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THE TWO-ON-ONE STOCHASTIC DUEL

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DEPARTMENT OF THE ARMY
US ARMY TRADOC SYSTEMS ANALYSIS ACTIVITY
WHITE SANDS MISSILE RANGE
NEW MEXICO 88002

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THE TWO-ON-ONE STOCHASTIC DUEL*

BY

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ABSTRACT

The one-on-one stochastic duel is extended to the general two-on-one duel for the first time. The state equations, win probabilities, means value and variance functions are derived. The case where one side has Erlang (2) firing times and the other is negative exponential is compared to the corresponding "Stochastic Lanchester" and Lanchester models to demonstrate their non-equivalence.

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1. INTRODUCTION

In this paper we extend the theory of stochastic duels to the two-on-one case. Previously (see Ancker [1]*) much work has been done on the marksman problem (one versus a passive target) and the one-on-one duel but, to our knowledge, this is the first general extension which does not necessarily assume all interfiring times have the negative exponential distribution (ned) (Clark [4]) or are constant (Ancker [2], Anderson [3]).

In the following sections the problem is rigorously formulated, and the state equations are derived and solved. Then we first assume all ned interfiring times as one example (this checks with previously derived "Stochastic Lanchester" equations) and secondly show an example with Erlang (2) interfiring times on the A side and ned interfiring times on the B side.

These two examples are then compared with the equivalent Lanchester Square Law solution to show that, for such small numbers involved, both the Lanchester and the "Stochastic Lanchester" approximations to a situation in which interfiring times are not all ned can be very poor.

2. THE MODEL

Two sides, A and B, conduct a continuous engagement until one or

* Numerals in brackets [ ], refer to the references listed at the end of the paper.

the other side is destroyed. There are two contestants on the A side, who fire continuously and independently of each other and who have identical random interfiring times, X_A, (with probability density function (pdf) $f_{X_A}(x)$) and identical kill probabilities, p_A, on each round fired. The B side has only one contestant whose random interfiring time is X_B with pdf $f_{X_B}(x)$ and kill probability p_B. All interfiring times are independent. Both sides start at time zero and each contestant fires his first round (if still alive) at one random interfiring time later. There are no time or ammunition limitations on the contest.

3. GENERAL SOLUTION

Our solution technique has three principal features. One, we shall first consider each contestant separately as a marksman firing at a passive target, and two, we shall concentrate only on firings which are kills rather than on every firing event. This is not necessary but simplifies the mathematics. This procedure entails knowing the solution to the problem of the marksman firing at a passive target which is exhaustively examined in reference [1] and which we shall consider known. Thus, if T_A is the time to a kill by one of the A's, we shall designate the distribution function (df) and the pdf of T_A as $F_A(t)$ and $f_A(t)$ respectively. Similarly for B, we have $F_B(t)$ and $f_B(t)$.

Finally, we shall superpose the three marksmen's firing sequences and use the backward recurrence time technique to write the state probability equations. This means that we observe the superposed sequence at the time t and, if we define Y to be the random variable time since the last event (kill) then the first order probability that the next event will be a kill by A and will occur in the interval $(t, t + \Delta)$ is given by

$$r_A(y)\Delta = \frac{f_A(y)}{F_A^c(y)} \Delta + o(\Delta) \tag{1}$$

for A, and by

$$r_B(y)\Delta = \frac{f_B(y)}{F_B^c(y)} \Delta + o(\Delta) \tag{2}$$

for B. $F_A^c(y)$ and $F_B^c(y)$ are the complementary distribution functions (cdf's) for the killing times of A and B respectively. See reference [1] for a discussion of the backward recurrence time technique. The r(y)'s are the instantaneous <u>kill rates</u> for each marksman. We note that

$$\left.\begin{aligned} f_A(y) &= r_A(y)\, e^{-\int_0^y r_A(\xi)d\xi} , \\ F_A^c(y) &= e^{-\int_0^y r_A(\xi)d\xi} , \end{aligned}\right\} \tag{3}$$

and similarly for B. We note that if there have been no events at time t, then y = t. The state random variables are defined to be,

$$\left.\begin{aligned} N_A(t) &= \text{the number alive on side A at time } t, \\ N_b(t) &= \text{the number alive on side B at time } t, \end{aligned}\right\} \qquad (4)$$

and the state probabilities are defined as

$$\left.\begin{aligned} p_{ij}(t) &= P[\text{i are alive on side A, and j are alive on side B at time t}], \\ &\text{and} \\ p_{ij}(t,y)dy &= P[\text{i are alive on side A, j are alive on side B at time t and it has been between y and y + dy time units since the last event}], \\ &\qquad i = 0,1,2 \\ &\qquad j = 0,1. \end{aligned}\right\} \qquad (5)$$

The relation between the two types of probabilities in (5) is $p_{ij}(t) = \int_0^t p_{ij}(t,y)dy$. The second form in equation (5) is necessary since in general, at any arbitrary time t, the state of the system is defined by the number alive on each side and the time to the last event. The backward recurrence time does not exist for the beginning state or terminal states. In fact, for our small system it is only needed for state $N_A(t) = 1$ and $N_B(t) = 1$, since at most two events terminate the system. In more complicated systems, more backward recurrence times must be accounted for, but we only need one since at most one event can precede termination.

The problem will be to derive the state probabilities and from these to obtain the win probabilities, P[A] and P[B], and the means and variances of the state random variables.

The state probabilities are obtained in the usual fashion. For example, in the first equation in (6) below, the probability of being in state $N_A(t) = 2$ and $N_B(t) = 1$ at time $t + \Delta$ is equal to the probability of being in the state at time t times the probability of remaining there in the interval $(t, t + \Delta)$, which is the probability that one A fails to kill, $(1 - r_A(t)\Delta)$, times the probability the second A fails, $(1 - r_A(t)\Delta)$, times the probability the B contestant fails, $(1 - r_B(t)\Delta)$. All these probabilities are independent by the statement of the problem. Hence,

$$\left.\begin{aligned}
&p_{21}(t+\Delta) = p_{21}(t)\ [1 - r_A(t)\Delta]^2\ [1 - r_B(t)\Delta] + o(\Delta)\\
&\text{and similarly,}\\
&p_{20}(t+\Delta) = p_{20}(t) + p_{21}(t)\ 2r_A(t)\Delta\ [1 - r_B(t)\Delta] + o(\Delta),\\
&p_{11}(t+\Delta, y+\Delta) = p_{11}(t,y)\ [1 - r_A(t)\Delta][1 - r_B(y)\Delta] + o(\Delta),\\
&p_{10}(t+\Delta) = p_{10}(t) + \int_0^t p_{11}(t,y)[1 - r_B(y)\Delta]r_A(t)\Delta dy + o(\Delta),\\
&p_{01}(t+\Delta) = p_{01}(t) + \int_0^t p_{11}(t,y)r_B(y)\Delta dy + o(\Delta).
\end{aligned}\right\}\qquad (6)$$

Rearranging terms, dividing by Δ and letting $\Delta \to 0$, we have that

$$\frac{dp_{21}(t)}{dt} = -\,p_{21}(t)\ [2r_A(t) + r_B(t)], \qquad (7)$$

$$\frac{dp_{20}(t)}{dt} = 2r_A(t)\ p_{21}(t), \qquad (8)$$

$$\left(\frac{\partial}{\partial t} + \frac{\partial}{\partial y} + r_A(t) + r_B(y)\right)\ p_{11}(t,y) = 0, \qquad (9)$$

$$\frac{dp_{10}(t)}{dt} = r_A(t)\int_o^t p_{11}(t,y)dy = r_A(t)p_{11}(t), \qquad (10)$$

$$\frac{dp_{01}(t)}{dt} = \int_o^t p_{11}(t,y)r_B(y)dy, \qquad (11)$$

with initial conditions,

$$\left.\begin{array}{ll} p_{21}(0) = 1, & \\ p_{ij}(0) = 0 & i \neq 2,\ j \neq 1, \end{array}\right\} \tag{12}$$

and boundary condition

$$p_{11}(t+\Delta,\ 0+\Delta)\Delta = p_{21}(t)r_B(t)\Delta + o(\Delta),$$

which, on dividing by Δ and letting $\Delta \to 0$, gives

$$p_{11}(t,0) = p_{21}(t)r_B(t). \tag{13}$$

Equations (7) and (8) are easily solved, in sequence, using initial condition (12), to give

$$p_{21}(t) = [\ F_A^c(t)]^2\ F_B^c(t) \tag{14}$$

and

$$p_{20}(t) = 2\int_o^t F_A^c(\xi)\ F_B^c(\xi)\ f_A(\xi)d\xi. \tag{15}$$

The general solution to equation (9) is

$$g(t-y)e^{-\int_o^y r_B(\xi)d\xi}\ e^{-\int_o^t r_A(\eta)d\eta} = g(t-y)\ F_B^c(y)\ F_A^c(t).$$

Using boundary condition (13), we find

$$g(t-y) = F_A^c(t-y)f_B(t-y)$$

and so,

$$p_{11}(t,y) = F_A^c(t)\ F_A^c(t-y)F_B^c(y)f_B(t-y). \tag{16}$$

Integrating out y from p_{11} (t,y),

$$p_{11}(t) = F_A^c(t)\ \int_o^t F_A^c(t-y)\ F_B^c(y)\ f_B(t-y)dy, \tag{17}$$

which satisfies initial conditions (12). Substituting (16) and (17), into equations (10 and (11), we have at once that,

$$p_{10}(t) = \int_o^t f_A(\xi)d\xi\ \int_o^\xi F_A^c(\xi-y)F_B^c(y)f_B(\xi-y)dy, \tag{18}$$

and

$$p_{01}(t) = \int_o^t F_A^c(\xi)d\xi\ \int_o^\xi F_A^c(\xi-y)f_B(y)f_B(\xi-y)dy, \tag{19}$$

which also satisfy initial conditions (12).

Equations (14), (15), (17), (18) and (19) are the equations of state for the model and sum to one for all t and along with equation (16) contain essentially all the information on the solution. Some other expressions of interest may be derived as follows.

The probabilities that A wins, P[A] and B wins, P[B] are

$$P[A] = p_{20}\ (\infty) + p_{10}(\infty) = 2\int_o^\infty F_A^c(\xi)\ f_A(\xi)d\xi$$

$$+ \int_o^\infty f_A(\xi)d\xi\ \int_o^\xi F_A^c(\xi-y)\ F_A^c(y)\ f_B(\xi,y)dy \tag{20}$$

and

$$P[B] = p_{01}(\infty) = \int_0^\infty F_A^c(\xi)d\xi \int_0^\xi F_A^c(\xi-y)\; f_B(y)\; f_B(\xi-y)dy, \tag{21}$$

and $P[A] + P[B] = 1$, as it should.

The means and variances may be calculated as follows:

$$E[N_A(t)] = \overline{n_A(t)} = 2(p_{21}(t) + p_{20}(t)) + 1(p_{11}(t) + p_{10}(t)), \tag{22}$$

$$E[N_B(t)] = \overline{n_B(t)} = 1(p_{21}(t) + p_{11}(t) + p_{01}(t)), \tag{23}$$

$$E[N_A^2(t)] = \overline{n_A^2(t)} = 4(p_{21}(t) + p_{20}(t)) + 1(p_{11}(t) + p_{10}(t)), \tag{24}$$

$$E[N_B^2(t)] = \overline{n_B^2(t)} = 1(p_{21}(t) + p_{11}(t) + p_{01}(t)) = \overline{n_B(t)}\ , \tag{25}$$

$$V[N_A(t)] = \overline{n_A^2(t)} - [\overline{n_A(t)}]^2 \tag{26}$$

and

$$V[N_B(t)] = \overline{n_B^2(t)} - [\overline{n_B(t)}]^2 = \overline{n_B(t)} - [\overline{n_B(t)}]^2. \tag{27}$$

The above are the general results for the stated problem. Examples for specific interfiring times follow.

4. EXAMPLES

Example 1

Let the interfiring times be

$$f_{X_A}(x) = \frac{1}{\mu_A} e^{-\frac{x}{\mu_A}} \quad , \quad f_{X_B}(x) = \frac{1}{\mu_B} e^{-\frac{x}{\mu_B}} \quad .$$

From prior work, reference [1]

$$f_A(t) = \frac{p_A}{\mu_A} e^{-\frac{p_A}{\mu_A} t} \quad , \quad f_B(t) = \frac{p_B}{\mu_B} e^{-\frac{p_B}{\mu_B}} \quad ,$$

and from equations (1) and (2) $r_A = \frac{p_A}{\mu_A}$, $r_B = \frac{p_B}{\mu_B}$ and neither is a function of time to the last event (this is a result of the no memory property of the negative exponential random variable).

Using $f_A(t)$, $f_B(t)$ and their cdf's in (14), (15), (17), (18) and (19) we have easily that,

$$\left.\begin{aligned}
p_{21}(t) &= e^{-(r_A + r_B)t} \, , \\
p_{20}(t) &= \frac{2r_A}{2r_A+r_B} [1-e^{-(2r_A+r_B)t}] , \\
p_{11}(t) &= \frac{r_B}{r_A} [e^{(r_A+r_B)t} - e^{-(2r_A+r_B)t}] , \\
p_{10}(t) &= \frac{r_B}{r_A+r_B} [1 - e^{-(r_A+r_B)t}] - \frac{r_B}{2r_A+r_B} [1 - e^{-(2r_A+r_B)t}] , \\
p_{01}(t) &= \frac{r_B^2}{r_A(r_A+r_B)} [1 - e^{-(r_A+r_B)t}] - \frac{r_B^2}{r_A(2r_A+r_B)} [1-e^{-(2r_A+r_B)t}] .
\end{aligned}\right\} \quad (28)$$

These results may also be obtained from the so-called "Stochastic Lanchester" equations which are a stochastic version of the famous deterministic Lanchester equations. In particular these are the two-on-one version of the "Square Law". For an exposition on these matters see, for example, Clark [4]. The state equations corresponding to (7) through (12) are quite similar but only involve ordinary differential equations and are somewhat simpler. We also have the following results from (20) and (21):

$$P[A] = \frac{r_A(2r_A+3r_B)}{(r_A+r_B)(2r_A+r_B)}, \tag{29}$$

$$P[B] = \frac{r_B^2}{(r_A+r_B)(2r_A+r_B)}, \tag{30}$$

and from (22) and (23),

$$\overline{n_A(t)} = \frac{r_A(5r_B+4r_A)}{(r_A+r_B)(2r_A+r_B)} + \frac{r_B^2}{r_A(r_A+r_B)}\, e^{-(r_A+r_B)t} + \frac{r_B(r_A-r_B)}{r_A(2r_A+r_B)}\, e^{-(2r_A+r_B)t} \tag{31}$$

$$\overline{n_B(t)} = \frac{r_B^2}{(r_A+r_B)(2r_A+r_B)} + \frac{r_B}{(r_A+r_B)}\, e^{-(r_A+r_B)t} + \frac{(2r_A-r_B)}{(2r_A+r_B)}\, e^{-(2r_A+r_B)t}, \tag{32}$$

and from (24) and (25),

$$\overline{n_A^2(t)} = \frac{r_A(8r_A+ 9r_B)}{r_A+r_B)(2r_A+r_B)} + \frac{r_B^2}{r_A(r_A+r_B)} e^{-(r_A+r_B)t}$$

$$+ \frac{r_B(3r_A-r_B}{r_A(2r_A+r_B)} e^{-(2r_A+r_B)t} \quad , \qquad (33)$$

$$\overline{n_B^2(t)} = \overline{n_B(t)} \; . \qquad (34)$$

The variances may now be calculated using equations (26) and (27).

Example 2

Let

$$f_{X_A}(x) = \frac{4}{\mu_A^2} x e^{-\frac{2x}{\mu_A}} \quad , \quad f_{X_B}(x) = \frac{1}{\mu_B} e^{-\frac{x}{\mu_B}} \quad .$$

Again from reference [1], setting $q_A = 1-p_A$,

$$f_A(t) = \frac{p_A}{\mu_A\sqrt{q_A}} [e^{-\frac{2}{\mu_A}(1 - \sqrt{q_A})t} - e^{-\frac{2}{\mu_A}(1 + \sqrt{q_A})t}],$$

$$f_B(t) = \frac{p_B}{\mu_B} e^{-\frac{p_B}{\mu_B}t} \quad .$$

In this case, since each contestant on side A has an Erlang (2) interfiring time (with the same mean, μ_A, as in example 1), $r_A(y)$ is no longer a constant but is a function of y, the backward recurrence time. Proceeding as before, after much integration and algebra we have the following results. Using the notation

$$\alpha_1 = \frac{2}{\mu_A}(1 - \sqrt{q_A}), \ \alpha_2 = \frac{2}{\mu_A}(1 + \sqrt{q_A}), \ r_B = \frac{p_B}{\mu_B},$$

then,

$$p_{21}(t) = \frac{e^{-r_B t}}{(\alpha_2-\alpha_1)^2}\,[\alpha_2 e^{\alpha_1 t} - \alpha_1 e^{-\alpha_2 t}]^2, \tag{35}$$

$$p_{20}(t) = \frac{2\alpha_1^2\alpha_2^2}{(\alpha_2-\alpha_1)^2}\{\frac{1}{\alpha_1(2\alpha_1+r_B)}[1-e^{-(2\alpha_1+r_B)t}] + \frac{1}{\alpha_2(2\alpha_2+r_B)}[1-e^{-(2\alpha_2+r_B)t}]$$

$$- \frac{(\alpha_1+\alpha_2)}{\alpha_1\alpha_2(\alpha_1+\alpha_2+r_B)}[1 - e^{-(\alpha_2+\alpha_2+r_B)t]}\}, \tag{36}$$

$$p_{11}(t) = \frac{r_B e^{-r_B t}}{(\alpha_2-\alpha_1)^2}\{\frac{(\alpha_2^2-\alpha_1^2)}{\alpha_1}e^{-\alpha_1 t} - \frac{(\alpha_2^2-\alpha_1^2)}{\alpha_2}e^{-\alpha_2 t} - (\alpha_2-\alpha_1)\,e^{-(\alpha_1+\alpha_2)t}$$

$$- \frac{\alpha_2^2}{\alpha_1}e^{-2\alpha_1 t} - \frac{\alpha_1^2}{\alpha_2}e^{-2\alpha_2 t}\}, \tag{37}$$

$$P_{10}(t) = \frac{r_B}{(\alpha_2-\alpha_1)^2} \{ \frac{(\alpha_2^2 - \alpha_1^2)}{(\alpha_1+r_B)} [1-e^{-(\alpha_1+r_B)t}] - \frac{(\alpha_2^2-\alpha_1^2)}{(\alpha_2+b)} [1-e^{-(\alpha_2+r_B)t}]$$

$$- \frac{\alpha_2^2}{(2\alpha_1+r_B)} [1-e^{-(2\alpha_1+r_B)t}] - \frac{\alpha_1^2}{(2\alpha_2+r_B)} [1-e^{-(2\alpha_2+r_B)t}]$$

$$+ \frac{(\alpha_1^2 + \alpha_2^2)}{(\alpha_1+\alpha_2+r_B)} [1-e^{-(\alpha_1+\alpha_2+r_B)t}]\}, \tag{38}$$

$$P_{01}(t) = \frac{r_B^2}{(\alpha_2-\alpha_1)^2} \{\frac{(\alpha_2^2-\alpha_1^2)}{\alpha_1(\alpha_1+r_B)} [1-e^{-(\alpha_1+r_B)t}] - \frac{(\alpha_2^2-\alpha_1^2)}{\alpha_2(\alpha_2+r_B)} [1-e^{-(\alpha_2+r_B)t}]$$

$$- \frac{\alpha_2^2}{\alpha_1(2\alpha_1+r_B)} [1-e^{-(2\alpha_1+r_B)t}] - \frac{\alpha_1^2}{\alpha_2(2\alpha_2+r_B)} [1-e^{-(2\alpha_2+r_B)t}]$$

$$+ \frac{(\alpha_1+\alpha_2)}{(\alpha_1+\alpha_2+r_B)} [1-e^{-(\alpha_1+\alpha_2+r_B)t}]\}, \tag{39}$$

$$P[A] = \frac{4\alpha_1\alpha_2(\alpha_1+\alpha_2)-2(\alpha_1^2+\alpha_2^2)r_B-(\alpha_1+\alpha_2)r_B^2}{(2\alpha_1+r_B)(2\alpha_2+r_B)(\alpha_1+\alpha_2+r_B)} + \frac{(\alpha_2+\alpha_1)r_B}{(\alpha_1+r_B)(\alpha_2+r_B)} \tag{40}$$

$$P[B] = \frac{(\alpha_1+\alpha_2)(\alpha_1+\alpha_2+r_B)r_B^2}{\alpha_1\alpha_2(\alpha_1+r_B)(\alpha_2+r_B)}$$

$$- \frac{r_B^2\{2(\alpha_1+\alpha_2)^3+[2(\alpha_1+\alpha_2)^2+\alpha_1^2+\alpha_1\alpha_2+\alpha_2^2]r_B+(\alpha_1+\alpha_2)r_B^2\}}{\alpha_1\alpha_2(2\alpha_1+r_B)(2\alpha_2+r_B)(\alpha_1+\alpha_2+r_B)}, \tag{41}$$

$$\overline{n_A(t)} = \frac{(\alpha_1+\alpha_2)r_B}{(\alpha_1+r_B)(\alpha_2+r_B)} - \frac{4\alpha_1\alpha_2(\alpha_1+\alpha_2)-(\alpha_1^2+\alpha_2^2)r_B}{(\alpha_2-\alpha_1)^2(\alpha_1+\alpha_2+r_B)} + \frac{\alpha_2^2(4\alpha_1-r_B)}{(\alpha_2-\alpha_1)^2(2\alpha_1+r_B)}$$

$$+ \frac{\alpha_1^2(4\alpha_2-r_B)}{(\alpha_2-\alpha_1)^2(2\alpha_2+r_B)} + \frac{(\alpha_1+\alpha_2)r_B^2}{\alpha_1(\alpha_2-\alpha_1)(\alpha_1+r_B)}\, e^{-(\alpha_1+r_B)t}$$

$$- \frac{(\alpha_1+\alpha_2)r_B^2}{\alpha_2(\alpha_2-\alpha_1)(\alpha_2+r_B)}\, e^{-(\alpha_2+r_B)t} + \frac{\alpha_2^2(\alpha_1-r_B)r_B}{\alpha_1(\alpha_2-\alpha_1)^2(2\alpha_1+r_B)}\, e^{-(2\alpha_1+r_B)t}$$

$$+ \frac{\alpha_1^2(\alpha_2-r_B)r_B}{\alpha_2(\alpha_2-\alpha_1)^2(2\alpha_2+r_B)}\, e^{-(2\alpha_2+r_B)t} - \frac{[2\alpha_1\alpha_2-(\alpha_1+\alpha_2)r_B]r_B}{(\alpha_2-\alpha_1)^2(\alpha_1+\alpha_2+r_B)}\, e^{-(\alpha_1+\alpha_2+r_B)t}, \tag{42}$$

$$\overline{n_B(t)} = r_B^2 \left\{ \frac{(\alpha_1+\alpha_2)(\alpha_1+\alpha_2+r_B)}{\alpha_1\alpha_2(\alpha_1+r_B)(\alpha_2+r_B)} + \frac{1}{(\alpha_2-\alpha_1)^2} \left[\frac{(\alpha_1+\alpha_2)}{(\alpha_1+\alpha_2+r_B)} - \frac{\alpha_2^2}{\alpha_1(\alpha_1+r_B)} \right.\right.$$

$$\left.\left. - \frac{\alpha_1^2}{\alpha_2(2\alpha_2+r_B)} \right]\right\}$$

$$+ \frac{e^{-r_B t}}{(\alpha_2-\alpha_1)^2} \left\{ \frac{(\alpha_2^2-\alpha_1^2)r_B}{(\alpha_1+r_B)}\, e^{-\alpha_1 t} - \frac{(\alpha_2^2-\alpha_1^2)r_B}{(\alpha_2+r_B)}\, e^{-\alpha_2 t} + \frac{\alpha_2^2(2\alpha_1-r_B)}{(2\alpha_1+r_B)}\, e^{-2\alpha_1 t} \right.$$

$$\left. + \frac{\alpha_1^2(2\alpha_2-r_B)}{(2\alpha_2+r_B)}\, e^{-2\alpha_2 t} - \frac{2\alpha_1\alpha_2(\alpha_1+\alpha_2)-(\alpha_1^2+\alpha_2^2)r_B}{(\alpha_1+\alpha_2+r_B)}\, e^{-(\alpha_1\alpha_2)t} \right\}, \tag{43}$$

$$
\begin{aligned}
\overline{n_A^2(t)} = {} & \frac{(\alpha_1+\alpha_2)r_B}{(\alpha_1+r_B)(\alpha_2+r_B)} + \frac{1}{(\alpha_2-\alpha_1)^2}\left[-\frac{8\alpha_1\alpha_2(\alpha_1+\alpha_2)-(\alpha_1^2+\alpha_2^2)r_B}{(\alpha_1+\alpha_2+r_B)} + \frac{\alpha_2^2(8\alpha_1-r_B)}{(2\alpha_1+r_B)}\right. \\
& \left. + \frac{\alpha_1^2(8\alpha_2-r_B)}{(2\alpha_2+r_B)}\right] + \frac{e^{-r_B t}}{(\alpha_2-\alpha_1)^2}\left\{\frac{(\alpha_2^2-\alpha_1^2)r_B^2}{\alpha_1(\alpha+r_B)}\, e^{-\alpha_1 t} - \frac{(\alpha_2^2-\alpha_1^2)r_B^2}{\alpha_2(\alpha_2+r_B)}\, e^{-\alpha_2 t}\right. \\
& + \frac{\alpha_2^2(3\alpha_1-r_B)r_B}{\alpha_1(2\alpha_1+r_B)}\, e^{-2\alpha_1 t} + \frac{\alpha_1^2(3\alpha_2-r_B)r_B}{\alpha_2(2\alpha_2+r_B)}\, e^{-2\alpha_2 t} \\
& \left. - \frac{[6\alpha_1\alpha_2-(\alpha_1+\alpha_2)r_B]r_B}{(\alpha_1+\alpha_2+r_B)}\, e^{-(\alpha_1+\alpha_2)t}\right\}, \qquad (44)
\end{aligned}
$$

$$
\overline{n_B^2(t)} = \overline{n_B(t)}\,, \qquad (45)
$$

and again the variances may be computed from equations (26) and (27).

5. COMPARISONS

The two examples given above are stochastic formulations of the Lanchester Square Law. It will be instructive to compare these results with the equivalent Lanchester results which are given below and are very well known, see for example, Clark [4]. Using our notation,

$$\overline{n_A(t)} = 2 \cosh \sqrt{r_A r_B}\, t - \sqrt{\frac{r_B}{r_A}} \sinh \sqrt{r_A r_B}\, t, \tag{46}$$

and

$$\overline{n_B(t)} = \cosh \sqrt{r_A r_B}\, t - 2\sqrt{\frac{r_A}{r_B}} \sinh \sqrt{r_A r_B}\, t. \tag{47}$$

This formulation is in fact deterministic and we are interpreting the variables as the mean values of the random variables $N_A(t)$ and $N_B(t)$. This is the generally accepted interpretation and indeed no other viewpoint seems reasonable.

In making our comparisions we should note that equation (46) has the following properties:

1. If $\frac{r_A}{r_B} < 1/4$, $\overline{n_A(t)}$ has a zero at

$$t = \frac{1}{\sqrt{r_A r_B}} \tanh^{-1} 2 \sqrt{\frac{r_A}{r_b}}$$

and becomes negative thereafter. Thus for t greater than this value we set $\overline{n_A(t)} = 0$ (we cannot have negative contestants on a side).

2. If $\frac{r_A}{r_B} > 1/4$, $\overline{n_A(t)}$ is always positive, but decreases monotonically from t = 0 to

$$t = \frac{1}{\sqrt{r_A r_B}} \tanh^{-1} 1/2 \sqrt{\frac{r_B}{r_A}}$$

where it has a minimum and then starts to increase. At this time we continue the function level at the value it has reached (our problem does not allow for reinforcements).

3. If $\frac{r_A}{r_B} = 1/4$, $\overline{n_A(t)}$ is asymptotic to zero as $t \to \infty$.

$\overline{n_B(t)}$ has complementary properties. It goes to zero where $\overline{n_A(t)}$ has a minimum; it has a minimum where $\overline{n_A(t)}$ has a zero; and it is asymptotic to zero as $t \to \infty$ for $\frac{r_A}{r_B} = 1/4$.

The comparisions we are making here are motivated by the fact that it is common in combat models where it is known that the interfiring times are not ned, to assume that they are ned and use the means of the true distributions. Thus, if μ is the mean of the true interfiring time distribution, the killing rate, r, is taken to be $\frac{p}{\mu}$ (done appropriately for each side) and either the "Stochastic Lanchester" (both sides ned) or the Lanchester equations are used with the appropriate $\frac{p}{\mu}$'s as the attrition coefficients.

The following comparative curves throw some light on this matter, at least at the two-on-one level. In all of these figures there are 3 parameters p_A, μ_A and r_B. For comparative purposes we shall combine the two paramters p_A and μ_A wherever possible into $\frac{p_A}{\mu_A} = r_A$ which, in fact, always occur together this way in the Lanchester (L) and the "Stochastic Lanchester" (SL) (all ned stochastic version) equations. However, in our example (on which the A side has Erlang (2) firing times and which we shall designate as the E model) they do not in general, occur together. So for comparative purposes we use p_A

and r_A to characterize that situation, even though r_A is not the instantaneous kill rate (which is a function of the backward recurrence time) but simply the ratio of two parameters.

In Figure 1 we let r_A and r_B be some typical fixed values, let p_A vary widely and observe $\overline{n_A(t)}$. This makes no difference in the L and SL models but causes considerable variation in the expected values over time in the E model. We observe that if r_A is held constant then as p_A increases and therefore μ_A increases (or $\frac{1}{\mu_A}$ the "firing rate" decreases) then the E model $\overline{n_A(t)}$ decreases. $\overline{n_A(t)}$ for the E model is always less than the SL model and diverges more as p_A increases (μ_A increases). The L model may be either greater than or less than either or both the SL and E models depending on the values of the parameters. The variations between models may be very substantial even in the early stages of the conflict.

In Figure 2 we have again plotted $\overline{n_A(t)}$ but this time p_A is fixed and we vary the parameters r_A and r_B such that the ratio $\frac{r_A}{r_B}$ is fixed. In all cases $\overline{n_A(\infty)}$ is different for each model but is the same within each model as the parameters vary. This is equivalent to only varying the mean firing times on each side, maintaining the kill ratio constant. What does change is that the time to near-convergence increases drastically as the r_A's and r_B's decrease. This is not unexpected. And again there is substantial variations, in $\overline{n_A(t)}$ among the models.

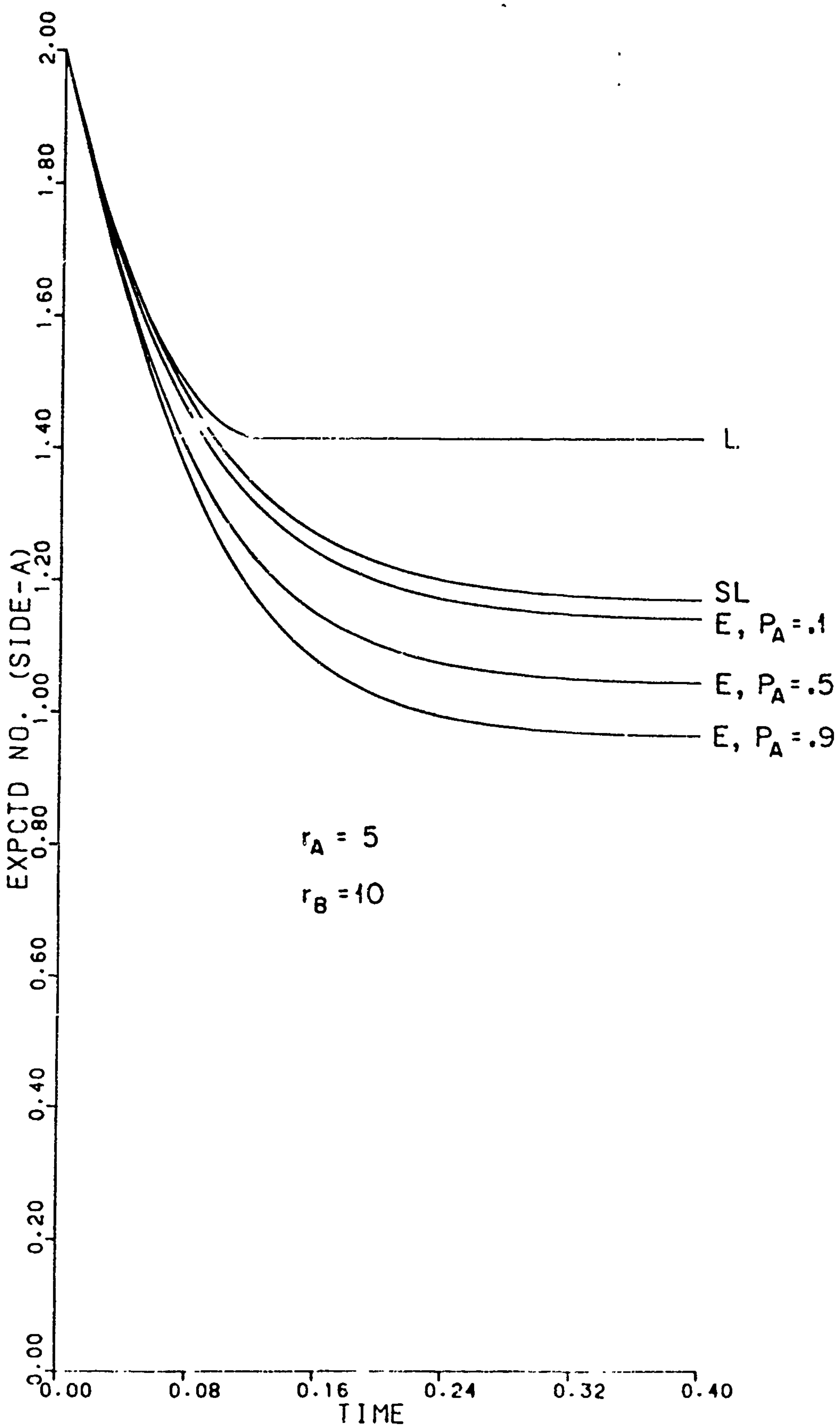
In Figure 3 we shift to plotting $\overline{n_B(t)}$ and hold the A side's parameters constant and vary r_B. In this case the approach to near-convergence on $\overline{n_B(\infty)}$ varies with r_B. As r_B increases the approach time decreases, at the same time $\overline{n_B(\infty)}$ itself increases. It is uniformly true that for $\overline{n_B(t)}$ the E model always is larger than the SL model and the L model may be located any where relatively depending on the value of the parameters. Again the variations among the models is quite substantial.

We note at this point, that Clark [4], has previously shown that the L and SL model vary and has derived the error term which he calls "bias".

Figure 4 is similar to Figure 1 in that r_A and r_B are fixed while p_A is varied, but here we are observing the standard deviation of $N_A(t)$. There is no plot for the Lanchester model since it is purely deterministic. As before varying p_A does not change anything in the SL model but the E model's standard deivation decreases as p_A increases. The difference may be considerable even in the early stages if the kill rates are large. Here we observe that the standard deviation for the E model may be greater than or less than the SL model depending on the kill rates.

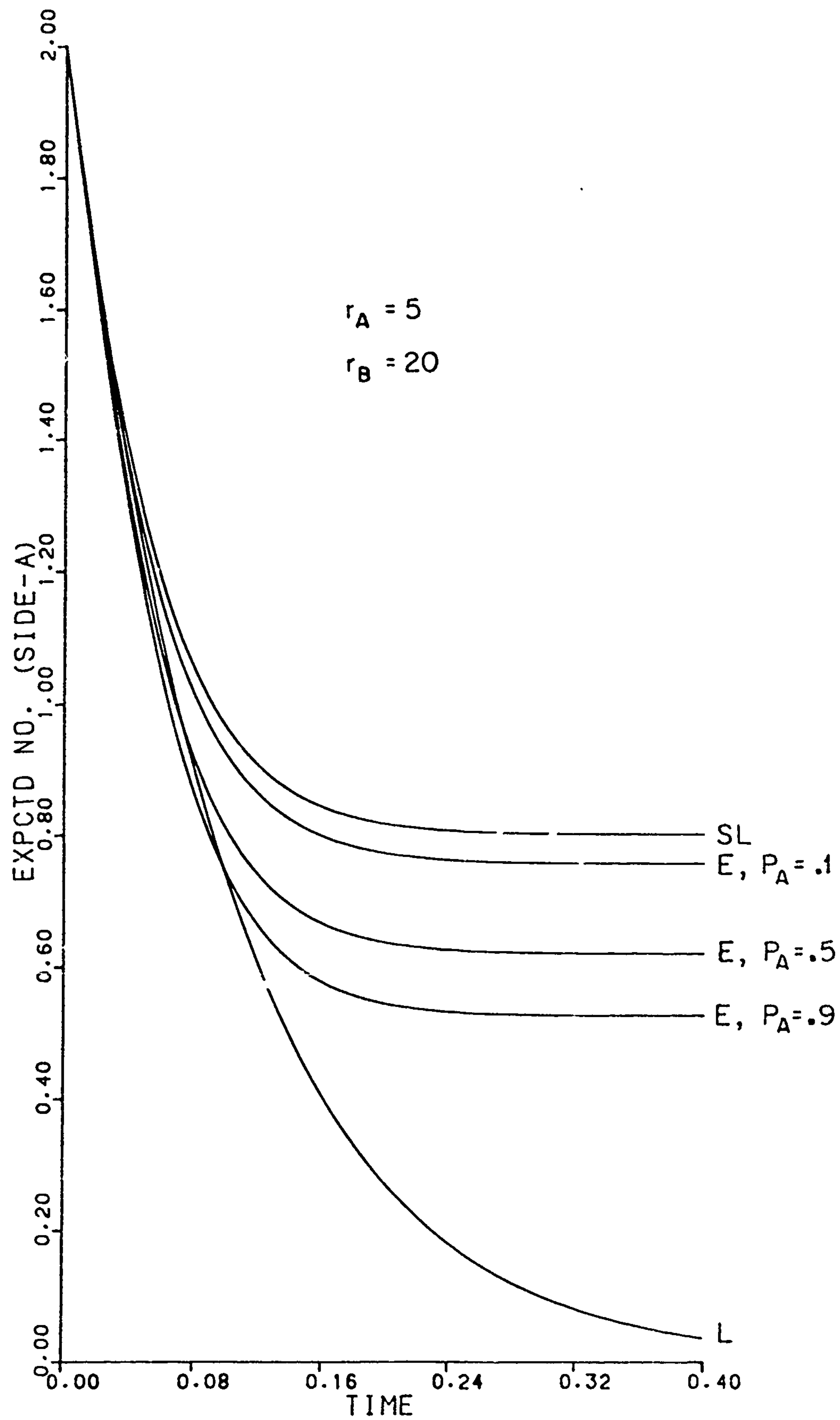
Figure 5 is similar to Figure 2 as we hold p_A fixed vary r_A and r_B such that r_A/r_B remain fixed and observe the standard deviation of $N_A(t)$. Again, as r_A and r_B vary each model is asymptotic at infinity to the same value of the standard derivation but the asymptotes vary between the SL and E models although not substantially. The time to near-convergence increases dramatically as the kill rates increase.

Finally, Figure 6 is similar to Figure 3 where the A side's parameters are held constant, r_B is varied and the standared deviation of $N_B(t)$ is observed. As before, the approach to near-convergence of the standard deviation of $N_B(t)$ increases as r_B increases. Substantial variation between models is again evident.



EXPCTD NO. (SIDE-A)
2.00
1.80
1.60
1.40
1.20
1.00
0.80
0.60
0.40
0.20
0.00
0.00
0.08
0.16
0.24
0.32
0.40
TIME
L
SL
E, P_A = .1
E, P_A = .5
E, P_A = .9
r_A = 5
r_B = 10

FIGURE 1(a)



2.00
1.80
1.60
1.40
1.20
1.00
0.80
0.60
0.40
0.20
0.00
EXPCTD NO. (SIDE-A)
$r_A = 5$
$r_B = 20$
SL
E, $P_A = .1$
E, $P_A = .5$
E, $P_A = .9$
L
0.00
0.08
0.16
0.24
0.32
0.40
TIME

FIGURE 1(b)

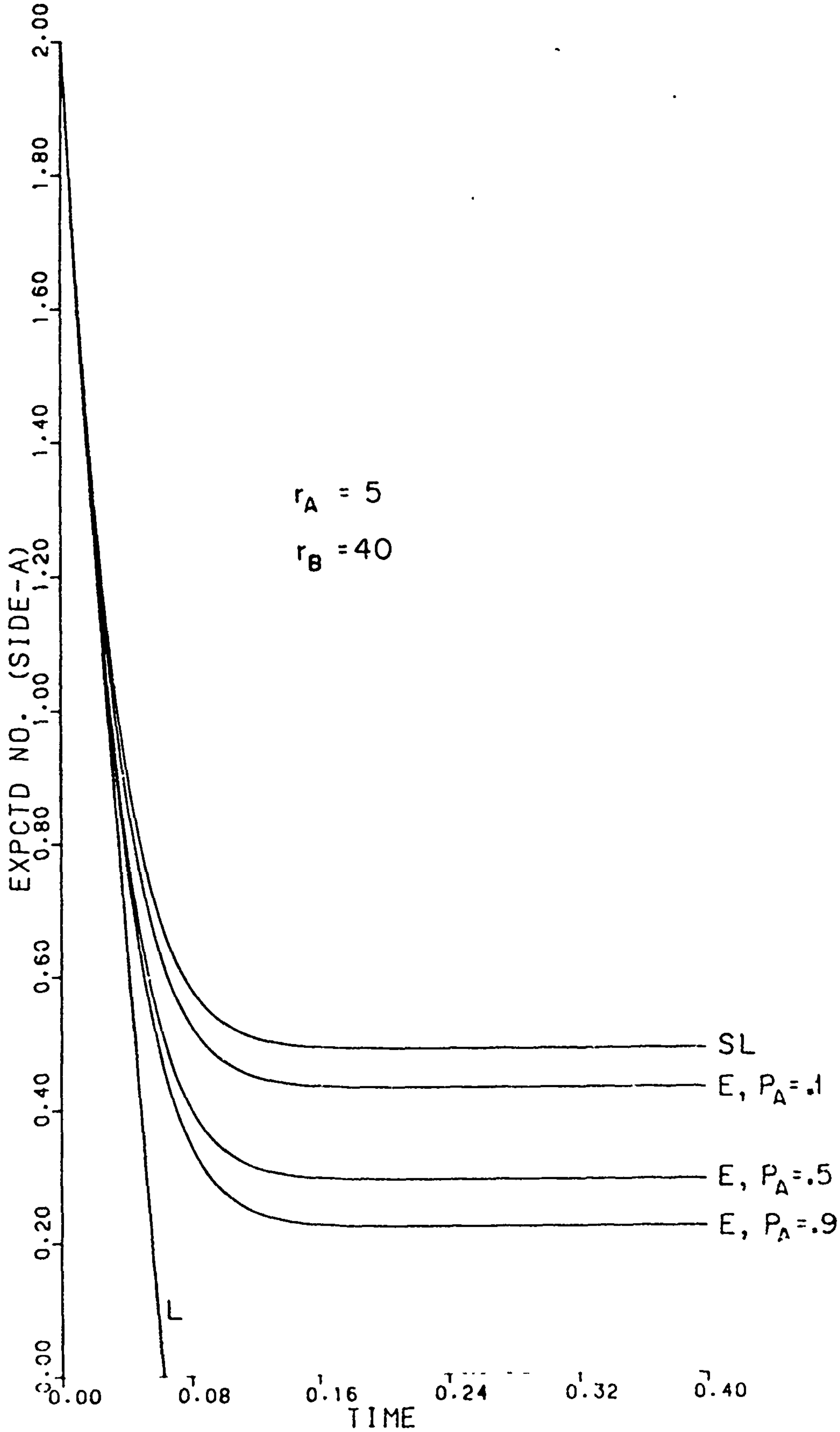

2.00
1.80
1.60
1.40
1.20
1.00
0.80
0.60
0.40
0.20
0.00
EXPCTD NO. (SIDE-A)
0.00
0.08
0.16
0.24
0.32
0.40
TIME
r_A = 5
r_B =40
SL
E, P_A=.1
E, P_A=.5
E, P_A=.9
L

FIGURE 1(c)

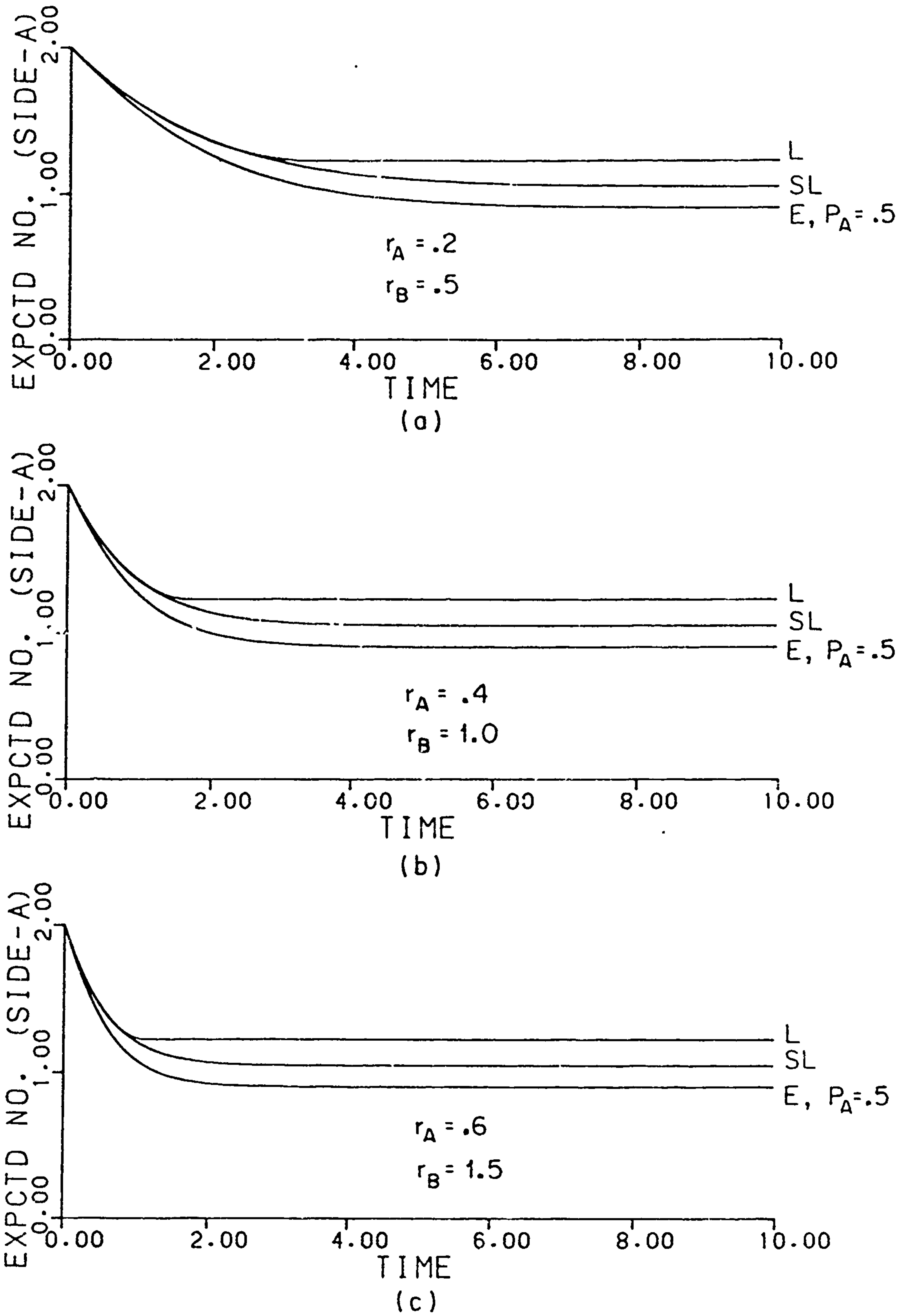
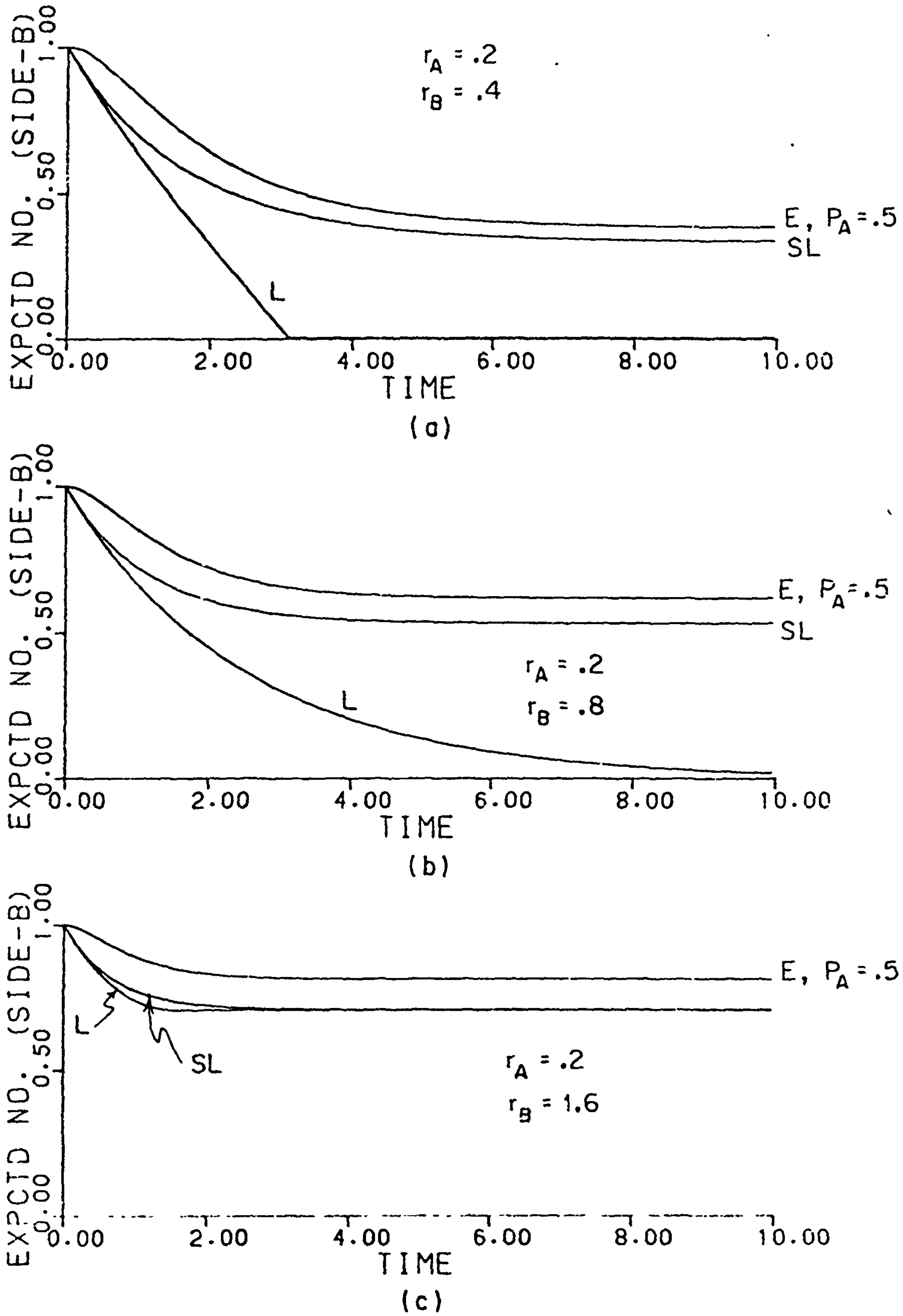
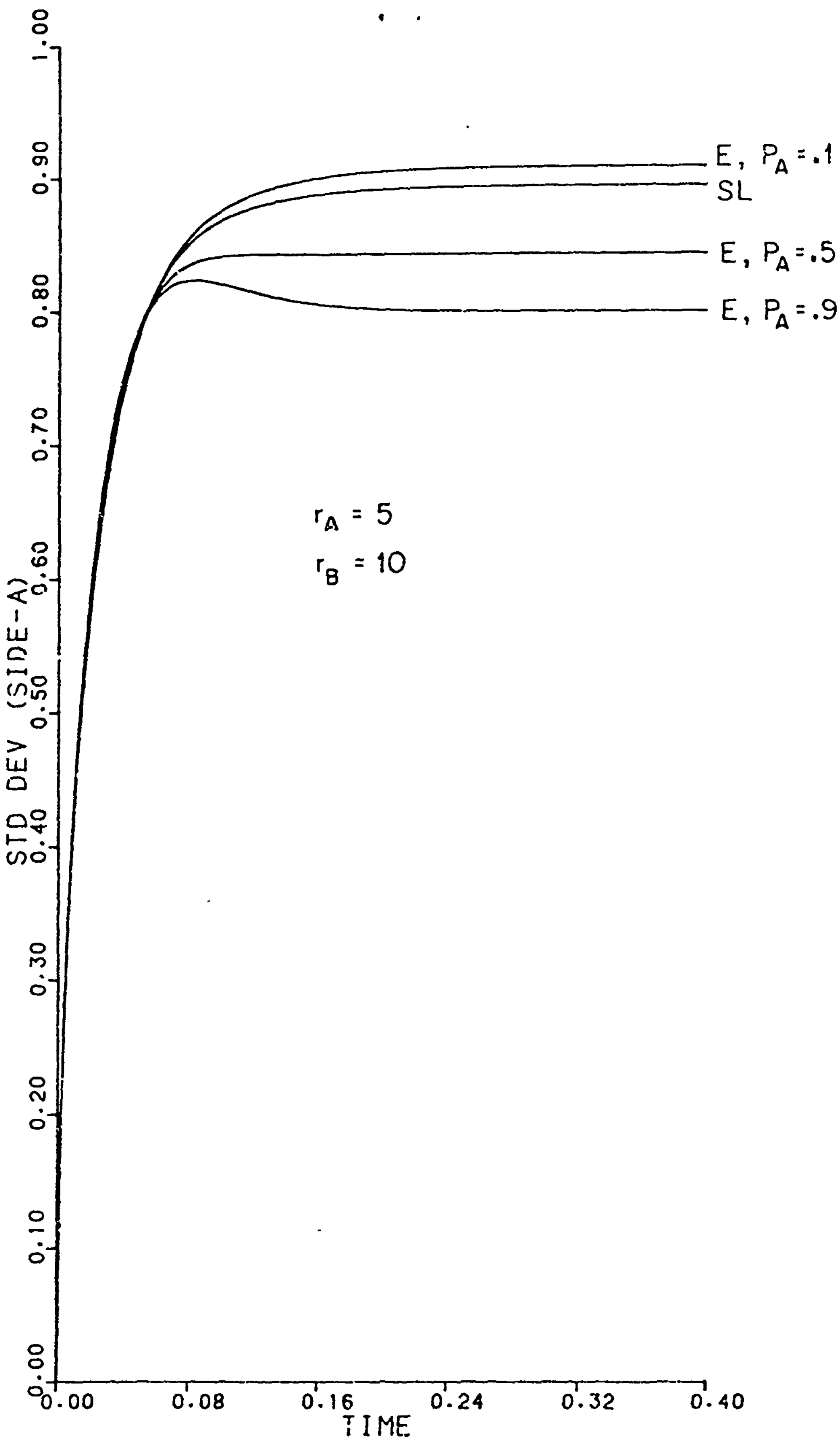

EXPCTD NO. (SIDE-A)
2.00
1.00
0.00
L
SL
E, P_A=.5
r_A = .2
r_B = .5
0.00
2.00
4.00
6.00
8.00
10.00
TIME
(a)
EXPCTD NO. (SIDE-A)
r_A = .4
r_B = 1.0
(b)
EXPCTD NO. (SIDE-A)
r_A = .6
r_B = 1.5
(c)

FIGURE 2



EXPCTD NO. (SIDE-B)
0.00
0.50
1.00
0.00
2.00
4.00
6.00
8.00
10.00
TIME
(a)
$r_A = .2$
$r_B = .4$
E, $P_A = .5$
SL
L
EXPCTD NO. (SIDE-B)
0.00
0.50
1.00
0.00
2.00
4.00
6.00
8.00
10.00
TIME
(b)
E, $P_A = .5$
SL
L
$r_A = .2$
$r_B = .8$
EXPCTD NO. (SIDE-B)
0.00
0.50
1.00
0.00
2.00
4.00
6.00
8.00
10.00
TIME
(c)
E, $P_A = .5$
L
SL
$r_A = .2$
$r_B = 1.6$

FIGURE 3



STD DEV (SIDE-A)
1.00
0.90
0.80
0.70
0.60
0.50
0.40
0.30
0.20
0.10
0.00
E, P_A=.1
SL
E, P_A=.5
E, P_A=.9
r_A = 5
r_B = 10
0.00
0.08
0.16
0.24
0.32
0.40
TIME

FIGURE 4 (a)

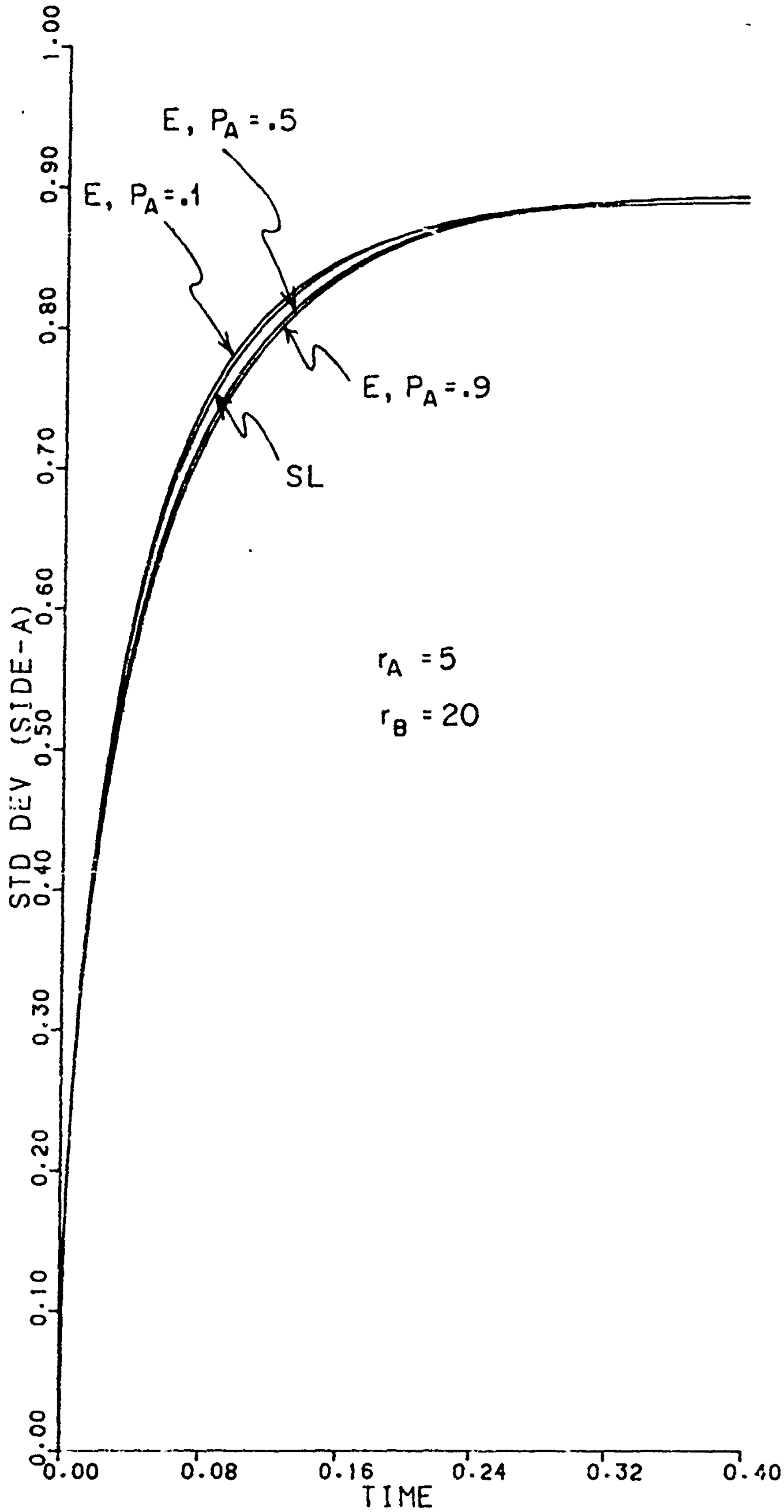

E, P_A =.5
E, P_A =.1
E, P_A =.9
SL
r_A = 5
r_B = 20
STD DEV (SIDE-A)
TIME
0.00
0.10
0.20
0.30
0.40
0.50
0.60
0.70
0.80
0.90
1.00
0.00
0.08
0.16
0.24
0.32
0.40

FIGURE 4(b)

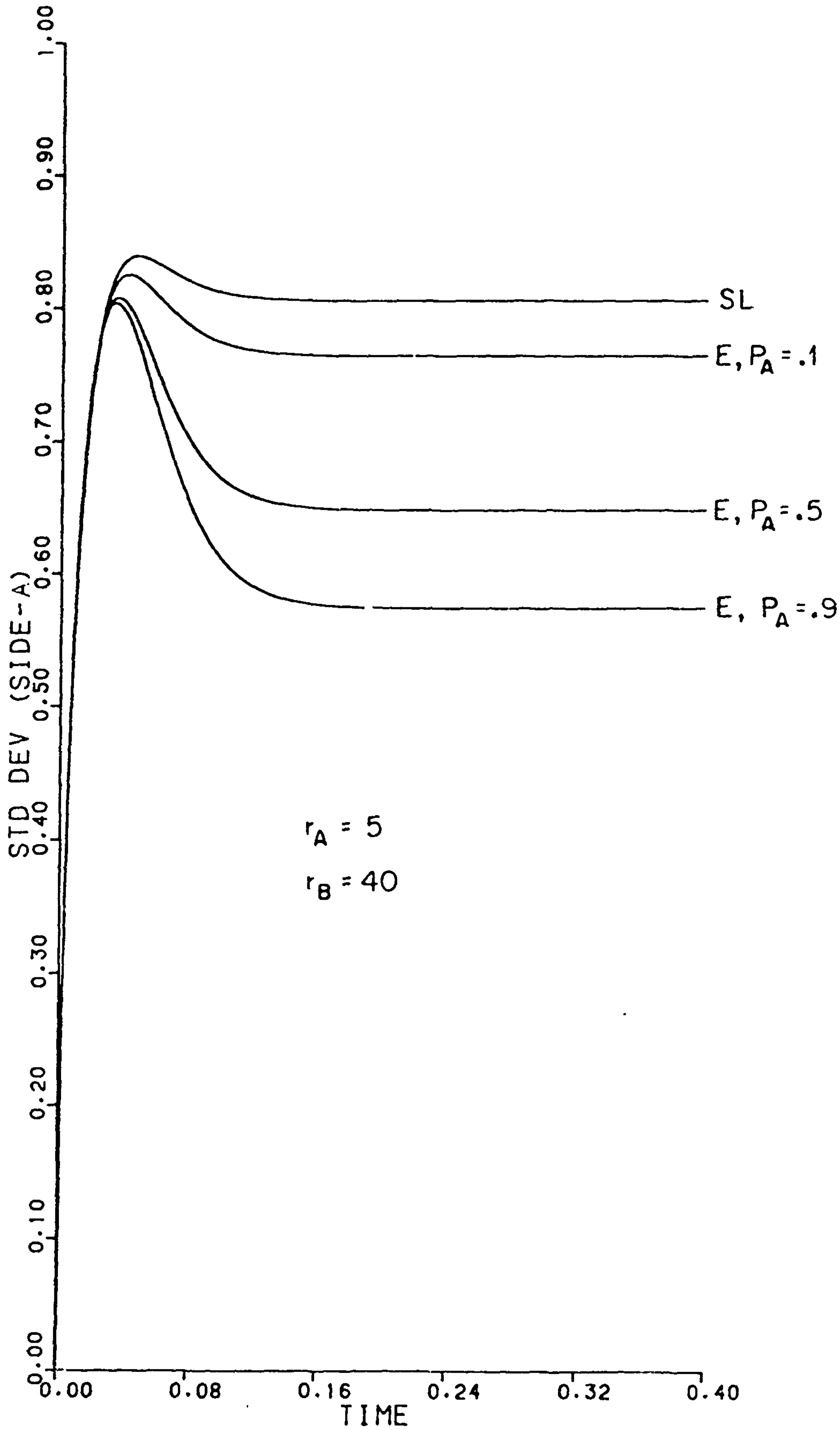

STD DEV (SIDE-A)
TIME
SL
E, P_A=.1
E, P_A=.5
E, P_A=.9
r_A = 5
r_B = 40
0.00
0.10
0.20
0.30
0.40
0.50
0.60
0.70
0.80
0.90
1.00
0.08
0.16
0.24
0.32
0.40

FIGURE 4(c)

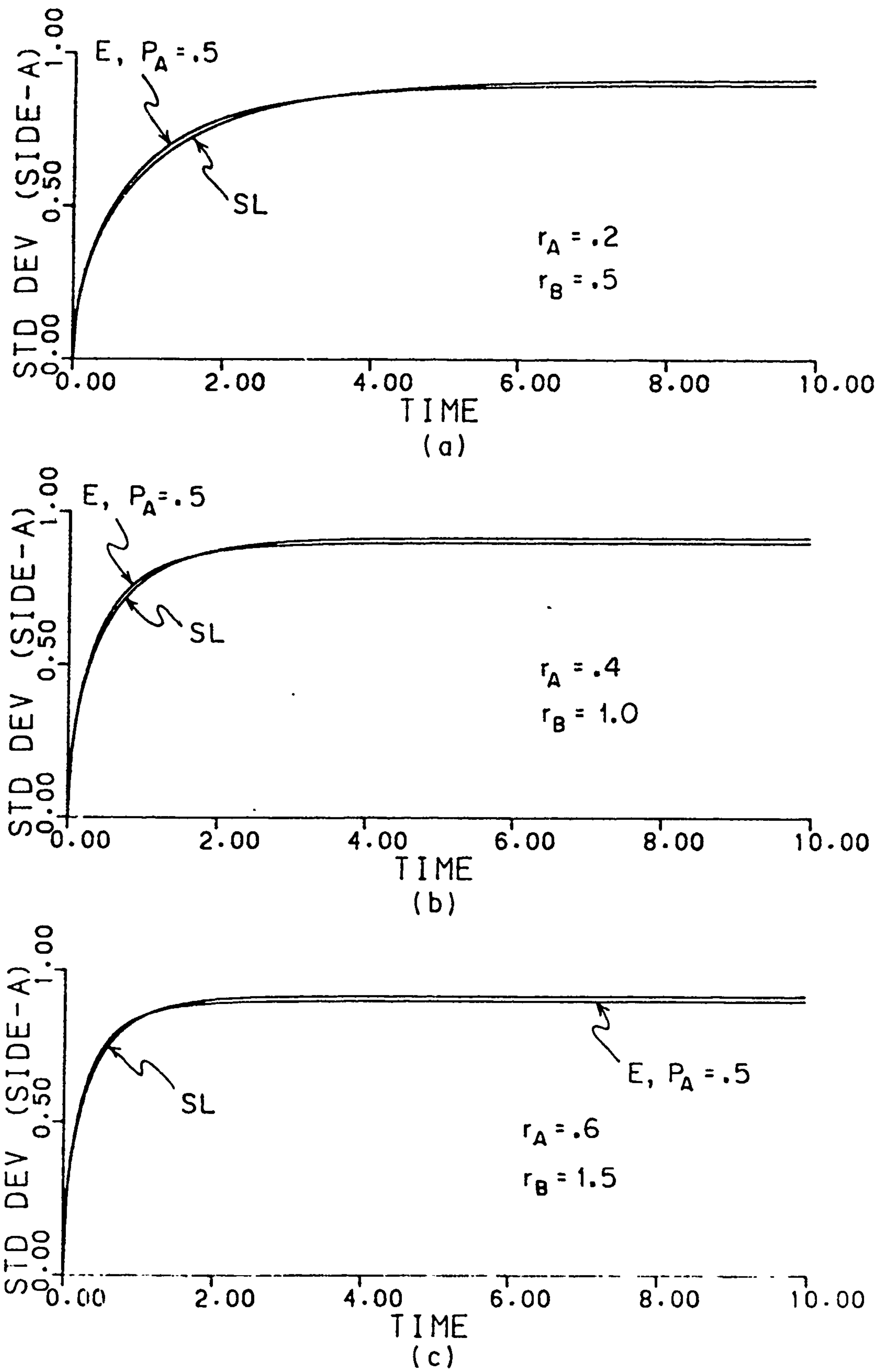

E, P_A = .5
SL
r_A = .2
r_B = .5
STD DEV (SIDE-A)
0.00
0.50
1.00
0.00
2.00
4.00
6.00
8.00
10.00
TIME
(a)
E, P_A = .5
SL
r_A = .4
r_B = 1.0
STD DEV (SIDE-A)
TIME
(b)
SL
E, P_A = .5
r_A = .6
r_B = 1.5
STD DEV (SIDE-A)
TIME
(c)

FIGURE 5

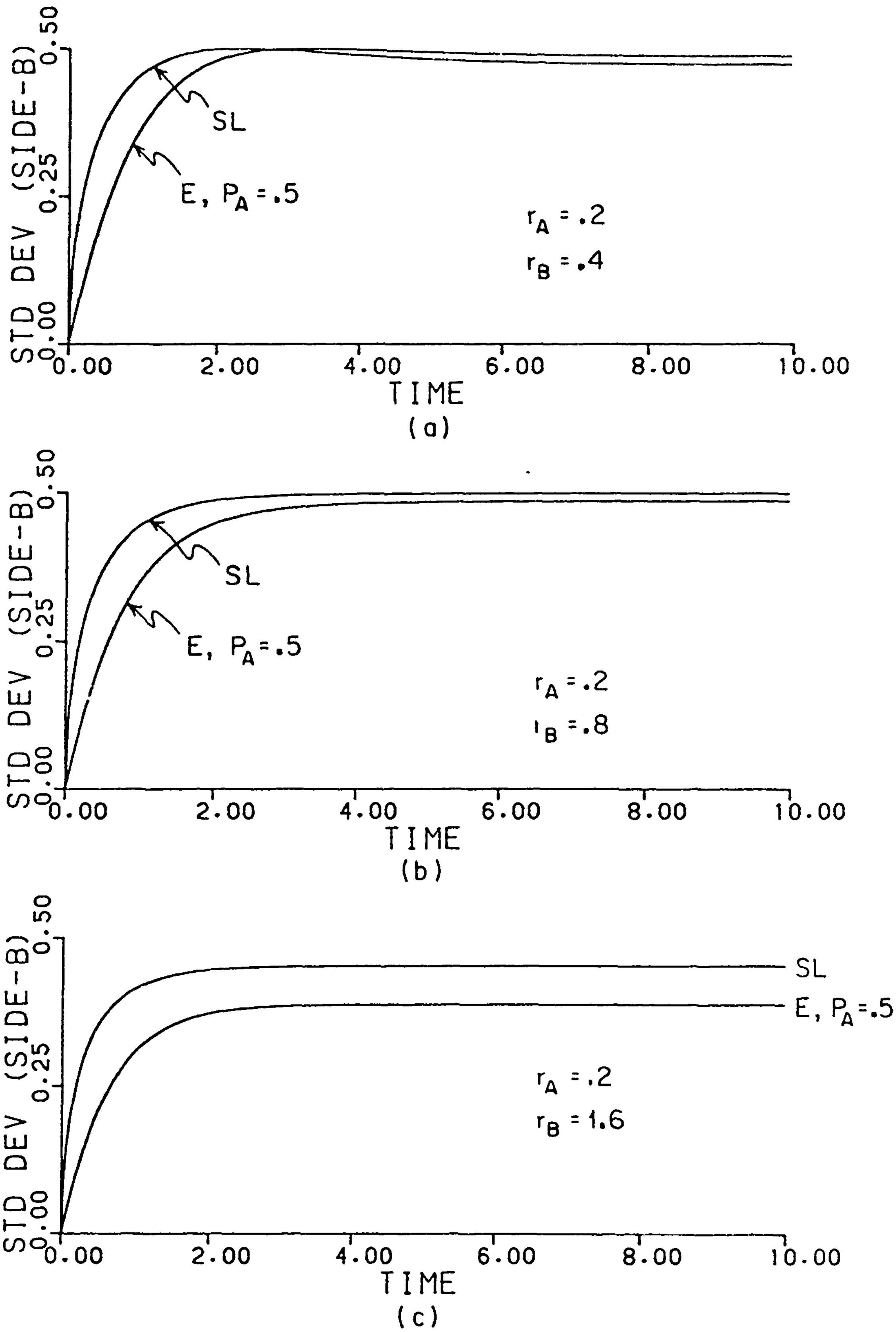

STD DEV (SIDE-B)
0.00
0.25
0.50
SL
E, P_A=.5
r_A = .2
r_B = .4
0.00
2.00
4.00
6.00
8.00
10.00
TIME
(a)
STD DEV (SIDE-B)
SL
E, P_A=.5
r_A = .2
r_B = .8
TIME
(b)
STD DEV (SIDE-B)
SL
E, P_A=.5
r_A = .2
r_B = 1.6
TIME
(c)

FIGURE 6

6. Conclusions

We have developed for the first time the general solution for the state probabilities for the two-on-one stochastic duel and from these derived the winning probabilities and the mean and standard deviation functions for the state random variables. These solutions are illustrated by two examples. In the first, all interfiring times are negative exponential (this is the so-called Stochastic Lanchester version of the two-on-one Square Law) and in the second example the side with two contestants had Erlang (2) interfiring times each with the same means as in example 1.

These two examples were then compared with the corresponding Lanchester Square Law solution to illustrate that all three of these models vary substantially from each other and neither the Stochastic Lanchester nor the Lanchester formulation is a satisfactory approximation to the general model at least at the two-on-one level.

REFERENCES

[1] Ancker, C.J. Jr., One-on-One Stochastic Duels, research monograph published by the Military Applications Section of the Operations Research Society of America, January 1982, 381 pp.

[2] Ancker, C.J., Jr., "The Status of Developments in the Theory of Stochastic Duels - II", Operations Research, Vol. 15, no. 3, May-June 1967, pp. 388-406.

[3] Anderson, D.W., "An Analysis of Stochastic Duels Involving Fixed Rates of Fire" Master of Science Thesis, The Naval Postgratuate School, Monterey, CA, March 1971, 58 pp., DDC #AD721237.

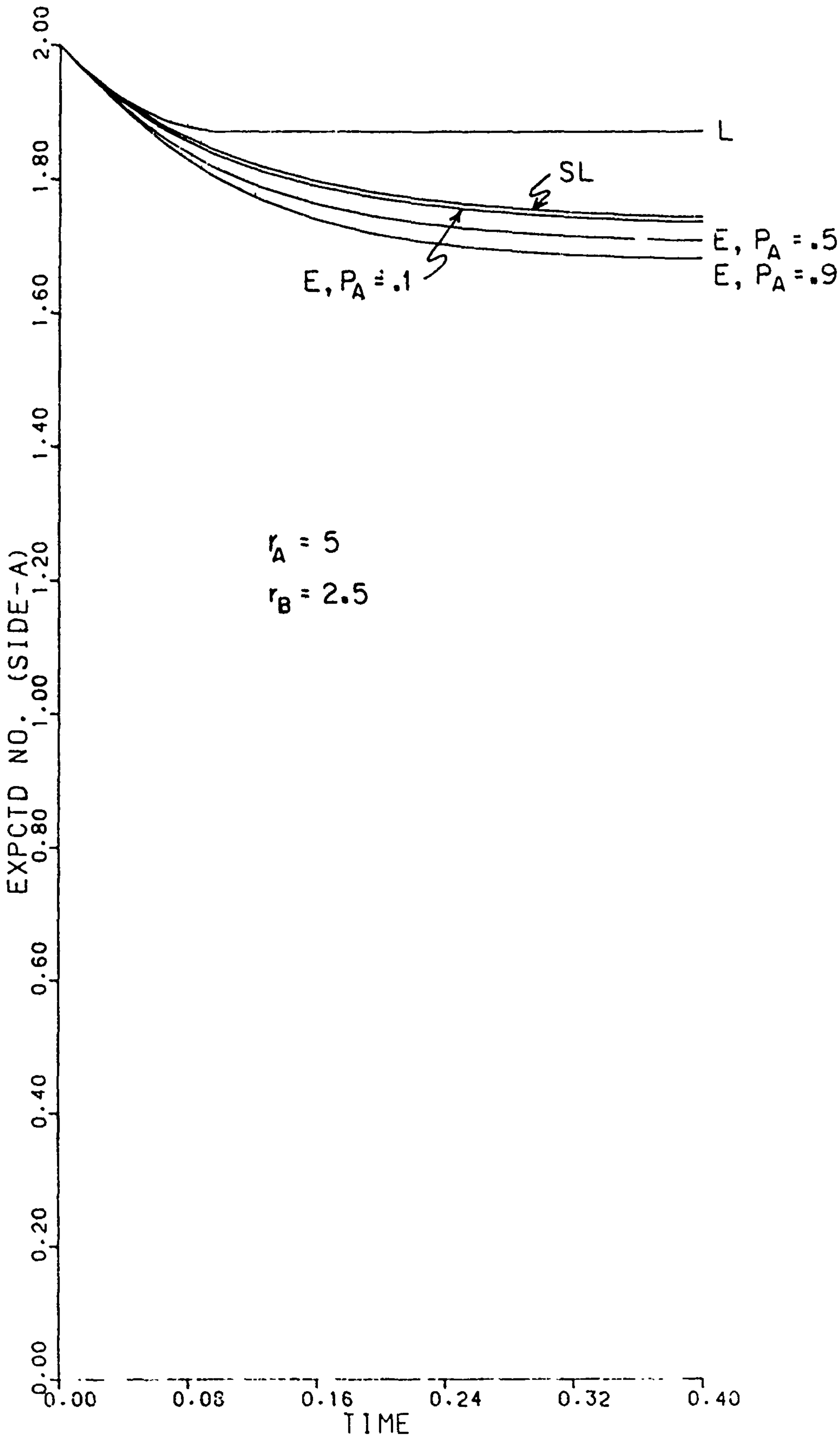
[4] Clark, G.M., "The Combat Analysis Model", Ph.D. thesis in Industrial Engineering, Ohio State University, 1969, 286 pp.

APPENDIX I

In this appendix we provide graphs supplementing the ones presented in the body of the paper by including more values of the parameters r_A, r_B, and p_A. The correspondence between the various sets in the text and the appendix are shown in the table below.

Text	Appendix
1	1
2	2,3
3	4,5
4	6
5	7,8
6	9,10

These figures demonstrate again that variations among the models are substantial.



2.00
1.80
1.60
1.40
1.20
1.00
0.80
0.60
0.40
0.20
0.00
EXPCTD NO. (SIDE-A)
0.00
0.08
0.16
0.24
0.32
0.40
TIME
L
SL
E, P_A = .5
E, P_A = .9
E, P_A = .1
r_A = 5
r_B = 2.5

FIGURE 1(a)

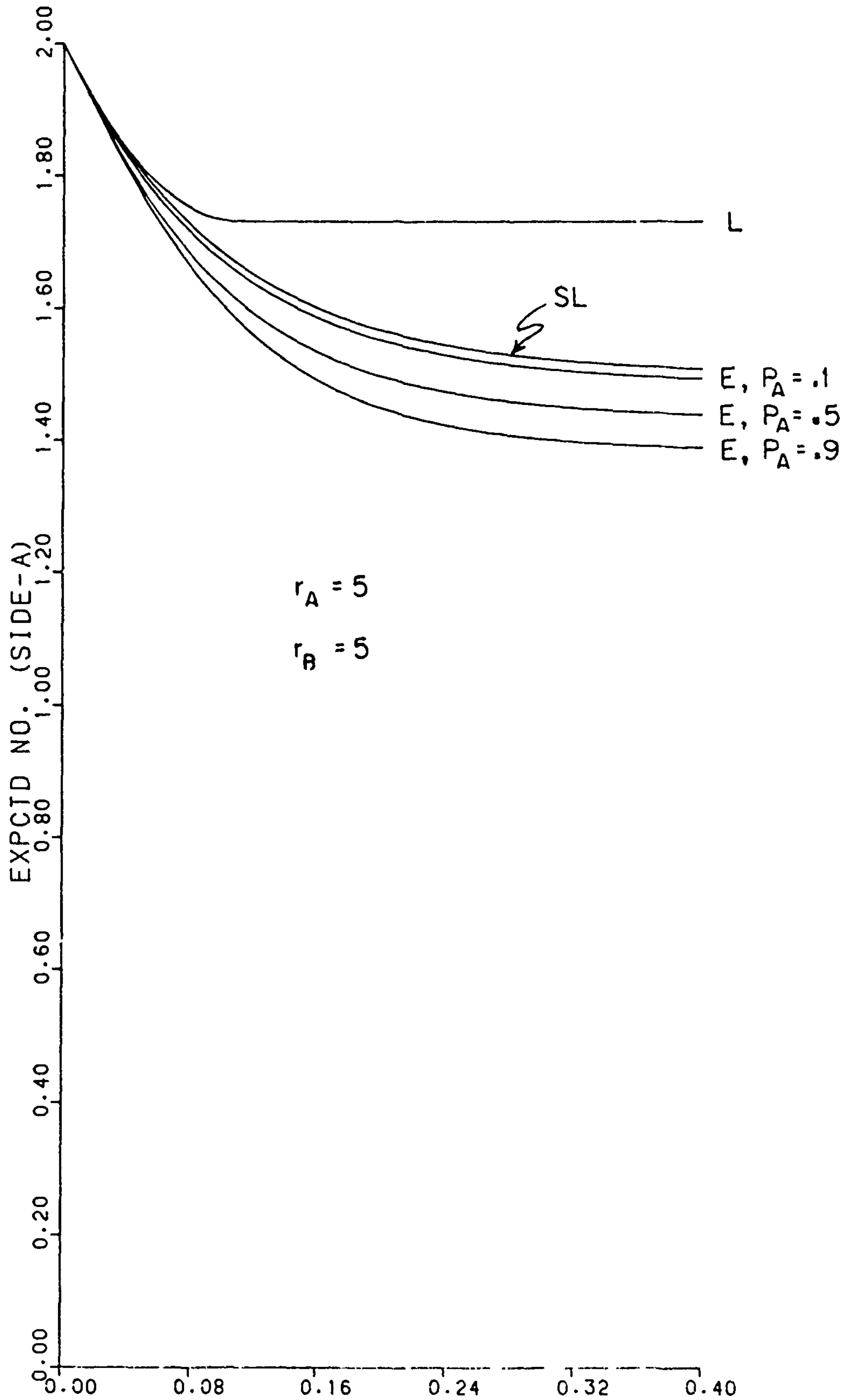

2.00
1.80
1.60
1.40
1.20
1.00
0.80
0.60
0.40
0.20
0.00
EXPCTD NO. (SIDE-A)
0.00
0.08
0.16
0.24
0.32
0.40
TIME
L
SL
E, P_A = .1
E, P_A = .5
E, P_A = .9
r_A = 5
r_B = 5

FIGURE 1(b)

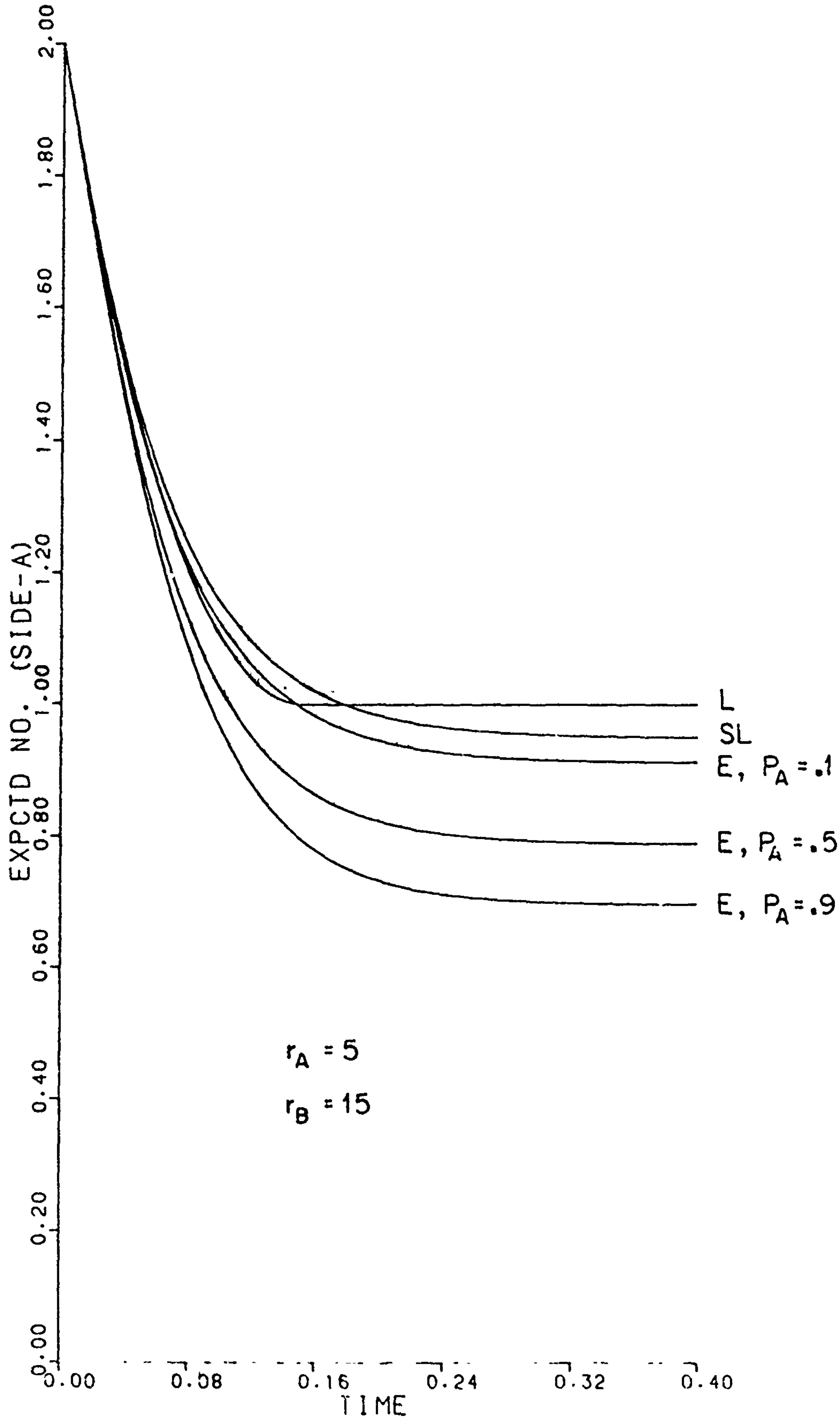
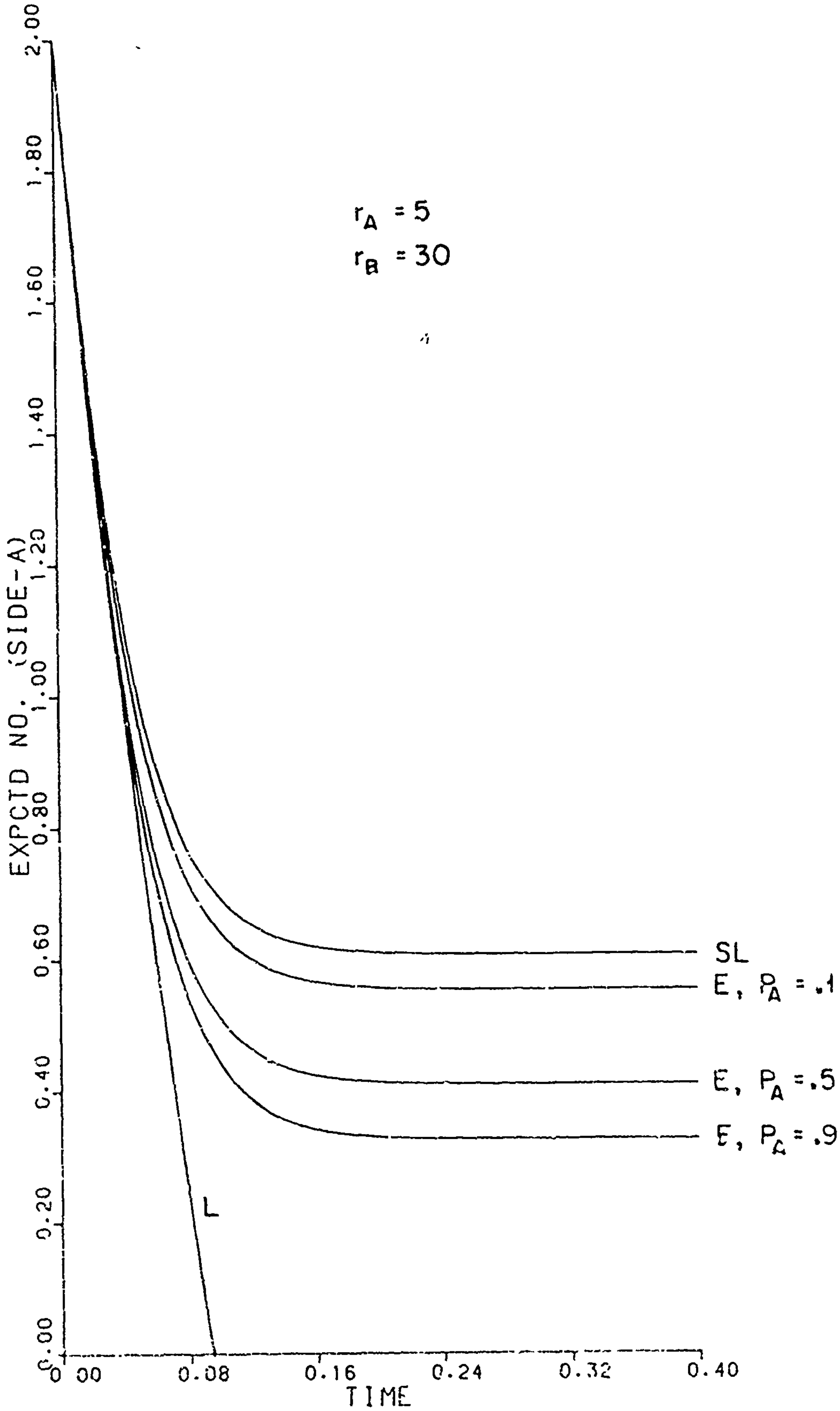
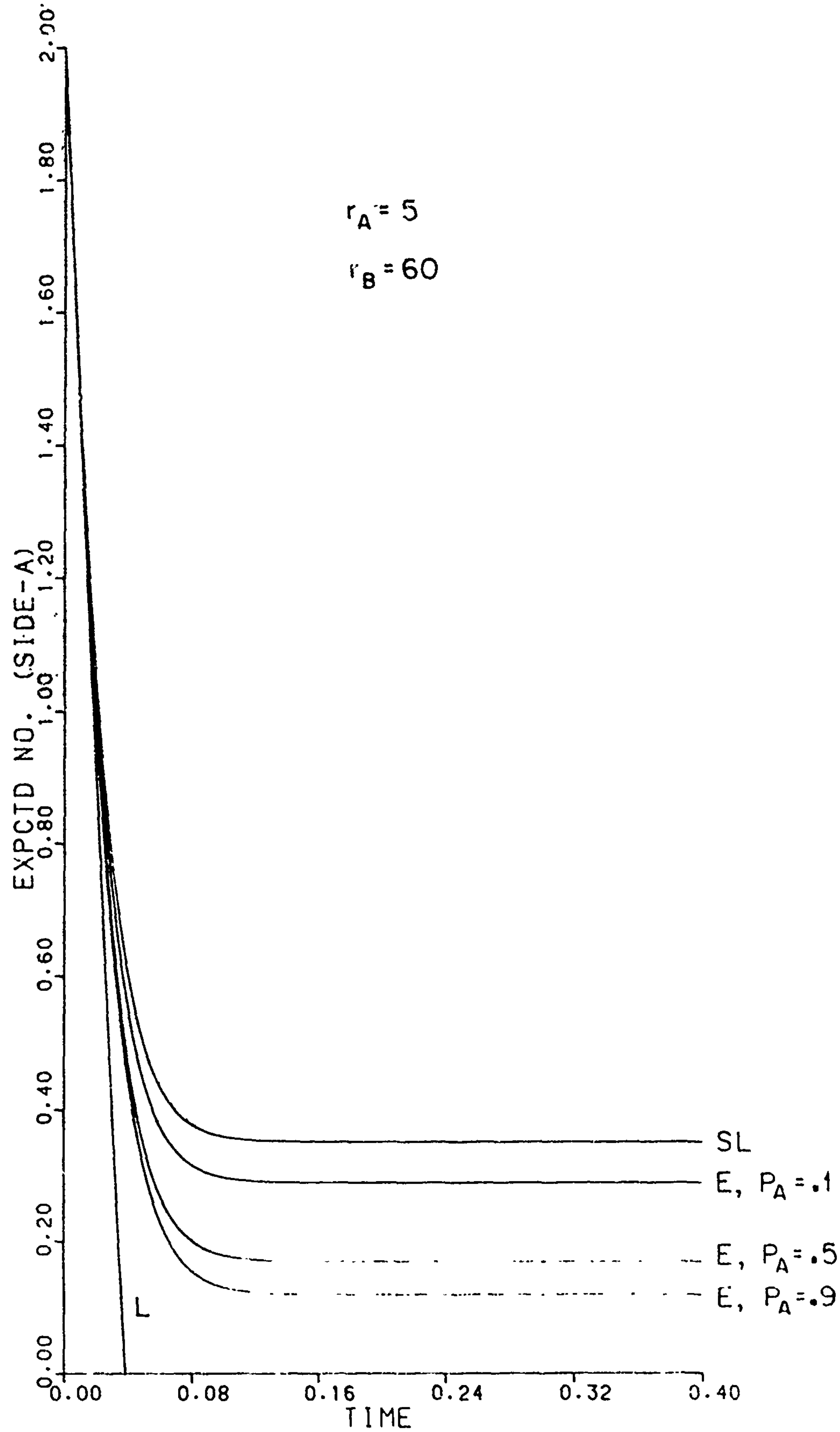

EXPCTD NO. (SIDE-A)
2.00
1.80
1.60
1.40
1.20
1.00
0.80
0.60
0.40
0.20
0.00
0.00
0.08
0.16
0.24
0.32
0.40
TIME
L
SL
E, P_A = .1
E, P_A = .5
E, P_A = .9
r_A = 5
r_B = 15

FIGURE 1(c)



$r_A = 5$
$r_B = 30$
EXPCTD NO. (SIDE-A)
2.00
1.80
1.60
1.40
1.20
1.00
0.80
0.60
0.40
0.20
0.00
SL
E, P_A = .1
E, P_A = .5
E, P_A = .9
L
0.00
0.08
0.16
0.24
0.32
0.40
TIME

FIGURE 1(d)



$r_A = 5$
$r_B = 60$
EXPCTD NO. (SIDE-A)
2.00
1.80
1.60
1.40
1.20
1.00
0.80
0.60
0.40
0.20
0.00
0.00
0.08
0.16
0.24
0.32
0.40
TIME
SL
E, $P_A = .1$
E, $P_A = .5$
E, $P_A = .9$
L

FIGURE 1(e)

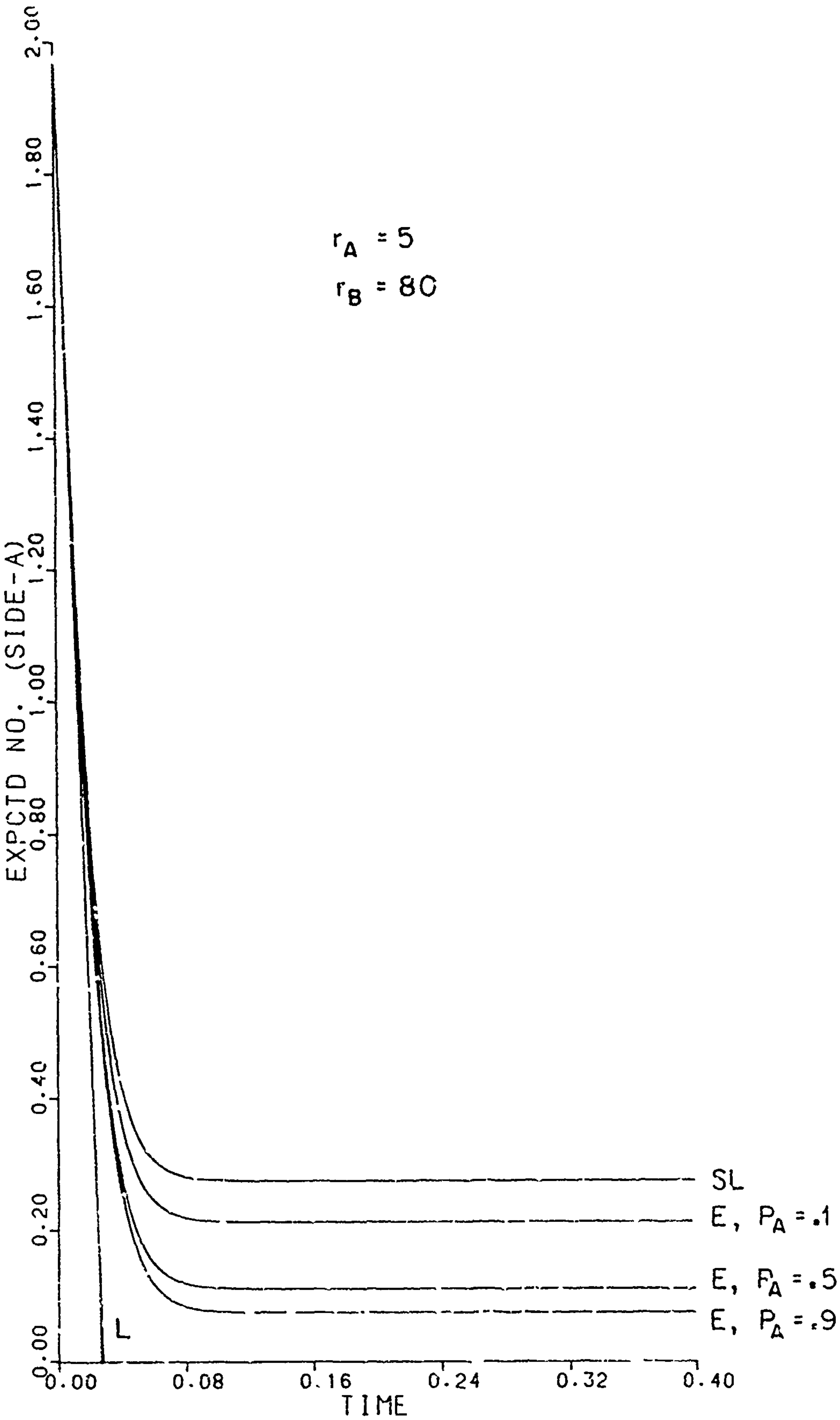
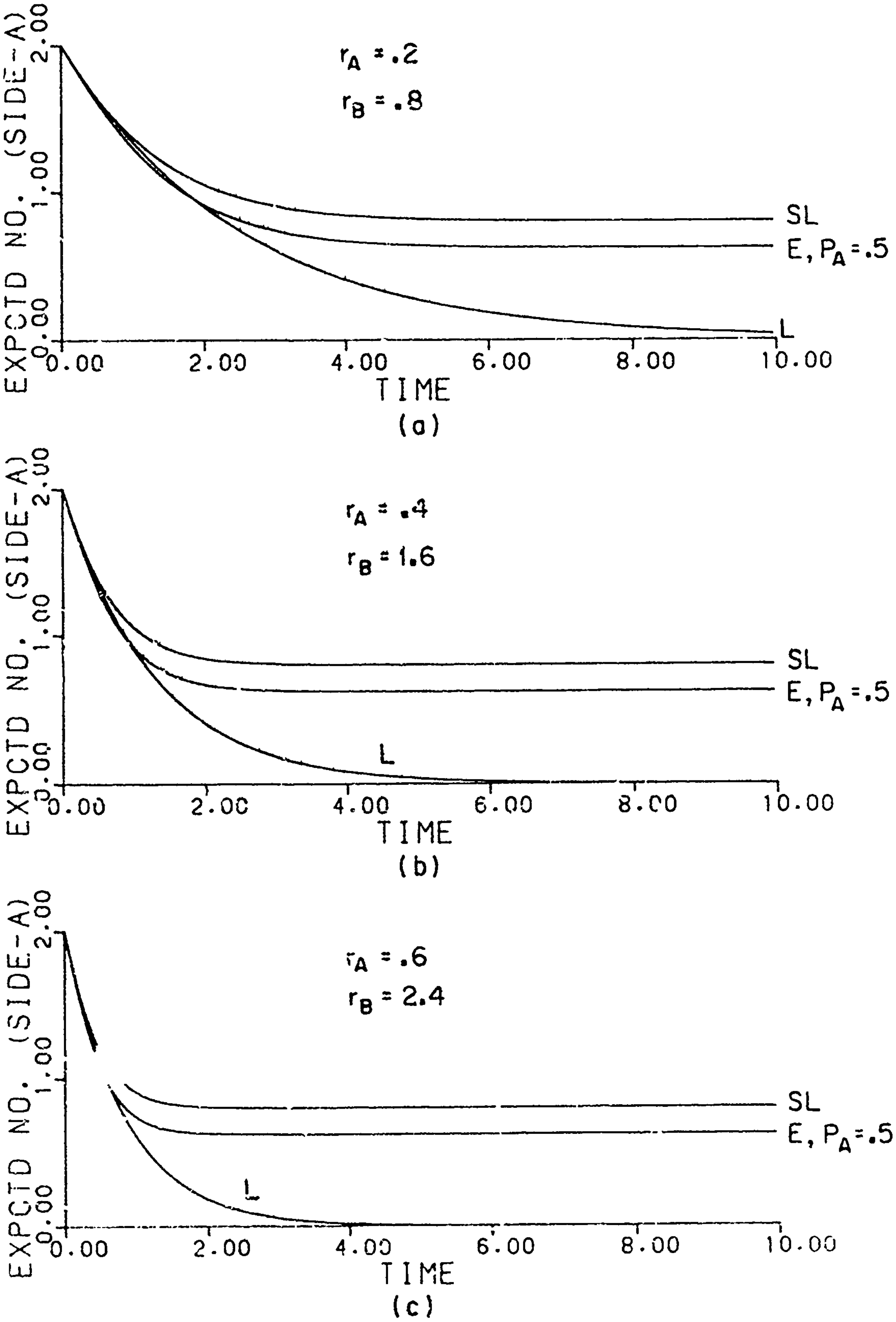
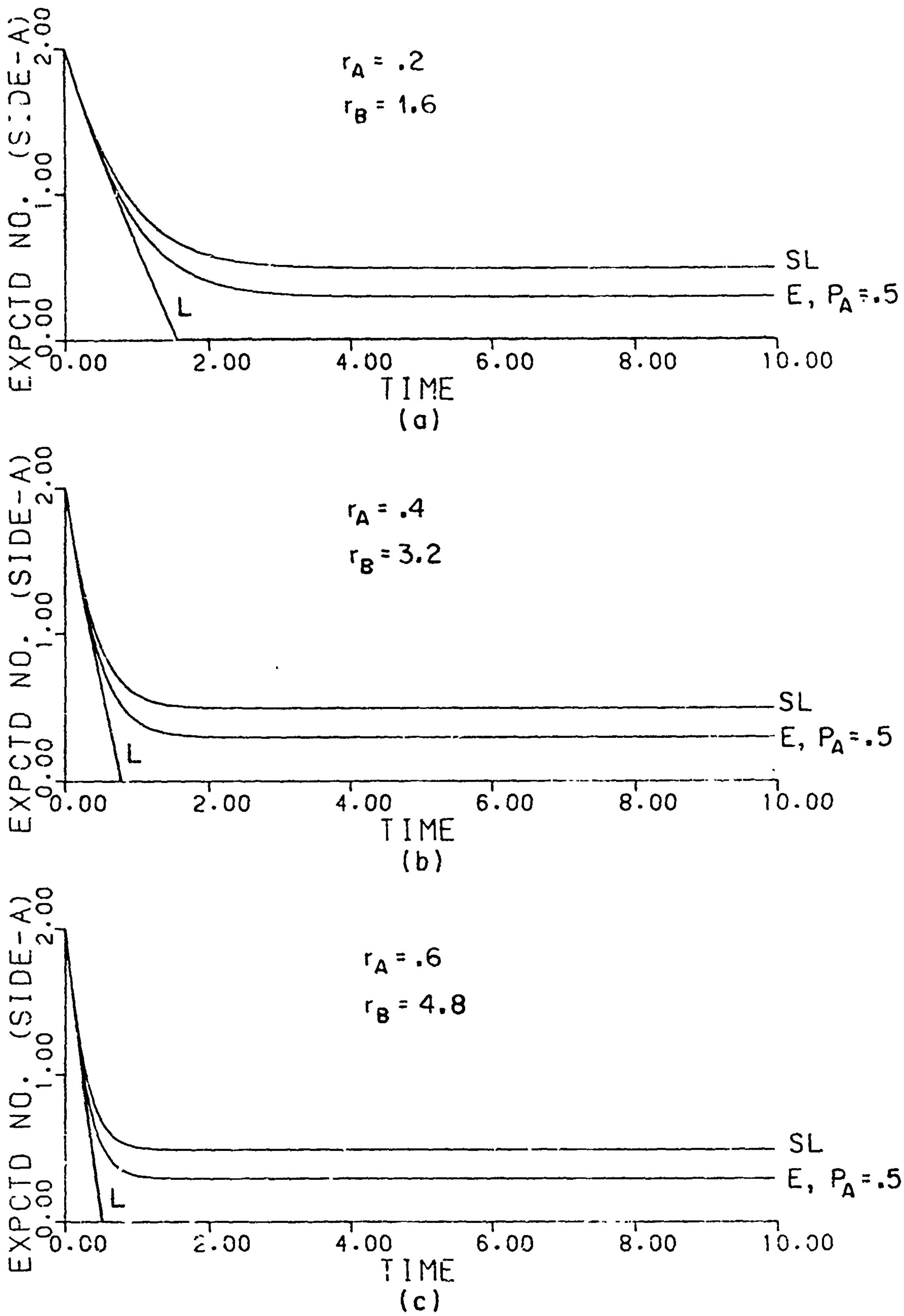

2.00
1.80
1.60
1.40
1.20
1.00
0.80
0.60
0.40
0.20
0.00
EXPCTD NO. (SIDE-A)
r_A = 5
r_B = 80
SL
E, P_A = .1
E, P_A = .5
E, P_A = .9
L
0.00
0.08
0.16
0.24
0.32
0.40
TIME

FIGURE 1(f)



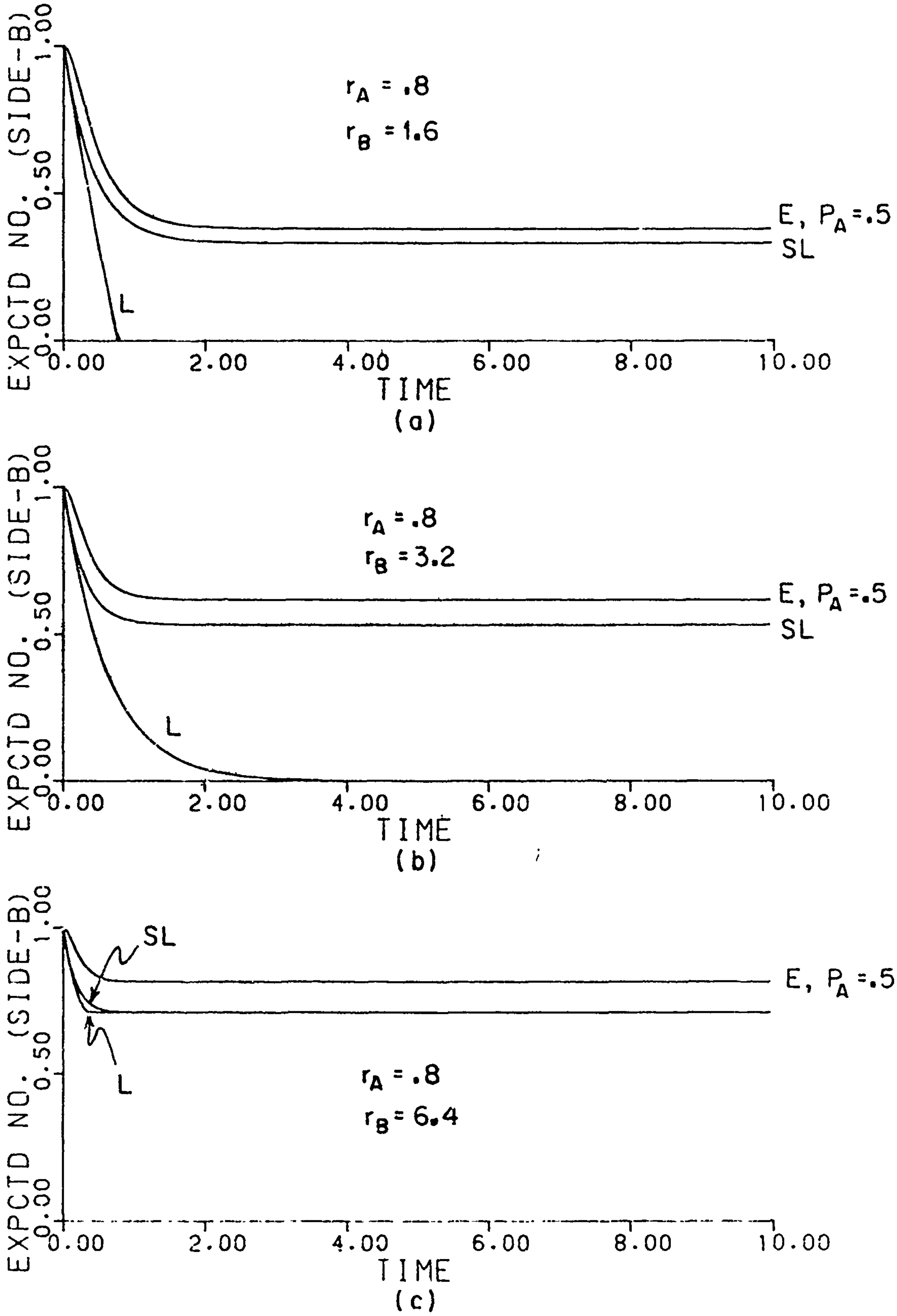
EXPCTD NO. (SIDE-A)
2.00
1.00
0.00
r_A = .2
r_B = .8
SL
E, P_A = .5
L
0.00
2.00
4.00
6.00
8.00
10.00
TIME
(a)
EXPCTD NO. (SIDE-A)
2.00
1.00
0.00
r_A = .4
r_B = 1.6
SL
E, P_A = .5
L
0.00
2.00
4.00
6.00
8.00
10.00
TIME
(b)
EXPCTD NO. (SIDE-A)
2.00
1.00
0.00
r_A = .6
r_B = 2.4
SL
E, P_A = .5
L
0.00
2.00
4.00
6.00
8.00
10.00
TIME
(c)

FIGURE 2



$r_A = .2$
$r_B = 1.6$
EXPCTD NO. (SIDE-A)
2.00
1.00
0.00
SL
E, P_A=.5
L
0.00
2.00
4.00
6.00
8.00
10.00
TIME
(a)
$r_A = .4$
$r_B = 3.2$
EXPCTD NO. (SIDE-A)
2.00
1.00
0.00
SL
E, P_A=.5
L
0.00
2.00
4.00
6.00
8.00
10.00
TIME
(b)
$r_A = .6$
$r_B = 4.8$
EXPCTD NO. (SIDE-A)
2.00
1.00
0.00
SL
E, P_A=.5
L
0.00
2.00
4.00
6.00
8.00
10.00
TIME
(c)

FIGURE 3



$r_A = .8$
$r_B = 1.6$
E, $P_A = .5$
SL
L
EXPCTD NO. (SIDE-B)
1.00
0.50
0.00
0.00
2.00
4.00
6.00
8.00
10.00
TIME
(a)
$r_A = .8$
$r_B = 3.2$
E, $P_A = .5$
SL
L
EXPCTD NO. (SIDE-B)
1.00
0.50
0.00
0.00
2.00
4.00
6.00
8.00
10.00
TIME
(b)
SL
E, $P_A = .5$
L
$r_A = .8$
$r_B = 6.4$
EXPCTD NO. (SIDE-B)
1.00
0.50
0.00
0.00
2.00
4.00
6.00
8.00
10.00
TIME
(c)

FIGURE 4

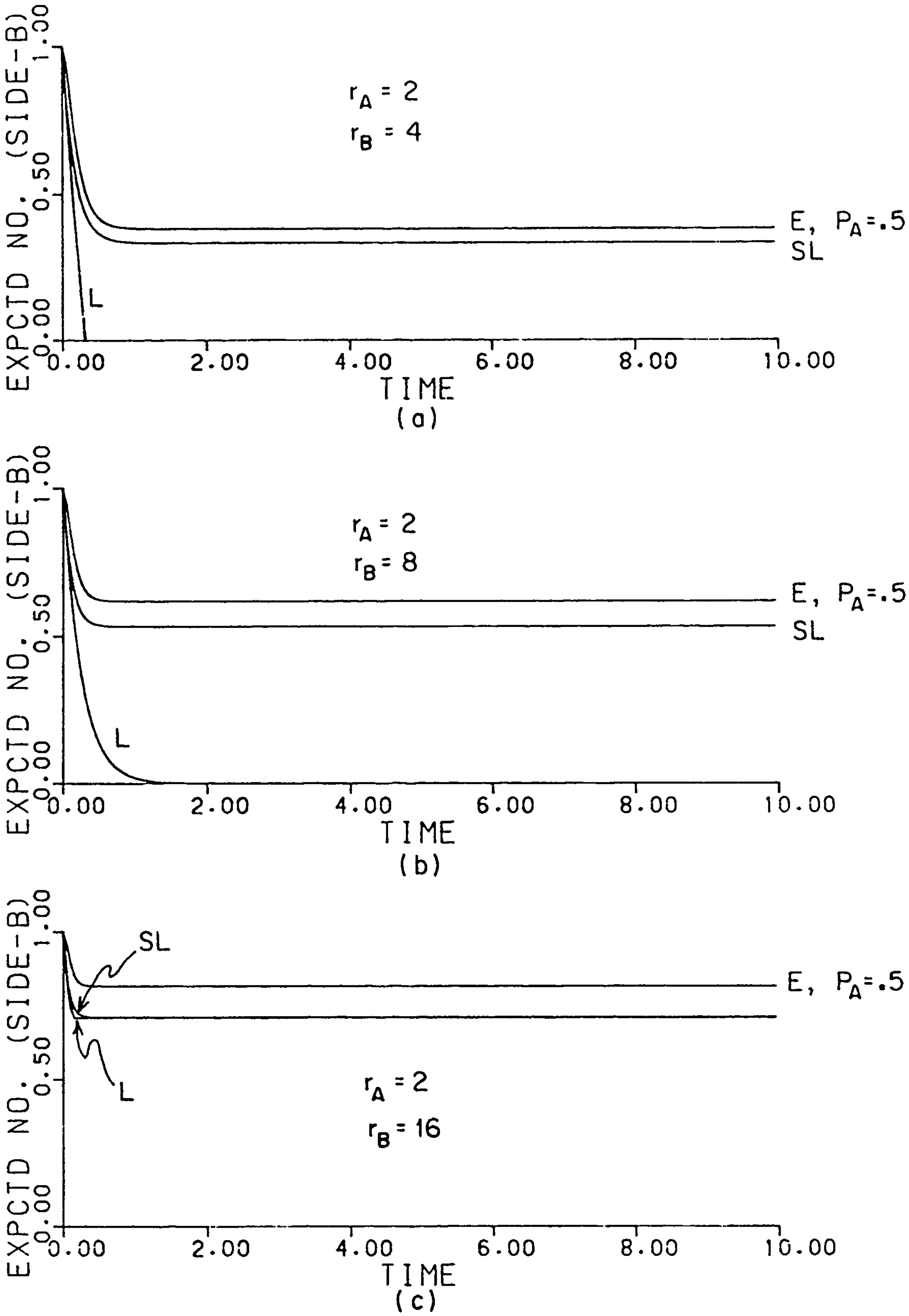
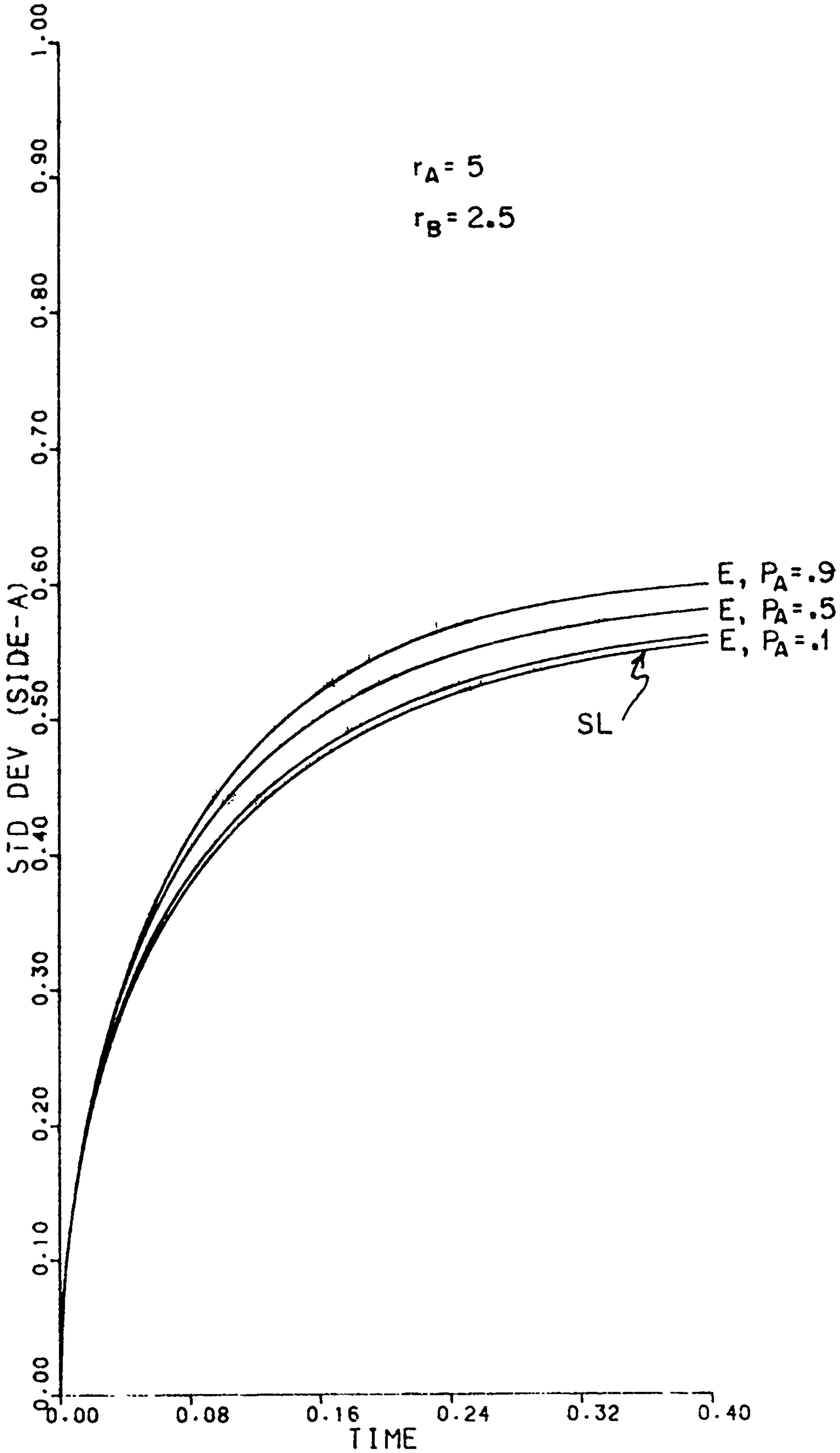

EXPCTD NO. (SIDE-B)
1.00
0.50
0.00
0.00
2.00
4.00
6.00
8.00
10.00
TIME
(a)
$r_A = 2$
$r_B = 4$
E, P_A=.5
SL
L
(b)
$r_A = 2$
$r_B = 8$
(c)
$r_A = 2$
$r_B = 16$

FIGURE 5



$r_A = 5$
$r_B = 2.5$
STD DEV (SIDE-A)
TIME
E, P_A=.9
E, P_A=.5
E, P_A=.1
SL
0.00
0.10
0.20
0.30
0.40
0.50
0.60
0.70
0.80
0.90
1.00
0.00
0.08
0.16
0.24
0.32
0.40

FIGURE 6(a)

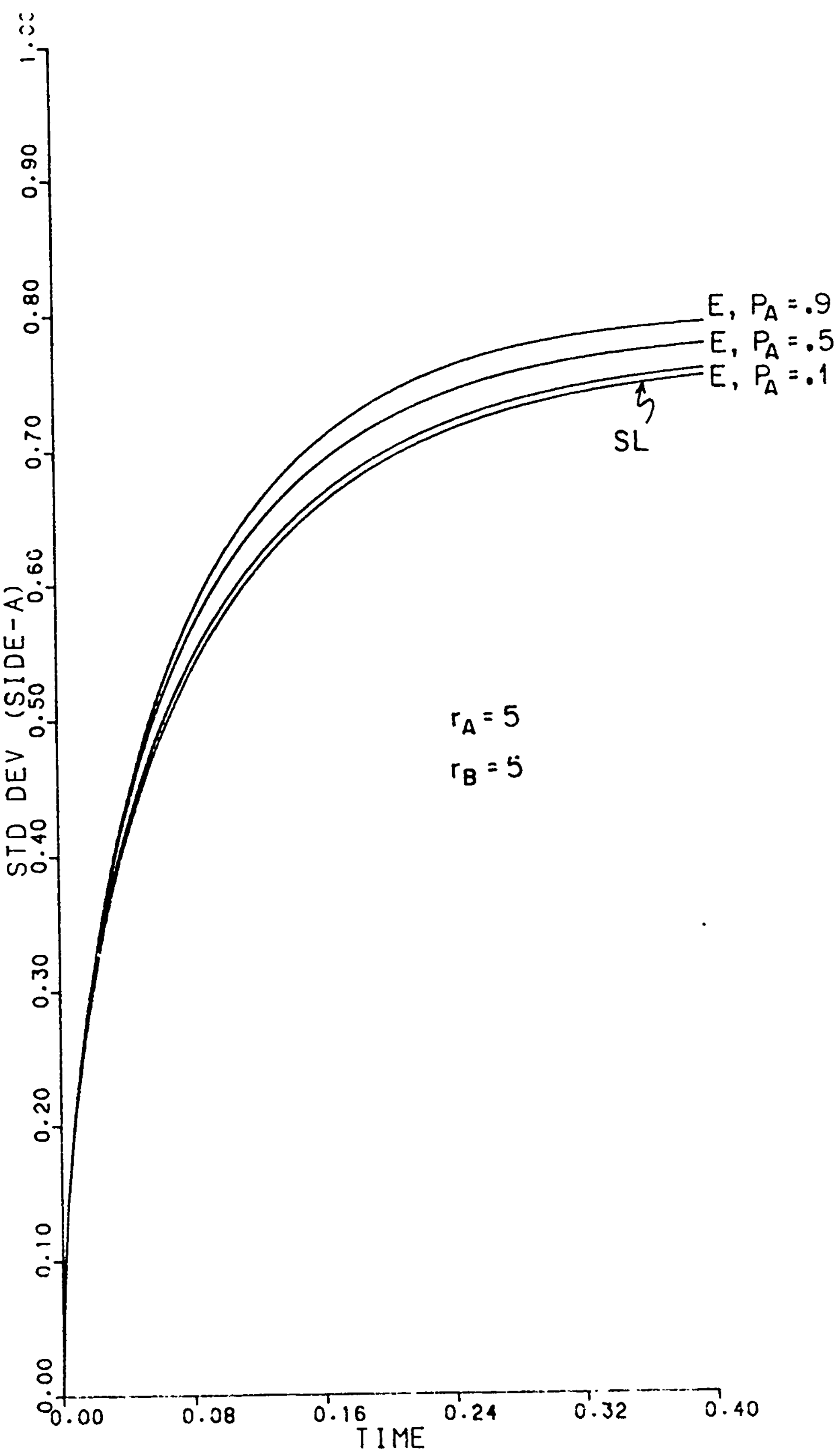
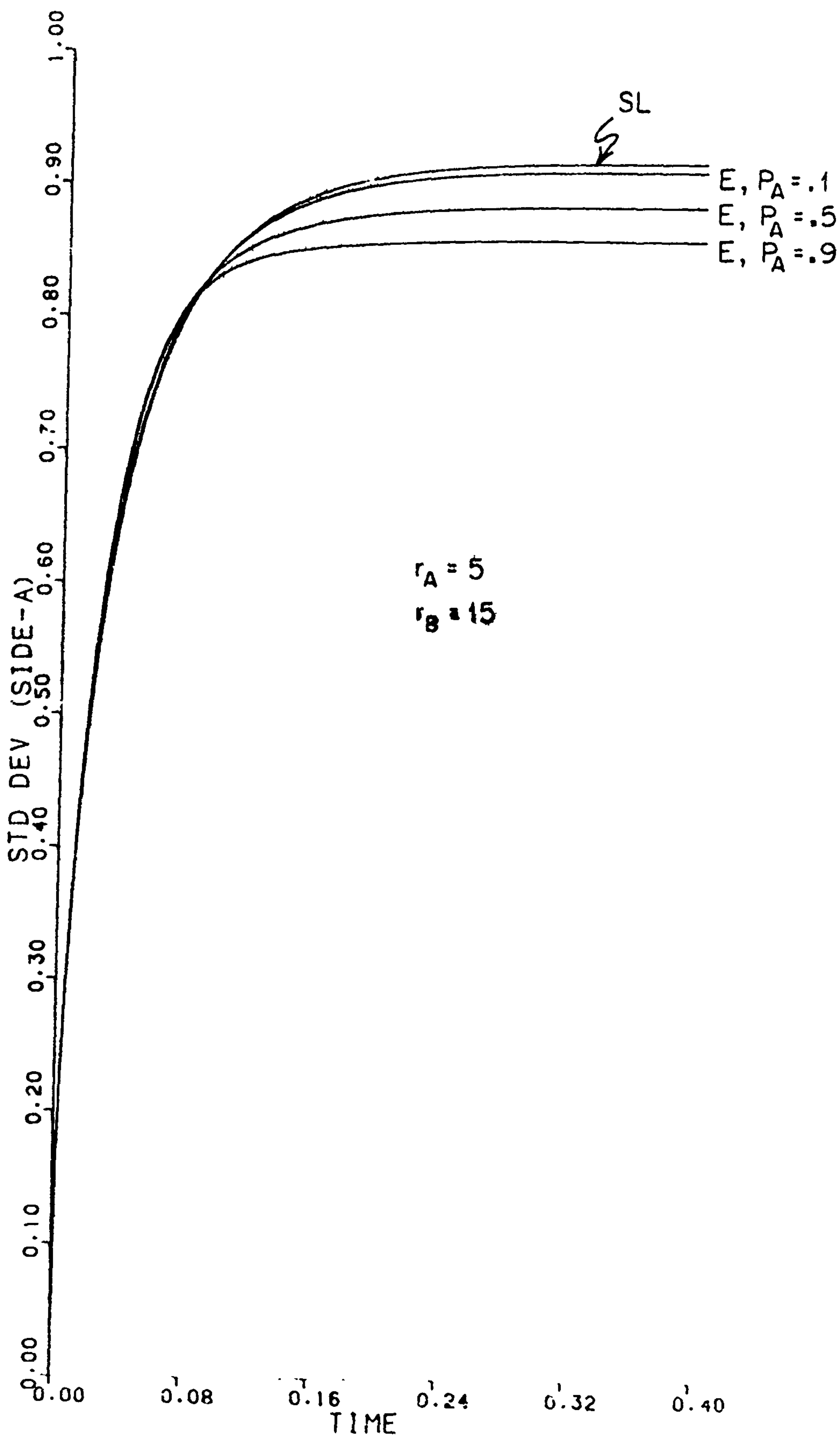
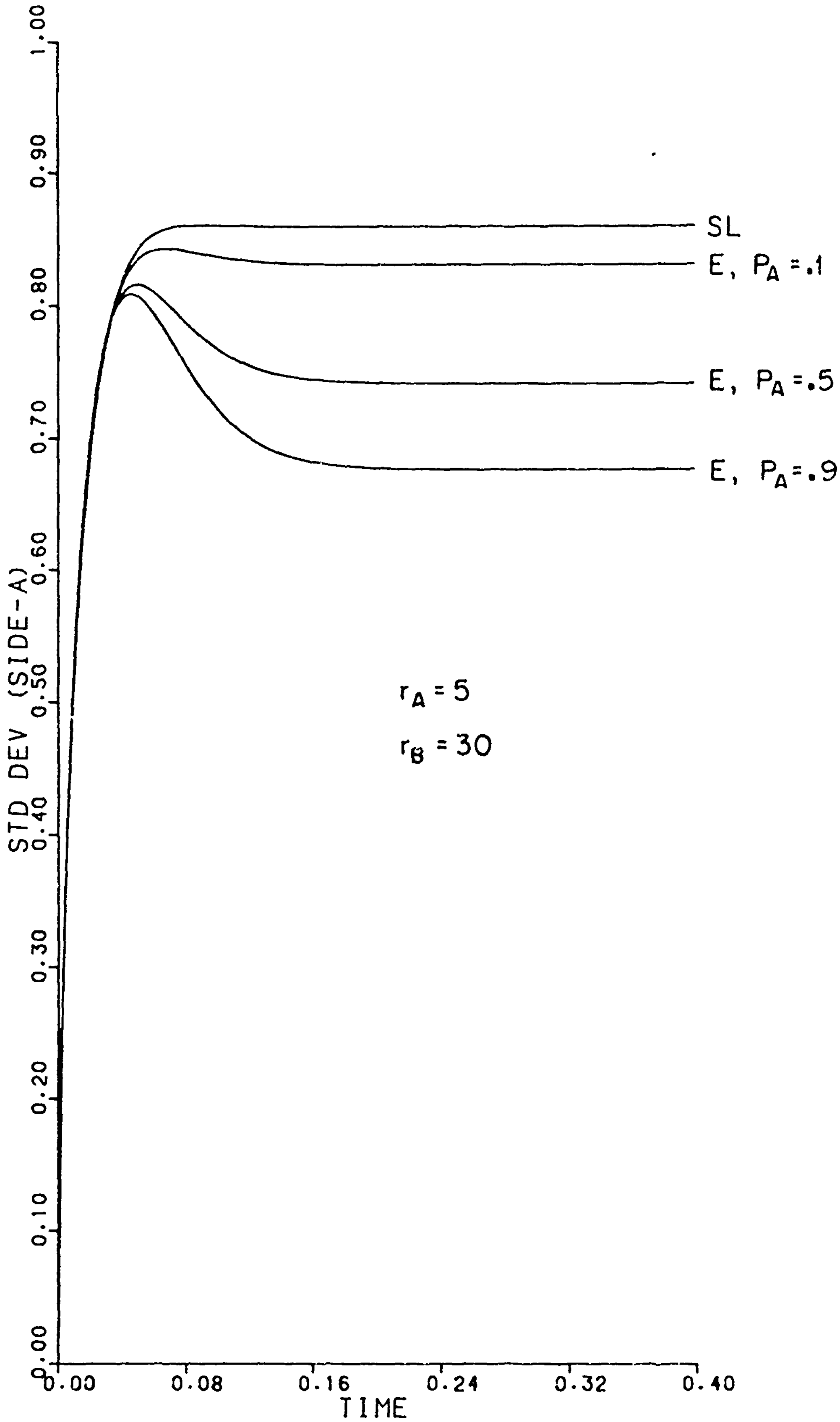

STD DEV (SIDE-A)
0.00
0.10
0.20
0.30
0.40
0.50
0.60
0.70
0.80
0.90
1.00
0.00
0.08
0.16
0.24
0.32
0.40
TIME
E, P_A = .9
E, P_A = .5
E, P_A = .1
SL
r_A = 5
r_B = 5

FIGURE 6(b)



STD DEV (SIDE-A)
0.00
0.10
0.20
0.30
0.40
0.50
0.60
0.70
0.80
0.90
1.00
SL
E, P_A=.1
E, P_A=.5
E, P_A=.9
r_A = 5
r_B = 15
0.00
0.08
0.16
0.24
0.32
0.40
TIME

FIGURE 6(c)



1.00
0.90
0.80
0.70
0.60
0.50
0.40
0.30
0.20
0.10
0.00
STD DEV (SIDE-A)
0.00
0.08
0.16
0.24
0.32
0.40
TIME
SL
E, P_A = .1
E, P_A = .5
E, P_A = .9
$r_A = 5$
$r_B = 30$

FIGURE 6(d)

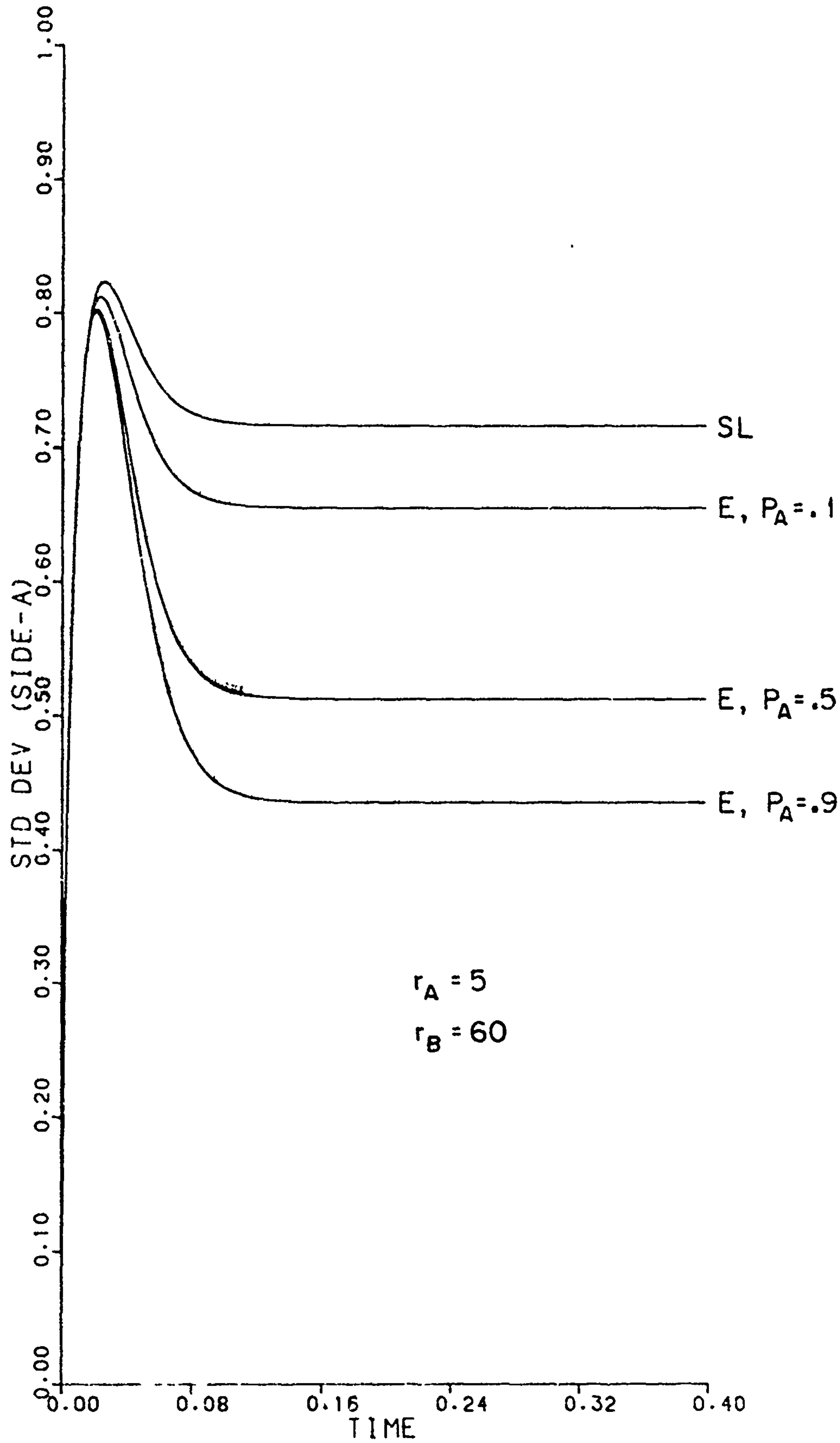

1.00
0.90
0.80
0.70
0.60
0.50
0.40
0.30
0.20
0.10
0.00
STD DEV (SIDE-A)
0.00
0.08
0.16
0.24
0.32
0.40
TIME
SL
E, P_A=.1
E, P_A=.5
E, P_A=.9
r_A = 5
r_B = 60

FIGURE 6(e)

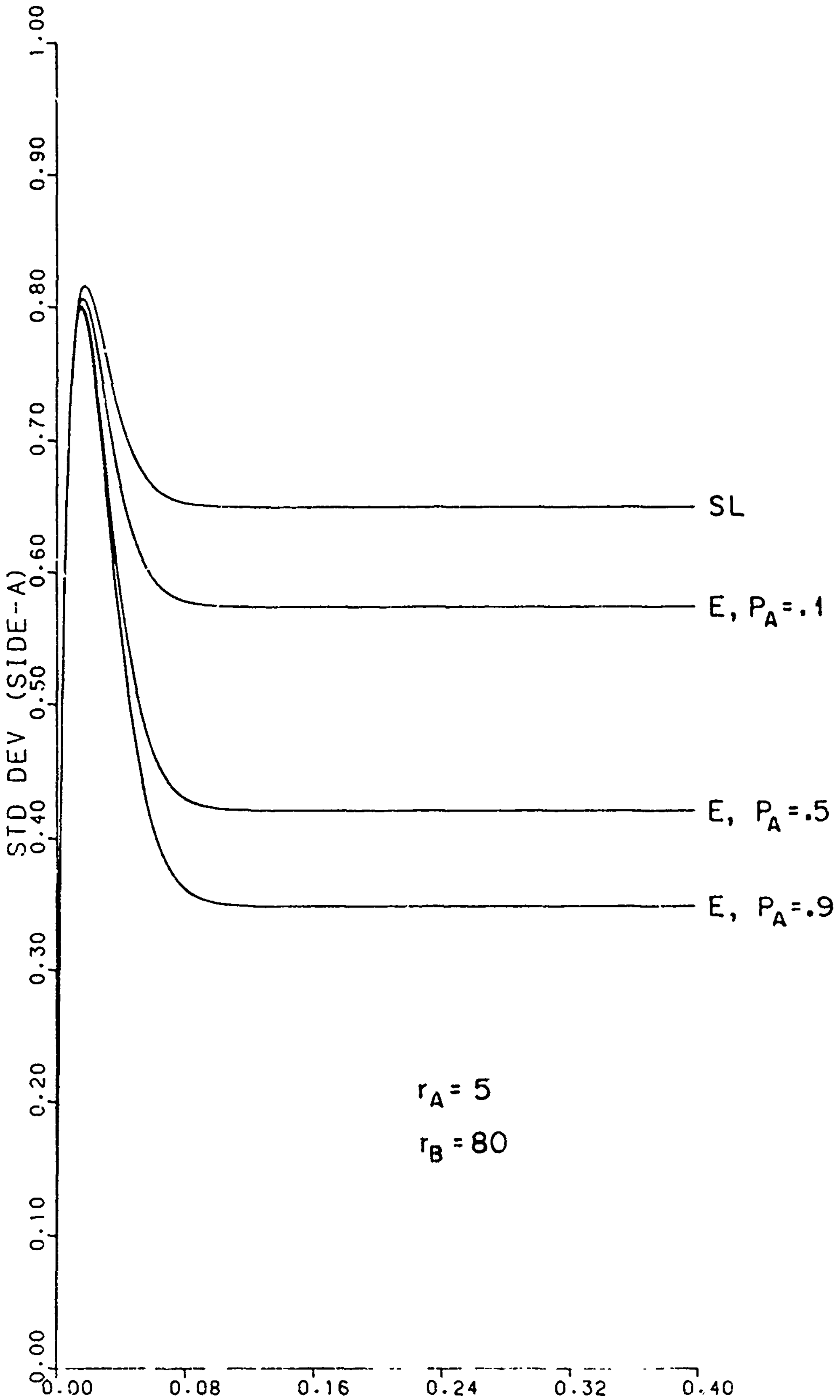

STD DEV (SIDE-A)
1.00
0.90
0.80
0.70
0.60
0.50
0.40
0.30
0.20
0.10
0.00
SL
E, P_A=.1
E, P_A=.5
E, P_A=.9
r_A= 5
r_B =80
0.00
0.08
0.16
0.24
0.32
0.40
TIME

FIGURE 6(f)

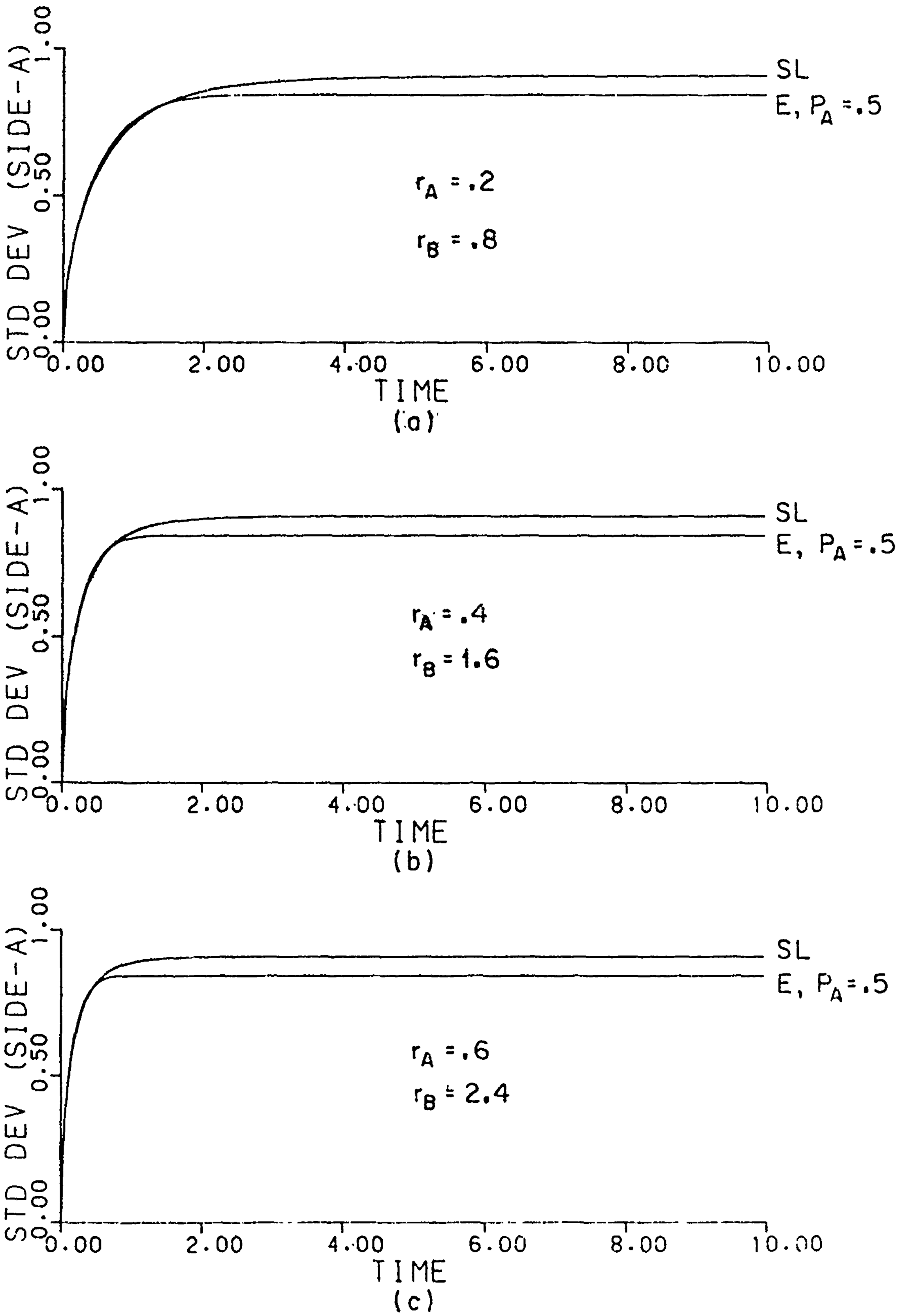

STD DEV (SIDE-A)
0.00
0.50
1.00
SL
E, P_A = .5
r_A = .2
r_B = .8
0.00
2.00
4.00
6.00
8.00
10.00
TIME
(a)
STD DEV (SIDE-A)
0.00
0.50
1.00
SL
E, P_A = .5
r_A = .4
r_B = 1.6
0.00
2.00
4.00
6.00
8.00
10.00
TIME
(b)
STD DEV (SIDE-A)
0.00
0.50
1.00
SL
E, P_A = .5
r_A = .6
r_B = 2.4
0.00
2.00
4.00
6.00
8.00
10.00
TIME
(c)

FIGURE 7

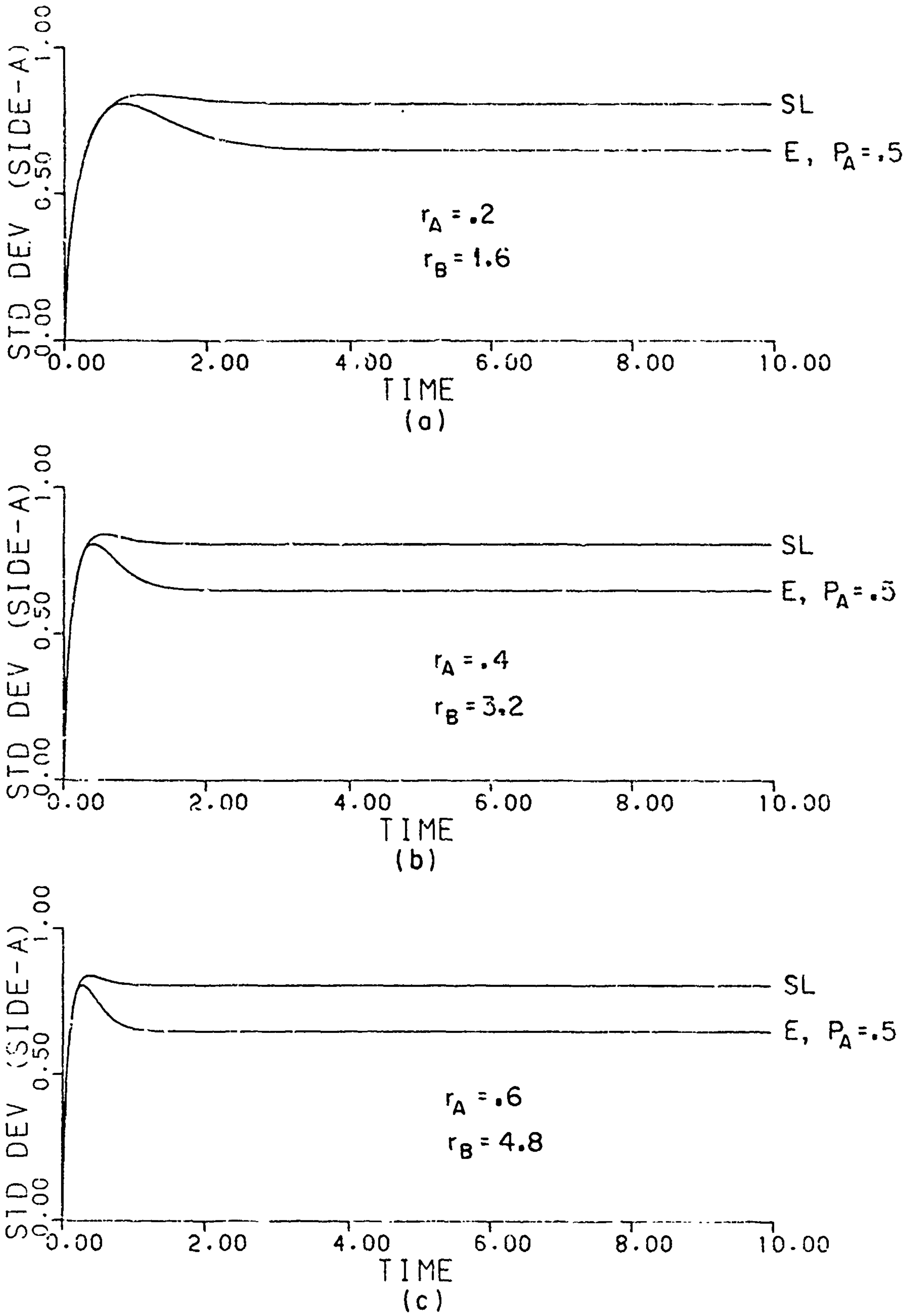

STD DEV (SIDE-A)
0.00
0.50
1.00
SL
E, P_A=.5
r_A = .2
r_B = 1.6
0.00
2.00
4.00
6.00
8.00
10.00
TIME
(a)
STD DEV (SIDE-A)
SL
E, P_A=.5
r_A = .4
r_B = 3.2
TIME
(b)
STD DEV (SIDE-A)
SL
E, P_A=.5
r_A = .6
r_B = 4.8
TIME
(c)

FIGURE 8

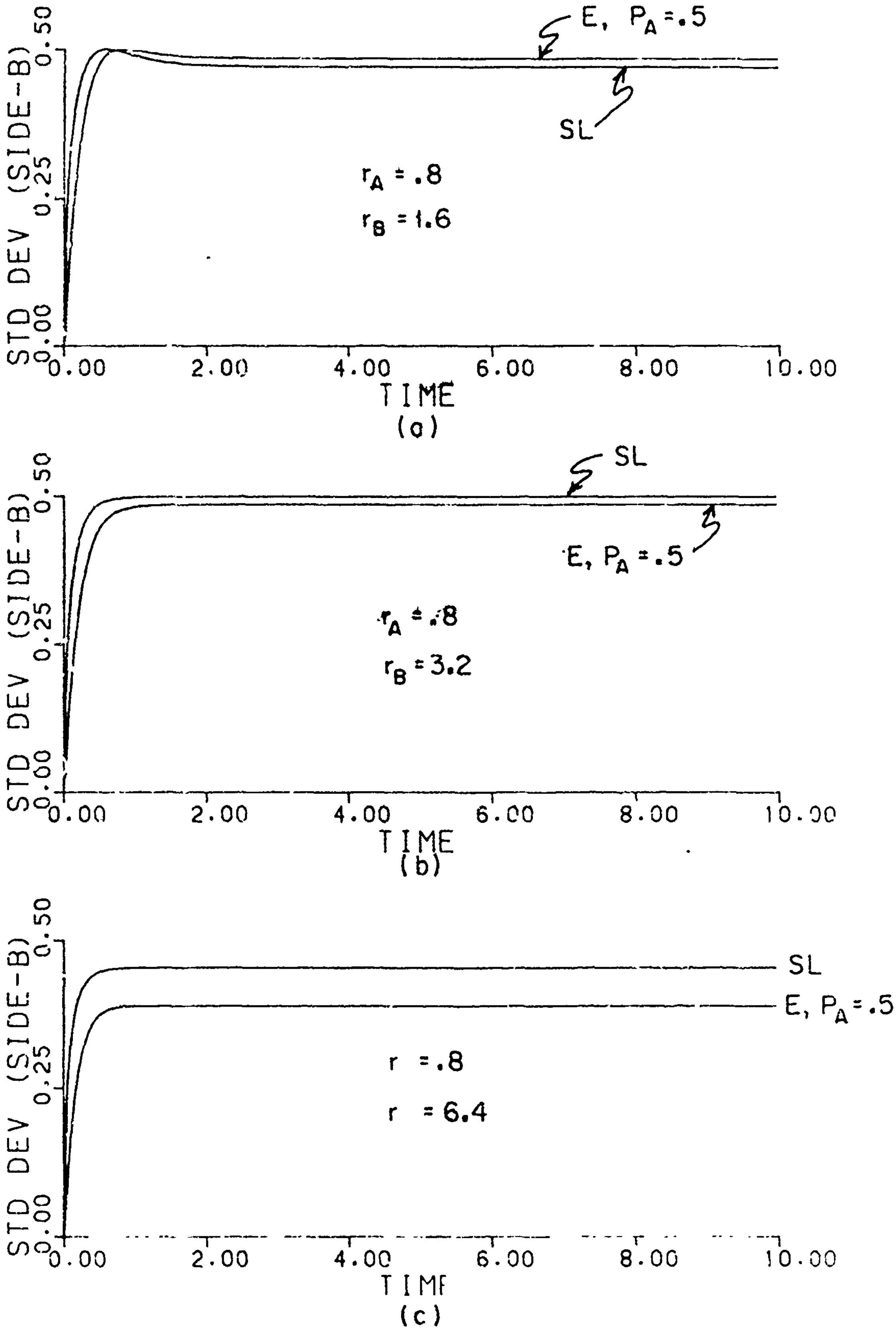
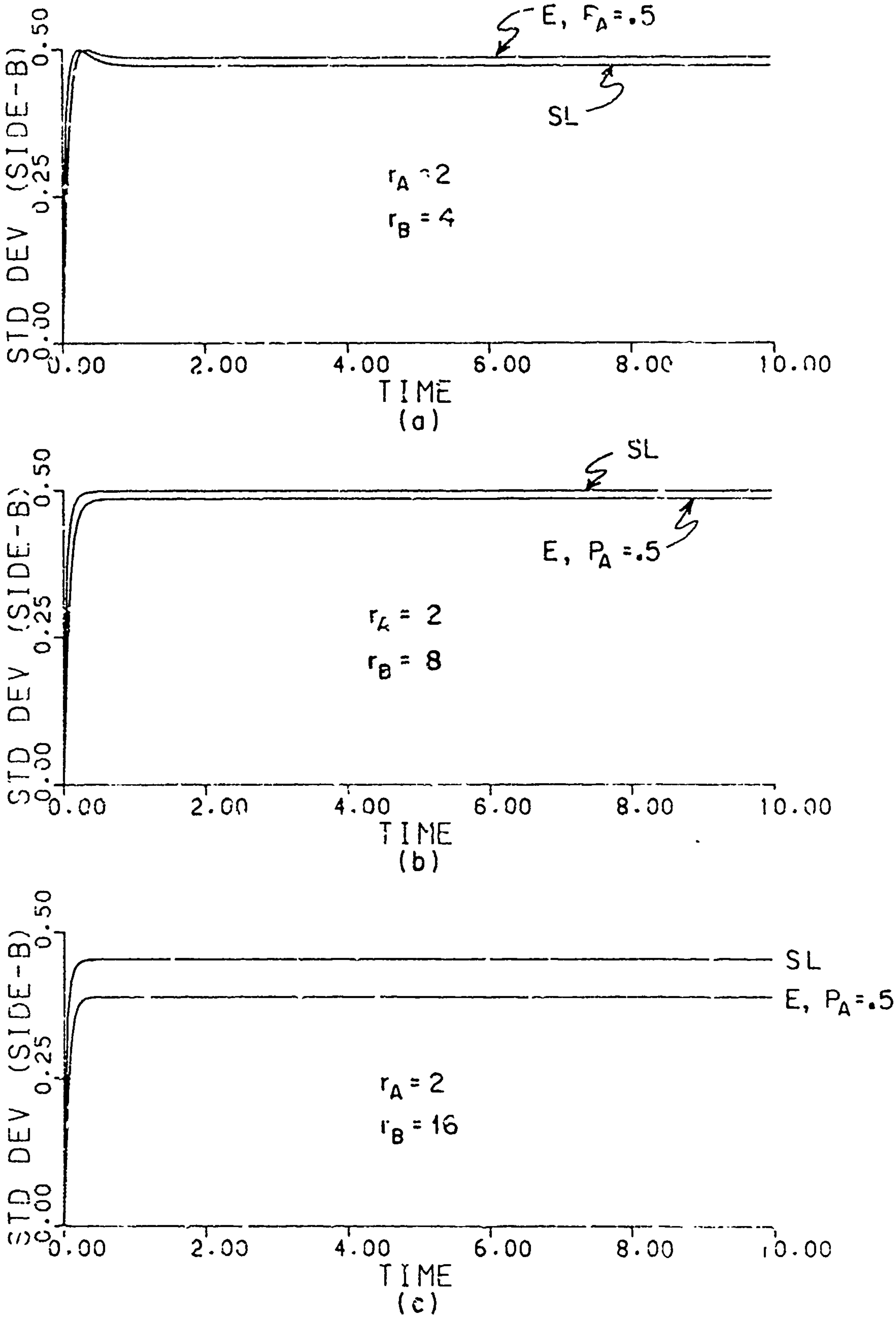

STD DEV (SIDE-B)
0.00
0.25
0.50
0.00
2.00
4.00
6.00
8.00
10.00
TIME
E, P_A =.5
SL
r_A = .8
r_B = 1.6
(a)
STD DEV (SIDE-B)
0.00
0.25
0.50
0.00
2.00
4.00
6.00
8.00
10.00
TIME
SL
E, P_A =.5
r_A = .8
r_B = 3.2
(b)
STD DEV (SIDE-B)
0.00
0.25
0.50
0.00
2.00
4.00
6.00
8.00
10.00
TIME
SL
E, P_A =.5
r =.8
r = 6.4
(c)

FIGURE 9



E, P_A=.5
SL
r_A = 2
r_B = 4
STD DEV (SIDE-B)
0.00
0.25
0.50
0.00
2.00
4.00
6.00
8.00
10.00
TIME
(a)
SL
E, P_A=.5
r_A = 2
r_B = 8
STD DEV (SIDE-B)
TIME
(b)
SL
E, P_A=.5
r_A = 2
r_B = 16
STD DEV (SIDE-B)
TIME
(c)

FIGURE 10

APPENDIX II

For the record, we give the results of the two-on-one duel with Erlang (2) interfiring time on both sides here. Let

$$f_{X_A}(x_A) = \frac{4}{\mu_A^2} x_A e^{-\frac{2x_A}{\mu_A}} , \quad f_{X_B}(x_B) = \frac{4}{\mu_B^2} x_B e^{-\frac{2x_B}{\mu_B}} .$$

Then from reference [1]

$$f_A(t) = \frac{p_A}{\mu_A \sqrt{q_A}} \left[e^{-\frac{2}{\mu_A}(1 - \sqrt{q_A})t} - e^{-\frac{2}{\mu_A}(1 + \sqrt{q_A})t} \right]$$

and $f_B(t)$ is the same with all A's replaced by B's.

Using the notation

$$\left.\begin{aligned} \alpha_1 &= \frac{2}{\mu_A}(1 - \sqrt{q_A}) , \quad \alpha_2 = \frac{2}{\mu_A}(1 + \sqrt{q_A}) , \\ \beta_1 &= \frac{2}{\mu_B}(1 - \sqrt{q_B}) , \quad \beta_2 = \frac{2}{\mu_B}(1 + \sqrt{q_B}) , \end{aligned}\right\} \qquad \text{II-1}$$

then

$$P_{21}(t) = \left(\frac{\alpha_2 e^{-\alpha_1 t} - \alpha_1 e^{-\alpha_2 t}}{\alpha_2 - \alpha_1}\right)^2 \left(\frac{\beta_2 e^{-\beta_1 t} - \beta_1 e^{-\beta_2 t}}{\beta_2 - \beta_1}\right), \qquad \text{II-2}$$

$$p_{20}(t) = \frac{2\alpha_1\alpha_2}{(\alpha_2-\alpha_1)^2(\beta_2-\beta_1)} \{\frac{\alpha_2\beta_2}{(2\alpha_1+\beta_1)}[1 - e^{-(2\alpha_1+\beta_1)t}] - \frac{\alpha_2\beta_1}{(2\alpha_1+\beta_2)}[1-e^{-(2\alpha_1+\beta_2)t}]$$

$$+ \frac{\alpha_1\beta_2}{(2\alpha_2+\beta_1)}[1-e^{-(2\alpha_2+\beta_1)t}] - \frac{\alpha_1\beta_1}{(2\alpha_2+\beta_2)}[1-e^{-(2\alpha_2+\beta_2)t}]$$

$$- \frac{\beta_2(\alpha_1+\alpha_2)}{(\alpha_1+\alpha_2+\beta_1)}[1-e^{-(\alpha_1+\alpha_2+\beta_1)t}] + \frac{\beta_1(\alpha_1+\alpha_2)}{(\alpha_1+\alpha_2+\beta_2)}[1-e^{-(\alpha_1+\alpha_2+\beta_2)t}]\}, \qquad \text{(II-3)}$$

$$p_{11}(t) = \frac{\beta_1\beta_2}{(\alpha_2-\alpha_1)^2(\beta_2-\beta_1)}(\alpha_2 e^{-\alpha_1 t} - \alpha_1 e^{-\alpha_2 t})\{\frac{\beta_2[(\alpha_2^2-\alpha_1^2)(\beta_2-\beta_1)+\alpha_2^3-\alpha_1^3]}{\alpha_1\alpha_2(\beta_1-\beta_2-\alpha_1)(\beta_1-\beta_2-\alpha_2)} e^{-\beta_1 t}$$

$$+ \frac{\beta_1[(\alpha_2^2-\alpha_1^2)(\beta_2-\beta_1)+\alpha_1^3-\alpha_2^3]}{\alpha_1\alpha_2(\beta_2-\beta_1-\alpha_1)(\beta_2-\beta_1-\alpha_2)} e^{-\beta_2 t} + \frac{\alpha_2(\alpha_1-\beta_2)}{\alpha_1(\beta_2-\beta_1-\alpha_1)} e^{-(\alpha_1+\beta_1)t}$$

$$+ \frac{\alpha_2(\beta_1-\alpha_1)}{\alpha_1(\beta_1-\beta_2-\alpha_1)} e^{-(\alpha_1+\beta_2)t} + \frac{\alpha_1(\beta_2-\alpha_2)}{\alpha_2(\beta_2-\beta_1-\alpha_2)} e^{-(\alpha_2+\beta_1)t}$$

$$+ \frac{\alpha_1(\alpha_2-\beta_1)}{\alpha_2(\beta_1-\beta_2-\alpha_2)} e^{-(\alpha_2+\beta_2)t}\}, \qquad \text{(II-4)}$$

$$
\begin{aligned}
P_{10}(t) = {} & \frac{\beta_1\beta_2}{(\alpha_2-\alpha_1)^2(\beta_2-\beta_1)} \Big\{ \frac{\beta_2[(\alpha_2^2-\alpha_1^2)(\beta_2-\beta_1)+\alpha_2^3-\alpha_1^3]}{(\alpha_1+\beta_1)(\beta_1-\beta_2-\alpha_1)(\beta_1-\beta_2-\alpha_2)} [1-e^{-(\alpha_1+\beta_1)t}] \\
& + \frac{\beta_1[(\alpha_2^2-\alpha_1^2)(\beta_2-\beta_1)+\alpha_1^3-\alpha_2^3]}{(\alpha_1+\beta_2)\beta_2-\beta_1-\alpha_1)(\beta_2-\beta_1-\alpha_2)} [1-e^{-(\alpha_1+\beta_2)t}] \\
& + \frac{\beta_2[(\alpha_2^2-\alpha_1^2)(\beta_1-\beta_2)+\alpha_1^3-\alpha_2^3]}{(\alpha_2+\beta_1)(\beta_1-\beta_2-\alpha_1)(\beta_1-\beta_2-\alpha_2)} [1-e^{-(\alpha_2+\beta_1)t}] \\
& + \frac{\beta_1[(\alpha_2^2-\alpha_1^2)(\beta_1-\beta_2)+\alpha_2^3-\alpha_1^3]}{(\alpha_2+\beta_2)(\beta_2-\beta_1-\alpha_1)(\beta_2-\beta_1-\alpha_2)} [1-e^{-(\alpha_2+\beta_2)t}] \\
& + \frac{\alpha_2^2(\alpha_1-\beta_2)}{(2\alpha_1+\beta_1)(\beta_2-\beta_1-\alpha_1)} [1-e^{-(2\alpha_1+\beta_1)t}] + \frac{\alpha_2^2(\beta_1-\alpha_1)}{(2\alpha_1+\beta_2)(\beta_1-\beta_2-\alpha_1)} [1-e^{-(2\alpha_1+\beta_2)t}] \\
& + \frac{\alpha_1^2(\alpha_2-\beta_2)}{(2\alpha_2+\beta_1)(\beta_2-\beta_1-\alpha_2)} [1-e^{-(2\alpha_2+\beta_1)t}] + \frac{\alpha_1^2(\beta_1-\alpha_2)}{(2\alpha_2+\beta_2)(\beta_1-\beta_2-\alpha_2)} [1-e^{-(2\alpha_2+\beta_2)t}] \\
& + \frac{1}{(\alpha_1+\alpha_2+\beta_1)} \left[\frac{\alpha_1^2(\beta_2-\alpha_2)}{(\beta_2-\beta_1-\alpha_2)} + \frac{\alpha_2^2(\beta_2-\alpha_1)}{(\beta_2-\beta_1-\alpha_1)}\right][1-e^{-(\alpha_1+\alpha_2+\beta_1)t}] \\
& + \frac{1}{(\alpha_1+\alpha_2+\beta_2)} \left[\frac{\alpha_1^2(\alpha_2-\beta_1)}{(\beta_1-\beta_2-\alpha_2)} + \frac{\alpha_2^2(\alpha_1-\beta_1)}{(\beta_1-\beta_2-\alpha_1)}\right][1-e^{-(\alpha_1+\alpha_2+\beta_2)t}]\Big\}, \qquad \text{(II-5)}
\end{aligned}
$$

$$p_{01}(t) = \frac{\beta_1^2\ \beta_2^2}{(\alpha_2-\alpha_1)^2(\beta_2-\beta_1)} \Big\{ \frac{(\alpha_2^2-\alpha_1^2)(\beta_2-\beta_1) + \alpha_2^3-\alpha_1^3}{\alpha_1(\alpha_1+\beta_1)(\beta_1-\beta_2-\alpha_1)(\beta_1-\beta_2-\alpha_2)} [1-e^{-(\alpha_1+\beta_1)t}]$$

$$+ \frac{(\alpha_2^2-\alpha_1^2)\ (\beta_2-\beta_1)+\alpha_1^3-\alpha_2^3}{\alpha_1(\alpha_1+\beta_2)(\beta_2-\beta_1-\alpha_2)(\beta_2-\beta_1-\alpha_2)} [1-e^{-(\alpha_1+\beta_2)t}]$$

$$+ \frac{(\alpha_2^2-\alpha_1^2)(\beta_1-\beta_2)+\alpha_1^3-\alpha_2^3}{\alpha_2(\alpha_2+\beta_1)(\beta_1-\beta_2-\alpha_1)(\beta_1-\beta_2-\alpha_2)} [1-e^{-(\alpha_2+\beta_1)t}]$$

$$+ \frac{(\alpha_2^2-\alpha_1^2)(\beta_1-\beta_2)+\alpha_2^3-\alpha_1^3}{\alpha_2(\alpha_2+\beta_2)(\beta_2-\beta_1-\alpha_1)(\beta_2-\beta_1-\alpha_2)} [1-e^{-(\alpha_2+\beta_2)t}]$$

$$- \frac{\alpha_2^2}{\alpha_1(2\alpha_1+\beta_1)(\beta_2-\beta_1-\alpha_1)} [1-e^{-(2\alpha_1+\beta_1)t}]$$

$$+ \frac{\alpha_2^2}{\alpha_1(2\alpha_1+\beta_2)(\beta_1-\beta_2-\alpha_1)} [1-e^{-(2\alpha_1+\beta_2)t}]$$

$$- \frac{\alpha_1^2}{\alpha_2(2\alpha_2+\beta_1)(\beta_2-\beta_1-\alpha_2)} [1-e^{-(2\alpha_2+\beta_1)t}]$$

$$+ \frac{\alpha_1^2}{\alpha_2(2\alpha_2+\beta_2)(\beta_1-\beta_2-\alpha_2)} [1-e^{-(2\alpha_2+\beta_2)t}]$$

$$+ \frac{(\alpha_1+\alpha_2)(\beta_2-\beta_1)-\alpha_1^2-\alpha_2^2}{(\alpha_1+\alpha_2+\beta_1)(\beta_2-\beta_1-\alpha_1)(\beta_2-\beta_1-\alpha_2)} [1-e^{-(\alpha_1+\alpha_2+\beta_1)t}]$$

(continued on next page)

$$+ \frac{(\alpha_1+\alpha_2)(\beta_2-\beta_1) + \alpha_1^2+\alpha_2^2}{(\alpha_1+\alpha_2+\beta_2)(\beta_1-\beta_2-\alpha_1)(\beta_1-\beta_2-\alpha_2)} [1-e^{-(\alpha_1+\alpha_2+\beta_2)t}] \; . \qquad (II-6)$$

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