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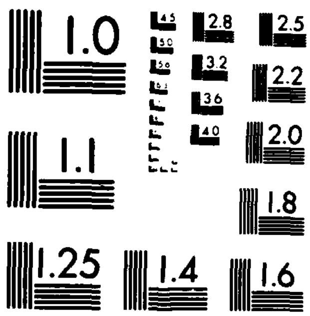

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THESIS

COMBAT ATTRITION ANALYSIS USING RENEWAL PROCESS

by

Cho, Joong Kun

March 1984

Thesis Advisor: J. G. Taylor

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This thesis uses renewal theory to investigate the Lanchester-type combat attrition process. The attrition process is analysed in detail and modelled as a so-called renewal process in which times between casualties are considered to be independently and identically distributed random variables. Other random variables that can be considered in the renewal process are examined, and the distrubutions of these random variables are determined in order to study the behaviour of attri-

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Combat Attrition Analysis Using Renewal Process

by

Joong Kun,Cho
Major, Republic of Korea Army
B.A., Korean Military Academy,1974

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Author: Cho, Joong Kun

Approved by: James G. Taylor
Thesis Advisor

James K. Hartman
Second Reader

Alan R. Washburn
Chairman, Department of Operations Research

Kneale T. Marshall
Dean of Information and Policy Sciences

ABSTRACT

This thesis uses renewal theory to investigate the Lanchester-type combat attrition process. The attrition process is analysed in detail and modelled as a so-called renewal process in which times between casualties are considered to be independently and identically distributed random variables. Other random variables that can be considered in the renewal process are examined, and the distributions of these random variables are determined in order to study the behaviour of attrition process. Examples with specific distribution functions are given for better understanding. Computer simulation is generated and compared with the attrition model developed. The total casualty occurence by total force is also discussed, through pooling of the single renewal processes. The total casualty occurence is shown to be a Poisson process and times between casualties to be approximately exponentially distributed for large numbers of combatants.

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I. INTRODUCTION

Today, deterministic-differential-equation models(that are commonly called Lanchester-type combat models) are widely used in military operations research for defense planning purposes, especially as concerns land combat. In combat analysis, such questions frequently arise as 'Who will win the battle ?, How long will the battle last ?, How many survivors will the winner have ?, and How do the force levels change over time ?',etc. There have been a lot of studies to estimate attrition in battle to answer such questions as these.

In 1914,F.W.Lanchester [Ref. 1] first proposed a mathematical model of combat in order to justify the principle of concentration of forces under "modern" conditions. He believed that in "ancient" times war was a series of one-to-one duels between men engaged in hand-to-hand combat using sword, axe, lance, etc. So there would be no advantage gained from concentration of forces. Lanchester reasoned that modern technology had changed the nature of this "ancient" warfare and now there was a decided advantage to "concentration of forces".[1] He hypothized that each side's casualty rate was proportional to the number of opposing firers. Implicit(but not yet explicit) in Lanchester's original work is the concept of an "attrition-rate coefficient" as the rate at which a single firer kills a particular enemy target type. Furthermore, Lanchester

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[1] Helmbold notes that the analysis of the data from 92 land combat battles suggests that victory in battle is primarily determined by the factors other than numerical superiority and challenges the ability of any model of combat which concentrates almost exclusively on numerical force size to yield a practically useful predictor of victory in battle.

thought that the number of survivors were the critical factor deciding the outcome of the battle in "modern" warfare.

Since Lanchester's original efforts to describe the dynamics of combat mathematically, many analysts - in an attempt to add reality to the combat description - have extended the theory to include additional factors. Many of these extentions were described by Dolansky [Ref. 2]. In his paper, Dolansky pointed out that the Lanchester attrition-rate coefficients were hard to determine for particular weapons, and, accordingly, coefficients of known magnitude had been assumed in most models. Thus, Dolansky concluded that the use of Lanchester-type models for prediction of battle results had been hampered by this inability to predict numerical values for attrition-rate coefficients. However, even if we may be in doubt as to the proper form of attrition relations, we may at least be able to make pragmatic assertions such as, 'If the Lanchester attrition is used for predictions, then by measuring certain factors and performing certain operations a prediction of such-and-such an aspect of combat can be obtained.' So, although there has been a continuing discussion among military operations analysts about the merits of Lanchester-type attrition-rate, it is still generally accepted as the basic founding for all the combat models studies and attrition-rate coefficient is still a core of these studies.

Bonder/Barfoot [Ref. 3] proposed that an attrition-rate coefficient be defined as the reciprocal of the expected time for an individual firer to kill an enemy target. A more thorough dicussion, however, of the justification for Bonder/Barefoot is to be found in Taylor [Ref. 4]. Barfoot [Ref. 5] suggested that <u>the attrition process be considered as a renewal process</u>. Thus, for a single Y firer, we will assume that each individual Y kills X targets according to an attrition process in which times to kills are independently and identically distributed (i.i.d.) random variables (r.v). We will also assume that each firer acts independently of any other firer, and similarly for targets. Such an attrition process turns out to be a renewal process in the theory of stochastic processes.

Once we introduce the theory of renewal process, we can consider many other random variables such as the time up to n-th kill, S_n; the number of kills in time t, N(t); the expected number of kills in time t, M(t), etc. Thus this thesis starts with consideration of Bonder/Taylor's concept of attrition-rate coefficient, and then studies the probability distribution of the time between casualties according to the theory of renewal process.

In this thesis, we will mainly focus on the fundamental Lanchester-type attrition paradigm which deals with homogeneous forces, because understanding this basic paradigm is essential for extentions to further complex models.

This thesis is organized in the following fashion. First, we review a few relevant facts about the Lanchester attrition models. Then we review basic ideas of renewal process and develop the random variables whose study is of our concern, followed by the study of these random variables. Then we will describe the combat attrition process by comparing the values of different random variables obtained through renewal process. Finally, the conclusion mainly contributes to the summary of the work and suggests further studies.

II. FUNDAMENTAL LANCHESTER-TYPE ATTRITION PARADIGM

A. THE BASIC PARADIGM

Let us consider combat between two homogeneous forces: a homogeneous X force opposed by a homogeneous Y force. We will focus on the force-on-force attrition process in the combat between these two homogeneous forces. The basic Lanchester-type paradigm for "modern" warfare assumes that <u>the casualty rate of such a homogeneous force is directly proportional to the number of enemy forces</u>, e.g. the X force casualty rate is given by

$$\frac{dx}{dt} = -ay \qquad \text{with} \quad x(0) = x_0 ,$$

and the Y force casualty rate is given by

$$\frac{dy}{dt} = -bx \qquad \text{with} \quad y(0) = y_0 ,$$

where 'a' denotes the rate at which a single Y firer kills X targets and is called a <u>Lanchester attrition-rate coefficient</u>. Here x_0 and y_0 denote the number of X and Y combatants (respectively) at the beginning of the battle. Many factors may affect the value of 'a' and 'b', but for the time being it is not essential that we be explicit about functional dependence of a and b. Let us next address the problem of computing a reliable numerical values for such attrition-rate coefficients.

Two approaches have been developed in the United States to determine the value of these coefficients as follows:

(1) A statistical estimate based on "combat" data generated by a detailed Monte Carlo combat simulation.

(2) An analytical submodel of attrition process for the particular combination of firer and target type.

In this thesis we will consider only the second approach. S.Bonder [Ref. 6] has called this approach <u>the use of a free-standing or independent analytical model</u>. The basic idea of this approach is to develop an analytical expression for each required kill rate by considering the single firer engaging a 'passive' target (i.e. one that doesn't fire back) and then tie all the attrition rates together in force-on-force combat with a Lanchester-type model.

Barfoot [Ref. 5] in his study has suggested <u>taking a Lanchester attrition-rate coefficient as the reciprocal of the expected time for an individual firer to kill an enemy 'target.'</u>[2] Within context of the above homogeneous force Lanchester-type combat model, this means

$$a = \frac{1}{E[T_{XY}]} ,$$

where T_{XY} denotes a r.v. representing the time for a single Y firer to kill an X target and E[T] denotes expected time to kill. Taylor also supported Barfoot's idea in his recent study [Ref. 7] by comparing times between kills that were both exponentially distributed and non-exponentially distributed, and using previous equation to determine the expected time to kill a target.

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[2] However, the justification of this approach is not accepted and is apparently somewhat controversial. But in case of exponentially distributed times between kills, a deterministic Lanchester-type model may indeed be considered to yield the mean course of combat. Anyway, this approach is accepted as general guide for further study of attrition-rate coefficient.

B. DETERMINATION OF EXPECTED TIME TO KILL

Bonder and Farrel [Ref. 8] have developed general methodology for determining the attrition-rate coefficient by determining the expected time to kill a target, E[T], for a wide spectrum of weapon system types. Morever, research since the mid-1960's has led to the development of several other methods for computing the expected time to kill a target. For present purposes, two methods will be of our interest,

(1) Method based on sum of component event times, and

(2) Method based on first-passage time in Semi-Markov process

If we assume the special case of tactical interest: namely, the case of <u>Markov-dependent fire</u>, Bonder [Ref. 3] has shown that E[T] turns out to be as follows:

$$E[T]=t_a + t_1 - t_h + \frac{(t_h+t_f)}{P(K|H)} + \frac{(t_m+t_f)}{P(H|M)}\left\{\frac{[1-P(H|H)]}{P(K|H)} + P(H|H) - P_1\right\}$$

where all symbols are defined in Table I This expession for E[T] holds for the following assumptions;

(1) Markov-dependent fire with parameter P_1, $P(H|H)$, and $P(H|M)$

(2) Geometric distribution for the number of hits required for a kill with parameter $P(K|H)$

In assessing the time factor to determine the attrition-rate coefficient, it turned out that target acquisition process and weapon system capability were most important factor from the formula. However, when we consider the time factor of the attrition process, target acquisition process would be the most important factor to study because it directly affects the required length of time to kill a target. So we will elaborate on target acquisition process for further use in attrition analysis.

TABLE I

Variables for Expected Time to Kill

time to acquire a target, t_a

time to fire first round after target acquired, t_1

time to fire a round following a hit, t_h

time to fire a round following a miss, t_m

time of flight of the projectile, t_f

probability of a hit on first round, P_1

probability of a hit on a round following a hit, $P(H|H)$

probability of a hit on a round following a miss, $P(H|M)$

probability of destroying a target given hit, $P(K|H)$

C. TARGET ACQUISITION PROCESS

Target acquisition is the initial step in engaging a target and a most deciding factor that affects the time to kill. An important distinction made in VRI's Lanchester-type combat models is whether the target acquisition process of a single "typical" firer type is a serial process or a parallel process [Ref. 4]. The two modes for the target acquisition process considered by VRI's models(including VECTOR-2) are as follows:

(1) Serial Acquisition

(2) Parallel Acquisition

Here a single firer using serial acquisition does not acquire targets while engaging another target, say target A. When such a firer ceases to engage a target A (due to kill, lost or any other reason), then he must acquire a new target all the way from the beginning. So it is assumed that he does not remember any acquisitions made prior to engaging the target A. On the other hand, a firer using parallel acquisition searchs for targets continuously, even while engaging target A and remembers those targets that have been acquired. Thus when such a firer finishes the engagement with target A, he can immediately shift engagement to the next target, provided that such a target was acquired and processed during or before the engagement of target A. For both methods, the VRI models assume that a firer never engages a killed target again.

1. Serial Acquisition

Now when we consider the basic Lanchester-type combat model in the previous section, a firer in serial process mode of target acquisition starts again from the beginning in searching a new target whenever the previous target has been killed. Thus we assume that target acquisition time is always needed to acquire a different target for engagement. Here it is assumed that the total-force kill rate is just the single-firer-kill-rate times the number of firers, e.g.

$$-\frac{dx}{dt} = ay \; ,$$

since each firer is independent from any other firer. Also, in the case of a serial process for each firer, the expected time to kill a target is the same as has been shown in the equation for E[T] in section B. A summary of attrition-rate coefficient results for serial acquisition is given in Table II .

TABLE II

Attrition Process Using Serial Acquisition Mode

$$\frac{dx}{dt} = - ay$$

$$a = \frac{1}{E[T_{XY}]}$$

T_{XY} = time for a Y firer to kill an X target

$$E[T] = t_a + t_1 - t_h + \frac{(t_h + t_f)}{P(K|H)} + \frac{(t_m + t_f)}{P(H|M)} \left\{ \frac{[1-p(H|H)]}{P(K|H)} + P(H|H) - P_1 \right\}$$

2. Parallel Acquisition

As we mentioned earlier, a firer in the parallel mode of acquisition continues to acquire new targets even while he is engaging a particular target. Once this target has been killed, then he can immediately shift fire to a new target which has been acquired already. In this case, total-force kill rate is given by the product of the kill rate of single firer against acquired targets and the expected number of firers who have already acquired one or more targets.

The summary results for attrition process using parallel acquisition is given in table III . Note here that f_{xy} is same as the probability that a single Y firer using parallel acquisition mode has available one or more acquired targets at which to fire at time t, because if he is firing we know that he already acquired the targets. Then it should also be noted that $f_{xy}\,y$ represents the expected number of Y firers who have already acquired one or more X targets.

TABLE III

Attrition Process Using Parallel Acquisition Mode

$$\frac{dx}{dt} = -\,f_{xy}\,a'y$$

f_{xy} : Prob{single Y firer using parallel acquisition is firing at X target at random time t }

$$f_{xy} = 1 - \exp\{-x \int_0^t R_{xy}(s)\,ds\}$$

$$a' = \frac{1}{E[T'_{XY}]}$$:Y firer kill rate against acquired target

T'_{XY} = Time for a Y firer to kill an acquired X target (conditional kill time)

R_{xy} : the rate at which a Y firer acquires X targets at time t when appropriate

$$E[T'] = t_1 - t_h + \frac{(t_h + t_f)}{P(K|H)} + \frac{(t_m + t_f)}{P(H|M)}\left\{\frac{[1-P(H|H)]}{P(K|H)} + P(H|H) - P_1\right\}$$

Now we may assume that it would take longer to kill a target for a firer using serial acquisition than a firer using parallel acquisition because of the time fraction of acquisition of target. This distinction between these two acquisition modes may be further studied as the case of a delayed renewal process. Here we assume that expected time to kill the first target would be obtained from the serial acquisition but following time to kill would be from the parallel acquisition, provided that the targets are acquired already. But in this thesis, we will only deal with the firer who uses the same acquisition mode to attrit all targets.

Before continuing, it is important that the reader understand both the basic Lanchester-type attrition model and also Bonder's approach of the attrition-rate coefficient as the reciprocal of expected time to kill a target. It must be recognized that this expected time to kill a target is primarily dependent on the acquisition process.

III. ANALYSIS OF THE COMBAT ATTRITION PROCESS

A. INTRODUCTION

The Lanchester attrition-rate coefficient is the rate at which a single firer kills a particular enemy target type in Lanchester-type combat. Development of technically-sound and scientifically-valid methodology for determining numerical values for Lanchester attrition-rate coefficient is an essential prerequisite for building a Lanchester-type combat model. The basic construct of Bonder/Barfoot methodology was to take a Lanchester attrition-rate coefficient as the reciprocal of the expected time for an individual firer to kill an enemy target as mentioned in the previous chapter. Taylor provided justification [Ref. 4] for taking Lanchester attrition-rate coefficient as the reciprocal of the expected time to kill. He started from the basic hypothesis that combat is a complex random process, but it contains enough regularity that the appropriate Lanchester-type equations are a good approximation to the mean course of combat. It is clear that the casualty rate is equal to the reciprocal of the expected time for a force to inflict a casualty, when the times between casualties are exponentially distributed. However, in case the times between casualties are no longer exponentially distributed, Taylor used the suggestion made by Bonder and Barfoot. Bonder [Ref. 3] and Barfoot [Ref. 5] suggested defining the Lanchester attrition-rate coefficient as the expected rate which a single firer kills enemy targets.

In the spirit of Bonder and Barfoot, Taylor provided more rigorous justification [Ref. 7] for the Lanchester attrition-rate coefficient that does not assume an exponential distribution for times between casualties. He considered the case where the initial force size of X and Y is large enough to insure a negligible probability that the battle is terminated before an attrition pattern could be established. He made no specific assumptions about the distribution of times between kills, but assumed that each individual Y force kills X targets according to an attrition process in which the times between kills are independently and identically distributed random variables. Thus in the parlance of the theory of stochastic processes, he said that such an attrition process is called a renewal process. Prior to this, attrition prediction has been difficult due to the inability to predict casualty patterns. By investigating the distribution of the times between casualties via renewal theory, combat attrition analysis becomes easier. Indeed, we know a great deal about the casualty pattern once we know the distribution of the interarrival time of kills. As we consider more variables from the renewal process, we find more information about the casualty pattern. Furthermore, we can track the development of the combat attrition more precisely by studying the probabilistic distribution function of those random variables that can be considered in the renewal process.

We assume here that the reader is aware of the concept of a counting process. When we think about combat between X and Y forces, we can assume that time between kills by a single firer has some distribution F. Now we are interested in observing the occurence of casualty and the number of casualties, N(t), that have happened in time interval (0,t]. Furthermore, when times between kills are i.i.d., we call this counting process as renewal process.

In this chapter, the study of attrition analysis will include the following :

- The r.v., $N(t)$, total number of casualties by time t, it's distribution, and properties.
- The expected number of casualties by time t.
- Theorems discussing casualty occurence.
- The limiting behavior of the casualty occurence.
- The age and excessive life of the casualty occurence.
- Examples with specific casualty distributions.
- Application of renewal process for casualty estimation.
- Total force attrition by pooling the single firer renewal processes.

B. REVIEW OF RENEWAL PROCESS

If the sequence of nonnegative random variables $\{X_1, X_2, \ldots,\}$ are i.i.d., then the counting process $\{N(t),\ t \geq 0\}$ is said to be a renewal process (or often called an ordinary renewal process). Thus, a renewal process is a counting process such that the time until the first event has some distribution F, the time between the first and the second event has, independently of the time of the first event, the same distribution F, and so on. So when a event occurs, we say that a renewal has taken place.

An example of a renewal process, let us suppose that we have an infinite supply of lightbulbs whose lifetimes are i.i.d.. Suppose also that we use a single lightbulb at a time and when it fails we immediately replace it with a new one. Under these conditions, $\{N(t),\ t \geq 0\}$ is a renewal process when $N(t)$ represents the number of lightbulbs that have failed by time t. Accordingly, if we assume that this failure is a kill by a single firer - X or Y -, then killing process also can be considered as a renewal process. Thus, renewal (or event or arrival) will be synonymous with kill or casualty occurence in the rest of this thesis. We now assume that casualty occurence is a kind of renewal process. If, in particular, the distribution of interarrival time of kill is exponential with probability distribution function e^{-x}, then this renewal process turns out to be a Poisson process with rate, . [Ref. 9]

Thus we may assume that X casualty occurence due to a single Y firer who uses the same acquisition mode to engage each X target can be explained by ordinary renewal process because we assume that times between casualties are all i.i.d. However, we often consider a counting process for which the first interarrival time has a different distribution than the remaining ones. So, we may think of a Y firer who at first uses serial acquisition mode to engage an X target, but once he has finished engaging an X target, then he may use parallel mode to engage the rest of the targets. So, in this case, we can assume that interarrival time of X casualty has different distribution between the first and the rest.

Formally, let $\{X_i, i \geq 1\}$ be a sequence of independent r.v. with X_1 having distribution G, and $\{X_i, i \geq 2\}$ having distribution F which is different from G. Then counting process $\{N_D(t), t\geq\}$ is said to be a delayed renewal process. So, we can think of using delayed renewal process for different acquisition mode, but here in this thesis, we will only deal with the case of ordinary renewal process for simplicity.

We now define for the renewal process a number of associated random variables whose study is the objective of this thesis. Table IV and Figure 3.1 give the intuitive interpretation of the random variables obtained in renewal process.

Here we can interpret each random variable as following:

- X_n is the time between the (n-1)st and n-th X force kill

TABLE IV

Random Variables in Renewal Process

r.v	interpretation
X_n	time between the (n-1) st and n-th renewal i.e. n-th interarrival time
S_n	time up to n-th renewal
N(t)	total number of renewals in (0, t]
M(t)	expected number of renewals in (0, t] known as <u>renewal function</u>
Z(t)	length of time measured from the last renewal until a given time t , called as <u>age or backward recurrence time</u>
Y(t)	length of time measured from time t until the next renewal, called as <u>excess life or forward recurrence time</u>
S(t)	length of renewal time at time t, called as <u>spread</u>

by a single Y firer.

- S_n is the time up to n-th kill.
- N(t) is the total number of kills in (0,t].

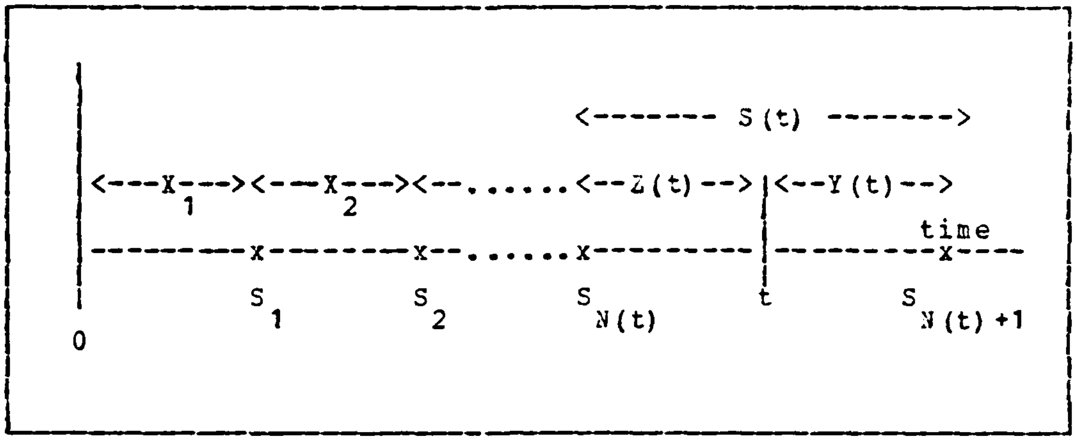

Figure 3.1 Renewal Process.

- $M(t)$ is average number of kills in $(0,t]$.
- $Z(t)$ is the time between the last kill and a given time t.
- $Y(t)$ is the length of time from time t up to the next kill.
- $S(t)$ is the length of lifetime of X force soldier at time t.

In this thesis, we will only deal with X force casualty by Y firer to avoid the confusion in notation.

C. DISTRIBUTION OF TOTAL NUMBER OF CASUALTIES

Suppose that for an ordinary renewal process a single Y firer uses the parallel acquisition mode to engage passive X targets from the beginning of combat, when we assume that targets are already acquired. Then, the distribution of $N(t)$, the total number of X force casualties by a single Y firer, can be obtained at least in theory by first noting the relationship that the number of kills by time t is greater than or equal to n if and only if the n-th kill occurs before or at time t.

Formally

$$N(t) \geq n \iff S_n \leq t \qquad (3.1)$$

From 3.1, we obtain

$$P\{N(t)=n\} = P\{N(t) \geq n\} - P\{N(t) \geq n+1\}$$
$$= P\{S_n \leq t\} - P\{S_{n+1} \leq t\}$$

Now, since the random variable X_i's, $(i \geq 1)$, are independent and have a common distribution F, it follows that $S_n = \sum_{i=1}^{n} X_i$ is distributed as F_n, which is n-fold convolution of F with itself [Ref. 10]. Here, F is the probability distribution of time between casualties. Therefore we obtain

$$P\{N(t)=n\} = F_n(t) - F_{n+1}(t) \text{ with } F_0(t) = 1$$

and

$$P\{N(t) \leq n\} = P\{S_n \geq t\} = 1 - F_n(t) \qquad (3.2)$$

Therefore the probability distribution of $N(t)$, which is total number of casualties by single Y firer, can be obtained explicitly for all n.

The simplest special case of equation 3.2 is obtained by taking the renewal process to be a Poisson process. Then S_n has the special Erlang distribution with n stages because each F is exponentially distributed. Also, it is known from the previous argument that <u>$N(t)$ has a Poisson distribution with mean t</u>.

D. EXPECTED NUMBER OF CASUALTIES

1. Renewal Function

Expected number of casualties, $E[N(t)]$, which is defined to be the mean number of casualties in the interval $(0,t]$, is known as the <u>mean value or renewal function, $M(t)$</u>.

From equation 3.1, the expected number of casualties in time t can be obtained as following:

$$M(t) = E[N(t)] = \sum_{k=1}^{\infty} k\, P\{N(t)=k\}$$

$$= \sum_{n=1}^{\infty} \sum_{k=n}^{\infty} P\{N(t)=k\}$$

$$= \sum_{n=1}^{\infty} P\{N(t) \geq n\}$$

$$= \sum_{n=1}^{\infty} P\{S_n \leq t\}$$

$$= \sum_{n=1}^{\infty} F_n(t) \qquad (3.3)$$

But here it is generally very difficult to find $M(t)$ directly from the distribution F. Thus, we will employ the Laplace Stieltzes Transform (L.S.T.) [Ref. 9] to find more conveniently an expression for $M(t)$. We generally need three steps to find $M(t)$.

Step 1:

From the definition of L.S.T., we know tnat L.S.T. of $M(t)$ is

$$M^*(s) = \int_0^{\infty} e^{-st}\, dM(t)$$

and also

$$F^*(s) = \int_0^{\infty} e^{-st}\, dF(t)$$

Step 2:

Then from equation 3.3

$$M^*(s) = \sum_{n=1}^{\infty} F_n^*(s)$$

$$= \sum_{n=1}^{\infty} [F^*(s)]^n \quad \text{for } s>0$$

The above is a sum of geometric series and can be written as follows:

$$M^*(s) = \frac{F^*(s)}{1-F^*(s)} \qquad (3.4)$$

or equivalently

$$F^*(s) = \frac{M^*(s)}{1+M^*(s)}$$

<u>**Step 3:**</u>

Since the L.S.T. of a function uniquely determines the function, the renewal function determines the distribution of the kill time and therefore determines the probability law of the casualty occurence. Specifically there is a one-to-one correspondence between interarrival time distribution F, and renewal function, M(t). So if we have the distribution F, then we can find $M(s)$ from the equation 3.4. But to find M(t),we have to transform M(s) to M(t) by inverse application of L.S.T.

Let us try the example. Remember that this example will be applied all the way through this chapter.

Example 3.1

Suppose that each interarrival time to kill X_i,$(i\geq 1)$ has exponential distribution with . Then L.S.T. of F is

$$F^*(s) = \int_0^{\infty} e^{-st}\,dF(t)$$

$$= \int_0^{\infty} e^{-st}\,\lambda e^{-\lambda t}\,dt$$

$$= \int_0^{\infty} \lambda e^{-(\lambda+s)t}\,dt$$

$$= \frac{\lambda}{\lambda+s}\int_0^{\infty} (\lambda+s)\,e^{-(\lambda+s)t}\,dt$$

$$= \frac{\lambda}{\lambda+s} \qquad \text{if } s>-\lambda$$

Then from equation 3.4

$$M^*(s) = \frac{F^*(s)}{1-F^*(s)} = \frac{\lambda/(\lambda+s)}{1-(\lambda/(\lambda+s))} = \frac{\lambda}{s}$$

where 1/s was an L.S.T. of F(t), which had value of t.[3] So $M(t) = \lambda t + c$ (some constant), but we know that $M(0) = 0$, because there would be no casualty at time 0. So, finally $M(t) = \lambda t$.

Let us be more specific about this example. Suppose that interarrival times of casualties are i.i.d. and exponentially distributed with mean 30 minutes. Then

a) What is the distribution of total X casualties at 5 hours later ?

b) What is the probability that 3rd X kill was occured at time t = 4 ?

c) What is the probability that at time t = 2, no X casualty occured ?

To answer these questions we know that $M(t) = 2t$ from above example because = 2, then number of X casualties in time t has Poisson distribution with mean 2t.

---

[3]Let $F(t) = t$ for $t \geq 0$. Then L.S.T. of F(t) is

$$F^*(t) = \int_0^\infty e^{-st} dF(t) = \int_0^\infty e^{-st} 1dt = -\frac{1}{s} e^{-st} \Big]_0^\infty$$

$$= 1/s \text{ for } s>0$$

Then

a) $P\{N(5)=n\} = \dfrac{e^{-10}(10)^n}{n!}$

b) $P\{N(4)=3\} = \dfrac{8^3 e^{-8}}{3!} = .0236$ and

c) $P\{N(2)=0\} = \dfrac{4^0 e^{-4}}{0!} = .0183$

2. Renewal Equation

Now an integral equation for $M(t)$ may be obtained by conditioning on the time of the first casualty occurence. Upon doing so, we obtain

$$M(t) = \int_0^{\infty} E[N(t) \mid X_1 = x]\, dF(x) \tag{3.5}$$

however

$$E[N(t) \mid X_1 = x] = \begin{cases} 0 & \text{if } x>t \\ 1 + M(t-x) & \text{if } x \le t \end{cases} \tag{3.6}$$

for if the first casualty occurs at time x, $x \le t$, then from this point on the process starts over again, and thus the expected number of casualties in $(o,t]$ is just 1 plus the expected number of casualties in time t-x from the beginning of an equivalent renewal process. Putting equation 3.6 in 3.5 yields

$$M(t) = \int_0^t [1+M(t-x)]dF(x)$$

$$= F(t) + \int_0^t M(t-x)\, dF(x) \tag{3.7}$$

where equation 3.7 is known as the renewal equation and F is a known function and M is an unknown function to be determined as a solution to the integral equation 3.7. Ross provided the solution for equation 3.7 [Ref. 11] as following and the solution is accepted true for every case.

Expected number of casualties up to time t is

$$M(t) = F(t) + \int_0^t F(t-x)\,dM(x) \qquad (3.8)$$

$$\text{where } M(x) = \sum_{n=1}^{\infty} F_n(x)$$

So it allows to find M(t) using only the known distribution function F, which is the distribution of time between casualties , thus avoiding finding individual F_n.

3. Renewal Density

Now let's define the renewal density as m(t) = dM(t)/dt. Remember that M(t) is the expected number of casualties up to time t. So m(t) may be regarded as the expected number of casualties per unit time; more precisely, the average number of casualties during the time interval from t to t+h is

$$M(t+h) - M(t) = \int_t^{t+h} m(x)\,dx \qquad (3.9)$$

Differentiating the renewal equation, we obtain the renewal equation for the density m(t)

$$m(t) = f(t) + \int_0^t f(t-x)\,m(x)\,dx \quad \text{for } t>0 \qquad (3.10)$$

So, this knowledge of the density f of interarrival time of casualties is sufficient to determine the density m(x) of casualty occurence.

E. LIMIT BEHAVIOUR OF CASUALTY OCCURENCE

We know that total number of casualties, $N(t)$, go to infinite as time goes to infinite, when we assume that force size is infinite. But it would be useful to know at what rate $N(t)$ approaches infinity. That is, we would like to be able to say something about limit of $N(t)/t$ as time goes to infinite.

From the definition of r.v. we know that

$$S_{N(t)} \le t \le S_{N(t)+1}$$

where $S_{N(t)+1}$ must be the time of the casualty after time t. By refer to [Ref. 10] we know that

$$\frac{N(t)}{t} \longrightarrow \frac{1}{u} \quad \text{as } t \longrightarrow \infty \tag{3.11}$$

where $u = E[X_i] = \int_0^\infty [1 - F(x)]dx$, which is the expected time between casualties.

Here the function $1/u$ is often called as <u>the rate of the renewal process</u>. Thus the average number of casualties per unit time converges to $1/u$. Then, how about the expected average number of casualties per unit time? Is it true that $M(t)/t$ also converges to $1/u$ as $t \longrightarrow \infty$? This result, known as the <u>elementary renewal theorem</u>, will be stated without proof [Ref. 11].

<u>Elementary Renewal Theorem</u>

$$\frac{M(t)}{t} \longrightarrow \frac{1}{u} \quad \text{as } t \longrightarrow \infty \tag{3.12}$$

Thus whatever the distribution F is, <u>the average number of casualties up to time t</u> is approximately (for large t),

$$M(t) \approx \frac{t}{u} \qquad (3.13)$$

Also, <u>the average number of casualties in the interval (t, t+h)</u> for h>0 is approximated for large t as following,

$$M(t+h) - M(t) \approx \frac{h}{u} \quad \text{as } t \to \infty \qquad (3.14)$$

for any distribution F with mean u.

Here equation (3.14) is known as <u>Blackwell's theorem</u> when F is not lattice.

From the example 3.1, where casualty occurence time is exponentially distributed with mean 30 minutes, the average number of X casualties up to time t is approximated by t/u, which is 2t. Then after 2 hours of combat, we can assume that the average number of X casualties by single Y firer will be 2t, which is 4 casualties.

<u>Another limiting result</u> which may prove to be useful in renewal process also concerns the r.v. N(t). It is shown [Ref. 12] that <u>N(t) has an asymptotically normal distribution with mean t/u and variance</u> $t\,\sigma^2/u^3$. Thus

$$\lim_{t \to \infty} P\{N(t) < y\} = \Phi\left(\frac{y - t/u}{\sqrt{\sigma^2 t/u^3}}\right) \qquad (3.15)$$

Suppose of the same example 3.1 that casualty occurence mean time is 30 minutes. Then u = 0.5 = E[T] = $1/\lambda$. So λ = 2. Also Var[T] = $1/\lambda^2$ = 1/4 = 0.25 hr^2 , which is 900 min^2 here. Let's assume that we had a 10 hours of combat action. Then expected number of X casualties is E[N(10)] = t/u = 10/0.5 = 20 and var[N(10)] = $t\sigma^2/u^3$ = $10 \times 0.25/(0.5)^3$ = 20.

Suppose of the question that how many X soldiers will be required in order to with probability of 95% that at least more than one X soldiers will survive after 10 hours of combat ? This question may be answered as following,

$$0.95 \le P\{N(10)<x\} = \Phi\left(\frac{x - 20}{\sqrt{20}}\right) = \Phi\left(\frac{x - 20}{4.472}\right)$$

Now from the normal table, 0.95 percentile of normal distribution is 1.645. So $1.645 \le (x - 20)/4.472$, where the answer turns out to be $x \ge 27.36$. So we can say that X force needs at least 28 soldiers as an initial force size to be 95% sure that at least more than one X soldier can survive after 10 hours of combat.

There also exist a <u>alternative normal approximation</u>, when we assume that for integer n,

$$P\{N(t)<n\} = P\{S_n > t\} \quad \text{where } S_n = \sum_{i=1}^{n} X_i$$

from equation 3.1.

$$\text{Then } P\{S_n > t\} = 1 - P\{S_n \le t\} = 1 - \Phi\left(\frac{t - nu}{\sigma\sqrt{n}}\right) \qquad (3.16)$$

So now we assume that S_n has normal distribution with mean nu and variance $n\sigma^2$. If we apply equation 3.16 to previous example,

$$0.95 \le P\{N(10)<x\} = 1 - \Phi\left(\frac{10 - 0.5x}{0.5\sqrt{x}}\right)$$

$$\text{Then } \Phi\left(\frac{10 - 0.5x}{0.5\sqrt{x}}\right) \le 0.05 \quad \text{or} \quad \frac{10 - 0.5x}{0.5\sqrt{x}} \le -1.645$$

If we solve for x, it turns out to be $x \ge 28.83$. So answer is almost same when we assume some rounding errors.

F. AGE AND EXCESS LIFE DISTRIBUTION OF CASUALTY OCCURENCE

As we mentioned earlier in this chapter, we can also consider such random variables as Y(t), which is the time from certain time t until next casualty occurence, which means residual lifetime of a soldier in combat from certain time t if he is still alive, and $Z(t)$, which is the time from t since the last casualty occurence (see Figure 3.2).

```
        S                              S
         N(t)                           N(t)+1
--------x------------|-------------x---------------
                     t

          Z(t)           Y(t)

               S(t)
```

Figure 3.2 Age and Excess Life of Casualty Occurence.

This means

$$Y(t) = S_{N(t)+1} - t$$

$$Z(t) = t - S_{N(t)}$$

where $Y(t)$ is called as *the excess life or forward recurrence time* and $Z(t)$ is called as *age or backward recurrence time*. Also

$$\begin{aligned} S(t) &= Z(t) + Y(t) \\ &= S_{N(t)+1} - t + t - S_{N(t)} \\ &= S_{N(t)+1} - S_{N(t)} \\ &= \sum_{i=1}^{N(t)+1} X_i - \sum_{i=1}^{N(t)} X_i \\ &= X_{N(t)+1} \qquad (3.17) \end{aligned}$$

Here $S(t)$ is often called as <u>spread</u> and $S(t)$ is also r.v. because $X_{N(t)+1}$ is r.v. too.

Suppose of the Poisson process that X_i is exponential, then $E[S(t)] = E[Y(t)] + E[Z(t)] = 1/\lambda + 1/\lambda = 2/\lambda = 2E[X_i]$. So, $S(t)$ can be assumed as the <u>length of the lifetime of the X force soldier</u> which is in combat at time t.

IV. THE SUPERPOSITION OF CASUALTY OCCURENCES

A. INTRODUCTION

In the previous chapter, we have considered in detail the theory of casualty occurence through the ordinary renewal process of a single Y force soldier. We deal in this chapter more briefly with the <u>superposition of several renewal processes, which is total X force casualty occurence behaviour</u>. When we assume multiple Y force soldiers who shoot at X force soldiers independently of each other, we may assume that all these Y soldiers kill X soldiers by independent casualty processes. So, we can say that there occur multiple independent ordinary renewal processes, simultaneous in time.

```
process 1 |------------x--------------x--------------x--
                       '              '              '
                       '              '              '
process 2 |-----x------'---x----------'----x---------'--
                '      '   '          '    '         '
                '      '   '          '    '         '
process 3 |--x--'------'---'------x---'----'-----x---'--
             '  '      '   '      '   '    '     '   '
             '  '      '   '      '   '    '     '   '
pooled       '  '      '   '      '   '    '     '   '
 output   |--x--x------x---x------x---x----x-----x---x--
```

Figure 4.1 Superposition of Casualty Occurences.

Suppose that we have 'p' independent casualty processes in operation simultaneously, all with the same P.D.F. of casualty occurence time. Consider the sequence of casualty occurences formed by pooling the individual processes. Fig 4.1 illustrates the special case, when p = 3. Formally, we can say that if $\{N_i(t), i \geq 0\}$, for i = 1,2, ... , k, are independent renewal processes and $N_p(t) = \sum_{i=1}^{k} N_i(t)$, then $\{N_p(t), t \geq 0\}$ is called <u>pooled process</u> which is total X force casualty occurence behaviour by k Y force soldiers.

B. SOME GENERAL PROPERTIES

1. <u>Normal Approximation of Pooled Output</u>

From casualty occurence processes in the previous chapter, we assumed that there exist a mean casualty occurence time. So let u_i and σ_i^2 be the mean casualty occurence time and corresponding variance by the i-th Y force soldier and let u(p) and $\sigma^2(p)$ be the mean casualty occurence time and corresponding variance of the total casualty occurence by k of Y force soldiers. Then the question is <u>how are these two values related each other</u> ?

From previous study, we know that 1) $N_i(t)$ is approximately normal with mean t/u_i and variance, $\sigma_i^2 t/u_i^3$. Therefore we can say that $N_p(t)$ is also approximately normal with mean $\sum_{i=1}^{k} t/ui$ and variance $\sum_{i=1}^{k} \sigma_i^2 t/u_i^3$. 2) Also assume now that either the pooled process is a renewal process or at least the analogous result holds, then approximately $N_p(t)$ is normal with mean t/u(p) and variance $\sigma^2(p)\, t/ u^3(p)$.

So, from 1) and 2), we must have

$$\frac{t}{u(p)} = t \sum_{i=1}^{k} \frac{1}{u_i} \qquad (4.1)$$

then $u(p) = \left(\sum_{i=1}^{k} \frac{1}{u_i}\right)^{-1}$ which is the harmonic mean of the u_i,

and $$\frac{{}^{2}(p)}{u^{3}(p)}\, t = t \sum_{i=1}^{k} \frac{{}_{i}^{2}}{u_i^{3}}\, t$$

then $$\sigma^2(p) = u^3(p) \sum_{i=1}^{k} \frac{\sigma_i^2}{u_i^3} = \left(\sum_{i=1}^{k} \frac{1}{u_i}\right)^{-3} \sum_{i=1}^{k} \frac{\sigma_i^2}{u_i^3} \qquad (4.2)$$

Also if we assume that $u_i = u$ and $\sigma_i =$ for all $i = 1, 2, \ldots, k$ which means all casualty occurences are independent and identical, then mean casualty occurence time and corresponding variance of the total casualty behaviour by k of Y force are as follows;

$$u(p) = \left(\frac{1}{u} k\right)^{-1} = \frac{u}{k}$$

$$\sigma^2(p) = \left(\frac{1}{u} k\right)^{-3} \frac{\sigma^2}{u^3} k = \frac{\sigma^2}{k^2} = \left(\frac{\sigma}{k}\right)^2 \qquad (4.3)$$

Also <u>total number of expected casualties,</u> $E[N_p(t)]$, can be obtained here by pooling individual casualty occurences as following;

$$E[N_p(t)] = \sum_{i=1}^{k} E[N_i(t)] = k\, E[N_i(t)] = k\, M(t) \qquad (4.4)$$

where $M(t)$ is expected number of casualty from single firer, which is t/u for large t.

Let's look at an example that deals with pooling of individual renewal processes.

Example 4.1

Suppose of Y force artillery shells from several sources are bombording the same X target. Each source hurls shells at the target at a rate of 40 shells per hour. Assume that the interarrival times of shells from each single source are uniformly distributed over an interval (0, a]. Then the question is 1) What is the probability that less than 825 shells will be hurled at the target from a single source during a 20 hour period and 2) If there are 5 sources from which such attacks are launched, what is the probability that more than 4125 (which is 5 times 825) shells are hurled at the target in a 20 hour period ?

To answer the first question, let λ = 40 shells/hour = 2/3 shells/min. Then $u = E[X] = 1/\lambda = 3/2$ min, when we assume poisson process. But interarrival times are uniformly distributed over an interval (0,a]. So, $u = E[X] = (0 + a)/2 = a/2$, where a = 3 minutes. Then $Var[X] = (a - 0)^2/12 = 3/4 \text{ min}^2$. Now let N(t) = number of shells hurled in (0, t] from a single source. Since N(t) is normal with mean t/u and variance $t\sigma^2/u^3$ from equation 3.15, for t = 20 hours = 1200 minutes, the probability turns out to be as following;

$$P\{N(1200)<825\} = \Phi\left(\frac{825 - \frac{1200}{3/2}}{\sqrt{\frac{1200 \cdot 3/4}{(3/2)^3}}}\right) = \Phi\left(\frac{825 - 800}{16.33}\right)$$

$$= \Phi(1.53) = 0.9370$$

and for second question,

Let $N_i(t)$ = number of shells hurled in (0 , t] by i-th source where i = 1,2, ... , 5 . and

$N_p(t) = \sum_{i=1}^{5} N_i(t)$ = total number of shells hurled in (0, t] by 5 sources.

Then $\{N_p(t), t \geq 0\}$ is the pooled process for which mean interarrival time is,

$$u(p) = \frac{u}{k} = \frac{3/2}{5} = \frac{3}{10} \text{ minutes.}$$

and corresponding variance is

$$\sigma^2(p) = (\sigma/k)^2 = \frac{3}{100} \text{ min}^2$$

Since $N_p(t)$ is normal with mean $t/u(p)$ and variance $\sigma^2(p)t/u^3(p)$ for $t = 1200$ minutes,

$$P\{N_p(1200) < 4125\} = \Phi\left(\frac{4125 - \frac{1200}{3/10}}{\sqrt{\frac{1200 \; 3/100}{(3/10)^3}}}\right) = \Phi\left(\frac{125}{36.51}\right)$$

$$= \Phi(3.42) = .9997$$

So, $P\{N_p(1200) \geq 4125\} = 0.0003$

2. Poisson Process of Pooled Output

Now let's assume that individual casualty processes are probabilistically identical with common underlying C.D.F. of casualty occurence time $F(x)$. Let $F_p(x)$ be the C.D.F of casualty occurence time of pooled process. Then the question is how is $F_p(x)$ related to $F(x)$? This question can be answered by using the idea of excess life. Let $Y_i(t)$ be the excess life of an X force soldier and $Y_p(t)$ be excess life of total X force. Then we know that $Y_p(t) = \min[Y_1(t), Y_2(t), \ldots, Y_k(t)]$ (see Fig 5.1). Then

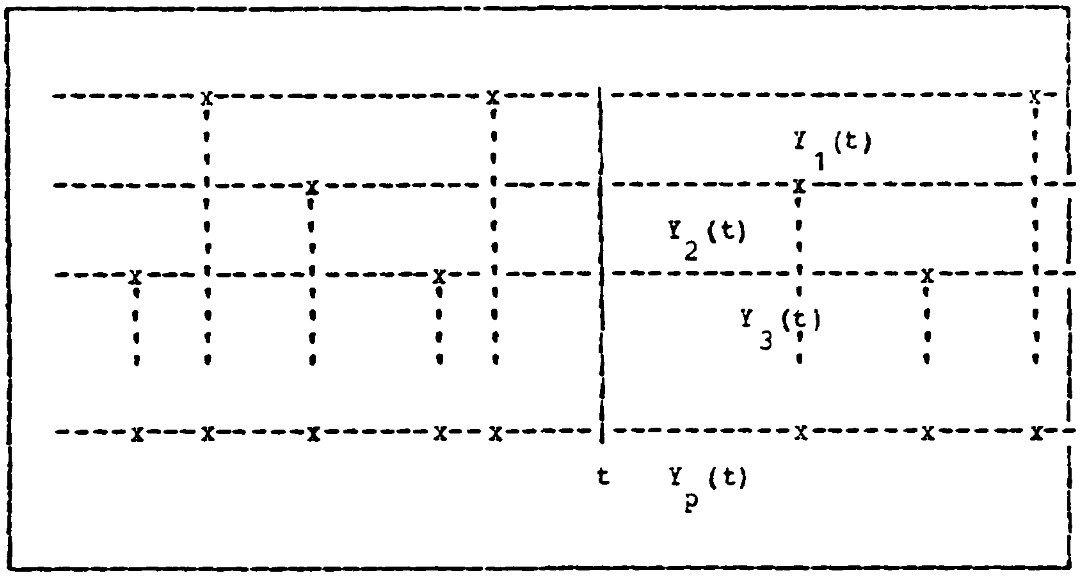

Figure 4.2 Superposition of Excess Life.

$$P\{Y_p(t)>y\} = P\{Y_i(t)>y\}^k \tag{4.5}$$

From previous study, we know that P.D.F of E.D. for individual renewal processes is $(1/u)[1 - F(x)]$ and then we can say that $\frac{1}{u(p)}[1 - F_p(x)]$ is P.D.F of E.D. for the pooled casualty process. Thus,

$$P\{Y(t)>y\} = \int_y^\infty \frac{1}{u}[1 - F(x)]dx \quad \text{then,}$$

$$P\{Y_p(t)>y\} = \int_y^\infty \frac{1}{u(p)}[1 - F_p(x)]dx$$

$$=\left[\frac{1}{u}\int_y^\infty [1 - F(x)]dx\right]^k \quad \text{from equation 4.5}$$

If we differentiate with respect to y

$$-\frac{1}{u(p)}[1 - F_p(y)] = k[\frac{1}{u}\int_y^{\infty}[1 - F(x)]dx]^{k-1}(-\frac{1}{u})[1 - F(y)]$$

Here we know that $u(p) = k/u$, so u's are all cancelled out. Then,

<u>C.D.F. of pooled output</u> is

$$F_p(y) = 1 - [\frac{1}{u}\int_y^{\infty}[1 - F(x)]dx]^{k-1}[1 - F(y)] \qquad (4.6)$$

For example, think of the exponential distribution of casualty occurence. Then $F(x) = 1 - e^{-\lambda x}$, and $u = 1/\lambda$. Then C.D.F of pooled casualty occurence is

$$F_p(y) = 1 - [\int_y^{\infty}[e^{-\lambda x}]dx]^{k-1}e^{-\lambda y}$$

$$= 1 - (e^{-\lambda y})^{k-1}e^{-\lambda y}$$

$$= 1 - e^{-\lambda k y} \qquad \text{for } y \geq 0$$

which follows that <u>pooled output is a Poisson process with rate k where k is the number of Y force combatants.</u>

Here, probably the most interesting properties of the pooled output refer to the 'local' behaviour where k is large. Khintchine [Ref. 13] has proposed that in the limit the numbers of renewals in non-overlapping intervals follow independent Poisson distributions, thus showing that **<u>in the limit the pooled output is a Poisson process</u>**. Also his proof does not require each interarrival time to be identically distributed. In attrition analysis in combat, it appears to be the same way, which means that if a large number of independent (and identical) casualty occurences are pooled, then the total force casualty occurence is approximately a Poisson process [Ref. 9] [Ref. 13]. Thus, we can say that <u>pooled casualty occurence by total Y comba-tants can be approximated by Poisson process if we assume a large number of Y forces</u>.

Thus, finally, from the basic theorem of Poisson process, we know that if $\{N(t), t \geq 0\}$ is Poisson process with rate >0, then the interarrival times $\{X_i, i \geq 1\}$ are i.i.d. exponential random variable with rate λ. Thus, from previous results, <u>total X casualty occurences is a Poisson process with rate $k\lambda$, where k is the number of Y force combatants</u>, and <u>interarrival times of X casualties can be considered to be exponentially distributed with parameter $k\lambda$</u>.

V. COMPARISON OF RENEWAL PROCESS WITH COMPUTER SIMULATION

In this chapter we will look at the results of some typical combat attrition process (Y firers with different number of passive X targets) obtained through the theoretical renewal process model which is developed in this thesis and computer simulation using random number generator.

As we mentioned earlier, total number of expected casualties by n Y firers turned out to be n M(t) for large t from the renewal theory, where we assumed that all casualty occurences were independent and identical. But it is clear that these expected casualties can not be bigger than the number of X targets. Now we will compare this result with simulated output to see whether we can find any interesting facts such as how many X targets will be most appropriate to apply renewal theory to combat attrition ? or what force ratio will be most applicable to renewal process ?, etc.

In order to facilitate the simulations, the assumptions were made that n firers shoot independently at m targets, and that the times to kill the targets are uniformly distributed between [0,1]. When a target is killed, all firers shooting at it are assumed to select a new target at random from those still surviving.

The procedure of the computer simulation is as follows (see Appendix A) ;

1. We generate the initial kill time for each firer by using pseudo random number generator, where we assume that every firer's kill time is uniformly distributed between [0,1].
2. Each target is indexed by integer number.
3. We need to decide which target does the firer shoot at ? The target selection rule is such that first we

generate a random uniform number between [0,1], then we take the integer value of (random number times the number of targets) plus one, which designates the index of target that is assigned to firer 1. Same procedures are repeated until all the firers are assigned a target, with some targets possibly having more than one firer. Thus at first time step random number for selecting the target is generated n times, because we have to assign a target to each firer. These targets are killed eventually as time proceeds.

Now, the target that will be killed first is the target which is assigned to the firer who has the minimum time to kill(= Tmin) among the firers, where this time to kill was already generated. So we now know which target is killed.

After the first target is killed, the number of targets killed is registered as one and the first time step ends. Now, in the second time step, the new kill times between [0,1] are generated again for all firers who were shooting at the first target. The firers who were still in the process of killing a target at the end of the first time step continue to engage the same target in the second time step, where time to kill in the second step is old kill time (generated value at first time step) less Tmin.

Once the second target is killed, the second time step ends and the number of targets killed is now registered as two. Now we proceed to the third time step, where same target selection rule and firing rule are applied until all the targets are killed.

After all the targets are killed, each time step length (which is the same as the time to kill the target in each time step) is stored, and this is one replication (see Appendix B). To obtain the expected value, 40 replications of the simulation were conducted and 6 data sets (n=3, m=6), (n=3,m=10), (n=3, m=14), and (n=10, m=6), (n=10, m=10), (n=10, m=14), were examined. After 40 replications, the data were evaluated to find the expected number of kills through order statistics. This expected number of kills was plotted against time. The results for n=3 and n=10 are shown in Fig 5.1.

The theoretical results can be obtained through two methods, which are discrete approximation and simulation. In this thesis, we used discrete approximation based on the renewal function modelled at equation 3.7. It's formula is as follows;

$$M(t) = \sum_{j=1}^{t} [\ 1 + M(t-j)\ ]\ P_j$$

where P_j represents the probability distribution of kill time.

Thus, expected number of kills obtained from the renewal process model using discrete approximation (see appendix C for program and data, where left column refers to the kill time and right column refers to the expected number of kills) is also plotted against time in Fig 5.1.

It was found through both cases that the expected number of kills by the renewal process model may only be justified through the assumption of an infinite number of targets because as m gets bigger, the expected kill value approaches the theoretical result. Also it was discovered that at the beginning of combat, the attrition process almost followed the theoretical renewal process model.

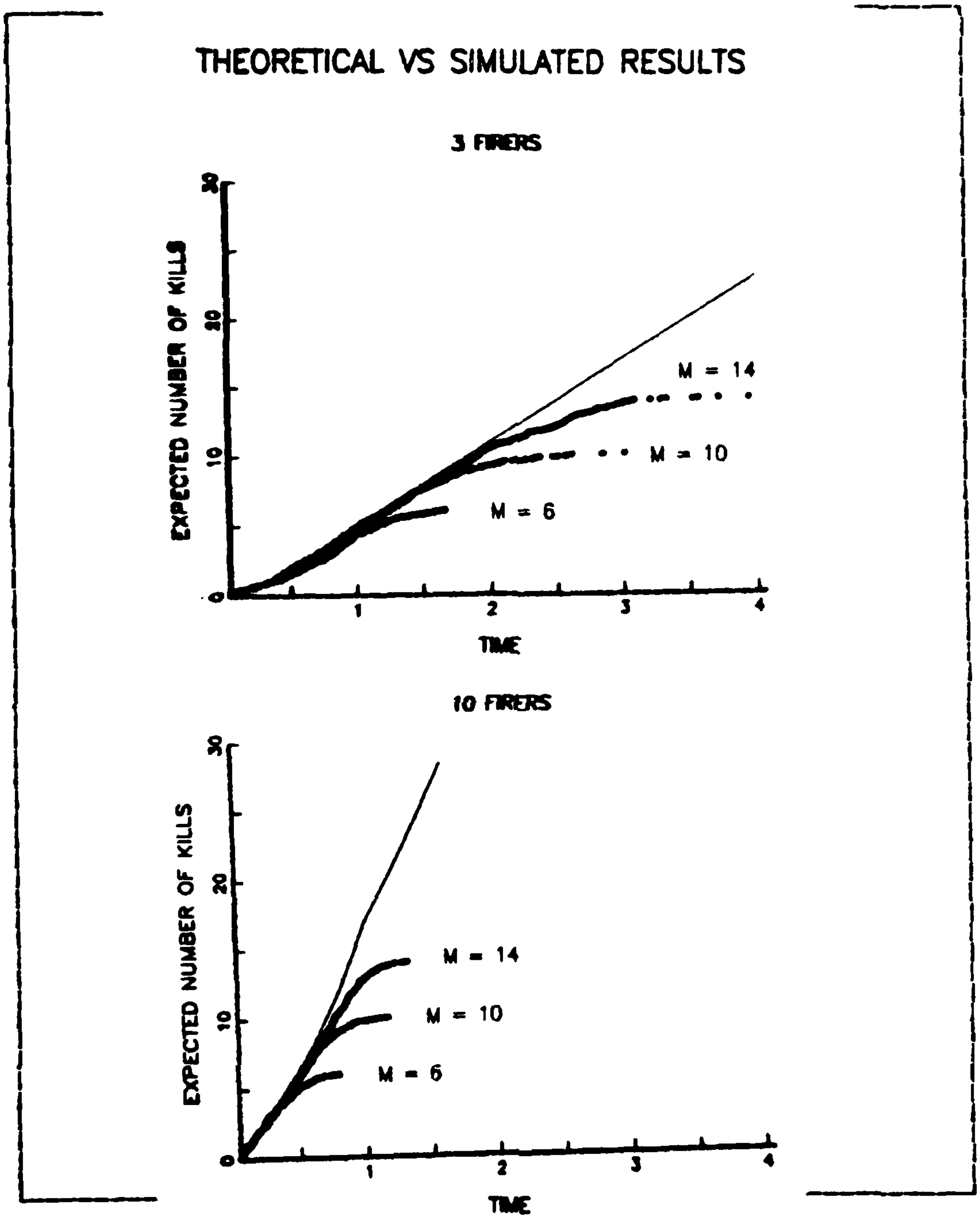

Figure 5.1 Expected Kills From Theoretical And Simulated.

VI. CONCLUSIONS

A. SUMMARY

Since Lanchester proposed his model, many studies have tried to model the combat attrition process mathematically. Recently, the main focus has been on calculation of numerical values for Lanchester attrition-rate coefficient. This numerical determination stands at the heart of casualty assessment in such models. There are three reasons for the importance of this attrition-rate coefficient methodology. First, Lanchester-type models are used in various U.S.Army and D.O.D. planning activities more widely than ever before. Second, Lanchester attrition-rate coefficient is a basic element of any Lanchester-type combat model, and that attrition-rate coefficient reflects the effective application of firepower. Finally, significant new developments have occured in methodology for developing more tactically realistic Lanchester attrition-rate coefficient and these important results have not been accessible to a very wide audience. In particular, a new approach for developing more realistic and mathematical Lanchester attrition-rate coefficient, that of computing combat attrition by using a renewal process, was proposed by Taylor recently. This thesis began with the assumption that the combat attrition process was a kind of renewal process, and that times between casualties were i.i.d. Thus, considering the problem from the standpoint of renewal theory, many new random variables were included and various distribution functions studied in order to understand better this type of casualty occurence paradigm.

This thesis only dealt with the case of homogeneous forces which were using the same acquisition mode to attrit the targets, which were passive X forces. Thus we could use an ordinary renewal process to study the behaviour of casualty occurences. We emphasized the probabilistic function of each r.v., so once we found the distribution of interarrival times of casualties, casualty analysis could be done by plugging distribution functions into the equation developed. Finding the distribution of interarrival times of casualties was the critical factor in continuing the casualty analysis.

The final conclusion is that <u>total casualty occurence in combat can be assumed to be a Poisson process</u>, <u>and thus times between casualties are exponentially distributed if we assume a large number of combatants</u>.

We have dealt here only with a passive X force(i.e. one that does not shoot back), perhaps an unrealistic assumption. But still this study permits better understanding of combat attrition, through the use of probability theory, than was previously available.

B. SUGGESTIONS FOR FURTHER STUDY

In this thesis, we have only dealt with passive targets. A more complex formulation may be required for real combat, since enemy targets shoot back and cause firer attrition.

In addition, a delayed renewal process could be applied for the case where different acquisition modes are used for the first shooting and for each subsequent shootings. A reinforcement of the forces could also be included. Casualty analysis could then be done using the theory of alternating renewal processes, once we know the rate of reinforcement.

APPENDIX A

FORTRAN PROGRAMMING FOR SIMULATING THE KILL TIME (N = 3)

```
      REAL U1(3),U2(3),U3,U4,TMIN,TIME(3),INTIME(40,14)
      INTEGER JTGT,FINDEX,L,M,N,KTGT,TGT(14),RTGT(14)
      DOUBLE PRECISION DSEED
      DSEED = 123457.D0 C
      DO 200 K = 1,40
        M = 14
        N = 3
        L = 1 C
        CALL GGUBS(DSEED,N,U1) C
        DO 100 I = 1,N
          TIME(I) = U1(I)
100     CONTINUE C
        DO 105 J = 1,M
          RTGT(J) = J
105     CONTINUE C
        DSEED = DSEED + 23.D0 C
        CALL GGUBS(DSEED,N,U2) C
        DO 110 I = 1,N
          TGT(I) = INT(U2(I) * FLOAT(M)) + 1
110     CONTINUE C
 10     CONTINUE
        TMIN = 2.
        DO 130 I = 1,N
          IF(TMIN.LE.TIME(I)) GO TO 130
          TMIN = TIME(I)
          KTGT = TGT(I)
          FINDEX = I
130     CONTINUE C
        INTIME(K,L) = TMIN C
        DO 125 J = 1,M
          IF(RTGT(J).NE.KTGT) GO TO 125
          JTGT = J
          RTGT(JTGT) = RTGT(M)
125     CONTINUE C
        M = M - 1
        L = L + 1 C
        DO 135 I = 1,N
          IF(KTGT.EQ.TGT(I)) GO TO 70
          TIME(I) = TIME(I) - TMIN
          GO TO 135
70        DSEED = DSEED + 12.D0
          CALL GGUBS(DSEED,1,U3)
          TIME(I) = U3
          DSEED = DSEED + 3.D0
          CALL GGUBS(DSEED,1,U4)
          TGT(I) = RTGT(INT(U4*FLOAT(M))+1)
135     CONTINUE C
        IF(M.GE.1) GO TO 10
200   CONTINUE
      WRITE(6,300)((INTIME(K,L),L=1,14),K=1,40)
      STOP
300   FORMAT(1X,14F5.2)
      END
```

APPENDIX B

SIMULATED DATA FOR KILL TIME (N = 3, M = 6)

0.2607	0.5056	0.2000	0.4392	0.1057	0.0649
0.3914	0.3860	0.1659	0.3315	0.0803	0.1646
0.1166	0.2987	0.0187	0.3911	0.0488	0.4744
0.4545	0.0154	0.1312	0.2905	0.1360	0.3672
0.4138	0.1269	0.0678	0.1365	0.1724	0.0101
0.0884	0.0730	0.2945	0.0284	0.7305	0.0820
0.4117	0.3926	0.0363	0.0697	0.0277	0.0151
0.0499	0.2410	0.3564	0.0033	0.1442	0.2979
0.4637	0.2188	0.1503	0.0546	0.5555	0.0230
0.7428	0.0736	0.3629	0.0375	0.0319	0.0322
0.2326	0.3066	0.2037	0.2982	0.1050	0.1466
0.1105	0.5015	0.2316	0.1160	0.2916	0.4032
0.0943	0.1942	0.2346	0.0342	0.2771	0.0218
0.5372	0.0104	0.3658	0.1481	0.0724	0.1435
0.0492	0.0554	0.1336	0.1750	0.5712	0.0609
0.2496	0.0227	0.1102	0.2098	0.2394	0.1294
0.4530	0.0027	0.1524	0.0486	0.0132	0.6276
0.2916	0.2342	0.0265	0.5775	0.3016	0.1875
0.2299	0.0756	0.5040	0.1737	0.0637	0.5797
0.7527	0.0343	0.1628	0.0063	0.3606	0.0877
0.2201	0.3036	0.1323	0.5570	0.0256	0.3332
0.6097	0.0538	0.0516	0.3331	0.1933	0.0139
0.7828	0.0719	0.1470	0.0156	0.0743	0.0118
0.1133	0.6589	0.0144	0.0712	0.1216	0.2284
0.0035	0.0602	0.4410	0.2291	0.1626	0.2609
0.0174	0.5669	0.1626	0.2062	0.1578	0.0334
0.6032	0.0008	0.2692	0.0024	0.0723	0.2653
0.3186	0.0492	0.3918	0.1603	0.1857	0.0369
0.5564	0.0105	0.0674	0.2037	0.0184	0.1283
0.5706	0.3844	0.3504	0.2022	0.0417	0.1048
0.2102	0.4294	0.1155	0.0393	0.1705	0.4165
0.0191	0.2929	0.4283	0.2152	0.1427	0.4954
0.1570	0.0601	0.3111	0.4439	0.0499	0.0877
0.1931	0.4658	0.0251	0.1568	0.0375	0.2395
0.1372	0.4183	0.1055	0.2174	0.0335	0.2340
0.0523	0.2950	0.4356	0.0280	0.1054	0.0326
0.3024	0.5260	0.0669	0.0079	0.1682	0.0748
0.0645	0.1504	0.0088	0.0741	0.5581	0.0228
0.0220	0.4350	0.2320	0.0207	0.2634	0.2070
0.1285	0.1379	0.1063	0.0695	0.0050	0.5903

APPENDIX C

DISCRETE APPROXIMATION FO RENEWAL FUNCTION

```
        $JOB
              REAL PO,M1(401),M(41),TIME(41)
              M1(1) = 0.
              PO = 1./100.
              DO 100 I = 1,400
                IF (I.GT.100) GO TO 200
                M1(I+1) = M1(I) + (1. + M1(I)) * PO
                GO TO 100
         200    M1(I+1) = M1(I) + (1. + M1(I)) * PO
            *   - (1. + M1(I-100)) * PO
         100  CONTINUE
              DO 110 J = 1,41
                TIME(J) = 0.1 * (J-1)
                M(J) = M1(1+10*(J-1))
                WRITE(6,*) TIME(J),M(J)
         110  CONTINUE
              STOP
              END 0                 $ENTRY
     0.0000000               0.0000000
     0.1000000               0.1046219
     0.2000000               0.2201895
     0.3000001               0.3478478
     0.4000001               0.4858622
     0.5000001               0.6446298
     0.6000001               0.8166940
     0.7000002               1.0067590
     0.8000002               1.2167040
     0.9000002               1.4486160
     1.0000000               1.7047900
     1.0999990               1.8783930
     1.1999990               2.0587170
     1.3000000               2.2452650
     1.3999990               2.4373700
     1.5000000               2.6341510
     1.6000000               2.8344850
     1.6999990               3.0369570
     1.8000000               3.2398290
     1.8999990               3.4409610
     2.0000000               3.6377740
     2.1000000               3.8320340
     2.1999990               4.0281400
     2.3000000               4.2256030
     2.4000000               4.4239440
     2.5000000               4.6227140
     2.6000000               4.8215160
     2.6999990               5.0200490
     2.8000000               5.2181320
     2.9000000               5.4157760
     3.0000000               5.6132290
     3.1000000               5.8109340
     3.2000000               6.0083100
     3.3000000               6.2070170
     3.4000000               6.4051470
     3.5000000               6.6032300
     3.6000000               6.8012370
     3.7000000               6.9991710
     3.8000000               7.1970600
     3.9000000               7.3949500
     4.0000000               7.5923790
```

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