ c_i,$$

where S^2 is the usual unbiased pooled sample estimator of σ^2 and c_i is the lower $a_i(a_i+b_i)^{-1}$ quantile of a t-distribution with $n_1+\ldots+n_k-k$ degrees of freedom, $i = 1,\ldots,k$.

Analogously, for the case of θ_0 unknown, let δ^{**} be given by

$$(3) \qquad \delta_i^{**}(X) = 0(1) \text{ iff } (n_0^{-1}+n_i^{-1})^{-1/2}(\bar{X}_i-\bar{X}_0)/S < (>)\ c_i,$$

where S^2 is now derived from $(\underline{X}_0,\underline{X}_1,\ldots,\underline{X}_k)$, and c_i is the lower $a_i(a_i+b_i)^{-1}$ quantile of a t-distribution with $n_0+n_1+\ldots+n_k-k-1$ degrees of freedom, $i = 1,\ldots,k$.

The main results to be proved below will confirm that these two procedures are minimax for their associated cases.

The problem of comparing k normal populations with a control has been considered by many authors. To mention a few of the earlier papers, Paulson (1952), Dunnett (1955), Gupta and Sobel (1958), and Tong (1969) have proposed and studied some natural procedures. Lehmann (1961) and Spjøtvoll (1972) have treated the problem with methods from the theory of testing hypotheses. Randles and Hollander (1971) and Miescke (1981) have derived optimal procedures under the Γ-minimax approach. An overview of this area of research can be found in Gupta and Panchapakesan (1979).

In many of the papers dealing with multiple comparisons with a control, the so-called indifference zones have been adopted, which means that wrong decisions with respect to parameters sufficiently close to θ_0 do not result in any loss. Thereby, intervals around θ_0 have to be specified which, together with certain other parameters to be chosen by the experimenter, make the proposed procedures look somewhat complicated.

Our approach to the problem may be more appealing to the experimenter because of its simplicity. There are only k pairs of losses to be chosen to determine the respective minimax procedure: $(a_1,b_1),\ldots,(a_k,b_k)$. These losses have a quite natural interpretation which facilitates the experimenter's choice of them. For each $i = 1,\ldots,k$, the ratio of a_i and b_i represents the relative importance of avoiding the two types of possible errors in the ith component decision problem. After these k ratios are determined, each pair may still be multiplied by an individual factor. These k factors may then be chosen in a way to reflect the relative importance of avoiding errors in the k component decisions.

The method used in this paper to prove minimaxity of δ^* and δ^{**} for their respective cases is an asymptotic extension of the standard method, where a procedure is found to be minimax if it is Bayes rule with respect to a least favorable prior. After two technical lemmas are proved in Section 2, minimaxity of δ^* in the case of a known σ_0 will be proved in Section 3, and the analogous result for δ^{**} will be derived in Section 4.

2. Two technical lemmas.

These are two main steps in the proof of minimaxity of δ^* which will be used later. Since they are common for both cases, where σ_0 is known or unknown, they are presented in this section to avoid repetitions. Also, one may get a fairly clear idea about the proofs to come by just looking at the two lemmas given below.

The first result holds in fact more generally for all k-decision problems under additive loss. It has been proved in the Γ-minimax approach in Miescke (1981). By allowing Γ to consist of all priors, it can be used also in the minimax approach. For convenience, let us state it below in a form suitable for the present context.

Lemma 1. A decision rule $\delta^M \in \mathcal{D}$ is minimax if there exists a sequence of priors $p_m(\theta,q)$, $\theta \in \mathbb{R}^k$, $q = \sigma^{-2} > 0$, $m = 1,2,\ldots$ such that for every $i \in \{1,\ldots,k\}$ the following holds true: For the ith component problem there exist Bayes rules δ_{im}^B with respect to $\tilde{p}_m$, $m \in \mathbb{N}$, such that

$$\text{(4)} \qquad \sup\{R^{(i)}((\theta,q), \delta_i^M) \mid \theta \in \mathbb{R}^k,\ q > 0\}$$

$$\le \liminf_{m\to\infty} r^{(i)}(\tilde{p}_m, \delta_{im}^B).$$

where $R^{(i)}$ and $r^{(i)}$ denote the risk function and the Bayes risk, respectively, for the ith component problem.

Lemma 1 can be used to reduce the k-decision problem under additive loss to k individual 1-decision problems, the only common link being the joint priors p_π, $\pi \in \Pi$. As can be anticipated, the second result will now be with respect to a single component problem. Since it may prove to be useful also in other situations, it is given below in a more general form than actually needed in the present context.

Consider the following situation. Let $\underline{Y}$ be a sample from a parametric family of probability distributions $\{P_\theta\}_{\theta \in \mathbb{R}}$, where we wish to test H_0: $\theta \le \theta_0$ versus H_1: $\theta > \theta_0$. Let the loss function be $L(\theta,1) = L_1(\theta) \ge 0$ if $\theta \le \theta_0$, $L(\theta,0) = L_2(\theta) \ge 0$ if $\theta > \theta_0$, and $L(\cdot,\cdot) = 0$ otherwise. This includes as a special case the 0-1 loss *function*, where $L_1 = L_2 = 1$.

Lemma 2. Let $\tilde{\pi}$ be a prior density w.r.t. a σ-finite measure μ defined on the Borel sets of $\mathbb{R}$, such that the following constant c exists and is not zero:

$$(5) \qquad c = \int_{-\infty}^{\theta_0} L_1(\theta)\,\tilde{\pi}(\theta)\,d\mu(\theta) + \int_{\theta_0}^{\infty} L_2(\theta)\,\tilde{\pi}(\theta)\,d\mu(\theta).$$

Let $\pi(\theta) = c^{-1}L_1(\theta)\tilde{\pi}(\theta)$ if $\theta < \theta_0$, and $\pi(\theta) = c^{-1}L_2(\theta)\tilde{\pi}(\theta)$ if $\theta \ge \theta_0$. Then the Bayes rules under $L(\cdot,\cdot)$ w.r.t. $\tilde{\pi}$ coincide with the Bayes rules under the 0-1 loss function w.r.t. π, and the Bayes risks are related to each other through

$$(6) \qquad r_L(\tilde{\pi}) = c\; r_{0,1}(\pi).$$

where the subscript of r indicates which loss function is assumed.

Proof: Let δ be a decision rule and assume, without loss of generality, that it is non-randomized. Under the loss function L, the Bayes risk of δ with respect to a prior π, for which $c > 0$ exists, is given by

$$(7) \qquad r_L(\pi, \delta) =$$

$$= \int_{-\infty}^{\theta_0} L_1(\theta) P_\theta(\delta(Y)=1)\pi(\theta)d\mu(\theta) + \int_{\theta_0}^{\infty} L_2(\theta) P_\theta(\delta(Y)=0)\pi(\theta)d\mu(\theta)$$

$$= c \left[\int_{-\infty}^{\theta_0} P_\theta(\delta(Y)=1)\tilde{\pi}(\theta)d\mu(\theta) + \int_{\theta_0}^{\infty} P_\theta(\delta(Y)=0)\tilde{\pi}(\theta)d\mu(\theta) \right]$$

$$= c\, r_{0,1}(\tilde{\pi}, \delta),$$

from which the assertions follow immediately.

The above lemma will be applied in Section 3 in the following way. Let $L_1(\cdot) = a$ and $L_2(\cdot) = b$, respectively. Consider a (normal) prior density π w.r.t. the Lebesgue measure, which is symmetric w.r.t. θ_0. Under 0-1 loss, its Bayes rule turns out to be very simple. It will be used later for the δ^B_{im}'s in (4). Under the loss function L, it is also Bayes rule w.r.t. the prior density $\tilde{\pi}$ given by $\tilde{\pi}(\theta) = 2b(a+b)^{-1}\pi(\theta)$ if $\theta \le \theta_0$, and $\tilde{\pi}(\theta) = 2a(a+b)^{-1}\pi(\theta)$ if $\theta > \theta_0$. In this case we have $c = 2ab(a+b)^{-1}$.

3. Known Control θ_0.

As a natural first step, let us derive the Bayes rules for the given k-decision problem with respect to the standard family of conjugate priors. Although they are interesting in their own, only the Bayes rule for the case of $a_i = b_i$, $i = 1,\ldots,k$, will prove to be useful for the problem under concern. Reconsidering this rule through Lemma 2 as a Bayes rule w.r.t. a non-symmetric prior, it will be used in connection with Lemma 1 to prove minimaxity of δ^*.

Following DeGroot (1970), ch. 9.6, let $q = \sigma^{-2}$ be the precision, and let $\Theta_1,\ldots,\Theta_k$ and Q denote the random parameters in the Bayes approach, which are assumed to have the following prior density w.r.t. the Lebesgue measure:

$$(8) \qquad p(\theta,q) = \prod_{i=1}^{k} p^{(i)}(\theta_i|q)\, g(q), \quad \theta \in \mathbb{R}^k,\ q > 0,$$

where $p^{(i)}(\cdot|q)$ is a $N(\mu_i,(\tau_i q)^{-1})$ density with known $\mu_i \in \mathbb{R}$ and $\tau_i > 0$, $i = 1,\ldots,k$, and where

$$(9) \qquad g(q) = \beta^{\alpha}\,\Gamma(\alpha)^{-1}\, q^{\alpha-1}\, e^{-\beta q}, \quad q > 0,$$

is the density of a Γ-distribution with known parameters $\alpha > 0$ and $\beta > 0$.

Standard analysis leads to the following posterior distributions at $X = x$. Given $Q = q$, $\Theta_1,\ldots,\Theta_k$ are independent $N((\tau_i\mu_i+n_i\bar{x}_i)(\tau_i+n_i)^{-1}, (q(\tau_i+n_i))^{-1})$, $i = 1,\ldots,k$, and marginally, Q follows a Γ-distribution with parameters $\alpha+2^{-1}n$ and β', where $n = n_1+\ldots n_k$ and

$$(10) \qquad \beta' = \beta+2^{-1}\sum_{i=1}^{k}\left\{\sum_{j=1}^{n_i}(x_{ij}-\bar{x}_i)^2 + \tau_i n_i(\tau_i+n_i)^{-1}(\bar{x}_i-\mu_i)^2\right\},$$

and where $\bar{x}_i$ denotes the sample mean of x_i, $i = 1,\ldots,k$.

For i [illegible] 1,...,k fixed, by looking at the posterior joint density of [illegible]$_i$ and [illegible], it can be seen that the posterior marginal density of [illegible]$_i$ is a t-distribution with [illegible] n + 2[illegible] degrees of freedom, with location parameter [illegible]$_i$ [illegible]$_i$+n_i[illegible]) [illegible]$_i$+n_i)$^{-1}$, and scale parameter [illegible], where [illegible]^2 = 2[illegible] ([illegible]$_i$+n_i[illegible]$^{-1}$.

The Bayes rule δ_i^B for the ith component problem can be found by minimizing the associated posterior expected loss. It is given by

$$(11) \qquad \delta_i^B(\cdot) = 1(0) \text{ iff } [\text{illegible}] \; (a_i + b_i)^{-1}$$

or, by using the results derived above,

$$(12) \qquad \delta_i^B(\cdot) = 1(0) \text{ iff } ([\text{illegible}]_i + n_i \bar{x}_i)([\text{illegible}]_i + n_i)^{-1} - \theta_0 \;[\text{illegible}]\; e_i,$$

where e_i is the lower $a_i(a_i+b_i)^{-1}$ quantile of a t-distribution with [illegible] degrees of freedom. Obviously, δ^B is then the overall Bayes rule for the original k-decision problem.

For the special case of [illegible]$_i$ [illegible], a_i = b_i = 1, i = 1,...,k, the Bayes rule turns out to be of a very simple form δ^0, say, where

$$(13) \qquad \delta_i^0(\cdot) = 1(0) \text{ iff } \bar{x}_i - \theta_0 \;[\text{illegible}]\; 0, \qquad i = 1,\ldots,k.$$

[illegible] is our candidate for the Bayes rules used on the right hand side of (4).

Under the prior distribution given by (8) and (9), assume for a moment that [illegible] is fixed. From the results stated just after (9), it is easy to see that δ_i^0 is Bayes rule for the ith component problem, and δ^0 is overall

[illegible]

[illegible]fore, in view of Lemma [illegible] and the remarks made at the end of Section [illegible], it follows that for every [illegible] $= 1,\ldots,$ [illegible] is also Bayes rule for the [illegible] under the loss function [illegible] with respect to the [illegible]

[illegible]

[illegible] the overall Bayes rule [illegible] does not depend on [illegible] the rules [illegible] under the loss function [illegible] with respect to the [illegible]

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[illegible] the main result of this section.

[illegible] the ... [illegible] value of the [illegible] is equal to [illegible]

[illegible] standard arguments show that for every [illegible] $1,\ldots,k$, the left hand [illegible] is equal to

(16) $$\sup R_i^1((\ldots,d),\ \ldots) \ldots \mathbb{R}^k,\ c \ldots 0 \quad a_i b_i (a_i + b_i)^{-1}.$$

On the right hand side of (4), let $\delta^B_m = \delta^0$, and let $\tilde{p}_m$ be equal to p as given in (14) and (15) with $\tau_1 = \dots = \tau_k = m$, $m \in \mathbb{N}$. Let $i \in \{1,\dots,k\}$ be fixed for the rest of the proof. We will show below that

$$(17) \qquad \lim_{m\to\infty} r_L^{(i)}(\tilde{p}_m, \delta_i^0) = a_i b_i (a_i + b_i)^{-1},$$

which clearly completes the proof since under the loss function L given in (1), δ_i^0 has been seen to be Bayes rule w.r.t. prior $\tilde{p}_m$, for every $m \in \mathbb{N}$.

Under the 0-1 loss function, δ_i^0 has also been seen to be Bayes rule w.r.t. prior p_m, say, which is equal to p as given in (8) and (9) with $\mu_1 = \dots = \mu_k = \mu_0$ and $\tau_1 = \dots = \tau_k = m$, and this for every $m \in \mathbb{N}$. In this setting, the Bayes risk of δ_i^0 can be written as

$$(18) \qquad r_{0,1}^{(i)}(p_m, \delta_i^0) = \int_0^\infty r_{0,1}^{(i)}(p_m, \delta_i^0 \mid q) g(q) dq, \text{ say, where at } q > 0,$$

$$r_{0,1}^{(i)}(p_m, \delta_i^0 \mid q) =$$

$$= \int_{-\infty}^{\mu_0} \Phi((n_i q)^{1/2}(\mu_i - \mu_0))\ (mq)^{1/2}\ \varphi((mq)^{1/2}(\mu_i - \mu_0))\ d\mu_i$$

$$+ \int_{\mu_0}^{\infty} \Phi((n_i q)^{1/2}(\mu_0 - \mu_i))\ (mq)^{1/2}\ \varphi((mq)^{1/2}(\mu_i - \mu_0))\ d\mu_i$$

$$= \int_{-\infty}^{0} \Phi((n_i/m)^{1/2}\, w)\ \varphi(w)\ dw + \int_0^\infty \Phi(-(n_i/m)^{1/2}\, w)\ \varphi(w)\ dw,$$

where φ and Φ denote the standard normal density and cumulative distribution function, respectively. Clearly for every $q > 0$, the sum of the last two integrals tends to 1/2 as m tends to infinity. Since the value of this sum is always between 0 and 1, uniformly in $q > 0$ and $m \in \mathbb{N}$, it follows by Lebesgues dominated convergence theorem that

$$(19) \qquad \lim_{m\to\infty} r_{0,1}^{(i)}(\rho_m, \delta_i^0) = 1/2.$$

Applying now Lemma 2, in the way described below of (13), we get

$$(20) \qquad r_L^{(i)}(\rho_m, \delta_i^0) = 2\, a_i b_i (a_i+b_i)^{-1}\, r_{0,1}^{(i)}(\rho_m, \delta_i^0).$$

From this it follows that (17) holds, and therefore the proof of the theorem is completed.

It should be pointed out that Lehmann (1957) has shown that the minimax-value of the ith component problem is equal to $a_i b_i (a_i+b_i)^{-1}$, $i = 1,\ldots,k$. Therefore from (16) it follows that δ_i^* is minimax for the ith component problem, $i = 1,\ldots,k$. It is a well known fact that student's t-test is minimax at the suitably chosen level of significance. However, this fact is of no use in the present context, since the overall minimax value may be less than the sum of the k minimax values of the k component problems.

As a final remark, let us mention that δ^* remains minimax if S^2, the pooled sample estimator of σ^2, is based on a subcollection of observations from X, and if $c_1,\ldots,c_k$ are properly adapted. However, such a modified procedure would have a strictly larger risk, except at $\theta_1 = \ldots = \theta_k = \theta_0$. This follows from the fact that for every $i \in \{1,\ldots,k\}$, δ_i^* is the uniformly most powerful unbiased test at its level, whereas the modified procedures' ith decision rule would only be an unbiased test at the same level of $b_i(a_i+b_i)^{-1}$. The modified procedure would thus be inadmissible. Whether or not δ^* is admissible remains an open question.

4. Unknown Control θ_0.

In this setting, an additional sample X_0 from population $\pi_0 \sim N(\theta_0, \sigma^2)$ is observed. The analogous results to Section 3 can be derived in a similar way. Therefore the treatment of this case will be rather concise.

First, let us find the Bayes rules w.r.t. the standard family of conjugate priors which is essentially the same as (8) and (9), but now with the product in (8) defined over the range $i = 0,1,\ldots,k$, since θ_0 is now an additional random parameter. Of course, $p^{(0)}(\cdot\,|\,q)$ is a $N(\mu_0,(\tau_0 q)^{-1})$ density with known $\mu_0 \in \mathbb{R}$ and $\tau_0 > 0$. From the results derived below of (9) it can be seen that the posterior distribution at $X = x$ has the following properties. For every $i = 1,\ldots,k$, given $Q = q$, $\theta_i - \theta_0$ has, marginally, a normal distribution with mean $\mu_{i0} = (\tau_i+n_i)^{-1}(\tau_i\mu_i+n_i x_i) - (\tau_0+n_0)^{-1}(\tau_0\mu_0+n_0 x_0)$ and variance $q^{-1}((\tau_i+n_i)^{-1} + (\tau_0+n_0)^{-1})$. The posterior marginal distribution of Q is a Γ-distribution with parameters $\alpha + 2^{-1}(n_0+n)$ and β'', where β'' is the analog to β' given in (10), where the first sum is now defined over the range $i = 0,1,\ldots,k$.

For $i = 1,\ldots,k$ fixed, by looking at the posterior joint density of $\theta_i - \theta_0$ and Q, it can be seen that the posterior marginal density of $\theta_i - \theta_0$ is a t-distribution with $n_0+n+2\alpha$ degrees of freedom with location parameter μ_{i0} and scale parameter σ_{i0}, where $\sigma_{i0}^2 = 2\beta''((\tau_i+n_i)^{-1} + (\tau_0+n_0)^{-1})(n_0+n+2\alpha)^{-1}$.

For the ith component problem the Bayes rule can be found by minimizing the associated posterior expected loss. It is given by

(21) $\delta_i^B(x) = 1(0)$ iff $P\{\theta_i \geq (<) \theta_0 \mid X = x\} \geq (<) b_i(a_i+b_i)^{-1}$

or, by using the results derived above,

(22) $\delta_i^B(x) = 1(0)$ iff $t_{i0} \geq (<) e_{i0}$,

where e_{i0} is the lower $a_i(a_i+b_i)^{-1}$ quantile of a t-distribution with $n_0+n+2\nu$ degrees of freedom.

For the special case of $\mu_i = \mu_0$, $n_0\tau_i = n_i\tau_0$, $a_i = b_i$, $i = 1,\ldots,k$, the Bayes rule turns out to be of the simple form δ^{00}, say, where

(23) $\delta_i^{00}(x) = 1(0)$ if $\bar{X}_i - \bar{X}_0 \geq (<) 0$, $i = 1,\ldots,k$.

Instead of following along the lines below of (13), there is a shorter way to prove minimaxity of δ^{**} in the present case. The main result of this section is

Theorem 2. Under the loss function (1), the multiple decision rule δ^{**}, as given in (3), is minimax. The minimax-value of the problem is equal to $\sum_{i=1}^{k} a_i b_i (a_i+b_i)^{-1}$.

Proof. Again, standard arguments show that for every $i = 1,\ldots,k$, the left hand side of (4) for $\delta_i^M = \delta_i^{**}$ is equal to

(24) $\sup\{R_L^{(i)}((\theta,q), \delta_i^{**}) \mid \theta \in \mathbb{R}^{k+1}, q > 0\} = a_i b_i (a_i+b_i)^{-1}$,

where the dimension of the θ-parameter space is now $k+1$.

On the right hand side of (4), instead of chosing δ^B_m to be δ^{00}, let rather $\delta^B_m = \delta^0$ as before in the proof of Theorem 1. As to the priors of $\theta_0, \theta_1, \ldots, \theta_k$, and Q, assume that $\theta_0 = \mu_0$ be a fixed known constant μ_0, say, and adopt the same priors for $\theta_1, \ldots, \theta_k$, and Q as have been used in the proof of Theorem 1. Then for every $i = 1, \ldots, k$, (17) holds true and δ_i^0 is Bayes rule with respect to prior p_m for all $m \in \mathbb{N}$. Therefore the proof of Theorem 2 is essentially the same as the proof of Theorem 1.

Concluding Remarks:

The remarks given at the end of Section 3 hold in an analogous form for the situation considered above. They are omitted for brevity.

For the proofs of the two theorems, the proper choice of priors was crucial. The relevant parameters θ_i were assumed to be independent, whenever the nuisance parameter $Q = q$ was fixed. In the unknown control case, an attempt to use the principle of (location) invariance may not lead to the desired results if one assumes that, apriori, $\theta_1 - \theta_0, \ldots, \theta_k - \theta_0$ are independent. This is due to the fact that at $X = x$, the posterior distribution of each $\theta_i - \theta_0$ would depend on all given observations. For the case of σ^2 known, Randles and Hollander (1971) have given an instructive example.

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Let $\pi_1, \dots, \pi_k$ be k normal populations with unknown means $\theta_1, \dots, \theta_k$, and a common unknown variance $\sigma^2 > 0$. Based on independent samples of sizes $n_1, \dots, n_k$, the populations are to be partitioned into two sets, where the first one contains all π_i with $\theta_i \geq \theta_0$, and where the other one contains the rest. At first it is assumed that θ_0 is known. Under an additive "a_i-b_i" loss function a minimax

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SECURITY CLASSIFICATION OF THIS PAGE (When Data Entered) procedure is derived which is of a simple natural form. The proof of minimaxity makes use of the Bayes approach and involves a sequence of nonsymmetric priors, which play a similar role as a least favorable prior in simpler problems. Analogous results are presented for the case that [illegible]$_0$ is not known. In this case, a control normal population is assumed to exist from which an additional sample of size n_0 can be drawn.

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