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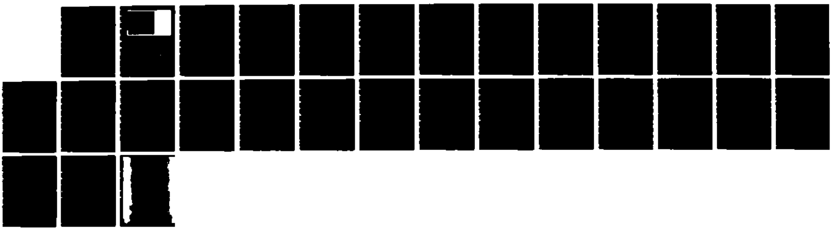

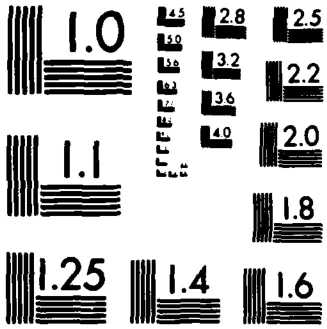
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ON THE CONTROLLED APPROXIMATION ORDER FROM CERTAIN SPACES OF SMOOTH BIVARIATE SPLINES

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ON THE CONTROLLED APPROXIMATION ORDER FROM CERTAIN SPACES OF SMOOTH BIVARIATE SPLINES

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ABSTRACT

Let Δ be the mesh in the plane obtained from a uniform square mesh by drawing in the north-east diagonal in each square. Let $\pi^{\rho}_{k,\Delta}$ be the space of bivariate piecewise polynomial functions in C^{ρ}, of total degree $\le k$, on the mesh Δ. It is demonstrated that the controlled approximation order from the linear span of all the box splines in $\pi^{\rho}_{k,\Delta}$ is

(1) $2k-2\rho$ if $2k-3\rho = 2$

(2) $2k-2\rho-1$ if $2k-3\rho = 3$ or 4

(3) $k + 1$ if $\rho = 0$

(4) $\min\{2k-2\rho-2,k\}$ if $2k-3\rho \ge 5$ and $\rho \ge 1$.

Thus the controlled approximation order problem is solved completely.

AMS (MOS) Subject Classifications: 41A15, 41A63, 41A25

Key Words: box splines, bivariate, controlled approximation order, pp, jump, quasi-interpolants, smooth, spline functions.

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SIGNIFICANCE AND EXPLANATION

This report continues the study of approximation by bivariate smooth splines on a three-direction mesh. Initiated by de Boor, DeVore and Höllig, box splines have proved useful in determining the approximation order from certain spaces of bivariate splines. By using box splines, de Boor and Höllig gave a sharp upper bound for the approximation order, and Jia got a sharp lower bound for it. But there is still a gap between these two bounds. While determining the exact value of the approximation order is still a formidable problem, Dahmen and Micchelli consider the so-called controlled approximation order from certain spaces of bivariate splines. In their study, Dahmen and Micchelli use a characterization result of Strang and Fix concerning controlled approximation. However, the result of Strang and Fix has been shown to be not true in their original sense. After adjusting the definition of controlled approximation order suitably, in another report, we obtain the desired characterization property for controlled approximation by box splines. Hereafter we shall refer to controlled approximation in the latter sense.

In this report, we determine completely the controlled approximation order from the span of all box splines of any given order and smoothness.

The responsibility for the wording and views expressed in this descriptive summary lies with MRC, and not with the author of this report.

ON THE CONTROLLED APPROXIMATION ORDER
FROM CERTAIN SPACES OF SMOOTH BIVARIATE SPLINES

Rong-Qing Jia

In this paper we study the controlled approximation order from certain spaces of smooth bivariate splines on a three-direction mesh. The work in this respect was initiated by [BD] and [BH_{1-3}], followed by [DM_{1-2}] and [J_{1-2}].

Following [BH_3] we first introduce some notations. Let

$$\Delta := \bigcup_{n\in\mathbb{Z}} \{x \in \mathbb{R}^2;\ x(1) = n,\ x(2) = n, \quad \text{or} \quad x(2) - x(1) = n\} \quad .$$

In other words, the mesh Δ is obtained from a uniform square mesh by drawing in the north-east diagonal in each square. Let

$$S := \pi^{\rho}_{k,\Delta} := \pi_{k,\Delta} \cap C^{\rho}$$

be the space of bivariate pp (piecewise polynomial) functions in C^{ρ}, of total degree $\leq k$, on the mesh Δ. Also, we denote by π_k the space of polynomials of total degree $\leq k$, and by π the space of all polynomials. We are interested in the approximation order m of S. In the case $\rho > (2k-2)/3$, the approximation order is $m = 0$ (see [BD]). In the case $\rho \leq (2k-2)/3$, it is known that

$$m(k)-2 \leq m \leq m(k) \quad ,$$

where $m(k) := \min\{2(k-\rho),\ k+1\}$ (see [BH_3] and [J_2]).

While determining the exact value of m is still a formidable problem, [DM_2] discuss the so-called **controlled** approximation order. This concept has been introduced by [S]. Here is the setup: Given a collection $\Phi = \{\phi_1,\ldots,\phi_N\}$ of certain locally supported functions on $\mathbb{R}^n$, we want to

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find, for any $f \in C^\infty(\mathbb{R}^n)$ and any $h > 0$, a nonnegative integer m and N multivariate sequences $w_i^h : \mathbb{Z}^n \to \mathbb{R}$ $(i = 1,\ldots,N)$ such that

$$\|f - \sum_{i=1}^{N} \sum_{j\in\mathbb{Z}^n} w_i^h(j)\phi_i(\tfrac{\cdot}{h} - j)\|_\infty \le \text{const } h^m \|f^{(m)}\|_\infty \tag{1}$$

and

$$\|w_i^h(\cdot)\|_\infty \le \text{const.}\|f\|_\infty \quad (i = 1,\ldots,N) \quad . \tag{2}$$

The largest value m with the above property is called the **controlled approximation order** of Φ. A characterization result for controlled approximation order has been stated by [FS]:

Theorem A. $\Phi = \{\phi_1,\ldots,\phi_n\}$ <u>has controlled approximation order bigger than</u> m <u>if and only if there exists a linear combination</u> B <u>of</u> $\phi_1,\ldots,\phi_N$ <u>and their translates for which the map</u>

$$T : p \mapsto \sum_{j\in\mathbb{R}^n} p(j)B(\cdot-j)$$

<u>is degree-preserving on</u> π_m.

Remark. A map T is said to be **degree preserving** on π_m if for any $p \in \pi_m$, $Tp-p$ has degree less than deg p. Let S_i be the shift operators on π_m:

$$S_i p := p(\cdot-e_i) \quad (i = 1,2) \quad .$$

If T commutes with $S_i (i = 1,2)$, then T is degree preserving on π_m if and only if T is a bijective map from π_m to π_m.

Recently, however, [J_3] produced a counterexample to Theorem A. This suggests that we should adjust the definition of controlled approximation suitably. We note that [DM_2] quote Theorem A in a different way. They require that the coefficients of the approximation be boundable locally. It turns out that if the requirement of (2) is replaced by

(2') There exists a positive constant R independent of h

such that

$$\operatorname{dist}(jh, \operatorname{supp} f) > R \text{ implies that } w_i^h(j) = 0 \ (i = 1,\ldots,N),$$

then Theorem A holds for any collection Φ of box splines (see [J_4]). Hereafter, we shall refer to controlled approximation in the latter sense.

We are interested in the case when Φ consists of all the box splines belonging to $\pi^\rho_{k,\Delta}$. We adapt the definition of box splines to suit our discussion. For $(r,s,t) \in \mathbb{Z}^3_+$, let $\Xi := (\xi_i)_1^{r+s+t}$ be the sequence given by

$$\xi_1 = \ldots = \xi_r = e_1 := (1,0),$$

$$\xi_{r+1} = \ldots = \xi_{r+s} = e_2 := (0,1),$$

$$\text{and} \qquad \xi_{r+s+1} = \ldots = \xi_{r+s+t} = e_3 := (1,1) \quad .$$

Then the box spline $M_\Xi := M_{r,s,t}$ is defined as the distribution given by the rule:

$$M_{r,s,t} : \phi \mapsto \int_{[0,1]^{r+s+t}} \phi\Big(\sum_{i=1}^{r+s+t} \lambda(i)\xi_i\Big)d\lambda$$

(see [BH_1]). Let

$$\Phi = \Phi_{k,\rho} := \{M_{r,s,t}; M_{r,s,t} \in \pi^\rho_{k,\Delta}\} \quad .$$

By $\tilde{m}(k,\rho)$ we denote the controlled approximation order of $\Phi_{k,\rho}$. It is known that

(i) (see [BH_1]) $\tilde{m}(k,\rho) = 2k-2\rho$ if $2k-3\rho = 2$

(ii) (see [DM_2]) $\tilde{m}(k,\rho) = 2k-2\rho-1$ if $2k-3\rho = 3$ or 4 .

If we denote by $\bar{m}(k,\rho)$ the approximation order of $\pi^\rho_{k,\Delta}$, then

$$\tilde{m}(k,\rho) \le \bar{m}(k,\rho) \quad .$$

In the case $2k-3\rho = 2$, [BH_1] point out that

$$2k-2\rho = \rho+2 \le \tilde{m}(k,\rho) = \bar{m}(k,\rho) = \rho+2 \quad .$$

Nevertheless, we must be careful in distinguishing the controlled approximation order from the approximation order. Indeed, we shall see that

$$\tilde{m}(5,1) = 5 < \bar{m}(5,1) = 6 \quad .$$

We will discuss this matter in more detail later.

In this paper we determine $\tilde{m}(k,\rho)$ completely. Our main result is that

(iii) $\tilde{m}(k,\rho) = k + 1$ if $\rho = 0$.

(iv) $\tilde{m}(k,\rho) = \min\{2k-2\rho-2,k\}$ if $2k-3\rho \geq 5$ and $\rho \geq 1$.

(Recall that $\tilde{m}(k,\rho) = 0$ if $2k-3\rho \leq 1$).

More generally, let Φ be a collection of bivariate box splines:

$$\Phi = \{M_u ; u \in U\}$$

with

$$U \subset \{(r,s,t) \in \mathbb{Z}^3 ; r,s,t \geq 0, \min\{r+s,s+t,t+r\} \geq 1\} \quad .$$

Then

$$M_u \in L_\infty \quad \text{for} \quad u \in U \quad .$$

Whenever convenient, we refer to the three components of $u \in U$ as r,s,t, respectively.

The following theorem gives a criterion for the controlled approximation order of Φ.

Theorem 1. *Let*

$$Q_m := \{(q_1,q_2) \in \mathbb{N}^2 ; q_1 + q_2 \leq m+1\} \quad .$$

Then $\Phi = \{M_u ; u \in U\}$ *has controlled approximation order* $\geq m$ *if and only if there exists a mapping* $b: K \to \mathbb{R}$ *such that*

$(1^\circ)_m$ $\sum_{r \geq q} b_u = 0$ *for any* q,s,t *with* $(q,s+t) \in Q_m$;

$(2^\circ)_m$ $\sum_{s \geq q} b_u = 0$ *for any* q,t,r *with* $(q,t+r) \in Q_m$;

$(3^\circ)_m$ $\sum_{t \geq q} b_u = 0$ *for any* q,r,s *with* $(q,r+s) \in Q_m$;

(4°) $\sum_{u \in U} b_u \neq 0$.

We notice that $(1^\circ)_m$, $(2^\circ)_m$ and $(3^\circ)_m$ imply that

$(5^\circ)_m$ $\quad b_u = 0$ for any $u = (r,s,t)$ with $r+s+t \le m$.

Indeed, if $u \in U$, then one of r, s and t is nonzero, say, $r \ge 1$. Now assume that $r+s+t \le m$. Then $r \le m-1$, for otherwise $s = t = 0$, contradicting that $u \in U$. Thus $(r,s+t)$ and $(r+1,s+t) \in \Omega_m$; hence $(1^\circ)_m$ implies that

$$\sum_{\lambda \ge r} b_{\lambda,s,t} = 0 \quad \text{and} \quad \sum_{\lambda \ge r+1} b_{\lambda,s,t} = 0 \ .$$

Therefore

$$b_u = \sum_{\lambda \ge r} b_{\lambda,s,t} - \sum_{\lambda \ge r+1} b_{\lambda,s,t} = 0 \ .$$

Before proving Theorem 1, we need to introduce some notation. Recall that

$$e_1 = (1,0),\ e_2 = (0,1),\ e_3 = (1,1) \ .$$

Let

$$\nabla_i f := f - f(\cdot - e_i) \ ,$$

$$D_i := D_{e_i} \ ,$$

i.e., D_i is the partial derivative with respect to the i-th component, $i = 1,2$, and $D_3 = D_1 + D_2$. It follows from [BH_1] that, for any function $a: \mathbb{Z}^2 \to \mathbb{R}$, we have

$$D_i\Big(\sum_{j \in \mathbb{Z}^2} a(j) M_{\Xi}(\cdot - j)\Big) = \sum_{j \in \mathbb{Z}^2} \nabla_i a(j) M_{\Xi \setminus e_i}(\cdot - j) \quad \text{if} \quad e_i \in \Xi \ .$$

We define, for any function $f : \mathbb{R}^2 \setminus \Delta \to \mathbb{R}$, and for $x \in \mathbb{R} \setminus \mathbb{Z}$,

$$\text{jump}_1 f(x) := \lim_{\varepsilon \downarrow 0} [f(x,\varepsilon) - f(x,-\varepsilon)]$$

$$\text{jump}_2 f(x) := \lim_{\varepsilon \downarrow 0} [f(\varepsilon,x) - f(-\varepsilon,x)]$$

$$\text{jump}_3 f(x) := \lim_{\varepsilon \downarrow 0} [f(x-\varepsilon,x+\varepsilon) - f(x+\varepsilon,x-\varepsilon)] \ .$$

Thus, as a function from $\mathbb{R}$ to $\mathbb{R}$, $\text{jump}_1 f$ represents the jump of f across the x_1-axis. For $\mathrm{jump}_2 f$ and $\mathrm{jump}_3 f$, we have a similar interpretation. With $j = (j_1,j_2) \in \mathbb{Z}^2$, one easily verifies the following formulae:

$$(3)\quad \mathrm{jump}_1 M_{r,s,t}(\cdot - j)(x) \quad \begin{array}{l} = M_r(x-j_1) \text{ if } j_2 = 0;\ r > 0, \text{ and } s+t = 1 \ ; \\ = -M_r(x-j_1-1) \text{ if } j_2 = -1;\ r > 0,\ s = 0 \text{ and } t = 1 \ ; \\ = -M_r(x-j_1) \text{ if } j_2 = -1;\ r > 0,\ s = 1 \text{ and } t = 0 \ . \\ = 0 \quad \text{otherwise} \ . \end{array}$$

Here M_r is the univariate B-spline of order r at a uniform mesh:

$$M_r(x) := r[0,\ldots,r]\ (\cdot - x)_+^{r-1}$$

For jump_2 and jump_3, we have similar formulae.

The proof of Theorem 1.

If $\Phi = \{M_u;\ u \in U\}$ has controlled approximation order $> m$, then by Theorem A, there exists B, a linear combination of M_u and their translates:

$$(4)\qquad B = \sum_{u \in U} \sum_{i \in I} a_{u,i} M_u(\cdot - i)$$

(here I is a finite subset of $\mathbb{Z}^2$) such that the mapping

$$T : p \to \sum_{j \in \mathbb{Z}^2} p(j) B(\cdot - j)$$

is degree-preserving on π_m. Set

$$(5)\qquad b_u := \sum_{i \in I} a_{u,i} \ .$$

We claim that b satisfies $(1^o)_m$, $(2^o)_m$, $(3^o)_m$ and (4^o). To this end we shall prove

$$(1^o)_{m,q_2} \quad \sum_{r > q_1} b_u = 0 \text{ for any } q_1,s,t \text{ with } s+t < q_2 \text{ and } 1 < q_1 < m+1-s-t$$

by induction on q_2. Then $(1^o)_{m,m}$ is just $(1^o)_m$. Notice that $(1^o)_{m,0}$ holds vacuously. Suppose that $(1^o)_{m,q_2}$ is true $(q_2 < m)$. We want to establish $(1^o)_{m,q_2+1}$. Consider

$$\mathrm{jump}_1[D_1^{q_1-1} D_2^{q_2}(\sum_{j\in\mathbb{Z}^2} p(j)B(\cdot-j))] \quad ,$$

where $(q_1,q_2) \in \mathbb{Z}_+^2$ with $q_1 > 1$ and $q_1 + q_2 < m$, and $p \in \pi_{q_1+q_2}$. Since $\sum_{j\in\mathbb{Z}^2} p(j)B(\cdot-j)$ is a polynomial, we have

$$\mathrm{jump}_1[D_1^{q_1-1} D_2^{q_2}(\sum_{j\in\mathbb{Z}^2} p(j)B(\cdot-j))] = 0 \quad . \tag{6}$$

On the other hand, (4) yields that

$$\mathrm{jump}_1[D_1^{q_1-1} D_2^{q_2}(\sum_{j\in\mathbb{Z}^2} p(j)B(\cdot-j))] \tag{7}$$

$$= \sum_{u\in U}\sum_{i\in I} a_{u,i}\{\mathrm{jump}_1[D_1^{q_1-1} D_2^{q_2}(\sum_{j\in\mathbb{Z}^2} p(j)M_u(\cdot-i-j))]\} \quad .$$

We now evaluate

$$J := \mathrm{jump}_1[D_1^{q_1-1} D_2^{q_2}(\sum_j p(j)M_{r,s,t}(\cdot-i-j))] \quad .$$

If $q_1 > r$, then

$$J = \mathrm{jump}_1[D_1^{q_1-1-r} D_2^{q_2}(\sum_j(\nabla_1^r p)(j)M_{0,s,t}(\cdot-i-j))] = 0 \quad ,$$

since, by (3), $\mathrm{jump}_1 M_{0,s',t'} = 0$, whatever s', t' might be. If $q_1 < r$, then

$$D_1^{q_1-1} D_2^{q_2}(\sum_j p(j)M_u(\cdot-i-j)) = D_2^{q_2}(\sum_j (\nabla_1^{q_1-1} p)(j)M_{r-q_1+1,s,t}(\cdot-i-j)) \quad .$$

There are two subcases: $q_2 < s$ and $q_2 > s$. If $q_2 < s$, then

$$J = \mathrm{jump}_1(\sum_j (\nabla_1^{q_1-1} \nabla_2^{q_2} p)(j)M_{r-q_1+1,s-q_2,t}(\cdot-i-j)) \quad .$$

By (3), $J \neq 0$ only if $(s-q_2,t) = (0,1)$ or $(1,0)$. We have, for $(s-q_2,t) = (0,1)$, that

$$J = \mathrm{jump}_1\Big(\sum_{j\in\mathbb{Z}^2} (\nabla_1^{q_1-1}\nabla_2^{q_2}p)(j)M_{r-q_1+1,0,1}(\cdot-i-j)\Big)$$

$$= \sum_{j_1\in\mathbb{Z}} (\nabla_1^{q_1-1}\nabla_2^{q_2}\nabla_3 p)(j)M_{r-q_1+1}(\cdot-i-j) \quad ,$$

by (3). Since $p \in \pi_{q_1+q_2}$, $\nabla_1^{q_1-1}\nabla_2^{q_2}\nabla_3 p$ is a constant. Thus

$$J = \nabla_1^{q_1-1}\nabla_2^{q_2}\nabla_3 p \quad \text{for} \quad (s,t) = (q_2,1) \quad \text{and} \quad q_1 < r \quad .$$

Similarly,

$$J = \nabla_1^{q_1}\nabla_2^{q_2}p \quad \text{for} \quad (s,t) = (q_2+1,0) \quad \text{and} \quad q_1 < r \quad .$$

Let us now consider the case $q_2 > s$. In this case

$$D_2^{q_2}\Big(\sum_j (\nabla_1^{q_1-1}p)(j)M_{r-q_1+1,s,t}(\cdot-i-j)\Big) = D_2^{q_2-s}\Big(\sum_j (\nabla_1^{q_1-1}\nabla_2^s p)(j)M_{r-q_1+1,0,t}(\cdot-i-j)\Big)$$

By the binomial theorem,

$$D_2^{q_2-s} = (D_3-D_1)^{q_2-s} = \sum_{n=0}^{q_2-s} (-1)^{q_2-s-n}\binom{q_2-s}{n}D_1^{q_2-s-n}D_3^n \quad .$$

Invoking (3) again, we see that

$$\mathrm{jump}_1\, D_1^{q_2-s-n}D_3^n\Big(\sum_j (\nabla_1^{q_1-1}\nabla_2^s p)(j)M_{r-q_1+1,0,t}(\cdot-i-j)\Big) \neq 0$$

only when $n = t-1$ and $q_2-s-n < r-q_1+1$. Also, we have, for $n = t-1 \in [0,q_2-s]$ and $q_2-s-n < r-q_1+1$, that

$$J = \mathrm{jump}_1\Big[(-1)^{q_2-s-t+1}\binom{q_2-s}{t-1}D_1^{q_2-s-t+1}D_3^{t-1}\Big(\sum_{j\in\mathbb{Z}^2} (\nabla_1^{q_1-1}\nabla_2^s p)(j)$$

$$M_{r-q_1+1,0,t}(\cdot-i-j)\Big)\Big]$$

$$= (-1)^{q_2-s-t+1}\binom{q_2-s}{t-1}\nabla_1^{q_1+q_2-s-t}\nabla_2^s\nabla_3^t p \quad .$$

If we interpret $\binom{-1}{-1}$ as 1, then the above results can be summarized as

$$\mathrm{jump}_1[D_1^{q_1-1}D_2^{q_2}(\sum_{j\in\mathbb{Z}^2} p(j)B(\cdot-j))] ,$$

where $(q_1,q_2) \in \mathbb{Z}_+^2$ with $q_1 > 1$ and $q_1 + q_2 < m$, and $p \in \pi_{q_1+q_2}$. Since $\sum_{j\in\mathbb{Z}^2} p(j)B(\cdot-j)$ is a polynomial, we have

$$\mathrm{jump}_1[D_1^{q_1-1}D_2^{q_2}(\sum_{j\in\mathbb{Z}^2} p(j)B(\cdot-j))] = 0 . \tag{6}$$

On the other hand, (4) yields that

$$\mathrm{jump}_1[D_1^{q_1-1}D_2^{q_2}(\sum_{j\in\mathbb{Z}^2} p(j)B(\cdot-j))] \tag{7}$$

$$= \sum_{u\in U}\sum_{i\in I} a_{u,i}\{\mathrm{jump}_1[D_1^{q_1-1}D_2^{q_2}(\sum_{j\in\mathbb{Z}^2} p(j)M_u(\cdot-i-j))]\} .$$

We now evaluate

$$J := \mathrm{jump}_1[D_1^{q_1-1}D_2^{q_2}(\sum_j p(j)M_{r,s,t}(\cdot-i-j))] .$$

If $q_1 > r$, then

$$J = \mathrm{jump}_1[D_1^{q_1-1-r}D_2^{q_2}(\sum_j(\nabla_1^r p)(j)M_{0,s,t}(\cdot-i-j))] = 0 ,$$

since, by (3), $\mathrm{jump}_1 M_{0,s',t'} = 0$, whatever s', t' might be. If $q_1 < r$, then

$$D_1^{q_1-1}D_2^{q_2}(\sum_j p(j)M_u(\cdot-i-j)) = D_2^{q_2}(\sum_j (\nabla_1^{q_1-1} p)(j)M_{r-q_1+1,s,t}(\cdot-i-j)) .$$

There are two subcases: $q_2 < s$ and $q_2 > s$. If $q_2 < s$, then

$$J = \mathrm{jump}_1(\sum_j (\nabla_1^{q_1-1}\nabla_2^{q_2}p)(j)M_{r-q_1+1,s-q_2,t}(\cdot-i-j)) .$$

By (3), $J \neq 0$ only if $(s-q_2,t) = (0,1)$ or $(1,0)$. We have, for $(s-q_2,t) = (0,1)$, that

$$J = \mathrm{jump}_1\Big(\sum_{j \in \mathbb{Z}^2} (\nabla_1^{q_1-1} \nabla_2^{q_2} p)(j) M_{r-q_1+1,0,1}(\cdot - i - j)\Big)$$

$$= \sum_{j_1 \in \mathbb{Z}} (\nabla_1^{q_1-1} \nabla_2^{q_2} \nabla_3 p)(j) M_{r-q_1+1}(\cdot - i - j) \quad ,$$

by (3). Since $p \in \pi_{q_1+q_2}$, $\nabla_1^{q_1-1} \nabla_2^{q_2} \nabla_3 p$ is a constant. Thus

$$J = \nabla_1^{q_1-1} \nabla_2^{q_2} \nabla_3 p \quad \text{for} \quad (s,t) = (q_2,1) \quad \text{and} \quad q_1 < r \quad .$$

Similarly,

$$J = \nabla_1^{q_1} \nabla_2^{q_2} p \quad \text{for} \quad (s,t) = (q_2+1,0) \quad \text{and} \quad q_1 < r \quad .$$

Let us now consider the case $q_2 > s$. In this case

$$D_2^{q_2}\Big(\sum_j (\nabla_1^{q_1-1} p)(j) M_{r-q_1+1,s,t}(\cdot - i - j)\Big) = D_2^{q_2-s}\Big(\sum_j (\nabla_1^{q_1-1} \nabla_2^s p)(j) M_{r-q_1+1,0,t}(\cdot - i - j)\Big)$$

By the binomial theorem,

$$D_2^{q_2-s} = (D_3 - D_1)^{q_2-s} = \sum_{n=0}^{q_2-s} (-1)^{q_2-s-n} \binom{q_2-s}{n} D_1^{q_2-s-n} D_3^n \quad .$$

Invoking (3) again, we see that

$$\mathrm{jump}_1 \, D_1^{q_2-s-n} D_3^n \Big(\sum_j (\nabla_1^{q_1-1} \nabla_2^s p)(j) M_{r-q_1+1,0,t}(\cdot - i - j)\Big) \neq 0$$

only when $n = t-1$ and $q_2-s-n < r-q_1+1$. Also, we have, for $n = t-1 \in [0, q_2-s]$ and $q_2-s-n < r-q_1+1$, that

$$J = \mathrm{jump}_1\Big[(-1)^{q_2-s-t+1} \binom{q_2-s}{t-1} D_1^{q_2-s-t+1} D_3^{t-1} \Big(\sum_{j \in \mathbb{Z}^2} (\nabla_1^{q_1-1} \nabla_2^s p)(j)$$

$$M_{r-q_1+1,0,t}(\cdot - i - j)\Big)\Big]$$

$$= (-1)^{q_2-s-t+1} \binom{q_2-s}{t-1} \nabla_1^{q_1+q_2-s-t} \nabla_2^s \nabla_3^t p \quad .$$

If we interpret $\binom{-1}{-1}$ as 1, then the above results can be summarized as

$$\mathrm{jump}_1[D_1^{q_1-1} D_2^{q_2}(\sum_{j\in\mathbb{Z}^2} p(j)M_u(\cdot-i-j))]$$

(8)

$$= (-1)^{q_2-s-t+1}\binom{q_2-s}{t-1}\nabla_1^{q_1+q_2-s-t}\nabla_2^s\nabla_3^t p \quad \text{for} \quad s+t \le q_2+1 \quad \text{and} \quad r+s+t > q_1+q_2 \quad ,$$

and 0 otherwise. Now (7) becomes

(9) $$\mathrm{jump}_1[D_1^{q_1-1} D_2^{q_2}(\sum_{j\in\mathbb{Z}^2} p(j)B(\cdot-j))]$$

$$= \sum_{\substack{s+t\le q_2+1\\ r+s+t>q_1+q_2}} (-1)^{q_2+1-s-t}\binom{q_2-t}{s-1}\nabla_1^{q_1+q_2-s-t}\nabla_2^s\nabla_3^t p \sum_{i\in I} a_{u,i}$$

$$= \sum_{\substack{s+t\le q_2+1\\ r+s+t>q_1+q_2}} b_{r,s,t}(-1)^{q_2+1-s-t}\binom{q_2-t}{s-1}\nabla_1^{q_1+q_2-s-t}\nabla_2^s\nabla_3^t p \quad .$$

Comparing (9) with (6) gives

(10) $$\sum_{s+t\le q_2+1} (-1)^{q_2+1-s-t}\binom{q_2-t}{s-1}\nabla_1^{q_1+q_2-s-t}\nabla_2^s\nabla_3^t p(\sum_{r>q_1+q_2-s-t} b_u) = 0 \quad .$$

If $s+t \le q_2$, then $(1^o)_{m,q_2}$ gives

$$\sum_{r>q_1+q_2-s-t} b_u = 0 \quad .$$

Moreover, $s+t = q_2+1$ implies that $(-1)^{q_2+1-s-t}\binom{q_2-t}{s-1} = 1$. Therefore, (10) becomes

(11) $$\sum_{s+t=q_2+1} \nabla_1^{q_1-1}\nabla_2^s\nabla_3^t\, p(\sum_{r>q_1} b_u) = 0 \quad .$$

For given (s_0,t_0) with $s_0+t_0 = q_2+1$, there exists $p \in \pi_{q_2+1}$ such that

$$\nabla_2^s\nabla_3^t P = 1 \quad \text{for} \quad (s,t) = (s_0,t_0)$$

$$= 0 \quad \text{for} \quad s+t = q_2+1 \quad \text{but} \quad (s,t) \neq (s_0,t_0)$$

(e.g., choose $P(x_1,x_2) := x_1^{t_0}(x_2-x_1)^{s_0}/s_0!t_0!$). Then we can find

$p \in \pi_{q_1+q_2}$ so that $\nabla_1^{q_1-1} p = P$. Now (11) yields that

$$\sum_{r \geq q_1} b_{r,s_0,t_0} = 0 \quad .$$

This proves $(1^o)_{m,q_2+1}$. By induction, $(1^o)_m$ has been proved. The proof of $(2^o)_m$ and $(3^o)_m$ is similar. As to (4^o), since $T : p \mapsto \sum_{j \in \mathbb{Z}^2} p(j)B(\cdot-j)$ is degree-preserving on π_m, we have

$$\sum_{j \in \mathbb{Z}^2} B(\cdot-j) \neq 0 \quad .$$

But

$$\sum_{j \in \mathbb{Z}^2} B(\cdot-j) = \sum_{u \in U} \sum_{i \in I} a_{u,i} \sum_{j \in \mathbb{Z}^2} M_u(\cdot-j)$$

$$= \sum_{u \in U} \sum_{i \in I} a_{u,i} = \sum_{u \in U} b_u \quad .$$

This proves (4^o).

Conversely, suppose that $(1^o)_m$, $(2^o)_m$, $(3^o)_m$ and (4^o) hold. We want to construct a linear combination B of $M_{r,s,t}$ and their translates such that

$$T : p \to \sum_{j \in \mathbb{Z}^2} p(j)B(\cdot-j)$$

is a degree-preserving map on π_m. Note that after multiplying by an appropriate constant, we may assume

$$\sum_{u \in U} b_u = 1 \quad .$$

Recall from $[J_2]$ that there exist constants $a_{\ell,d-1} (\ell = 0,1,\ldots,d-2)$ such that for any polynomial f of degree $< d-1$,

$$\sum_{i \in \mathbb{Z}} f(i)M_{i,d-1} = \sum_{i \in \mathbb{Z}} f(i)\Big(\sum_{\ell=0}^{k-2} a_{\ell,d-1}M_{i,d}(\cdot+\ell)\Big)$$

where $M_{i,d}$ is the i-th B-spline of order d:

$$M_{i,d}(x) := d[i,\dots,i+d](\cdot-x)_+^{d-1} ,$$

(see [J_2; Lemma 1]). We define B_u in terms of these a as

$$B_u(x_1,x_2) := \sum_{i=0}^{r-1}\sum_{\lambda_i=0}^{i-1}\sum_{j=0}^{s-1}\sum_{\mu_j=0}^{j-1}\sum_{k=0}^{t-1}\sum_{\nu_k=0}^{k-1}\Big[\prod_{i=1}^{r-1} a_{\lambda_i,i}\prod_{j=1}^{s-1} a_{\mu_j,j}\prod_{k=1}^{t-1} a_{\nu_k,k}x$$

$$M_u\Big(x_1+\sum_{i=0}^{r-1}\lambda_i+\sum_{k=0}^{t-1}\nu_k,\ x_2+\sum_{j=0}^{s-1}\mu_j+\sum_{k=0}^{t-1}\nu_k\Big)\Big] .$$

These $B_{r,s,t}$ have the following property:

Lemma <u>(cf. [J_2; Lemma 2])</u>. <u>For any bivariate polynomial</u> p <u>of degree</u> $< r+s+t$, <u>we have</u>

(i) $D_1^r D_2^s[\sum_{j\in\mathbb{Z}^2} p(j)(\sum_{\lambda>t} b_{r,s,\lambda}B_{r,s,\lambda}(\cdot-j))] = 0,$ if $\sum_{\lambda>t} b_{r,s,\lambda} = 0$;

(ii) $D_1^r D_3^t[\sum_{j\in\mathbb{Z}^2} p(j)(\sum_{\lambda>s} b_{r,\lambda,t}B_{r,\lambda,t}(\cdot-j))] = 0,$ if $\sum_{\lambda>s} b_{r,\lambda,t} = 0$;

(iii) $D_2^s D_3^t[\sum_{j\in\mathbb{Z}^2} p(j)(\sum_{\lambda>r} b_{\lambda,s,t}B_{\lambda,s,t}(\cdot-j))] = 0$ if $\sum_{\lambda>r} b_{\lambda,s,t} = 0$.

Proof. Since $\sum_{\lambda>t} b_{r,s,\lambda} = 0$, summation by parts gives

$$\sum_{\lambda>t} b_{r,s,\lambda}B_{r,s,\lambda} = \sum_{\ell>t}\Big(\sum_{\lambda=t}^{\ell} b_{r,s,\lambda}\Big)(B_{r,s,\ell}-B_{r,s,\ell+1}) .$$

By [J_2; Lemma 2],

$$D_1^r D_2^s\Big[\sum_{j\in\mathbb{Z}^2} p(j)(B_{r,s,\ell}-B_{r,s,\ell+1})(\cdot-j)\Big] = 0$$

for any polynomial p of degree $< r+s+\ell$. This proves (i). One proves (ii) and (iii) in the same way.

In the following construction we use only those M_u for which $r+s+t > m$. In other words, we may assume that $u \in U$ implies $r+s+t > m$. Let

$$B := \sum_{u\in U} b_u B_u \quad .$$

We claim that

$$T : p \to \sum_{j\in\mathbb{Z}^2} p(j)B(\cdot-j)$$

is a degree-preserving mapping on π_m. As we did in [J_2; Lemma 4], we first prove that T carries π_m into π_m by showing that

$$D_1^{q_1}D_2^{q_2}[\sum_j p(j)B(\cdot-j)] \in \pi_0 \tag{12}$$

$$\text{for any } (q_1,q_2) \in \mathbb{Z}_+^2 \text{ with } q_1+q_2 = \deg p < m \quad .$$

Let

$$E_1 := \{u \in U;\ r > q_1 \text{ and } s > q_2\}$$

$$E_2 := \{u \in U;\ r \le q_1 \text{ and } s \le q_2\}$$

$$E_3 := \{u \in U;\ r \le q_1 \text{ and } s > q_2\}$$

$$E_4 := \{u \in U;\ r > q_1 \text{ and } s \le q_2\} \quad .$$

To prove (12), it suffices to show that

$$D_1^{q_1}D_2^{q_2}[\sum_j p(j) \sum_{u\in E_i} b_u B_u(\cdot-j)] \in \pi_0$$

for each $i = 1,2,3,4.$

Case $i = 1$. In this case, $r > q_1$ and $s > q_2$; hence

$$D_1^{q_1}D_2^{q_2}[\sum_j p(j)M_u(\cdot-j)] = \sum_j \nabla_1^{q_1}\nabla_2^{q_2}p(j)M_{r-q_1,s-q_2,t}(\cdot-j) = \nabla_1^{q_1}\nabla_2^{q_2}p \quad ,$$

since $\nabla_1^{q_2}\nabla_2^{q_2}p(j)$ is independent of j. Moreover, $B_{r,s,t}$ is a linear combination of $M_{r,s,t}$ and its translates; therefore

$$D_1^{q_1}D_2^{q_2}[\sum_j p(j) \sum_{(r,s,t)\in E_1} b_u B_u(\cdot-j)] = \text{const.}$$

Case $i = 2$. In this case, $q_1+q_2 < m$ implies that $r+s < m$; hence $t > 1$. Thus $\sum_{t>1} b_{r,s,t} = 0$ by $(2^o)_m$, and therefore by the Lemma we have

$$D_1^{q_1} D_2^{q_2}[\sum_j p(j) \sum_{(r,s,t)\in E_2} b_{r,s,t} B_{r,s,t}(\cdot - j)]$$

$$= \sum_{\substack{r<q_1 \\ s<q_2}} D_1^{q_1-r} D_2^{q_2-s} [D_1^r D_2^s \sum_{t>1} b_{r,s,t} \sum_j p(j)\, B_{r,s,t}(\cdot - j)] = 0 .$$

Case $i = 3$. In this case (see [J2]),

$$(13) \qquad D_1^{q_1} D_2^{q_2} = D_1^r D_3^t H_{r,t} + D_1^r D_2^s G_{r,s}$$

$$+ D_1^r [\sum_{\ell=q_1+q_2-r-s+1}^{t-1} (-1)^{q_1-r-\ell} \binom{q_1-r}{\ell} D_3^\ell D_2^{q_1+q_2-r-\ell}] ,$$

where $H_{r,t}$ and $G_{r,s}$ are polynomials in D_1 and D_2. Furthermore,

$$H_{r,t} = 0 \quad \text{for} \quad r+t > q_1$$

$$G_{r,s} = 0 \quad \text{for} \quad r+s > q_1+q_2 .$$

Denote by A_u the third term on the right-hand side of (13). Since $q_1+q_2-r-s+1 < \ell < t-1$ implies that $t > \ell$ and $s > q_1+q_2-r-\ell$, we have

$$A_{ut}[\sum_j p(j) B_u(\cdot - j)] \in \pi_0 .$$

Thus, by the Lemma, the hypotheses $(1^o)_m$, $(2^o)_m$ and $(3^o)_m$ of Theorem 1 yield

$$D_1^{q_1} D_2^{q_2}[\sum_j p(j) \sum_{u\in E_3} b_u B_u(\cdot - j)]$$

$$= \sum_{u\in E_3} b_u (D_1^r D_3^t H_{r,t} + D_1^r D_2^s G_{r,s} + A_u)[\sum_j p(j) B_u(\cdot - j)]$$

$$= \sum_{t+r<q_1} H_{r,t} D_1^r D_3^t [\sum_{s>q_2} b_u \sum_j B_u(\cdot - j)]$$

$$+ \sum_{\substack{r+s<q_1+q_2 \\ s>q_2}} G_{r,s} D_1^r D_2^s [\sum_{t>q_1+q_2-r-s} b_u \sum_j B_u(\cdot - j)] + \text{const} \in \pi_0 .$$

Case $i = 4$. The argument is similar to that in the case $i = 3$.

We have proved (12), and therefore conclude that T carries π_m into π_m. To finish the proof, we observe that for any $p \in \pi_{q_1+q_2}$, $\nabla_1^{q_1}\nabla_2^{q_2}p$ is a constant, therefore

$$\nabla_1^{q_1}\nabla_2^{q_2} - Tp = \sum_j p(j)(\nabla_1^{q_1}\nabla_2^{q_2}B(\cdot-j))$$

$$= \sum_j (\nabla_1^{q_1}\nabla_2^{q_2}p)(j)B(\cdot-j) = \nabla_1^{q_1}\nabla_2^{q_2}p \quad .$$

This shows that p and Tp have the same leading coefficients, hence $p - Tp$ is a polynomial of degree < deg p. This completes the proof of Theorem 1.

Now we are in a position to prove our main result.

Theorem 2 <u>The controlled approximation order</u> $\tilde{m}(k,\rho)$ <u>of</u> $\Phi_{k,\rho}$ <u>is</u>

(i) $2k - 2\rho$ if $2k-3\rho = 2$;

(ii) $2k-2\rho-1$ if $2k-3\rho = 3$ or 4 ;

(iii) $k+1$ if $\rho = 0$

(iv) $\min\{2k-2\rho-2,k\}$ if $2k-3\rho \geq 5$ and $\rho \geq 1$.

Proof. Although (i) has already been proved by [BH_1], and (ii) has already been proved by [DM_2], we still give a proof for them to illustrate our method.

If $2k-3\rho = 2$, then $\rho = 2\mu-2$ for some integer μ and $k = 3\mu-2$. Thus $M_u \in \pi_{k,\Delta}^{\rho}$ is equivalent to $u = (\mu,\mu,\mu)$. For $m = 2\mu-1$, we choose $b_{\mu,\mu,\mu} = 1$. This b certainly satisfies all the hypotheses of Theorem 1. But, for $m = 2\mu$, $(1^o)_m$ implies $b_{\mu,\mu,\mu} = 0$. Hence $\Phi_{k,\rho} = \{M_{\mu,\mu,\mu}\}$ has controlled approximation order $2\mu = 2k-2\rho$.

If $2k-3\rho = 3$, then $\rho = 2\mu-1$ for some integer k and $k = 3\mu$. Thus

$$\Phi_{k,\rho} = \{M_{\mu+1,\mu+1,\mu}, M_{\mu,\mu+1,\mu+1}, M_{\mu+1,\mu,\mu+1}\} \quad .$$

For $m = 2\mu$, we choose

$$b_{\mu+1,\mu+1,\mu} = 1 \quad \text{and} \quad b_{\mu,\mu+1,\mu+1} = b_{\mu+1,\mu,\mu+1} = 0 \ .$$

Then b satisfies $(1^\circ)_m$, $(2^\circ)_m$, $(3^\circ)_m$ and (4°) in Theorem 1. But, for $m = 2\mu+1$, $(1^\circ)_m$ implies that $b_{\mu+1,\mu+1,\mu} = 0$; similarly, $b_{\mu,\mu+1,\mu+1} = b_{\mu+1,\mu,\mu+1} = 0$. Therefore $\Phi_{k,\rho}$ has controlled approximation order $2\mu+1 = 2k-2\rho-1$.

If $2k-3\rho = 4$, then $\rho = 2\mu-2$ for some integer μ and $k = 3\mu-1$. Then

$$\Phi_{k,\rho} = \{M_{\mu+1,\mu,\mu}, M_{\mu,\mu+1,\mu}, M_{\mu,\mu,\mu+1}, M_{\mu,\mu,\mu}\} \ .$$

For $m = 2\mu$, we choose

$$b_{\mu+1,\mu,\mu} = b_{\mu,\mu+1,\mu} = b_{\mu,\mu+1,\mu} = \frac{1}{2}, \ b_{\mu,\mu,\mu} = -\frac{1}{2} \ .$$

This b satisfies $(1^\circ)_m$, $(2^\circ)_m$, $(3^\circ)_m$ and (4°). But, for $m = 2\mu+1$, $(1^\circ)_m$ implies $b_{\mu,\mu,\mu+1} = 0$; similarly, $b_{\mu+1,\mu,\mu} = b_{\mu,\mu+1,\mu} = 0$. Then invoking $(1^\circ)_m$ again, we have $b_{\mu+1,\mu,\mu} + b_{\mu,\mu,\mu} = 0$; hence $b_{\mu,\mu,\mu} = 0$. This shows that $\Phi_{k,\rho}$ has controlled approximation order $2\mu+1 = 2k-2\rho-1$.

In case (iii), $\rho = 0$. If we had talked about the approximation order, the result would be trivial. However, for controlled approximation order, this result is not trivial: We must exhibit a map $b : K \to \mathbb{R}$ such that $(1^\circ)_k$, $(2^\circ)_k$, $(3^\circ)_k$ and (4°) hold. Let

$$b_u := \begin{cases} 1 & \text{if} \quad r+s+t = k+2 \quad \text{and} \quad \min\{r,s,t\} \geq 1 \\ -1 & \text{if} \quad r+s+t = k+1 \quad \text{and} \quad \min\{r,s,t\} \geq 1 \\ 0 & \text{otherwise} \ . \end{cases}$$

Then, for fixed r, s with $r+s < k$, we have

$$b_{r,s,\lambda} = \begin{cases} 1 & \text{if} \quad \lambda = k+2-(r+s) \\ -1 & \text{if} \quad \lambda = k+1-(r+s) \\ 0 & \text{otherwise} \ . \end{cases}$$

Hence

$$\sum_{\lambda} b_{r,s,\lambda} = 0 \ .$$

This proves $(3^\circ)_k$. Also, one proves $(1^\circ)_k$ and $(2^\circ)_k$ in the same fashion. As to (4°), we observe that

$$\sum b_u = \sum_{r+s=k+1} b_u + \sum_{r+s\le k} b_u \quad .$$

The second sum on the right is 0, while $r+s = k+1$ and $\min\{r,s,t\} \ge 1$ implies that $t = 1$. But $b_{r,s,t} = 1$ for $r+s = k+1$ and $t = 1$. Hence $\sum b_u = k$, which verifies (4°). Thus $\tilde{m}(k,\rho) = k+1$ for $\rho = 0$.

Now we turn to the new result (iv). If $k \le 2\rho+2$, then it is shown in $[J_2]$ that

$$\tilde{m}(k,\rho) \ge 2k-2\rho-2 \quad .$$

If $k > 2\rho+2$, it is also proved there that

$$\tilde{m}(k,\rho) \ge k \quad .$$

Thus we always have

$$\tilde{m}(k,\rho) \ge \min\{2k-2\rho-2,k\} \quad .$$

It remains to prove

$$\tilde{m}(k,\rho) \le \min\{2k-2\rho-2,k\} \quad .$$

First, we prove $\tilde{m}(k,\rho) \le k$. Suppose to the contrary that $\tilde{m}(k,\rho) > k$. Let

$$U := \{u;\ r+s+t = k+1 \ \text{ or } \ k+2, \ \text{ and } \ \min\{r+s,s+t,t+r\} \ge \rho+2\} \quad .$$

Then, by Theorem 1, $\Phi_{k,\rho}$ has the same controlled approximation order as

$$\Phi_U := \{M_u;\ u \in U\}$$

has. By Theorem 1, there exists a function $b : U \to \mathbb{R}$ such that $(1^\circ)_k$, $(2^\circ)_k$ and $(3^\circ)_k$ and (4°) hold; i.e.,

$$\sum_r b_u = 0 \quad \text{for any} \quad s,t \quad \text{with} \quad s+t \le k \quad ,$$

$$\sum_s b_u = 0 \quad \text{for any} \quad r,t \quad \text{with} \quad r+t \le k \quad ,$$

$$\sum_t b_u = 0 \quad \text{for any} \quad t,s \quad \text{with} \quad t+s \le k \quad ,$$

$$\sum_{u\in U} b_u \ne 0 \quad .$$

We claim that $(1^\circ)_k$, $(2^\circ)_k$ and $(3^\circ)_k$ imply that all $b_{r,s,t} = 0$.

Since $\rho \geq 1$, we have $2k \geq 3\rho+5 \geq 8$; hence $k \geq 4$. Thus

$$\min\{r+s, s+t, t+r\} \leq 2(r+s+t)/3 \leq (2k+4)/3 \leq k \quad .$$

Suppose $b_u \neq 0$ for some u. Without loss of any generality, we may assume $s+t \leq k$. Then there exist s_0 and t_0 such that $b_{r,s_0,t_0} \neq 0$, but $b_u = 0$ for all (s,t) with $s+t < s_0+t_0$. Note that $s_0+t_0 \geq \rho+2 \geq 3$; hence $s_0 \geq 2$ or $t_0 \geq 2$. For the triple (r,s_0,t_0), there are two possibilities: $r+s_0+t_0 = k+2$ or $k+1$. If $r+s_0+t_0 = k+2$, then $r+s_0 \leq k$ or $r+t_0 \leq k$. If $r+t_0 \leq k$, then by Theorem 1,

$$b_{r,s_0,t_0} + b_{r,s_0-1,t_0} = 0 \quad .$$

But by the choice of (s_0,t_0), $b_{r,s_0-1,t_0} = 0$; hence $b_{r,s_0,t_0} = 0$. Similarly, if $r+s_0 \leq k$, then by Theorem 1,

$$b_{r,s_0,t_0} + b_{r,s_0,t_0-1} = 0 \quad .$$

Again by the choice of (s_0,t_0), $b_{r,s_0,t_0-1} = 0$; hence $b_{r,s_0,t_0} = 0$. Now assume $r+s_0+t_0 = k+1$. In this case, Theorem 1 gives

$$b_{r+1,s_0,t_0} + b_{r,s_0,t_0} = 0 \quad .$$

But $(r+1)+s_0+t_0 = k+2$; hence by what we have proved, $b_{r+1,s_0,t_0} = 0$. Therefore $b_{r,s_0,t_0} = 0$. This shows that all $b_u = 0$. Thus there is no b satisfying $(1^\circ)_k$, $(2^\circ)_k$, $(3^\circ)_k$ and (4°) simultaneously. Hence

$$\tilde{m}(k,\rho) \leq k \quad .$$

In particular, we have proved, for $k \geq 2\rho+2$,

$$\tilde{m}(k,\rho) = k \quad .$$

Finally, we want to treat the case $k < 2\rho+2$. As did [J_2], we set

$$\sigma := 2\rho+2-k, \; k' := k-3\sigma, \; \rho' := \rho-2\sigma \quad .$$

Then $\rho' \geq 1$ and $k' = 2\rho'+2$. We claim that

$$\min\{r,s,t\} \geq \sigma \quad .$$

Indeed, we have

$$\min\{s,t\} < (s+t)/2 < (k+2-r)/2 \quad ;$$

hence

$$\rho+2 < \min\{r+s,r+t\} < r + (k+2-r)/2 = (k+2+r)/2 \quad .$$

It follows that

$$r > 2(\rho+2)-(k+2) = 2\rho+2-k = \sigma \quad .$$

Also, one proves $s > \sigma$ and $t > \sigma$ in the same fashion. Let

$$U' := \{u;\ r+s+t < k'+2 \ \text{ and } \ \min\{r+s,s+t,t+r\} > \rho'+2\} \quad .$$

Let F be the mapping given by

$$F((r,s,t)) = (r-\sigma,s-\sigma,t-\sigma) \quad .$$

Then F maps U to U'. F is injective, obviously. F is also surjective, since $u \in U'$ implies that $(r+\sigma,s+\sigma;t+\sigma) \in U$. Then $b : U \to \mathbb{R}$ satisfies $(1^o)_m$, $(2^o)_m$, $(3^o)_m$ and (4^o) if and only if $b \circ F$ satisfies $(1^o)_{m-2\sigma}$, $(2^o)_{m-2\sigma}$, $(3^o)_{m-2\sigma}$ and (4^o). Therefore

$$\tilde{m}(k,\rho)-2\sigma < k' \quad .$$

We conclude that

$$\tilde{m}(k,\rho) < 2\sigma+k' = 2k-2\rho-2 \quad .$$

This finishes the proof of Theorem 2.

Remark. We have seen that $\pi^1_{5,\Delta}$ has approximation order 6 but controlled approximation order only 5. The latter fact means that we cannot find a finite linear combination B of M_u ($M_u \in \pi^1_{5,\Delta}$) and their translates such that the mapping

$$T_B : p \mapsto \sum_{j\in\mathbb{Z}^2} p(j)B(\cdot-j)$$

is degree-preserving on π_5. Nevertheless, there exists $B \in \pi^1_{5,\Delta}$ with compact support such that T_B is degree-preserving on π_5. This can be proved by using local interpolation on triangles. Denote by $[x]$ the linear functional of evaluation at x; i.e., $[x]\,f := f(x) = f(x_1,x_2)$. For $j = (j_1,j_2) \in \mathbb{Z}^2$, let

$$\lambda_{1,j} := [j],\ \lambda_{2,j} := [j]D_1,\ \lambda_{3,j} := [j]D_2$$

$$\lambda_{4,j} := [j]D_1^2,\ \lambda_{5,j} := [j]D_1D_2,\ \lambda_{6,j} := [j]D_2^2$$

$$\lambda_{7,j} := [j+(\tfrac{1}{2},0)]D_2,\ \lambda_{8,j} := [j+(0,\tfrac{1}{2})]D_1,\ \lambda_{9,j} := [j+(\tfrac{1}{2},\tfrac{1}{2})](D_1-D_2)\ .$$

From [BZ] we know that there exist $B_{i,j} \in \pi^1_{5,\Delta}$ $(i = 1,\ldots,9;\ j \in \mathbb{Z}^2)$ with compact support such that $B_{i,j} = B_{i,0}(\cdot-j)$ and

$$\lambda_{i_1,j}B_{i_2,k} = \delta_{i_1i_2}\delta_{jk}\ .$$

(Here δ denotes the usual Kronecker sign.) Then for any $p \in \pi_5$,

$$p = \sum_i \sum_j (\lambda_{i,j}p)B_{i,j} = \sum_i \sum_j (\lambda_{i,j}p)B_{i,0}(\cdot-j)\ .$$

From the above formula we can easily deduce that there exists $B \in \pi^1_{5,\Delta}$ with compact support such that T_B is degree-preserving on π_5.

We conjecture that, for any k and ρ, if $m+1$ is the approximation order of $\pi^\rho_{k,\Delta}$, then there exists $B \in \pi^\rho_{k,\Delta}$ with compact support such that the mapping T_B is degree-preserving on π_m.

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Let Δ be the mesh in the plane obtained from a uniform square mesh by drawing in the north-east diagonal in each square. Let $\pi^{\rho}_{k,\Delta}$ be the space of bivariate piecewise polynomial functions in C^{ρ}, of total degree $\le k$, on the mesh Δ. It is demonstrated that the controlled approximation order from the linear span of all the box splines in $\pi^{\rho}_{k,\Delta}$ is

(1) $2k-2\rho$ if $2k-3\rho = 2$

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