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by

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### Cumulative Damage Threshold Crossing Models

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J. Sethuraman and Thomas R. Young  
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#### Abstract

This paper describes a model for survival functions called the cumulative damage threshold crossing model. Under this model, an item, consisting of a large number of components which suffer damage at regular moments of time, fails as soon as the maximum cumulative damage to some component crosses a certain threshold. Under various conditions on the damage distribution, two plausible new survival distributions

$$S_1(t) = \exp\{-\exp(\alpha + \beta t^{\frac{1}{2}} - \gamma t^{-\frac{1}{2}})\}, \quad t > 0, \quad -\infty < \alpha < \infty, \quad \beta > 0 \quad \text{and} \quad \gamma > 0,$$

and

$$S_2(t) = \exp\{-\beta t(1/t - \gamma)^{-\alpha}\}, \quad 0 < t < 1/\gamma, \quad \alpha > -1, \quad \beta > 0 \quad \text{and} \quad \gamma > 0$$

One arise. ~~The~~ survival function  $S_1(\cdot)$  is shown to fit several data sets in the area of cancer studies, failure times of insulating fluids and failure times of air conditioning units.

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## 1. Introduction.

The achievements of Reliability Theory depend largely on successful modelling of survival functions of lifetimes. The famous 'shock model' of Esary, Marshall and Proschan (1973) can be described as follows. An item is buffeted by shocks which arise according to a Poisson process. Each shock causes a random damage and the damages accumulate. The item fails as soon as the cumulated damage exceeds a certain threshold. Further generalizations of this model have appeared in the literature, e.g. A-Hameed and Proschan (1973, 1975). Only qualitative properties of survival functions under this model were studied in these papers. The survival functions were not fitted to actual data.

The classical results of Gnedenko (1943) on the three types of asymptotic distributions for an extreme observation lend themselves naturally to the modelling of survival functions. That is why in many applications, survival functions are assumed to be Weibull

$$(1.1) \quad S_4(t) = \exp\{-(\alpha t)^\beta\}, \quad t > 0, \quad \alpha > 0, \quad \beta > 0,$$

or to be the shifted Weibull

$$(1.2) \quad S_3(t) = \exp\{-[\alpha(t-\gamma)]^\beta\}, \quad t > \gamma, \quad \alpha > 0, \quad \beta > 0, \quad -\infty < \gamma < \infty.$$

There are many variations to the use of Gnedenko's (1943) results in modelling survival distributions. One can introduce dependence among the observations, one can have spare parts which instantaneously replace parts that fail and thus increase system life, etc. A partial list of references that contain such extensions is, Ashar (1960), Birnbaum, Esary and Saunders (1961), Davis (1952), Flehinger (1962), Harris (1970), Laurent (1958), Schafer and Finkelstein (1962), Sen, Bhattacharyya and Suh (1973) and Weiss (1961).

In this paper we suggest a different model which can be described in broad terms as follows. An item consists of a large number of components. At regular moments of time,  $t = 1, 2, \dots$ , random damages occur to the components and the damages accumulate. The item fails as soon as the cumulative damage to some component crosses a certain threshold.

It is easy to produce examples where such a model would apply. In cancer studies one may define the onset of cancer as the time at which one of the cells becomes mutant. Thus one may postulate that an organism consists of a large number of cells, that the environment produces random malevolent effects on the cells and that a cell becomes mutant when the cumulative malevolent effect exceeds a certain threshold. Similar models can be postulated in other situations like machines with a large number of parts, electronic equipment with large numbers of transistors, diodes, etc.

For the simplest form of our model, we present some asymptotic results in Section 2. This leads to two new survival functions

$$(1.3) S_1(t) = \exp\{-\exp(\alpha + \beta t^{\frac{1}{2}} - \gamma t^{-\frac{1}{2}})\}, \quad t > 0, \quad -\infty < \alpha < \infty, \quad \beta > 0, \quad \gamma > 0$$

and

$$(1.4) S_2(t) = \exp\{-\beta t(1/t - \gamma)^{-\alpha}\}, \quad 0 < t < 1/\gamma, \quad \alpha > -1, \quad \beta > 0, \quad \gamma > 0.$$

The proofs of these results employ results on rates of convergence for moderate and large deviations.

In Section 3 we fit our survival functions  $S_1(\cdot)$  and  $S_2(\cdot)$  as well as the classical ones  $S_3(\cdot)$  and  $S_4(\cdot)$  to a variety of data. The data come from cancer studies in England and Wales, Doll (1971), from studies on insulating fluids, W. Nelson (1972, 1975), and from air conditioning systems on Boeing aircraft, Proschan (1963). It is gratifying to see that the new survival function  $S_1(\cdot)$  fits nearly all these data better than the others.

The survival functions  $S_1(\cdot)$ ,  $S_2(\cdot)$  and  $S_3(\cdot)$  each contain three parameters  $\alpha$ ,  $\beta$  and  $\gamma$  while  $S_4(\cdot)$  contains only two parameters  $\alpha$  and  $\beta$ . The survival functions  $S_2(\cdot)$ ,  $S_3(\cdot)$  and  $S_4(\cdot)$  may have either an increasing failure rate (IFR) or a decreasing failure rate (DFR). For example,  $S_2(\cdot)$  is IFR if  $\alpha \geq 0$  and  $S_3(\cdot)$  and  $S_4(\cdot)$  are each IFR if  $\beta \geq 1$ . On the other hand,  $S_1(\cdot)$  is a more flexible survival function.  $S_1(\cdot)$  may, depending upon the values of its parameters, be an IFR survival distribution or it may have an increasing failure rate for small and large values of  $t$  while having a decreasing failure rate for moderate values of  $t$ .

## 2. The survival functions $S_1(\cdot)$ and $S_2(\cdot)$ .

Consider an item that consists of  $k$  components. The  $i^{\text{th}}$  component suffers damage  $Z_{it}$  at time  $t = 1, 2, \dots$ . Let  $S_{it} = \sum_{j=1}^t Z_{ij}$  be the cumulative damage to component  $i$  at time  $t$ ,  $i = 1, \dots, k$ , and let  $S_{kt}^* = \max_{1 \leq i \leq k} S_{it}$  be the maximum cumulative damage suffered by any component. The cumulative damage threshold crossing model that we propose in this paper postulates that the life of the item is given by

$$(2.1) \quad L = \inf\{t: S_{kt}^* \geq d\},$$

where  $d$  is the threshold.

We make the following simplifying assumption. The random damages  $Z_{11}, Z_{12}, \dots, Z_{21}, Z_{22}, \dots, \dots, Z_{k1}, Z_{k2}, \dots$  are independent and identically distributed with common distribution function  $F(\cdot)$ . One could relax this by assuming that the vectors  $(Z_{11}, \dots, Z_{k1}), (Z_{12}, \dots, Z_{k2}), \dots$  are independent and identically distributed or follow a special stochastic process. One could also define the life of the item by

$$L = \inf\{t: \phi(S_{1t}, \dots, S_{kt}) \geq d\},$$

where  $\phi(\cdot, \dots, \cdot)$  is a function of the individual damages representing the damage to the item. We propose to examine these generalizations in a future paper.

Reverting our attention to the simple model, we obtain the asymptotic distribution of  $S_{kt}^*$  as  $t \rightarrow \infty$  and  $k = k(t) \rightarrow \infty$  at a certain rate in Theorems 1 and 2. In Theorem 1 we assume, among other conditions, that  $F(\cdot)$  has a finite variance and obtain a double exponential asymptotic distribution. In Theorem 2 we assume, among other conditions, that  $F(\cdot)$  has a support that is unbounded above and a tail that decreases to 0 like a power of  $x^{-1}$  and obtain a Weibull asymptotic distribution. By inverting these distributions to obtain the approximate distribution of the lifelength  $L$  we are led to the survival functions  $S_1(\cdot)$  and  $S_2(\cdot)$  of (1.1) and (1.2).

Theorem 1. Let the mean  $\mu$  and variance  $\sigma^2$  of  $F(\cdot)$  be finite. Further, for some  $q$  in  $(0,1)$ , let  $F(\cdot)$  satisfy

$$(2.2) \quad 1 - F(z) = o(z^{-(2+q^2)} (\log z)^{(1+q^2)/2})$$

and

$$(2.3) \quad \int_z^\infty u^2 dF(u) = o(1/\log z)$$

as  $z \rightarrow \infty$ . Let  $k = k(t)$  be an integer valued function satisfying

$$(2.4) \quad k \leq t^{q^2/2}$$

for large  $t$ . Set

$$(2.5) \quad \alpha_{kt} = \sigma(t/2 \log k)^{1/2}$$

and

$$(2.6) \quad \beta_{kt} = \mu t + \sigma(2t \log k)^{\frac{1}{2}} - (1/2)\alpha_{kt}(\log \log k + \log 4\pi).$$

Then, for  $-\infty < y < \infty$ , as  $t \rightarrow \infty$ ,

$$(2.7) \quad P\{S_{kt}^* \leq \alpha_{kt}y + \beta_{kt}\} \rightarrow \exp\{-\exp(-y)\}.$$

Proof: Since the  $Z_{ij}$ 's have finite variance, the random variables  $S_{it}$  are approximately normally distributed. The maximum of normally distributed random variables has an asymptotic double exponential distribution. Conditions (2.2)-(2.6) allow us to approximate moderate deviation probabilities of  $S_{it}$  adequately and then to use standard extreme value theory to establish (2.7), as shown below.

Theorem 1 of Amosova (1978) on moderate deviations states that under conditions (2.2) and (2.3),

$$(2.8) \quad P\{S_{1t} > \mu t + x\sigma t^{\frac{1}{2}}\} = (1-\phi(x))(1+o(1))$$

uniformly for  $x$  in  $(0, q(\log t)^{\frac{1}{2}})$ , as  $t \rightarrow \infty$ , where  $\phi(\cdot)$  is the standard normal distribution function.

Notice that under condition (2.4), for any  $y$ ,

$$\alpha_{kt}y + \beta_{kt} = \mu t + x_{kt}\sigma t^{\frac{1}{2}}$$

where

$$(2.9) \quad x_{kt} = (2 \log k)^{\frac{1}{2}} + (2 \log k)^{-\frac{1}{2}}\{y - (1/2)(\log \log k + \log 4\pi)\} < q(\log t)^{\frac{1}{2}}$$

for large  $t$ . Thus

$$kP\{S_{1t} > \alpha_{kt}y + \beta_{kt}\} = k(1-\phi(x_{kt}))(1+o(1)) \rightarrow e^{-y}$$

as  $t \rightarrow \infty$  from (2.8) and a standard result on the maximum from the normal distribution, for instance see Galambos (1978, Sec. 2.2.3). Hence

$$\begin{aligned} \log P\{S_{kt}^* \leq \alpha_{kt}y + \beta_{kt}\} &= k \log P\{S_{1t} \leq \alpha_{kt}y + \beta_{kt}\} \\ &\sim -kP\{S_{1t} > \alpha_{kt}y + \beta_{kt}\} + -e^{-y}, \end{aligned}$$

which implies (2.7).  $\square$

In the next theorem we assume that the support of  $F(\cdot)$  is unbounded above and has a tail that goes to zero like a power of  $x^{-1}$ . A distribution  $F(\cdot)$  can satisfy the conditions of both Theorem 1 and Theorem 2. For such  $F(\cdot)$ , one can obtain two different asymptotic distributions, because  $k = k(t)$  may tend to  $\infty$  at different rates.

Theorem 2. Let  $F(\cdot)$  have a finite mean  $\mu$  and satisfy

$$(2.10) \quad 1 - F(x) = x^{-\alpha}h(x)$$

where  $h(\cdot)$  is a slowly varying function and  $\alpha > 1$ . Let

$$(2.11) \quad \gamma_t = \inf\{x: x^{-\alpha}h(x) \leq 1/t\}$$

and

$$(2.12) \quad \delta_t = \mu t.$$

Let  $k = k(t) \rightarrow \infty$  so that

$$(2.13) \quad t^\alpha/k = o(1)$$

at  $t \rightarrow \infty$ . Then, for  $x > 0$ ,

$$(2.14) \quad P\{S_{kt}^* \leq \gamma_{kt}x + \delta_t\} \rightarrow \exp(-x^{-\alpha})$$

as  $t \rightarrow \infty$ .

Proof: Condition (2.10) implies that the support of  $F(\cdot)$  is unbounded above. The fact that  $\alpha > 1$  implies that  $F(\cdot)$  has a finite mean. There are two more uses of Condition (2.10) in the proof. The asymptotic distribution of a maximum from  $F(\cdot)$  will be  $\exp(-x^{-\alpha})$  (see for instance Galambos (1978, Theorem 2.4.3)); in fact, for  $x > 0$ ,

$$(2.15) \quad \log F^t(\gamma_t x) = t \log F(\gamma_t x) \sim -t(1-F(\gamma_t x)) + -x^{-\alpha}$$

as  $t \rightarrow \infty$ .

The second use of (2.10) comes in approximating large deviation probabilities of  $S_{1t}$ . Let  $q > \mu$ . Theorem 1 of S. V. Nagaev (1982) states that under Condition (2.10)

$$(2.16) \quad P\{S_{1t} > x\} = t(1-F(\frac{x}{t})) (1+o(1))$$

as  $t \rightarrow \infty$ , uniformly for  $x > qt$ . From the definition of  $\gamma_t$  in (2.11) and from (2.15) it can be seen that

$$(2.17) \quad \gamma_t \rightarrow \infty \text{ as } t \rightarrow \infty \text{ and } \gamma_t = t^{1/\alpha} s(t)$$

where  $s(t)$  is slowly varying. Thus

$$(2.18) \quad \gamma_{kt} \sim k^{1/\alpha} \gamma_t$$

Condition (2.13) relating  $k$  to  $t$  can be rewritten as  $k^{1/\alpha} \geq pt$  for some  $p > 0$ . Hence, for any  $x > 0$  and  $q > 0$ ,

$$\gamma_{kt} x + \delta_t \sim k^{1/\alpha} \gamma_t x + t\mu > pt\gamma_t x + \mu t > qt$$

for large  $t$  since  $\gamma_t \rightarrow \infty$ . From (2.16) and (2.15), it follows that, for  $x > 0$ ,  $\log P\{S_{kt}^* \leq \gamma_{kt} x + \delta_t\} = k \log P\{S_{1t} \leq \gamma_{kt} x + \delta_t\} \sim -kP\{S_{1t} > \gamma_{kt} x + \delta_t\} \sim -kt(1-F(\gamma_{kt} x)) + -x^{-\alpha}$  as  $t \rightarrow \infty$ . This establishes (2.14).  $\square$

We will now show how the survival functions  $S_1(\cdot)$  and  $S_2(\cdot)$  of (1.1) and (1.2) arise naturally from Theorems 1 and 2. From the definition of  $L$  in (2.1),

$$(2.19) \quad P\{L \geq t\} = P\{S_{kt}^* < d\}.$$

Under the conditions of Theorem 1, this is approximated by

$$\exp\{-\exp((d/\sigma)((2 \log k)/t)^{1/2} - (\mu/\sigma)(2t \log k)^{1/2} + g(k))\}$$

where

$$g(k) = 2 \log k - (1/2)(\log \log k + \log 4\pi).$$

Since  $\log k \leq (1/2)q^2 \log t$ , we can, after ignoring terms involving  $\log t$ , approximate the survival function by

$$\exp\{-\exp(\alpha + \beta t^{1/2} - \gamma t^{-1/2})\}.$$

The parameters  $\alpha$ ,  $\beta$ ,  $\gamma$  satisfy  $-\infty < \alpha < \infty$ ,  $\beta > 0$  and  $\gamma > 0$ , since in practice one would assume that  $E(Z_{11}) > 0$  and  $d > 0$ . This is the survival function  $S_1(\cdot)$ . Loosely speaking, the assumptions required for this are that the damage distribution  $F$  possess moments of order  $2 + q^2$  and that  $k$  tend to  $\infty$  like  $t^{q^2/2}$  or slower.

Under the conditions of Theorem 2, by specializing condition (2.13) to read

$$k = pt^\alpha$$

for some finite  $p$ , the survival function in (2.19) can be approximated by

$$\exp\left\{-\left(\frac{d-\delta}{\gamma kt}\right)^{-\alpha}\right\} = \exp\left\{-p^\alpha t^{\frac{\alpha}{1+\alpha}} (d-ut)^{-\alpha/s(t)}\right\}$$

where  $s(t)$  is a slowly varying function of  $t$ . As before we ignore the slowly varying function and approximate the survival function by

$$\exp\{-\beta t(1/t - \gamma)^{-\alpha}\}$$

for  $0 < t < 1/\gamma$  and where  $\beta > 0$ ,  $\gamma > 0$  and  $\alpha > 1$ . This leads to the survival function  $S_2(\cdot)$ . Loosely speaking the assumptions required for this are that the tail of the damage distribution be slowly varying of order  $\alpha > 1$  and that  $k$  tend to  $\infty$  like  $t^\alpha$  or faster.

We will now give a counterexample to Theorem 1 to show that when condition (2.4) relating  $k$  to  $t$  is violated the asymptotic distribution of  $S_{kt}^*$  need not be the double exponential or even one of the other two extreme value distributions. Let  $Z_{11}, Z_{12}, \dots$  be independent and identical distributed with  $P(Z_{11} = 0) = P(Z_{11} = 1) = 1/2$ . Let  $k = [2^t \log 2]$ , where  $[\cdot]$  is the largest integer function. Then conditions (2.2) and (2.3) are fulfilled but not condition (2.4). Note that  $S_{kt}^*$  is integer valued and satisfies  $0 \leq S_{kt}^* \leq t$ . Further,

$$P\{S_{kt}^* \leq t-1\} = (1 - 1/2^t)^k \rightarrow 1/2,$$

and

$$P\{S_{kt}^* \leq t-2\} = (1 - (t+1)/2^t)^k \rightarrow 0,$$

as  $t \rightarrow \infty$ . Thus the limiting distribution of  $S_{kt}^* - t$  exists and takes on values  $-1$  and  $0$  with probabilities  $1/2$  and  $1/2$ .

In the above, by using Theorem 2, we have given a model for the survival function  $S_2(\cdot)$  only when  $\alpha > 1$ . However, it is clear from the form of  $S_2(\cdot)$  that it is a survival function for  $\alpha > -1$ . When we fit this distribution in Section 3 we will allow the range of  $\alpha$  to be  $(-1, \infty)$ .

By computing derivatives it is easy to see that  $S_2(\cdot)$  is IFR for  $\alpha > 0$ , exponential for  $\alpha = 0$  and DFR for  $-1 < \alpha < 0$ . Similarly, it can be shown that  $S_1(\cdot)$  is IFR when  $\beta\gamma \geq (9+6\sqrt{3})/16 = 1.212$ . If  $\beta\gamma < (9+6\sqrt{3})/16$ , then the failure rate of  $S_1(\cdot)$  initially increases, then decreases and finally becomes an increasing function.

### 3. Fitting $S_1(\cdot)$ and $S_2(\cdot)$ to data.

Many authors have proposed probability models to explain observed cancer incidence rates. Creasy (1981) ascribes the most popular model to Armitage and Doll (1961) in which a multistage theory of cancer development was proposed. This leads to the Weibull survival function  $S_4(\cdot)$ . Later, Doll (1971) extended this model to allow for an initial dormant period, which lead to the shifted Weibull survival function  $S_3(\cdot)$ . The cumulative damage threshold crossing model proposed in this paper suggests two new survival functions,  $S_1(\cdot)$  and  $S_2(\cdot)$ .

Doll (1971) gives data on incidence rates ( $\times 100,000$ ) in age groups of 0-4, 5-9, ..., 70-74 years, for several types of cancer in England and Wales from 1961 to 1963. Denoting the failure rate of  $S_j(\cdot)$  by  $r_j(\cdot)$ , we estimated the parameters of  $S_j(\cdot)$  by minimizing

$$(3.1) \quad \sum_{i=1}^{16} (r(t_i) - r_j(t_i))^2$$

where  $t_i$  is the midpoint of the  $i^{\text{th}}$  age group and  $r(t_i)$  is the corresponding observed incidence rate. The Nelder-Mead algorithm was used for this minimization, as in L. S. Nelson (1973), which included the constraints on the parameters. The residual sum of squares, namely the minimum value attained by (3.1), will be denoted by  $R_j$ ,  $j = 1, \dots, 4$ . This is to be compared with

the corrected sum of squares,  $R_0$ , given by

$$(3.2) \quad R_0 = \sum_{i=1}^{16} (r(t_i) - \bar{r})^2$$

where  $\bar{r}$  is the average observed incidence rate.

The data were classified into seven types of cancer as indicated in Table 1. The category 'other' represents cancer of the bone, testes, rectum or prostate. Table 1 gives a summary of these results. For all cancer types except leukemia, Model 1 provided the best fit.

W. Nelson (1972, 1975) examined data on time until failure to insulate when charges of several thousand volts were applied to parallel plates separated by an insulating fluid. The parameters of the survival function  $S_j(\cdot)$  were estimated by minimizing the von-Mises statistic

$$(3.3) \quad \sum_{i=1}^n \frac{(\hat{F}_n(t_i) - S_j(t_i))^2}{S_j(t_i)(1 - S_j(t_i))}$$

where  $t_1, \dots, t_n$  are the observed times of failures arranged in increasing order, and  $\hat{F}_n(t_i) = \frac{n-i}{n}$ . The minimum value of (3.3) will be denoted by  $V_j$ ,  $j = 1, \dots, 4$ . The results are reported in Table 2. Once again,  $S_1(\cdot)$  seems to give the best fit.

The third set of data analyzed comes from Proschan (1963) on times to failure of Boeing 720 aircraft air conditioning units. The following modified von-Mises statistic was minimized in order to estimate the parameters of  $S_1(\cdot)$  and  $S_3(\cdot)$ :

$$(3.4) \quad \sum_{i=1}^n \frac{(\hat{G}_n(t_i) - S_j(t_i))^2}{G_n(t_i)\hat{G}_n(t_i)}$$

where again  $t_1, \dots, t_n$  are the observed times of failure arranged in increasing order, and  $\bar{G}_n(t_i) = 1 - G_n(t_i) = \frac{n-i+\frac{1}{2}}{n}$ . The minimum value of (3.4) will be denoted by  $W_j$ ,  $j = 1, 3$ . The results are reported in Table 3 and  $S_1(\cdot)$  gave the better fit.

The first data set on cancer appears to have an IFR distribution, while the second and third data sets indicate DFR distributions. In fact, Proschan (1963) rejects the hypothesis of constant failure rate in favor of a DFR distribution for the pooled failure times of all of the air conditioners. As fitted to the air conditioner data,  $S_1(t)$ , has a decreasing failure rate on the observed range from  $t = 11.5$  to  $t = 231.7$ . It is heartening to see that  $S_1(\cdot)$  fares well in all the cases considered.

Table 1. Results of fitting  $S_1(\cdot), \dots, S_4(\cdot)$  to cancer incidence data from Doll (1971).

$S_1(\cdot)$		$S_2(\cdot)$		$S_3(\cdot)$		$S_4(\cdot)$		
$R_1^*$	parameter estimates	$R_2^*$	parameter estimates	$R_3^*$	parameter estimates	$R_4^*$	parameter estimates	
Leukemia (1862 males, $R_0^* = 1.61E-7$ )								
α β γ	5.21E-9	-1.52E+1 1.14 1.00E-4	6.38E-9	4.14 1.00E-12 1.01E-4	4.66E-9	2.95E-3 1.52E+1 -1.64E+2	6.39E-9	4.69E-3 5.16 --
	Stomach (9274 males, $R_0 = 8.34E-6$ )							
	α β γ	3.80E-8	6.80 6.30E-6 8.93E+1	8.27E-8	4.05 1.00E-11 7.57E-5	7.86E-8	6.79E-3 4.93 2.50	8.22E-8
Skin (9012 males, $R_0 = 6.89E-6$ )								
α β γ		8.54E-9	-6.89 7.81E-1 2.98E+1	1.00E-8	4.09 7.61E-12 9.14E-5	9.48E-9	6.32E-3 5.83 -1.07E+1	1.01E-8
	Lung (26676 males, $R_0 = 4.88E-5$ )							
	α β γ	4.60E-6	5.15 7.23E-6 6.55E+1	5.85E-6	2.75 8.32E-9 8.35E-8	5.76E-6	7.13E-3 3.63 2.50	5.85E-6
Other (10565 males, $R_0 = 2.25E-5$ )								
α β γ		2.41E-8	1.18E+1 3.57E-2 1.35E+2	2.97E-8	6.75 9.30E-17 1.04E-4	2.86E-8	8.77E-3 7.52 2.50	2.93E-8
	All Male (57389 males, $R_0 = 2.89E-4$ )							
	α β γ	2.34E-6	7.98 4.82E-7 8.36E+1	4.41E-6	3.73 2.56E-10 8.94E-5	4.23E-6	9.62E-3 4.61 2.50	4.39E-6
Breast (21017 females, $R_0 = 7.76E-6$ )								
α β γ		1.89E-7	2.53 4.49E-10 4.77E+1	3.96E-7	1.83 2.47E-7 5.79E-9	2.24E-7	4.35E-3 2.16 1.75E+1	3.96E-7

\* $R_0$  is the corrected sum of squares, (3.2), and  $R_j$ ,  $j = 1, \dots, 4$  are the minimized residual sum of squares, (3.1).

Table 2. Results of fitting  $S_1(\cdot), \dots, S_4(\cdot)$  to data on failure times of insulators from W. Nelson (1972, 1975).

$S_1(\cdot)$		$S_2(\cdot)$		$S_3(\cdot)$		$S_4(\cdot)$	
$V_1^*$	parameter estimates	$V_2^*$	parameter estimates	$V_3^*$	parameter estimates	$V_4^*$	parameter estimates
32 KV (n = 15)							
a B Y	0.107		0.238		0.191		0.205
	-0.523		-0.513		0.051		0.047
	0.140		0.235		0.424		0.464
	0.923		1.469E-3		0.218		--
34 KV (n = 19)							
a B Y	0.341		0.296		0.295		0.303
	-0.317		-0.283		0.095		0.094
	0.231		0.190		0.675		0.692
	1.512		6.357E-3		0.082		--
36 KV (n = 15)							
a B Y	0.200		0.315		0.249		0.328
	1.200		-0.058		0.335		0.295
	0.183		0.321		0.836		0.940
	2.640		3.738E-2		0.256		--

\* $V_j$ ,  $j = 1, \dots, 4$  are the minimum values of the von-Mises statistic, (3.3).

Table 3. Results of fitting  $S_1(\cdot)$  and  $S_3(\cdot)$  to 213 failures of Boeing air conditioners from Proschan (1963).

$S_1(\cdot)$		$S_3(\cdot)$	
$W_1^*$	Parameter estimates	$W_3^*$	Parameter estimates
0.3055		0.5532	
	0.2945		0.1170
	0.0728		0.8978
	8.7581		1.0000

\* $W_1$  and  $W_3$  are the minimum values of the modified von-Mises statistic, (3.4).

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20. ABSTRACT (Continue on reverse side if necessary and identify by block number)

This paper describes a model for survival functions called the cumulative damage threshold crossing model. Under this model, an item, consisting of a large number of components which suffer damage at regular moments of time, fails as soon as the maximum cumulative damage to some component crosses a certain threshold. Under various conditions on the damage distribution, two plausible new survival distributions

$$S_1(t) = \exp\{-\exp(\alpha + \beta t^{\frac{1}{2}} - \gamma t^{-\frac{1}{2}})\}, \quad t > 0, \quad -\infty < \alpha < \infty, \quad \beta > 0 \quad \text{and} \quad \gamma > 0,$$

and

$$S_2(t) = \exp\{-\beta t(1/t - \gamma)^{-\alpha}\}, \quad 0 < t < 1/\gamma, \quad \alpha > -1, \quad \beta > 0 \quad \text{and} \quad \gamma > 0$$

arise. The survival function  $S_1(\cdot)$  is shown to fit several data sets in the area of cancer studies, failure times of insulating fluids and failure times of air conditioning units.

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