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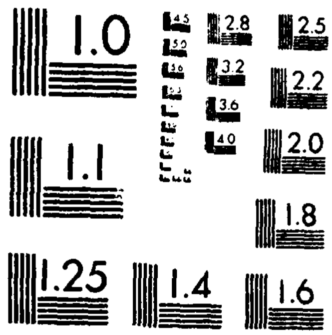

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SUPERSONIC FLOWS AROUND A CIRCULAR CONE WITH OR WITHOUT BLOWING ON THE SURFACE

by

E. Carafoli, C. Berbente and P. Marinescu

FOREIGN TECHNOLOGY DIVISION

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SUPERSONIC FLOWS AROUND A CIRCULAR CONE WITH OR WITHOUT BLOWING ON THE SURFACE

E. Carafoli, C. Berbente and P. Marinescu

SUMMARY

This article studies the flow around a straight circular cone with or without blowing in the presence of a supersonic current parallel to the axis of symmetry. First we consider the case of a solid circular cone for which a nearly exact solution is found for the equation of motion. Thus formulae are obtained for simple calculation to determine the angle of the shock wave, the fields of speeds and pressures. The results obtained are very consistent with the data of other authors using numerical methods and with experimental data. This fact made it possible to extend the method to the case of flow around a permeable porous cone with suction or normal injection on the surface.

The results obtained may be applied directly to the calculation of the fuselage of supersonic and jet aircraft, or the thermal protection of the tip of aircraft by sonic injection and rockets, or the thermal protection at the tip of aircraft by injection of a liquid or gas. Likewise the use of suctions through the walls reduces the intensity of the shock wave for a certain period of the flight, eliminating a series of harmful effects of the latter.

1. MOTION AROUND A SOLID CONE WITHOUT BLOWING

A circular cone is considered located in a supersonic current of velocity U_∞ (fig. 1).

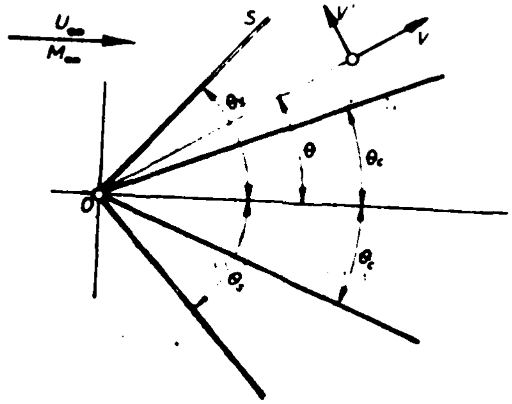

Figure 1

Designating by $V(\theta)$ the velocity component according to the radius of the vector, we may write $V_R = V(\theta) = V$,

$$V_\theta = \frac{dV}{d\theta} = V', \qquad (1)$$

$$\frac{d^2V}{d\theta^2} = V''.$$

Since the movement is conical, the velocity field is given by the Taylor-Maccoll equation

$$\left(1 - \frac{V'^2}{a^2}\right)V'' + V'\operatorname{ctg}\theta + \left(2 - \frac{V'^2}{a^2}\right)V = 0, \qquad (2)$$

in which a represents the local velocity of sound, given by the relation

$$a^2 = a_\infty^2 - \frac{\gamma - 1}{2}\left(U_\infty^2 - V^2 - V'^2\right), \qquad (3)$$

where γ is the ratio of the specific heat (γ=1.4 for air).

Introducing the velocity of interference V, given by the equation

$$v - V = U_\infty \cos\theta, \ (v' = V' + U_\infty \sin\theta, \quad v'' = V'' + U_\infty \cos\theta) \tag{4}$$

equation (2) becomes

$$\left(1 - \frac{V'^2}{a^2}\right) v'' + v' \operatorname{ctg}\theta + \left(2 - \frac{V'^2}{a^2}\right) v = 0. \tag{5}$$

Since it is not simple to solve equations (2) or (5) we propose to linearize them by replacing the nonlinear term V'^2/a^2 by the function $t^2(\theta)$

$$\frac{V'^2}{a^2} \approx t^2, \tag{6}$$

suitably chosen, thus the solution which results will have a simpler form leading at the same time to more exact results.

Consequently, instead of equation (5) we will solve equation

$$(1 - t^2)\, v'' + v' \operatorname{ctg}\theta + (2 - t^2) v = 0, \tag{7}$$

whose general solution is:

$$\frac{v}{U_\infty} = (CI + D) \cos\theta, \tag{8}$$

where the coefficients C and D are constant quantities, while I is the result of the following integrals:

$$I = \int \frac{d\theta}{H \cos^2\theta}, \quad \ln H = \int \frac{\operatorname{ctg}\theta \, d\theta}{1 - t^2}. \tag{9}$$

From equation (8) we deduce immediately

$$\frac{v'}{U_\infty} = \frac{C}{H \cos\theta} + \frac{v}{U_\infty} \operatorname{tg}\theta. \tag{10}$$

to determine the constants C and D we use the boundary conditions with ($\theta=\theta_c$) and for the shock wave ($\theta=\theta_s$). Since the speed normal to the surface of the cone is zero, we will have

$$\theta = \theta_c, \quad V'_c = v'_c - U_\infty \sin\theta_c = 0. \tag{11}$$

From the equation of conservation of masses and impulses we have the relationship

$$\theta = \theta_s, \quad v_s = 0, \tag{12a}$$

$$\theta = \theta_s, \quad v'_s = U_\infty(1 - k)\left(1 - \frac{1}{K_s^2}\right), \tag{12b}$$

where

$$k = \frac{\gamma - 1}{\gamma + 1}, \quad K_s = M_\infty \sin\theta_s. \tag{13}$$

Applying conditions (11), (12a) and (12b) we then deduce:

$$\frac{C}{H_c \cos\theta_c} - (CI_c + D)\sin\theta_c = \sin\theta_c, \tag{14a}$$

$$CI_s + D = 0, \tag{14b}$$

$$\frac{C}{H_s \cos\theta_s} - (CI_s + D)\sin\theta_s = (1 - k)\left(1 - \frac{1}{K_s^2}\right)\sin\theta_s. \tag{14c}$$

Starting from them we obtain

$$\frac{v}{U_\infty} = C(I - I_s)\cos\theta \quad (D = -CI_s), \tag{15a}$$

$$C = \frac{H_c \sin\theta_c \cos\theta_c}{1 + H_c(I_s - I_c)\sin\theta_c \cos\theta_c}, \tag{15b}$$

$$\sin^2\theta_s = \frac{1}{M_\infty^2} + \frac{C \operatorname{tg}\theta_s}{(1 - k)H_s}, \tag{15c}$$

$$\frac{v'}{U_\infty} = C\left(\frac{1}{H\cos\theta} - (I - I_s)\sin\theta\right). \tag{16}$$

where H_c, H_s, I_c, I_s represent values of H and I for $\theta=\theta_c$ and $\theta=\theta_s$. To determine the function t^2, we may observe that (fig. 2) the function v'^2/a^2 and its first derivative become equal to zero for $\theta=\theta_c$, but the flow after the shockwave has a normal subsonic component. Even in the case when the motion is incompressible we will have $t^2=0$, and we obtain from equation (9)

$$H_0 = \sin\theta, \quad I_0 = \frac{1}{\cos\theta} + \ln \operatorname{tg}\frac{\theta}{2}. \tag{17}$$

We may note the fact that relation (17) furnishes results close to the real ones which makes it possible for us to choose a simple expression for t^2 in the form

$$t^2 = b(x - 1), \quad x = \left(\frac{\sin\theta}{\sin\theta_c}\right)^n, \quad n = 1, 2, \ldots, \tag{18}$$

where

$$b = \frac{t_s^2}{x_s - 1}, \quad t_s^2 = \frac{V_s'^2}{a_s^2} = \frac{kK_s^2 + 1 - k}{(1+k)K_s^2 - k}, \tag{19}$$

the values of t_s^2 being determined on the basis of the boundary conditions (12a) and (12b).

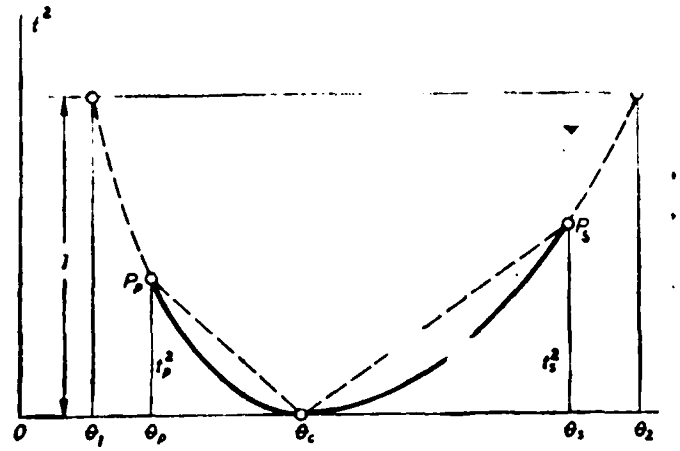

Figure 2

Expression (18) as resulting from the applications, gives nearly exact values and leads to practically applicable expressions. These expressions are simple, being established in the form of a correction to the incompressible solution (17).

Introducing expression (18) in (9) we have

$$\ln H = \int \frac{\mathrm{ctg}\, d\theta}{1 - t^2} = \frac{1}{n} \int \frac{dx}{x[1 - b(x-1)]}. \tag{20}$$

The integral (20) is solved immediately and results in the expression:

$$\frac{1}{H} = \frac{(1 - \sigma x)^\nu}{(\sin\theta)^{n\nu}} = \frac{(\sin\theta)^\sigma (1 - \sigma x)^\nu}{\sin\theta}, \tag{21}$$

in which some constants are given up which can be included in the general constant C and we note

$$\sigma = \frac{b}{1 + b}, \quad n\nu = 1 - \sigma, \quad \sigma x < 1. \tag{22}$$

To calculate the integral I we observe that expression (21) may be expanded in the form

$$\frac{1}{H} = \frac{[1-(1-\sin\theta)]^{\sigma}(1-\sigma x)^{\nu}}{\sin\theta} = \frac{[1-\delta_{\sigma}(1-\sin\theta)](1-\delta_{\nu}\sigma x)}{\sin\theta} \approx$$

$$\approx \frac{1-\delta_{\sigma}}{\sin\theta} + \delta_{\sigma} - \frac{\sigma\delta_{\nu}(1-\delta_{\sigma})}{\sin^{n}\theta_{c}}\sin^{n-1}\theta - \frac{\sigma\delta_{\nu}\delta_{\sigma}}{\sin^{n}\theta_{c}}\varepsilon\sin^{n}\theta, \qquad (23)$$

where δ_{σ} and δ_{ν} are constants chosen suitably to represent better the development of the two binomials of expression (23). It is observed that fo $\delta_{\sigma} = \delta_{\nu} = 0$ we obtain expression (17) and H_0. Consequently δ_{σ} and δ_{ν} represent correction terms before the basic integrals H_0 and I_0. Likewise it is possible to put the function H in a simpler form, retaining only the important terms of (23) and certain terms in $1/\sin\theta$ and terms in $\sin^{n-1}\theta$ which have a larger coefficient before the final term. Consequently we may write

$$\frac{1}{H} = \frac{1-\delta}{\sin\theta} + \varepsilon\sin^{n-1}\theta, \qquad (24)$$

where the coefficients δ and ε are determined from the condition that the function (24) is obtained from the points $\theta=\theta_C$ and $\theta=\theta_S$, where the values of H in these points are given by (21):

$$\frac{1}{H_c} = \left(\frac{1-\sigma}{\sin^{n}\theta_c}\right)^{\nu}, \qquad \frac{1}{H_s} = \left(\frac{1-\sigma x_s}{\sin^{n}\theta_s}\right)^{\nu}. \qquad (25)$$

From conditions $\theta=\theta_C$ and $\theta=\theta_S$, we have

$$\frac{1}{H_c} = \frac{1-\delta}{\sin\theta_c} + \varepsilon\sin^{n-1}\theta_c, \qquad \frac{1}{H_s} = \frac{1-\delta}{\sin\theta_s} + \varepsilon\sin^{n-1}\theta_s. \qquad (26)$$

and consequently we obtain

$$1-\delta = \frac{H_s x_s\sin\theta_c - H_c\sin\theta_s}{H_c H_s(x_s-1)}; \quad \varepsilon = \frac{1}{H_c\sin^{n-1}\theta_c} - \frac{1-\delta}{\sin^{n}\theta_c}. \qquad (27)$$

We emphasize the fact that the expression (24) is only used to calculate the integral I, the single place where it is necessary that the deviations are the smallest, as follows from the integration.

With expression (24) obtained for 1/H the integral (9) may be obtained easily in the form

$$I = (1-\delta)\, I_0 + \varepsilon \int \frac{\sin^{n-1}\theta\, d\theta}{\cos^2\theta}. \qquad (28)$$

As may be seen, the basic term of the integral I is I_0, the others being correction terms, a fact which makes it possible to simplify the calculations. Thus all the above expressions are easy to manipulate and lead to results which practically correspond to the numerical calculations.

The verifications carried out show that the same results are valid for the complete flow conditions in which the shock wave remains attached, including the transonic conditions, corresponding to n=4 and leading to an expression for I deduced from (28) in the form

$$I = (1 - \delta)\, I_0 + \varepsilon\left(\cos\theta + \frac{1}{\cos\theta}\right). \qquad (29)$$

Determination of the angle of the shock wave. To determine the angle θ_s of the shock wave, we will apply equation (15c) which leads to the expression

$$\sin^2\theta_s = \frac{1}{M_\infty^2} + \frac{H_c \sin\theta_c \cos\theta_c \operatorname{tg}\theta_s}{(1-k) H_s [1 + H_c (I_s - I_c) \sin\theta_c \cos\theta_c]}. \qquad (30)$$

This expression may be solved in a simpler or more complicated manner, through numerical examples. This is accomplished more easily by using the method of successive iterations which leads but after the second approximation to exact values. For this purpose we will seek the most suitable starting expression for the real value using for H in the first approximation the expression given by (17), i.e. $H_c = \sin\theta_c$, $H_s =$ sine θ_s. We will write

$$\sin^2\theta_s^* \approx \frac{1}{M_\infty^2} + \frac{\sin^2\theta_c}{1-k} \cdot \frac{1}{1 + (I_{0s} - I_{0c}) \sin^2\theta_c \cos\theta_c} \cdot \frac{\cos\theta_c}{\cos\theta_s^*}. \qquad (31)$$

Starting from this and observing that the last two factors of the second term of the right hand side of equation (31) are mutually compensated, tending to the value 1, we will write in the first approximation

$$\sin^2\theta_s^* = \frac{1}{M_\infty^2} + \frac{\sin^2\theta_c}{1-k}. \qquad (32)$$

In the above equations (31) and (32) the index* was introduced to designate the starting value θ_s^*. After calculation of θ_s^* from (32) we will introduce this value in (15b) and (21) and we recalculate the value of θ_s from (15c) in the form

$$\sin^2\theta_s = \frac{1}{M_\infty^2} + \frac{C^* \operatorname{tg}\theta_s^*}{(1-k)H_s^*}. \tag{33}$$

With this value of θ_s, we will calculate the new characteristics of motion C, H, I.

Calculations of pressures: After the pressure drop by the shock wave (transformation accompanied by losses), the gas undergoes isentropic compression, reaching a maximum pressure at $\theta=\theta_c$

Designating by p_s the immediate pressure after the shock wave we have

$$\frac{p_s}{p_\infty} = \frac{2\gamma}{\gamma+1} M_\infty^2 \sin^2\theta_s - \frac{\gamma-1}{\gamma+1} = (1+k)K_s^2 - k = \lambda, \tag{34}$$

a ratio which we designate by λ, to simplify the writing of the following formulae.

After the shock wave the transformation of the gases are isentropic, the ratio of the pressure p under the current angle θ and the pressure p_∞ may be written in the form

$$\frac{p}{p_\infty} = \frac{p}{p_s}\frac{p_s}{p_\infty} = \lambda\left(\frac{W^2 - V^2 - V'^2}{W^2 - V_s^2 - V_s'^2}\right)^{\frac{\gamma}{\gamma-1}} \tag{35}$$

where W represents the maximum speed, given by the equation /7/

$$\frac{W^2}{V_\infty^2} = 1 + \frac{1-k}{kM_\infty^2}. \tag{36}$$

From the boundary conditions in (12a) and (12b) we obtain

$$\frac{W^2 - V_s^2 - V_s'^2}{V_\infty^2} = \frac{1-k}{M_\infty^2}\left(1 + \frac{1-k}{kK_s^2}\right)\lambda. \tag{37}$$

In this case, for a certain angle θ and for the surface of the solid cone ($\theta=\theta_c$, $V=V_c$, $V_c'=0$) we have

$$\frac{p_r}{p_\infty} = \lambda \left[\frac{1 + \frac{\gamma - 1}{2} M_\infty^2 \left(1 - \frac{V^2 + V'^2}{U_\infty^2}\right)}{\lambda\left(k + \frac{1-k}{K_s^2}\right)} \right]^{\frac{\gamma}{\gamma - 1}} \tag{38}$$

$$\frac{p}{p_\infty} = \lambda \left[\frac{1 + \frac{\gamma - 1}{2} M_c^2 \left(1 - \frac{V_c^2}{U_\infty^2}\right)}{\lambda\left(k + \frac{1-k}{K_s^2}\right)} \right]^{\frac{\gamma}{\gamma - 1}} \tag{39}$$

Knowing the ratio of pressures, from the equations (35) and (38) it is possible to calculate the pressure coefficients C_p defined by the relation

$$C_p = \frac{2}{\gamma M_\infty^2}\left(\frac{p}{p_\infty} - 1\right). \tag{40}$$

Verification and comparison. The comparisons are made for two important parameters of the flow, specifically the angle of the shock wave θ_s (equation 33) and the pressure coefficient on the body C_{pc} (equations 39 and 40).

From figures 3 and 4 we observe that the results obtained practically for θ_s and C_{pc} coincide with the values obtained by numerical integration of equations (2), (3), (4), for a large interval of mach numbers (M_∞=1.25-5.7) and the deviations (θ_c=10°-50°) covering the entire supersonic and hypersonic range are moderate.

It is observed that the method is applicable also in cases when the flow conditions are transonic, as occurs, for instance, at M_∞=1.25, θ_c=20° or at M_∞=2, θ_c=40°. In the latter case the flow is very close to the point of detachment of the shock wave ($\theta_{detachment}$=41.5°).

We may therefore observe the satisfactory consistency of the analytical formulae and the results obtained by numerical integration, as well as the experimental data, which implies a very high precision for the distribution of velocities.

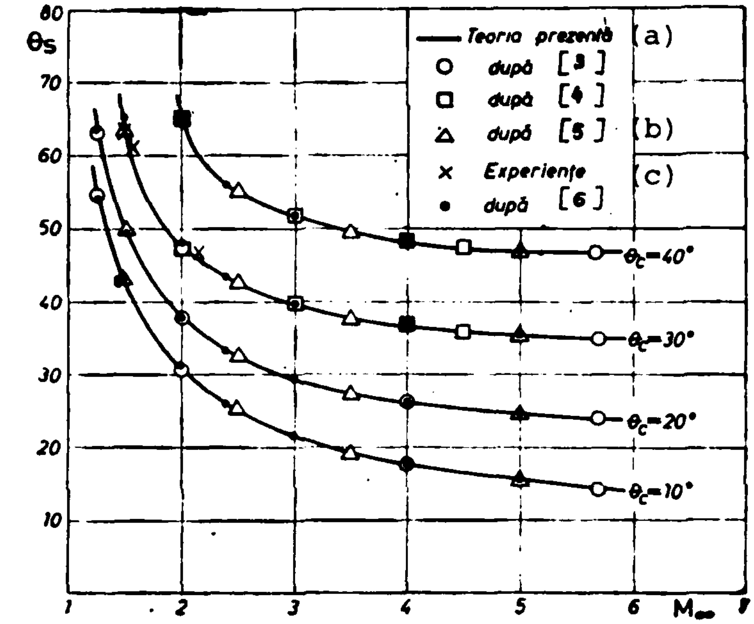

Figure 3: Key: (a) present theory; (b) according to; (c) according to experiment.

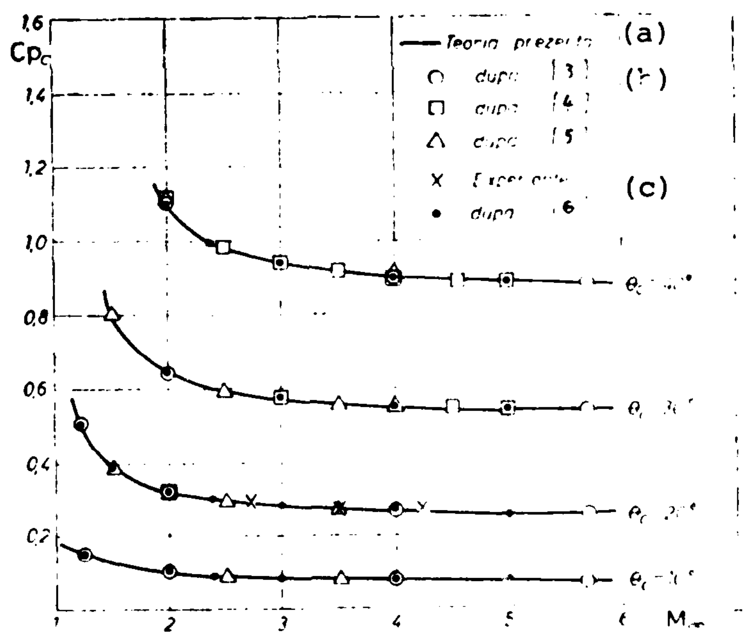

Figure 4: Key: (a) present theory; (b) according to; (c) experiments according to.

2. FLOW AROUND A POROUS PERMEABLE CONE WITH SUCTION[1]

It is assumed that through the walls of a cone, with a half-angle at the vertex $\theta=\theta_p$ (fig. 5) the suction is carried out at normal speed W_p. On the vector radius $\theta=\theta_c$, the normal speed is equal to zero and therefore the cone forms with this angle a straight definite shaft with solid equivalent by analogy to the previous case.

Designating by β a coefficient in the form

$$\beta = 1 - \frac{W_p}{U_\infty \sin\theta_p}, \quad 0 \leqslant \beta \leqslant 1, \tag{41}$$

for the characteristic of suction, and taking the boundary conditions just as in the previous case, with the difference that the walls are porous we will have

$$\theta = \theta_p, \quad V'_p = v'_p - U_\infty \sin\theta_p = W_p, \tag{42}$$

resulting in

$$v'_p = \beta U_\infty \sin\theta_p \tag{43}$$

and consequently we find for C the expression

$$C = \frac{\beta H_p \sin\theta_p \cos\theta_p}{1 + H_p(I_s - I_p)\sin\theta_p \cos\theta_p} = \frac{H_c \sin\theta_c \cos\theta_c}{1 + H_c(I_s - I_c)\sin\theta_c \cos\theta_c}, \tag{44}$$

obtained in the same way as expression (15b).

We will consider three important situations of the calculation:

a. Direct problem. The angle θ_p and suctions W_p or the parameter $\beta(0\leq\beta\leq 1)$ are given, together with the Mach number of the flow M .

From equation (44) we calculate the angle θ_c^0 of the equivalent cone, using for H the expression $H^0=\sin\theta$.

We will obtain successively, observing that the denominator does not differ much from unity,

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[1] A similar article, "Supersonic Cone with Blowing", E. Carafoli and C. Berbente was published in The Aeronautical Quarterly, London, 1976.

$$\sin\theta_c^0 \sin 2\theta_c^0 \approx \beta \sin\theta_p \sin 2\theta_p, \tag{45a}$$

$$2\sin\theta_c^0\sqrt{1-\sin^2\theta_c^0} \approx \beta \sin\theta_p \sin 2\theta_p, \tag{45b}$$

$$\sin^2\theta_c^0 \approx 1 - \sqrt{1-\beta\sin\theta_p \sin 2\theta_p}. \tag{45c}$$

With the value of θ_c given by (45c) the problem is solved in the case of a solid cone and we obtain the values of H_p^0, H_c^0, C^0, I_s^0, I_c^0. Then we calculate the expression for the angle θ_c in two approximations of the equation (44) in the form

$$\frac{\sin 2\theta_c}{\sin 2\theta_p} = \beta \frac{H_p^0\, 2 + H_c^0(I_s^0 - I_c^0)\sin 2\theta_c^0}{H_c^0\, 2 + H_p^0(I_s^0 - I_p^0)\sin 2\theta_p}. \tag{46}$$

This new value of θ_c deduced from (46) is sufficiently accurate and no more corrections are needed. The upper index (0) was introduced for designating the first approximation.

b. Indirect problem. The parameters β and θ_c are given, and we seek the angle θ_p, which requires the implementation of a definite suction of parameter β. In this case we calculate from (45b) the value of the angle θ_p through a first approximation, which is designated by θ_p^0. We will obtain just like for (45c)

$$\sin^2\theta_p^0 = 1 - \sqrt{1 - \frac{1}{\beta}\sin\theta_c \sin 2\theta_c}\,. \tag{47}$$

The correction of the angle θ_p is accomplished using once again the equation (44) in the form

$$\frac{\sin 2\theta_p}{\sin 2\theta_c} = \frac{1}{\beta}\frac{H_c}{H_p^0}\,\frac{2 + H_p^0(I_s - I_p^0)\sin 2\theta_p^0}{2 + H_c(I_s - I_c)\sin 2\theta_c}. \tag{48}$$

The necessary speed of suction is calculated by means of relation (41)

$$W_p = (\beta - 1)\, U_\infty \sin\theta_p. \tag{49}$$

c. Mixed problem. We observe that the angle of the solid equivalent cone θ_c may be used as a parameter for the suction. This means

This means that θ_p may indicate the intensity of the section by choosing the angle $0<\theta_c<\theta_p$. In this case the flow is determined and there only remains the calculation of the suction speed for $\theta=\theta_p$ from equations (42) and (16)

$$\frac{W_p}{U_\infty} = C\left[\frac{1}{H_p\cos\theta_p} - (I_p - I_s)\sin\theta_p\right] - \sin\theta_p . \qquad (50)$$

It is found that it is simpler to solve this in the third case.

3. CONE AROUND A POROUS CONE WITH INJECTION

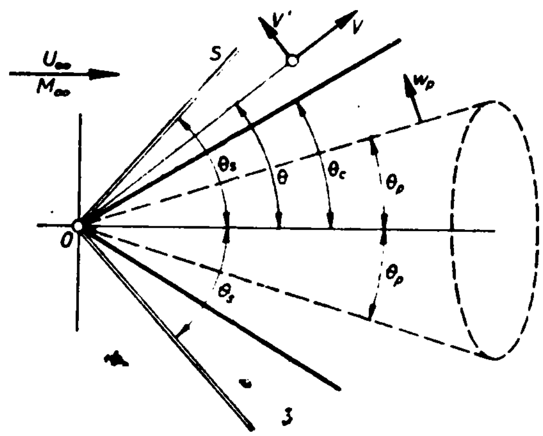

Fig. 6

If through the porous wall a fluid is injected at a constant speed ($W_p>0$), for example for the purpose of thermal protection, a surface of separation will occur between the injected fluid and because of it a current to infinity, located under an angle $\theta_c>\theta_p$ (fig. 6). On this surface of separation, the normal speed component is cancelled ($V_c'=0$) and may therefore be defined also in the case of suction as an equivalent cone. However we should observe that the radial speed component, that is V, may present a drop at $\theta=\theta_c$. This drop occurs even if the fluid injected is identical to the external one, but it has a different stagnation pressure. Consequently the conical surface $\theta=\theta_c$ represents the conical layer which maintains a stable form if the pressure in these two flows, internal and external of the cone θ_c are equal ($p_{ci}=p_{ce}$).

The external motion around the equivalent cone of half-angle θ_c has been studied previously (paragraph 1). We will now study the motion of the inside of the equivalent cone, which may be treated independently of the external one. The problem is therefore reduced to the study of the motion inside a solid cone of half-angle θ_c after the injection of a fluid through a porous cone of half-angle θ_p coaxial with the solid one (fig. 7).

Flow inside a solid cone following a subsonic blowing through a coaxial internal porous cone.

Let us consider a porous cone of half-angle $\theta=\theta_p$ inside a solid cone of half-angle θ_c (fig. 7). The fluid injection is carried out through the porous wall along the normal, at a constant speed W_p more than 0, along the radius. In the following we will assume that the speed W_p is subsonic ($t_p^2 < 1$). The problem is bound with the solution of equation (2) which we shall linearize by introducing the function t^2 in the form

$$(1 - t^2) V'' + V' \operatorname{ctg}\theta + (2 - t^2) V = 0. \tag{51}$$

The difference for the previous case for the solid cone consists in the fact that the function t^2 must be considered on the descending branch ($\theta < \theta_c$ fig. 2) and the speed of sound is obtained from the expression

$$a^2 = a_p^2 + \frac{\gamma - 1}{2} (V_p^2 + W_p^2 - V^2 - V'^2), \tag{52}$$

where θ_p represents the speed of sound $\theta=\theta_p$.

The general solution of equation (51) is similar to that given for equation (8). We will write

$$V = (GI + F) \cos\theta, \tag{53}$$

where the function I is given by (9), by the constants G and F are determined from the following boundary conditions

$$\theta = \theta_p, \quad V_p' = W_p, \quad a = a_p, \tag{54a}$$

$$\theta = \theta_c, \quad V_c' = 0. \tag{54b}$$

We obtain immediately from these two equations (54):

$$G = \frac{W_p}{\sin\theta_p} \frac{H_c H_p \sin 2\theta_c \sin 2\theta_p}{2H_c \sin 2\theta_c - 2H_p \sin 2\theta_p + (I_c - I_p) H_c H_p \sin 2\theta_c \sin 2\theta_p}, \qquad (55.a)$$

$$F = G\left(\frac{2}{H_c \sin 2\theta_c} - I_c\right). \qquad (55.b)$$

For the function t^2 we will choose an expression of the form

$$t^2 = b\left(\frac{1}{x} - 1\right), \quad x = \left(\frac{\sin\theta}{\sin\theta_c}\right)^m, \quad m = 1, 2, 3, \ldots, \qquad (56)$$

which is similar to the form (18) but for negative n.

The coefficient b is determined from the equations

$$b = \frac{t_p^2 x_p}{1 - x_p}, \quad t_p^2 = \frac{V'^2}{a_p^2} = \frac{W_p^2}{a_p^2}. \qquad (57)$$

For $t^2=0$, we obtain expression (17) for the functions H_0 and I_0 in the form

$$H_0 = \sin\theta, \quad I_0 = \frac{1}{\cos\theta} + \ln \mathrm{tg}\,\frac{\theta}{2}. \qquad (58)$$

For this case which corresponds to an incompressible motion, the constants which we shall designate as F_0 and G_0 are obtained from the expressions (55). The same movement may take place if a liquid is injected for cooling purposes. Let us consider the angle $\theta=\theta_1$ corresponding to $t_1^2=1$ (fig. 2) where θ_1 represents the boundary value which is never reached in subsonic blowing, that is we will always have $\theta_p > \theta_1$. Taking

$$\sigma = \left(\frac{\sin\theta_1}{\sin\theta_c}\right)^m = \frac{b}{1+b}, \quad m\nu = 1 - \sigma, \qquad (59)$$

we can write

$$\ln H = \frac{1-\sigma}{m}\int \frac{dx}{x-\sigma}; \quad \frac{1}{H} = \frac{\text{const.}}{(x-\sigma)^\nu} \qquad (60)$$

or choosing suitably the constants

$$\frac{1}{H} = \frac{1}{(\sin\theta)^{m\nu}\left(1 - \frac{\sin^m\theta_1}{\sin^m\theta}\right)^\nu} = \frac{1}{\sin\theta\left(1 - \frac{\sin^m\theta_1}{\sin^m\theta}\right)^\nu}. \qquad (61)$$

Since ν and σ have low values, the expression (61) may be expanded in suitable manner, which, just like expression (23) will be used only for integration.

Thus using suitably chosen constants δ_ν, δ_σ, we may write

$$\frac{1}{H} \approx \frac{1}{\sin\theta}\,\frac{1-\delta_\sigma(1-\sin\theta)}{1-\delta_\nu \dfrac{\sin^m\theta_1}{\sin^m\theta}} \approx \frac{1-\delta_\sigma}{\sin\theta} + \delta_\sigma + \frac{2\delta_\nu\delta_\sigma \sin^m\theta_1}{\sin^m\theta} + \frac{2\delta_\nu(1-\delta_\sigma)\sin^m\theta_1}{(\sin\theta)^{m+1}}. \tag{62}$$

Just as in the previous case we can consider a suitable value for δ_σ and δ_ν for the purpose of a better representation of expression (61). Observing that the first and last terms of (62) are preponderant, the above expression may be written with sufficiently good approximation in a form similar to (24):

$$\frac{1}{H} = \frac{1-\delta'}{\sin\theta} + \frac{\varepsilon'}{(\sin\theta)^{m+1}}. \tag{63}$$

The constants δ' and ε' are determined from the conditions that at the ends of the interval $\theta=\theta_p$ and $\theta=\theta_c$, the expression (63) coincides with (61) that is H will have two values H_p and H_c.

$$H_c = (\sin\theta_c)^{m\nu}(1-\sigma)^\nu, \quad H_p = (\sin\theta_p)^{m\nu}\left(1-\frac{\sin^m\theta_1}{\sin^m\theta_p}\right)^\nu. \tag{64}$$

By this means we may easily deduce that

$$1-\delta' = \frac{H_p\sin\theta_c - H_c x_p \sin\theta_p}{H_c H_p(1-x_p)}, \quad \varepsilon' = \sin^m\theta_c\left[\frac{\sin\theta_c}{H_c} - (1-\delta')\right]. \tag{65}$$

Consequently, the integral I will have the form

$$I = (1-\delta')I_0 + \varepsilon'\int \frac{d\theta}{\cos^2\theta\,\sin^{m+1}\theta}. \tag{66}$$

Results very near to exact ones are obtained for m=3 when the expression (66) leads to

$$I = (1-\delta')I_0 + \varepsilon'\left(2\,\mathrm{tg}\,\theta + \frac{1}{3}\,\mathrm{tg}^3\theta - \mathrm{ctg}\,\theta\right). \tag{67}$$

Calculation of pressures. We designate that p_p the pressure on the internal cone $\theta=\theta_p$ and p_c the pressure on the external cone $\theta=\theta_c$. Designating by p the pressure at a current point, we will have

$$\frac{p}{p_p}=\left(\frac{W_i^2-V^2-V'^2}{W^2-V_p^2-V_p'^2}\right)^{\frac{\gamma}{\gamma-1}},\quad \frac{p_c}{p_p}=\left(\frac{W_i^2-V_c^2}{W_i^2-V_p^2-V_p'^2}\right)^{\frac{\gamma}{\gamma-1}}, \qquad (68)$$

where the maximum speed W_i is given by the expression.

$$W_i^2=V_p^2+V_p'^2+\frac{2a_p^2}{\gamma-1}. \qquad (69)$$

From here we calculate the pressure coefficient

$$C_p=\frac{2}{\gamma M_p^2}\left(\frac{p}{p_p}-1\right), \qquad (70)$$

M_p being the Mach number at $\theta=\theta_p$.

Determination of the lines of current. It is useful to know the form of the lines of current in the region inside the two cones in view of multiple applications. In the first row we will give the expression of the angle α of inclination of the total speed, on the side of the surface of the porous cone

$$\operatorname{tg}\alpha=\frac{W_p}{V_p}=\frac{2/(H_p\sin 2\theta_p)-2/(H_c\sin 2\theta_c)+(I_c-I_p)}{2/(H_c\sin 2\theta_c)-(I_c-I_p)}\operatorname{tg}\theta_p. \qquad (71)$$

We will determine below the function of the current Ψ using the equation of the lines of current in spherical coordinates, for the special case of conical movement

$$\frac{dR}{V}=\frac{Rd\theta'}{V'}. \qquad (72)$$

The function of the current is introduced in the form

$$\psi=R^2\int_{\theta_c}^{\theta}\frac{\rho}{\rho_1}\sin\theta V d\theta=R^2\int_{\theta_c}^{\theta}\frac{\rho}{\rho_1}(GI+F)\sin\theta\, d\theta, \qquad (73)$$

where for $\theta=\theta_c$ we obtain the line of current $\Psi=0$, while ρ_1 represents the reference density and it may be shown that $\rho_1=\rho_c$. In case of incompressible flow, $\rho=\rho_1$=constant, we have $t^2=0$ and we obtain for the function of the current the following expression

$$\psi_0 = R^2\int_{\theta_c}^{\theta} (G_0 I_0 + F_0)\sin\theta d\theta = \frac{R^2}{2}\left[G_0\left(\sin^2\theta \ln \operatorname{tg}\frac{\theta}{2} - \cos\theta\right) + \right.$$

$$\left. + F_0 \sin^2\theta + \text{const}\right]. \qquad (74)$$

where the constant is determined from the condition $\Psi_0=0$.

For the general case on the basis of the expressions (67) and (73) observing that we can find for ρ_1 the average value and therefore the ratio ρ/ρ_1, of small variations ($\rho/\rho_1 \approx 1$), we have

$$\psi = \frac{R^2}{2}\left[(1-\delta')G\left(\sin^2\theta \ln \operatorname{tg}^3\frac{\theta}{2} - \cos\theta\right) + \right.$$

$$\left. + \frac{2}{3}G\varepsilon'(\operatorname{tg}\theta - 2\sin 2\theta) + F\sin^2\theta + \text{const.}\right]. \qquad (75)$$

Thus we obtain the analytical expression for the function of the current, which may be applied without any difficulty.

Examples of calculation and choice of the injection parameters.

In case when we consider a motion in two coaxial cones (Fig. 7) of which we know the half-angle of the solid rod $\theta = \theta_c$ and the half-angle of the internal permeable cone $\theta = \theta_p$ together with the speed of sound a_p at the level of the porous wall $(\theta - \theta_p)$ for the blowing speed W_p along the normal, t_p^2 may be calculated along the walls according to the relation

$$t_p^2 = \frac{W_p^2}{a_p^2} \quad (t_p^2 < 1),$$

which is implemented if the blowing is subsonic. In this case all the relationships of the calculation are explicit and there is no difficulty in their application.

In case the fluid is injected inside a current from infinity (fig. 6) the cone $\theta=\theta_c$ represents the surface of separation between two flows (one internal) ($\theta \leq \theta_c$) and the other external ($\theta > \theta_c$). We may disregard the eddy effect, located at $\theta=\theta_c$, as well as the heat exchange, because the latter is generally of secondary nature and represents only a correction which may be added subsequently. The problem is solved by considering simultaneously two flows, the external one whose solution has the form (8) and the internal one given by equation (53).

The stabilization of the quantities characterizing the flow leads to the following situations of calculation:

a. Direct problem. The characteristics of the external current are given: U_∞, M_∞, p_∞, angle θ_p, the speed of sound a_p, the speed of injection W_p and pressure p_p, and we seek how to determine the position of the equivalent cone θ_c and the position of the shock wave θ_c, at the same time as the entire field of motion. Together with the constants C, D, G, F we have six unknowns for which we have six equations given by the two conditions $\theta=\theta_s$ (equations 12a and 12b), three conditions for $\theta=\theta_c$ (equation 11 and 54b for the external flow and the internal flow and equality of pressures) and the condition $\theta=\theta_p$. Taking into account the expressions of the integration constants (15a, 15b, 55a and 55b) the problem is reduced to finding the angle of the solid cone θ_c and the angle of the shock wave θ_s from the system of equations:

$$\sin^2\theta_s = \frac{1}{M_\infty^2} + \frac{\mathrm{tg}\,\theta_s}{(1-k_e)H_{se}} \cdot \frac{H_{ce}\sin 2\theta_c}{2 + H_{ce}(I_{se}-I_{ce})\sin 2\theta_c}, \tag{76}$$

$$\frac{p_e}{p_\infty} = \lambda\left[\frac{1+\frac{\gamma-1}{2}M_\infty^2\left(1-\frac{V_{ce}^2}{U_\infty^2}\right)}{\lambda\left(k_e+\frac{1-k_e}{k_s^2}\right)}\right]^{\frac{\gamma_e}{\gamma_e-1}} = \frac{p_p}{p_\infty}\left(\frac{W_i^2-V_{ci}^2}{W_i^2-V_p^2-V_p'^2}\right)^{\frac{\gamma_i}{\gamma_i-1}} \tag{77}$$

where the indices e and i refer to the external and internal flows respectively. From expression (77) we observe the appearance of a new important parameter; the ratio p_p/p.

The system of equations (76) and (77) is easily solved by attempts observing that at the value chosen for the angle θ_c($\theta c \quad \theta_p$), both the internal and external flows are completely determined. This initial value is modified until equation (77) is verified.

b. Indirect problem. The quantities are given which characterize the internal flow a_p, W_p, p_p, θ_p and the angle θ_c and we attempt to determine the characteristics of the external flow with surface of separation at $\theta=\theta_c$.

Since the internal flow is totally determined and therefore the angle θ_c is given, there remains the determination of the constants C and D along with the parameters of the external flow M_∞, U_∞, p_∞, and the angle θ_s, that is six unknowns. In this case we have four equations given by the two conditions at $\theta=\theta_s$ (equations 12a and 12b) and two conditions at $\theta=\theta_c$ (equations 11 and 77). The result is that two of the parameters of the external flow must be chosen. If we give M and U then the angle θ_s is calculated from the equations (15c) or (76) while the pressure p and therefore the stagnation pressure from the external flow are obtained from (77) (before and after the shock wave). In this case in the indirect problem the equation of calculation becomes explicit. We may observe that the speed U may now be determined from the equations referring to the external flow, as representing only the stages of movement. Consequently in the indirect problem we cannot consider the values M_∞ and p_∞ as known simultaneously. If the values are given for p_∞ and U_∞, then M_∞ and the angle θ_s are determined from trials by means of the system of equations (76) and (77).

Comparison with other methods. Figures 8 and 9 show the diagrams of the results obtained on the basis of the analytical methods proposed and are compared with values determined by numerical integration by Emanuel /2/. We find a very good consistency both as regards the angle of blowing α, (formula 71) and the ratio of pressures p_p/p . The parameters chosen were θ_p, t_p^2, and θ_c for which we considered three values:

$$\theta_p = 10^\circ, \quad t_p^2 = 0{,}375, \quad \theta_c = 20^\circ,$$
$$\theta_p = 10^\circ, \quad t_p^2 = 0{,}50, \quad \theta_c = 30^\circ$$
$$\theta_p = 10^\circ, \quad t_p^2 = 0{,}125, \quad \theta_c = 55^\circ$$

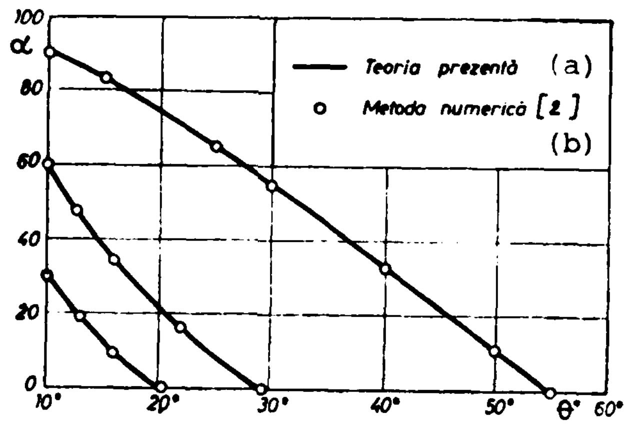

Figure 8: Key: (a) present theory; (b) numerical method /2/.

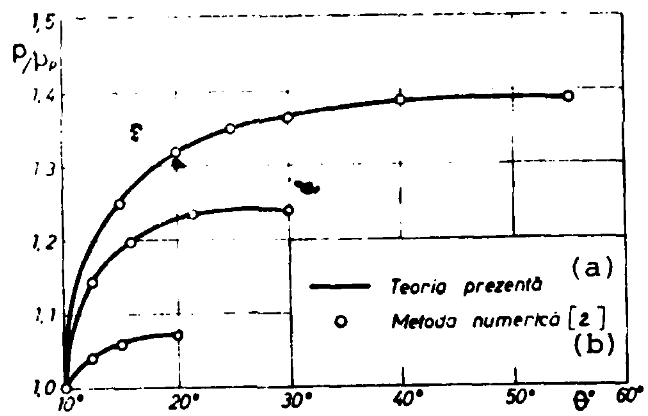

Figure 9: Key: (a) present theory; (b) numerical method /2/.

For these values values of α were found according to (71) and p_p/p_∞ (formula 77) while $\gamma_e=\gamma_i=1.4$. The consistency is good also in the case of larger values for t_p^2.

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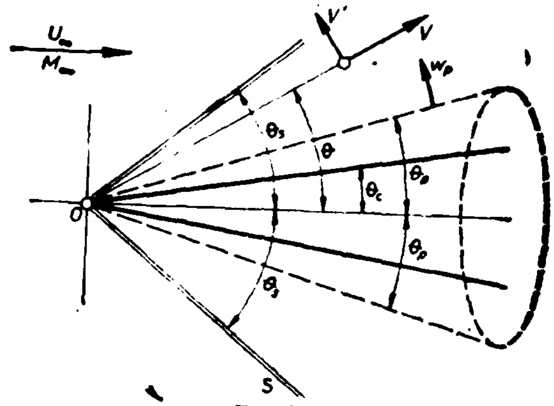
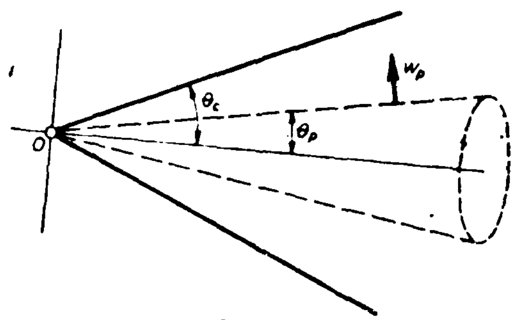


U_∞
M_∞
V'
V
w_p
θ_s
θ
θ_c
θ_p
θ_p
θ_s
O
S

Fig. 5



w_p
θ_c
θ_p
O

Fig. 7

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