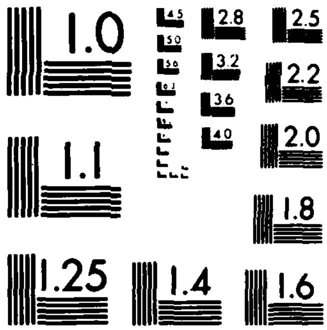
AD-A142 844 HOPF BIFURCATION IN TWO-COMPONENT FLOW(U) WISCONSIN UNIV-MADISON MATHEMATICS RESEARCH CENTER M RENARDY ET AL. MAY 84 MRC-TSR-2689 DAAG29-80-C-0041 1/1

UNCLASSIFIED F/G 12/1 NL



MICROCOPY RESOLUTION TEST CHART
NATIONAL BUREAU OF STANDARDS-1963-A

2

MRC Technical Summary Report #2689

HOPF BIFURCATION IN TWO-COMPONENT FLOW

M. Renardy and D. D. Joseph

Mathematics Research Center
University of Wisconsin—Madison
610 Walnut Street
Madison, Wisconsin 53705

May 1984

(Received November 14, 1983)

Approved for public release
Distribution unlimited

Sponsored by

U. S. Army Research Office
P. O. Box 12211
Research Triangle Park
North Carolina 27709

National Science Foundation
Washington D.C. 20550

UNIVERSITY OF WISCONSIN-MADISON
MATHEMATICS RESEARCH CENTER

HOPF BIFURCATION IN TWO-COMPONENT FLOW

M. Renardy[1] and D. D. Joseph[2]

Technical Summary Report #2689
May 1984

ABSTRACT

The stability of viscosity-stratified bicomponent flow has been studied by long wave asymptotics, by short wave asymptotics and numerically. These studies have shown that interfacial instabilities arise from the viscosity difference between the two fluids. If the surface tension between the fluids is non-zero, then Hopf type bifurcations leading to traveling interfacial waves are expected. In this paper, we prove a rigorous theorem establishing the existence of bifurcating solutions of this nature.

AMS (MOS) Subject Classifications: 35B32, 35Q10, 76E05, 76V05

Key Words: Two-Component Flow, Hopf bifurcation

Work Unit Number 1 (Applied Analysis)

[1]Department of Mathematics and MRC, University of Wisconsin, Madison, WI
[2]Department of Aerospace Engineering, University of Minnesota, Minneapolis, MN 55455.

Sponsored by:
[1]The United States Army under Contract No. DAAG29-80-C-0041. This material is based upon work supported by the National Science Foundation under Grant Nos. MCS-7927062, Mod. 2 and MCS-8215064.
[2]The United States Army under Contract No. DAAG29-82-K-0029 and the Fluid Mechanics branch of the National Science Foundation.

SIGNIFICANCE AND EXPLANATION

Studies employing both asymptotic and numerical methods have shown that, in the flow of two fluids with different viscosities, interfacial instabilities can arise at any Reynolds number. Among the geometries that have been studied are plane Couette and Poiseuille flow, pipe flow and flow between rotating cylinders.

The question thus arises what happens in those cases where the interface is unstable. Which new flow patterns that would be stable can develop? C. S. Yih has conjectured that there may be traveling interfacial waves. More recent analysis by Hooper and Boyd has shown that there is a crucial difference between the cases of zero and non-zero surface tension between the two fluids. If the surface tension is zero, then waves of arbitrarily short wave length are unstable. In this case, standard methods of bifurcation theory are not applicable, and it seems possible that no smooth interface, steady or unsteady, would be stable. In fact, we believe that this might be a mechanism for the formation of emulsions.

If, on the other hand, surface tension is non-zero, then bifurcation theory is applicable and Yih's conjecture could be true if the bifurcation turns out to be supercritical. In this paper, we study plane Couette flow with periodic boundary conditions imposed in the streamwise direction. We consider a "basic" solution with a flat interface parallel to the walls. We prove the existence of bifurcating periodic solutions (interfacial waves) at a point where the flat interface becomes unstable.

The responsibility for the wording and views expressed in this descriptive summary lies with MRC, and not with the authors of this report.

HOPF BIFURCATION IN TWO-COMPONENT FLOW

M. Renardy[1] and D. D. Joseph[2]

1. Introduction

The stability of two-component parallel shear flows has been analyzed by long-wave asymptotics [5], [16], short-wave asymptotics [6], [13] and numerically [11], [13]. These studies show that, if the fluids have different viscosities, then instabilities can arise at all Reynolds numbers.

This raises the question of possible alternative flow patterns which might be stable. Yih [16] has conjectured that wavy interfaces might develop. The analysis of Hooper and Boyd [6] reveals a crucial difference between the cases of zero and non-zero surface tension between the fluids. If the surface tension is zero, then sufficiently short waves are always unstable, i.e., there is an infinite number of unstable modes. This situation is very much unlike the usual problems of bifurcation theory, and we believe it is possible that no smooth interface, steady or unsteady, would be stable in this situation. (In reality, of course, the surface tension is not zero, but there will be instability for very short waves when the surface tension is small and the Reynolds number is large. We think that this instability mechanism may be relevant in the formation of emulsions.)

In the case of non-zero surface tension, however, one can establish a bifurcation theorem. If the bifurcation turns out supercritical, this provides a basis for Yih's conjecture. For the sake of simplicity, we confine attention to plane Couette flow, but it is clear that similar techniques can be applied to more complicated geometries such as

[1]Department of Mathematics and MRC, University of Wisconsin, Madison, WI 53705.
[2]Department of Aerospace Engineering, University of Minnesota, Minneapolis, MN 55455

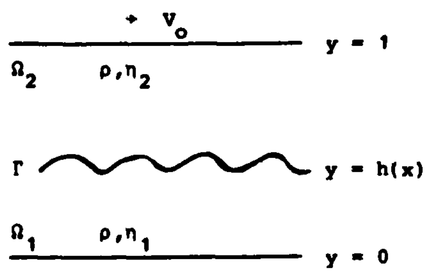
Sponsored by:
[1]The United States Army under Contract No. DAAG29-80-C-0041. This material is based upon work supported by the National Science Foundation under Grant Nos. MCS-7927062, Mod. 2 and MCS-8215064.
[2]The United States Army under Contract No. DAAG29-82-K-0029 and the Fluid Mechanics branch of the National Science Foundation.

concentric flow in pipes or between rotating cylinders. We consider plane Couette flow of two fluids with equal density, but different viscosities, and an interface parallel to the plates. Periodic boundary conditions are imposed in the streamwise direction. Evidently, this configuration is stable at rest. If there is a flow, however, then instabilities can develop [6], [16]. Since, however, surface tension will damp short waves, there can always be only a finite number of unstable modes. Generically, as the flow rate is increased, one specific mode will be first to become unstable. Since the eigenvalues are complex, one expects a bifurcation of the Hopf type [7], leading to traveling interfacial waves.

The Hopf bifurcation theorem in infinite dimensions [4], [8-10], [14] relies on coercive estimates for the linearized equations. For one-component free surface flows such estimates were derived by Beale [2], [3], and we shall, in Chapter 3, derive analogous estimates for two-component flow. Our proof differs from Beale's and is slightly simpler. Using these coercive estimates, we can then establish a bifurcation theorem in Chapter 4. In Chapters 5-8, we outline an algorithm for the computation of bifurcating solutions.

2. Formulation of the Problem

We consider two-dimensional flow of two fluids with different viscosities and equal densities between parallel plates.



The motion in each fluid is described by the Navier-Stokes equations:

$$(2.1) \qquad \left.\begin{aligned} \rho(\dot{\underline{u}} + (\underline{u} \cdot \nabla)\, \underline{u}) &= \eta_1 \Delta \underline{u} - \nabla p, \\ \nabla \cdot \underline{u} &= 0\, , \end{aligned}\right\} \quad 0 < y < h(x),$$

$$(2.2)\qquad \left.\begin{aligned} &\rho(\dot{\underline{v}} + (\underline{v}\cdot\nabla)\underline{v}) = \eta_2\,\Delta\underline{v} - \nabla q, \\ &\nabla\cdot\underline{v} = 0\ , \end{aligned}\right\}\quad h(x) < y < 1.$$

We have no slip conditions at the walls:

$$(2.3)\qquad \underline{u} = 0 \quad \text{at} \quad y = 0,$$

$$(2.4)\qquad \underline{v} = (V_o, 0) \quad \text{at} \quad y = 1.$$

Across the interface, there must be continuity of velocity and traction. This leads to the equations

$$(2.5)\qquad \underline{u} = \underline{v} \quad \text{at} \quad y = h(x),$$

$$(2.6)\qquad \eta_1\left\{(1-h'^2)\left(\frac{\partial u_2}{\partial x} + \frac{\partial u_1}{\partial y}\right) + 2h'\left(\frac{\partial u_2}{\partial y} - \frac{\partial u_1}{\partial x}\right)\right\}$$

$$= \eta_2\left\{(1-h'^2)\left(\frac{\partial v_2}{\partial x} + \frac{\partial v_1}{\partial y}\right) + 2h'\left(\frac{\partial v_2}{\partial y} - \frac{\partial v_1}{\partial x}\right)\right\} \text{ at } \quad y = h(x),$$

$$(2.7)\qquad 2\eta_1\frac{\partial u_2}{\partial y} - p - 2h'\eta_1\left(\frac{\partial u_2}{\partial x} + \frac{\partial u_1}{\partial y}\right) + h'^2\left(2\eta_1\frac{\partial u_1}{\partial x} - p\right)$$

$$= 2\eta_2\frac{\partial v_2}{\partial y} - q - 2h'\eta_1\left(\frac{\partial v_2}{\partial x} + \frac{\partial v_1}{\partial y}\right) + h'^2\left(2\eta_2\frac{\partial v_1}{\partial x} - q\right)$$

$$+ T\frac{h''}{(1+h'^2)^{1/2}} \qquad \text{at} \quad y = h(x).$$

Here T is the surface tension parameter. Finally, we have the kinematic boundary condition

$$(2.8)\qquad \dot{h} + u_1 h' = u_2.$$

We are interested in solutions to (2.1) - (2.8) which have a given period L in the x-direction and are periodic in t. We denote by Ω_1 the set $\{(x,y)\,|\,0 < x < L,\ 0 < y < h(x)\}$, by Ω_2 the set $\{(x,y)\,|\,0 < x < L,\ h(x) < y < 1\}$ and by Γ the interface $\{(x,y)\,|\,0 < x < L,\ y = h(x)\}$. The spaces $H^k(\Omega_1)$, $H^k(\Omega_2)$, $H^k(\Gamma)$ consist of those functions which have k square integrable derivatives and satisfy periodic boundary conditions in the x-direction.

3. The linearized problem

In this chapter, we obtain coercive estimates on the linear problem, which we shall need later. We put $V_o = 0$ and linearize (2.1) - (2.8) at the rest state $\underline{u} = \underline{v} = 0$, with a flat interface $h = h_o$. We include inhomogeneous terms in (2.1), (2.2), (2.6) and (2.7). $\frac{\partial}{\partial t}$ is replaced by λ. This leads to the problem:

$$\left.\begin{aligned} \lambda\rho\underline{u} &= \eta_1 \Delta\underline{u} - \nabla p + \underline{f}_1, \\ \nabla\cdot\underline{u} &= 0 \ , \end{aligned}\right\} \quad 0 < y < h_o \tag{3.1}$$

$$\left.\begin{aligned} \lambda\rho\underline{v} &= \eta_2 \Delta\underline{v} - \nabla q + \underline{f}_2 \\ \nabla\cdot\underline{v} &= 0 \ , \end{aligned}\right\} \quad h_o < y < 1 \ , \tag{3.2}$$

$$\underline{u} = 0 \quad , \quad y = 0, \tag{3.3}$$

$$\underline{v} = 0 \quad , \quad y = 1, \tag{3.4}$$

$$\underline{u} = \underline{v} \quad , \quad y = h_o, \tag{3.5}$$

$$\eta_1\left(\frac{\partial u_2}{\partial x} + \frac{\partial u_1}{\partial y} \right) - \eta_2\left(\frac{\partial v_2}{\partial x} + \frac{\partial v_1}{\partial y} \right) = f_3, \ y = h_o \ , \tag{3.6}$$

$$2\eta_1 \frac{\partial u_2}{\partial y} - p - 2\eta_2 \frac{\partial v_2}{\partial y} + q - Th'' = f_4 \quad , \ y = h_o, \tag{3.7}$$

$$\lambda h = u_2 \quad , \quad y = h_o. \tag{3.8}$$

We seek solutions periodic in x with period L, and of course the same periodicity is assumed for $\underline{f}_1$, $\underline{f}_2$, f_3 and f_4. Our goal is the following estimate:

Theorem 3.1:

Let $\gamma > 0$. For $\mathrm{Re}\,\lambda > \gamma$, the following estimate holds for solutions of (3.1) - (3.8).

$$\begin{aligned} &\|\underline{u}\|_{H^2(\Omega_1)} + \|\underline{v}\|_{H^2(\Omega_2)} + \|p\|_{H^1(\Omega_1)} + \|q\|_{H^1(\Omega_2)} + |\lambda|\,\|\underline{u}\|_{L^2(\Omega_1)} \\ &+ |\lambda|\,\|\underline{v}\|_{L^2(\Omega_2)} + \|h\|_{H^{5/2}(\Gamma)} + |\lambda|\,\|h\|_{H^{3/2}(\Gamma)} < C\,\Big[\|\underline{f}_1\|_{L^2(\Omega_1)} + \|\underline{f}_2\|_{L^2(\Omega_2)} \\ &+ \|f_3\|_{H^{1/2}(\Gamma)} + |\lambda|^{1/4}\|f_3\|_{L^2(\Gamma)} + \|f_4\|_{H^{1/2}(\Gamma)}\Big] \end{aligned} \tag{3.9}$$

Proof:

We multiply (3.1) by $\bar{\underline{u}}$ (the complex conjugate of $\underline{u}$), and (3.2) by $\bar{\underline{v}}$, add them and integrate over the domain. Integrating by parts, and using the boundary and interface conditions, we find

$$\lambda\rho(\|\underline{u}\|^2_{L^2} + \|\underline{v}\|^2_{L^2}) + \tfrac{1}{2}\eta_1 \langle \underline{u}, \underline{u} \rangle + \tfrac{1}{2}\eta_2 \langle \underline{v}, \underline{v} \rangle + \bar{\lambda}\, T\, \|h'\|^2_{L^2}$$

(3.10)

$$= (\underline{f}_1, \bar{\underline{u}})_{L^2} + (\underline{f}_2, \bar{\underline{v}})_{L^2} + \int_0^L f_3 \bar{u}_1(y=h_o)\, dx + \int_0^L f_4\, \bar{u}_2\, (y = h_o)\, dx$$

Here $\langle \underline{u}, \underline{u} \rangle = \int (\nabla\underline{u} + (\nabla\underline{u})^T) : (\nabla\bar{\underline{u}} + (\nabla\bar{\underline{u}})^T)$.

Since $\underline{u}$ is divergence free, u_2 has a trace on the interface whose $H^{-1/2}$- norm can be bounded by $\|\underline{u}\|_{L^2(\Omega_1)}$ [15], hence the last term on the right side can be bounded by $\|f_4\|_{H^{1/2}(\Gamma)} \|\underline{u}\|_{L^2(\Omega_1)}$. The third term is bounded by

$$\|f_3\|_{L^2(\Gamma)} \|\underline{u}\|_{L^2(\Gamma)} \le \|f_3\|_{L^2(\Gamma)} \|\underline{u}\|^{1/2}_{L^2(\Omega_1)} \|\underline{u}\|^{1/2}_{H^1(\Omega_1)} .$$

We assume that the right side of (3.9) is bounded by a constant of order one, and we wish to bound the left side. From (3.10), we immediately obtain bounds for $\|\underline{u}\|_{H^1}$, $\|\underline{v}\|_{H^1}$, $\|h\|_{H^1}$ (we have used Korn's inequality here).

In the following, we make repeated use of the following estimates:

(3.11)
$$\|\underline{u}\|_{H^2} + \|\underline{v}\|_{H^2} + \|p\|_{H^1} + \|q\|_{H^1}$$

$$\le C(\|\underline{f}_1\|_{L^2} + \|\underline{f}_2\|_{L^2} + \|f_3\|_{H^{1/2}} + |\lambda|\,\|h\|_{H^{3/2}} + |\lambda|\,\|\underline{u}\|_{L^2} + |\lambda|\,\|\underline{v}\|_{L^2}),$$

(3.12)
$$\|h\|_{h^{5/2}} \le C(\|\underline{u}\|_{H^2} + \|\underline{v}\|_{H^2} + \|p\|_{H^1} + \|q\|_{H^1} + \|f_4\|_{H^{1/2}}),$$

(3.13)
$$\|h\|_{H^{3/2}} \le \|h\|^{2/3}_{H^1} \|h\|^{1/3}_{H^{5/2}} .$$

Estimate (3.11) follows from (3.1)-(3.6) and (3.8), which form an elliptic system in the sense of Agmon, Douglis and Nirenberg [1]. (see the remarks at the end of this chapter). (3.12) is a trivial consequence of (3.7), and (3.13) follows from the convexity property of Sobolev norms.

<u>Lemma 3.2.</u>:

For any $\varepsilon > 0$, the following quantities can be estimated by the right hand side of (3.9):

$$\|h\|_{H^1}|\lambda|^{1-\varepsilon},\ \|\underline{u}\|_{L^2}|\lambda|^{1-\varepsilon},\ \|\underline{v}\|_{L^2}|\lambda|^{1-\varepsilon},\ \|\underline{u}\|_{H^1}|\lambda|^{1/2-\varepsilon},\ \|\underline{v}\|_{H^1}|\lambda|^{1/2-\varepsilon},\ (\|\underline{u}\|_{H^2}+\|p\|_{H^1})$$
$$|\lambda|^{-1/2-\varepsilon},\ (\|\underline{v}\|_{H^2}+\|q\|_{H^1})\ |\lambda|^{-1/2-\varepsilon}.$$

Proof of the lemma:

We prove by induction that

$(3.14)_n$ $\quad \|h\|_{H^1} \leq c|\lambda|^{2^n/3^n-1}$,

$(3.15)_n$ $\quad \|\underline{u}\|_{L^2}, \|\underline{v}\|_{L^2} \leq C\ |\lambda|^{2^n/3^n-1}$,

$(3.16)_n$ $\quad \|\underline{u}\|_{H^1}, \|\underline{v}\|_{H^1} \leq C\ |\lambda|^{1/2(2^n/3^n-1)}$,

$(3.17)_n$ $\quad \|\underline{u}\|_{H^2} + \|p\|_{H^1}, \|\underline{v}\|_{H^2} + \|q\|_{H^1} \leq C\ |\lambda|^{1/2\ +\ 2^n/3^n}$.

Obviously the lemma follows by letting $n \to \infty$. We already have (3.14)-(3.16) for $n = 0$. By combining (3.11) - (3.13), we obtain

$$(3.18)\qquad \|\underline{u}\|_{H^2} + \|p\|_{H^1} + \|\underline{v}\|_{H^2} + \|q\|_{H^1} \leq C\ (\|\underline{f}_1\|_{L^2} + \|\underline{f}_2\|_{L^2}$$
$$+ \|f_3\|_{H^{1/2}} + |\lambda|\ \|\underline{u}\|_{L^2} + |\lambda|\ \|\underline{v}\|_{L^2} + |\lambda|\ \|h\|_{H^1}{}^{2/3}\ .$$
$$(\|\underline{u}\|_{H^2} + \|\underline{v}\|_{H^2} + \|p\|_{H^1} + \|q\|_{H^1} + \|f_4\|_{H^{1/2}})^{1/3}\ .$$

With $\beta = \|\underline{u}\|_{H^2} + \|p\|_{H^1} + \|\underline{v}\|_{H^2} + \|q\|_{H^1}$, we find, using $(3.14)_n$ - $(3.16)_n$

(3.19) $\quad \beta \leq C\,(1 + |\lambda|^{2^n/3^n} + |\lambda|^{2^{n+1}/3^{n+1} + \frac{1}{3}}\,(\beta + c')^{1/3})$.

From this, $(3.17)_n$ follows easily.

Next, we wish to show that $(3.14)_{n+1}$ - $(3.16)_{n+1}$ follow from $(3.17)_n$. Using (3.8), the trace theorem and the convexity property of Sobolev norms, we find

(3.20) $\quad \|h\|_{H^1} \leq \frac{C}{|\lambda|} \sqrt{\|\underline{u}\|_{H^2}} \sqrt{\|\underline{u}\|_{H^1}}$.

Moreover (3.10) implies

(3.21) $\quad \|\underline{v}\|_{H^1} + \|\underline{u}\|_{H^1} \leq C(\|\underline{u}\|_{L^2} + \|\underline{v}\|_{L^2} + |\lambda|^{-1/4} \|\underline{u}\|_{L^2}^{1/2}\|\underline{u}\|_{H^1}^{1/2})^{1/2}$,

and

(3.22) $\quad \|\underline{v}\|_{L^2} + \|\underline{u}\|_{L^2} \leq C\left[\|h\|_{H^1} + |\lambda|^{-1/2}(\|\underline{u}\|_{L^2} + \|\underline{v}\|_{L^2} + |\lambda|^{-1/4}\|\underline{u}\|_{L^2}^{1/2} \|\underline{u}\|_{H'}^{1/2})^{1/2}\right]$.

With $x = \|\underline{u}\|_{H^1} + \|\underline{v}\|_{H^1}$, $y = \|\underline{u}\|_{L^2} + \|\underline{v}\|_{L^2}$ and using $(3.17)_n$, we find that (3.21), (3.22) take the following form

(3.23) $\quad x \leq C(y^{1/2} + |\lambda|^{-1/8}\, x^{1/4}\, y^{1/4})$,

(3.24) $\quad y \leq C(|\lambda|^{2^{n-1}/3^n - \frac{3}{4}}\, x^{1/2} + |\lambda|^{-1/2}\, y^{1/2} + |\lambda|^{-5/8}\, y^{1/4}\, x^{1/4})$.

From (3.23) it follows that $x \leq Cy^{1/2}$ or $x \leq C|\lambda|^{-1/6}\, y^{1/3}$, depending on whether the first or second term on the right is bigger. In the first case,

(3.24) yields

(3.25) $\quad y \leq C\,(|\lambda|^{2^{n-1}/3^n - \frac{3}{4}}\, y^{1/4} + |\lambda|^{-1/2}\, y^{1/2} + |\lambda|^{-5/8}\, y^{3/8})$

From this $(3.15)_{n+1}$ is immediate. In the second case, we get

(3.26) $\quad y \leq C(|\lambda|^{2^{n-1}/3^n - \frac{3}{4} - \frac{1}{12}}\, y^{1/6} + |\lambda|^{-1/2}\, y^{1/2} + |\lambda|^{-3/4 - \frac{1}{24}}\, y^{1/3})$

From this, $(3.15)_{n+1}$ is also immediate. From $(3.15)_{n+1}$ and (3.21) follows $(3.16)_{n+1}$, and using (3.20) and $(3.17)_n$ we find $(3.14)_{n+1}$.

To proceed further, we take difference quotients in the x - direction. These satisfy the same equations (3.1) - (3.8) with the f's replaced by their difference quotients. From (3.10), we then see that the H^1-norms of all x-derivatives of $\underline{u}$ and $\underline{v}$ can be estimated by terms of order 1. The divergence condition now yields

$$\|u_2\|_2, \|v_2\|_2 \leq C.$$

Equation (3.20) now yields $\|h\|_{H^1} \leq C|\lambda|^{-5/4+\varepsilon}$, and (3.21), (3.22) now imply

$\|\underline{u}\|_{L^2} + \|\underline{v}\|_{L^2} \leq \frac{C}{|\lambda|}$. The rest follows easily. It remains to prove (3.11). For this, we only need to show that (3.1), (3.2) with boundary conditions (3.3) - (3.6) and (3.8) from an elliptic system in the sense of Agmon, Douglis and Nirenberg [1]. We can formally regard (3.1), (3.2) as being posed in the same domain by mapping the strip occupied by fluid 2 onto the strip occupied by fluid 1. We do this in such a way that the interface is mapped onto itself and the solid boundaries are mapped onto each other. This yields a system for the six unknowns (u_1, u_2, p, v_1, v_2, q), which are now defined on the same domain. That the Stokes equations are an elliptic system is well known. The same holds of course for two sets of Stokes equations. It is also well known that Dirichlet boundary conditions satisfy the complementing condition. Showing the complementing nature of the interface condition is a straight forward calculation, which we omit.

Remarks:

We have so far only given estimates for a solution that was assumed to exist and have the regularity implied by the left hand side of (3.9). Such estimates show that (3.1) - (3.8) for $\mathrm{Re}\,\lambda \geq \gamma > 0$ is solvable for a closed set of f's (in the topology indicated by the right side of (3.9)). Solvability for a dense set of f's can be shown in a straight forward manner by separation of variables. From this we see that in fact, for any λ with $\mathrm{Re}\,\lambda > 0$, (3.1) - (3.8) has a unique solution. (3.9) holds uniformly in any closed subset of any right half-plane, if this subset contains no eigenvalues. Moreover, by compactness, the number of eigenvalues is countable, and there can only be finitely many in any bounded set. (3.10) implies that all eigenvalues have negative real parts.

4. Bifurcation to travelling waves

It is convenient to use a domain mapping which takes the domain occupied by each fluid to a fixed one. The most straight forward way to construct such a mapping is to stretch or contract vertical lines. We shall in addition transform the velocity fields in such a way that the divergence condition is preserved and (2.8) reduces the linearized form even in the nonlinear case. In doing this, we essentially follow Beale [2].

Let $y = h_o$ be the flat interface of the rest state, and let $y = h(x,t)$ be the actual interface, which we assume periodic in x with period L.

Let P be any linear extension operator that maps functions $h(\zeta) \in H^s$, into functions $\tilde{h}(\zeta,\eta)$ such that $\tilde{h}(\zeta,h_o) = h(\zeta)$ and $\tilde{h} \in H^{s+1/2}$ (there are many ways to construct P). For simplicity, we also assume that P takes $h \equiv h_o$ to $\tilde{h} \equiv h_o$. Let then $f_o(\eta)$ be a C^∞ - function of η such that $f_o = 1$ near $\eta = h_o$ and $f_o = 0$ near $\eta = 0$ and $\eta = 1$. Define $\bar{h}(\zeta,\eta,t) = \tilde{h}(\zeta,\eta,t) \cdot f_o(\eta) + h_o(1-f_o(\eta))$

We now define

$$(4.1) \qquad \Theta(\zeta,\eta,t) = (\zeta,\eta \cdot \frac{\bar{h}(\zeta,\eta,t)}{h_o}) .$$

Evidently, Θ maps the strip $0 < \eta < h_o$ to Ω_1, and the strip $h_o < \eta < 1$ to Ω_2.

The velocities are transformed as follows:

$$(4.2) \qquad u_i(\Theta(\zeta,\eta,t)) = \frac{\partial\Theta_i}{\partial\zeta_j} \tilde{u}_j \,/\, J,$$

where J is the jacobian of Θ. Of course, $\underline{v}$ in Ω_2 is transformed in the same way.

Explicitly, (4.2) reads

$$(4.3) \qquad \underline{u} = (\frac{\bar{h}}{h_o} + \eta \frac{h_\eta}{h_o})^{-1}(\tilde{u}_1, \eta \frac{\bar{h}_\zeta}{h_o} \tilde{u}_1 + (\frac{\bar{h}}{h_o} + \eta \frac{\bar{h}_\eta}{h_o}) \tilde{u}_2).$$

Formula (4.2) is set up in such a way that $\underline{u}$, $\underline{v}$ are divergence-free in the x,y-plane, if $\tilde{\underline{u}}$, $\tilde{\underline{v}}$ are divergence-free in the ζ, η-plane. Moreover, (2.8) now assumes the simple form

(4.4) $\dot{h}(\zeta,t) = \tilde{u}_2(\zeta,h_o,t)$.

It is also clear that (2.5) does not change its form, i.e. $\underline{u} = \underline{v}$ at $y = h(x)$ ε becomes $\tilde{\underline{u}} = \tilde{\underline{v}}$ at $\eta = h_o$. The boundary conditions at the walls are also preser When these substitutions are inserted into (2.1) - (2.8), we obtain a new set of equations, which we do not write down explicitly. We shall refer to these new equ as (2.1)* - (2.8)*. If we have a flat interface $h = h_o$, then of course (4.3) reads $\underline{u} = \tilde{\underline{u}}$, and (2.1)* - (2.8)* have the same form as (2.1) - (2.8).

Plane Couette flow is the following solution of (2.1) - (2.8):

$$\hat{h}(x) \equiv h_o,\ \hat{u}_1 = \frac{\eta_2 V_o}{\eta_1 + h_o(\eta_2-\eta_1)}\, y,\ \hat{u}_2 = 0,\ \hat{v}_1 = \frac{\eta_1 V_o + V_o h_o(\eta_2-\eta_1)}{\eta_1 + h_o(\eta_2-\eta_1)},$$

$$\hat{v}_2 = 0,\ \hat{p} = 0,\ \hat{q} = 0.$$

We can linearize at this solution, and obtain a set of linearized equations analog (3.1) - (3.8). As usual, we call λ an eigenvalue, if the homogeneous linearized has nontrivial solutions. The estimates in chapter 3 imply that, for $V_o = 0$(rest is a countable sequence of eigenvalues, all in a sector of the left half plane bou away from the imaginary axis. All these eigenvalues have finite multiplicity and index zero. Estimates like those in Chapter 3 can easily be extended to the linea at Couette flow with finite V_o. If we linearize (2.1) - (2.8) at this flow, then are a number of terms perturbing (3.1) - (3.8). All these terms are relatively co except the one resulting from $u_1 h'$ in (2.8). This latter term vanishes in a fr moving with the fluid on the interface. Standard perturbation theory [12] can now to show that estimates like in Chapter 3 hold for λ in any closed set that lies right half plane and contains no eigenvalues. However, there can, and as [6], [16 there will be a finite number of eigenvalues with positive real parts if V_o is l enough. Generically, there will be a critical value V_{oc}, such that, for $V_o < V_c$ eigenvalues have negative real parts, but a pair of simple complex conjugate eigenvalues crosses the imaginary axis transversally as V_o increases past V_{oc}.

denote these imaginary eigenvalues by $\pm i\omega_o$.

We introduce the following substitutions in (2.1)* - (2.8)*:

$$\tau = \omega t,\ \underline{u}^* = \underline{\tilde{u}} - \underline{\hat{u}}(V_o),\ \underline{v}^* = \underline{\tilde{v}} - \underline{\hat{v}}(V_o),\ h^* = (h-h_o)\frac{\omega}{\omega_o}\ .$$

We use the following notation for function space: $H^k_{\tilde{\Omega}_1}$ denotes the space of functions defined on $-\infty < \zeta < \infty$, $0 \le \eta \le h_o$, which have period L in ζ and H^k-regularity. Similarly, we define $H^k_{\tilde{\Omega}_2}$. Finally $H^k_{\tilde{\Gamma}}$ is the set of k times differentiable periodic functions depending on ζ alone.

$H^k(X)$ denotes the spaces of all 2π-periodic functions defined on $-\infty < \tau < \infty$, taking values in X, and having k square integrable derivatives.

For the analysis of (2.1)* - (2.8)*, we choose the following space V:

$$V = \{(\underline{u}^*, \underline{v}^*, p, q, h^*) |\ \underline{u}^* \in H^1(H^2_{\tilde{\Omega}_1}) \cap H^2(L^2_{\tilde{\Omega}_1}),\ \underline{v}^* \in H^1(H^2_{\tilde{\Omega}_2}) \cap H^2(L^2_{\tilde{\Omega}_2}),$$

$$p \in H^1(H^1_{\tilde{\Omega}_1}),\ q \in H^1(H^1_{\tilde{\Omega}_2}),\ h^* \in H^1(H^{5/2}_{\tilde{\Gamma}}) \cap H^2(H^{3/2}_{\tilde{\Gamma}});\ \operatorname{div} \underline{u}^* = 0,\ \operatorname{div} \underline{v}^* = 0,$$

$$\iint_{\tilde{\Omega}_1} p + \iint_{\tilde{\Omega}_2} q = 0,\ \underline{u}^* = 0 \text{ at } \eta = 0,\ \underline{v}^* = 0 \text{ at } \eta = 1,\ \underline{u}^* = \underline{v}^* \text{ at}$$

$$\eta = h_o,\ \omega_o \frac{\partial h^*}{\partial \tau} = \underline{u}^* \text{ at } \eta = h_o,\ \int_o^L h^*(\zeta)d\zeta = 0\}$$

Functions in this space have sufficient regularity such that all the nonlinearities in (2.1)* - (2.8)* are defined. The coercive estimates of Chapter 3 and the implicit function theorem can be used to prove the following result (the details of the proof are analogous to the standard proof of the Hopf bifurcation theorem and are omitted here).

<u>Theorem 4.1</u>:

Assume that, at $V_o = V_{oc}$, there is a pair of algebraically simple complex conjugate eigenvalues $\pm i\omega_o$, $\omega_o \neq 0$, and no other eigenvalue is an integral multiple of $i\omega_o$. Moreover, assume that those eigenvalues cross the imaginary axis transversally, i.e. if $\lambda(V_o)$ denotes the branch of eigenvalues for which $\lambda(V_{oc}) = i\omega_o$, then

$\frac{d}{dV_o} \operatorname{Re} \lambda(V_o)|_{V_o = V_{oc}} \neq 0$. Then there is an analytic branch of nontrivial time-periodic solutions $(\underline{u}^*(\varepsilon), \underline{v}^*(\varepsilon), p(\varepsilon), q(\varepsilon), h^*(\varepsilon)) =: Y(\varepsilon)$, such that $Y(\varepsilon) \in V$ is a solution of (2.1)* - (2.8)* for $V_o = V_o(\varepsilon)$ with temporal frequency $\omega = \omega(\varepsilon)$. For $\varepsilon = 0$, we have $V_o = V_{oc}$, $\omega = \omega_o$ and $Y = 0$. This branch of periodic solutions is unique except for phase shift or changes of parametrization.

If, at $V_o = V_{oc}$, all eigenvalues other than $\pm i\omega_o$ have negative real parts and we have $\frac{d}{dV_o} \operatorname{Re} \lambda(V_o)|_{V_o = V_{oc}} > 0$, then the bifurcating periodic solutions are stable if $V_o(\varepsilon) > V_{oc}$ for small $\varepsilon \neq 0$, and unstable if $V_o(\varepsilon) < V_{oc}$.

Remark:

It is easy to show higher regularity of the bifurcating solutions by choosing function spaces of higher regularity for the bifurcation analysis.

5. Reduction of the bifurcation problem to local form

In the previous two sections, we have provided the analytical tools and the estimates needed to establish a bifurcation theorem. In the following, we now describe an algorithm for calculating approximations to this bifurcating solution.

As before, we study bifurcation from plane Couette flow, and we consider the velocity of the upper plate as the bifurcation parameter. Plane Couette flow is the following solution of (2.1) - (2.8):

$$\hat{h}(x) = h_o,$$

$$\hat{u}_1 = \frac{\eta_2 V_o y}{\eta_1 + h_o(\eta_2 - \eta_1)},$$

$$\hat{u}_2 = 0, \quad \hat{p} = 0, \tag{5.1}$$

$$\hat{v}_1 = \frac{[\eta_1 y + h_o(\eta_2 - \eta_1)] V_o}{\eta_1 + h_o(\eta_2 - \eta_1)},$$

$$\hat{v}_2 = 0, \quad \hat{q} = 0.$$

For the bifurcation problem, it is convenient to introduce new variables representing the perturbation of this solution. We therefore replace u_1 and v_1 by $u_1 + \hat{u}_1$, $v_1 + \hat{v}_1$, where $\hat{u}_1$, $\hat{v}_1$ are given by the formula (5.1) in the regions $0 < y < h(x,t)$ and $h(x,t) < y < 1$, respectively. Moreover, we set

$$h(x,t) = h_o + \delta(x,t), \tag{5.2}$$

and $\delta(x,t)$ has zero mean value as a function of x. With this change of variables, (2.1) and (2.2) take the form

$$\begin{aligned} &\rho\, [\, \dot{\underline{u}} + \hat{u}_1 \frac{\partial \underline{u}}{\partial x} + \underline{e}_x\, u_2 \frac{\partial \hat{u}_1}{\partial y} + (\underline{u}\cdot\nabla)\, \underline{u}] = \eta_1 \Delta \underline{u} - \nabla p, \quad 0 < y < h(x,t), \\ &\operatorname{div}\ \underline{u} = 0\ , \end{aligned} \tag{5.3}$$

$$\begin{aligned} &\rho\, [\, \dot{\underline{v}} + \hat{v}_1 \frac{\partial \underline{v}}{\partial x} + \underline{e}_x\, v_2 \frac{\partial \hat{v}_1}{\partial y} + (\underline{v}\cdot\nabla)\, \underline{v}] = \eta_2\, \Delta \underline{v} - \nabla q, \quad h(x,t) < y < 1, \\ &\operatorname{div}\ \underline{v} = 0\ . \end{aligned} \tag{5.4}$$

On the walls, we have

$$\begin{aligned} \underline{u} &= 0 \quad \text{at} \quad y = 0\ , \\ \underline{v} &= 0 \quad \text{at} \quad y = 1\ . \end{aligned} \tag{5.5}$$

In the normal and shear stress conditions, the terms introduced by $\hat{u}_1$, $\hat{v}_1$ are such that they cancel. Moreover, $h' = \delta'$, so h can be replaced by δ. Across the interface $y = h(x,t)$, we thus obtain the following conditions resulting from (2.6) and (2.7)

$$\begin{aligned} &(1 - \delta'^2)\ \eta_1 (\frac{\partial u_2}{\partial x} + \frac{\partial u_1}{\partial y}) + 2\eta_1\ \delta' (\frac{\partial u_2}{\partial y} - \frac{\partial u_1}{\partial x}) \\ &= (1 - \delta'^2)\ \eta_2\ (\frac{\partial v_2}{\partial x} + \frac{\partial v_1}{\partial y}) + 2\eta_2\ \delta' (\frac{\partial v_2}{\partial y} - \frac{\partial v_1}{\partial x}) \end{aligned} \tag{5.6}$$

$$2\eta_1 \frac{\partial u_2}{\partial y} - p - 2\delta' \eta_1 \left(\frac{\partial u_2}{\partial x} + \frac{\partial u_1}{\partial y}\right) + \delta'^2 \left(2\eta_1 \frac{\partial u_1}{\partial x} - p\right)$$

$$(5.7) \qquad = 2\eta_2 \frac{\partial v_2}{\partial y} - q - 2\delta'\eta_2 \left(\frac{\partial v_2}{\partial x} + \frac{\partial v_1}{\partial y}\right) + \delta'^2 \left(2\eta_2 \frac{\partial v_1}{\partial x} - q\right)$$

$$+ T\,\delta'' / (1+\delta'^2)^{1/2}$$

The condition (2.5) is replaced by

$$(5.8) \qquad u_1 - v_1 = \hat{v}_1(h_o+\delta) - \hat{u}_1(h_o+\delta) = \frac{(\eta_1-\eta_2)\delta V_o}{\eta_1+h_o(\eta_2-\eta_1)} ,$$

$$(5.9) \qquad u_2 = v_2 .$$

Equation (5.8) shows that, for $\eta_1 \neq \eta_2$ and $V_o \neq 0$, $\delta(x,t)$ can be eliminated and expressed by the jump in the x-component of velocity at $y = h_o + \delta(x,t)$. Finally, equation (2.8) assumes the form

$$(5.10) \qquad \dot{\delta} + \hat{u}_1(h_o)\,\delta' + u_1\,\delta' + \frac{\eta_2 V_o \delta\delta'}{\eta_1+h_o(\eta_2-\eta_1)} = u_2 .$$

Our bifurcation problem is now given by (5.1) - (5.10). The null solution corresponds to plane Couette flow. We consider δ in (5.6), (5.7) and (5.10) as having been eliminated from (5.8), and look for solutions ($\underline{u}$, $\underline{v}$, p,q) which are periodic in x.

6. The spectral problem and its adjoint

The spectral problem for the stability of the null solution is

$$(6.1) \qquad \begin{aligned} &\rho[\lambda\underline{u}+\hat{u}_1 \frac{\partial \underline{u}}{\partial x} + \underline{e}_x\, u_2\hat{u}_1'] + \nabla p - 2\eta_1 \operatorname{div} \underline{\underline{D}}\,[\underline{u}] = 0 \quad 0 < y < h_o \\ &\operatorname{div}\, \underline{u} = 0 , \end{aligned}$$

$$(6.2) \qquad \begin{aligned} &\rho[\lambda\underline{v}+\hat{v}_1 \frac{\partial \underline{v}}{\partial x} + \underline{e}_x\, v_2\hat{v}_1'] + \nabla q - 2\eta_2 \operatorname{div} \underline{\underline{D}}\,[\underline{v}] = 0 \quad h_o < y < 1 , \\ &\operatorname{div}\, \underline{v} = 0 . \end{aligned}$$

Here $\underline{\underline{D}}\,[\underline{u}]$ is the symmetric part of $\nabla\underline{u}$. On the walls we have

$$\underline{u} = 0 \quad \text{at} \quad y = 0\,, \tag{6.3}$$
$$\underline{v} = 0 \quad \text{at} \quad y = 1.$$

From (5.8), we have $\delta = k(u_1 - v_1)$, where $k = \dfrac{\eta_1+(\eta_2-\eta_1)h_o}{V_o(\eta_1-\eta_2)}$.

By inserting this into the remaining interface conditions and linearizing, we find the following conditions at $y = h_o$:

$$u_2 - v_2 = 0, \tag{6.4}$$

$$2\eta_1\, D_{12}\,[\underline{u}] - 2\eta_2\, D_{12}[\underline{v}] = 0, \tag{6.5}$$

$$-p + 2\eta_1\, D_{22}\,[\underline{u}] + q - 2\eta_2\, D_{22}[v] - Tk(u_1''-v_1'') = 0, \tag{6.6}$$

$$(\lambda+\hat{u}(h_o)\,\frac{\partial}{\partial x}\,)k(u_1-v_1) - u_2 = 0. \tag{6.7}$$

We turn next to the computation of the spectral problem, which is adjoint to (6.1) - (6.7). We multiply (6.1) by $\overline{\underline{u}}^*$, (6.2) by $\overline{\underline{v}}^*$, the complex conjugates of the adjoint velocities, and integrate the resulting expressions over their domain of definition. We assume periodicity in x with period L, and integrate by parts using periodicity, solenoidality and (6.3) to derive

$$\begin{aligned}
&\int_{\Omega_1} \{\rho\lambda\overline{\underline{u}}^* - \rho\,\hat{u}_1\,\frac{\partial\overline{\underline{u}}^*}{\partial x} + \rho\,\overline{u}_1^*\,\hat{u}_1'\underline{e}_y - 2\eta_1\,\mathrm{div}\,\underline{\underline{D}}\,[\overline{\underline{u}}^*]\}\cdot\underline{u}\,dxdy \\
&+\int_{\Omega_2} \{\rho\lambda\overline{\underline{v}}^* - \rho\hat{v}_1\,\frac{\partial\overline{\underline{v}}^*}{\partial x} + \rho\,\overline{v}_1^*\,\hat{v}_1'\,\underline{e}_y - 2\eta_2\,\mathrm{div}\,\underline{\underline{D}}\,[\overline{\underline{v}}^*]\}\cdot\underline{v}\,dxdy \\
&-\int_{\Omega_1} p\,\mathrm{div}\,\overline{\underline{u}}^* - \int_{\Omega_2} q\,\mathrm{div}\,\overline{\underline{v}}^* \\
&= -\int_0^L \{(-q+2\eta_2\,D_{22}\,[\underline{v}])\,\overline{v}_2^* - (-p + 2\eta_1\,D_{22}[\underline{u}])\overline{u}_2^* \\
&+ 2\eta_2\overline{v}_1^*\,D_{12}[\underline{v}] - 2\eta_2\,v_1\,D_{12}[\overline{v}^*] - 2\eta_2\,v_2\,D_{22}[\overline{\underline{v}}^*]
\end{aligned} \tag{6.8}$$

$$+ 2\eta_1 u_1 D_{12}[\underline{\bar{u}}^*] - 2\eta_1 \bar{u}_1^* D_{12}[\underline{u}] + 2\eta_1 u_2 D_{22}[\underline{\bar{u}}^*]\}dx \ .$$

By considering special forms of $\underline{u}$, $\underline{v}$, p, q, we find that in $0 < y < h_o$, we have

$$(6.9) \qquad \rho\lambda\underline{\bar{u}}^* - \rho\, \hat{u}_1 \frac{\partial \underline{\bar{u}}^*}{\partial x} + \rho\, \underline{e}_y\, \bar{u}_1^*\, \hat{u}_1' - 2\eta_1 \operatorname{div} \underline{\underline{D}}\,[\underline{\bar{u}}^*] = -\nabla\, \bar{p}^*,$$

$$\operatorname{div} \underline{\bar{u}}^* = 0,$$

whilst in $h_o < y < 1$,

$$(6.10) \qquad \rho\lambda\underline{\bar{v}}^* - \rho\, \hat{v}_1 \frac{\partial \underline{\bar{v}}^*}{\partial x} + \rho\, \underline{e}_y\, \bar{v}_1^*\, \hat{v}_1' - 2\eta_2 \operatorname{div} \underline{\underline{D}}\,[\underline{\bar{v}}^*] = -\nabla \bar{q}^*,$$

$$\operatorname{div} \underline{\bar{v}}^* = 0 \ .$$

We insert (6.9) and (6.10) back into (6.8), and compute

$$+ \int_{\Omega_1} \underline{u} \cdot \nabla\bar{p}^* dxdy + \int_{\Omega_2} \underline{v}\cdot\nabla\bar{q}^*\, dxdy$$

$$= \int_0^L (\bar{p}^*\, u_2 - \bar{q}^*\, v_2)\, dx \ .$$

This term is added to the right hand side of (6.8), leading to

$$(6.11) \qquad 0 = \int_0^L \{(-q+2\eta_2\, D_{22}\,[\underline{v}])\bar{v}_2^* + (p-2\eta_1\, D_{22}[\underline{u}])\bar{u}_2^*$$

$$+ (\bar{q}^* - 2\eta_2\, D_{22}[\underline{\bar{v}}^*])\, v_2 + (-\bar{p}^* + 2\eta_1\, D_{22}\, [\underline{\bar{u}}^*])u_2$$

$$+ 2\eta_2\, \bar{v}_1^*\, D_{12}[\underline{v}] - 2\eta_1\, \bar{u}_1^*\, D_{12}[\underline{u}]$$

$$+ 2\eta_1\, u_1\, D_{12}\, [\underline{\bar{u}}^*] - 2\eta_2\, v_1\, D_{12}[\underline{\bar{v}}^*]\ \}\ dx \ .$$

We use (6.6) to reduce the first line of (6.11), (6.4) for the second line and (6.5) for the third line. Thus we find

$$(6.12)\quad 0 = \int_0^L \{(-q+2\eta_2\, D_{22}\,[v])\,(\bar{v}_2^* - \bar{u}_2^*) - T\,\delta''\bar{u}_2^*$$

$$+ u_2(\bar{q}^* - 2\eta_2\, D_{22}[\bar{v}^*] - \bar{p}^* + 2\eta_1\, D_{22}[\underline{\bar{u}}^*])$$

$$+ (\bar{v}_1^* - \bar{u}_1^*)\, 2\eta_1\, D_{12}\,[\underline{u}]$$

$$+ (u_1 - v_1)\cdot 2\eta_1\, D_{12}[\bar{u}^*]$$

$$+ v_1(2\eta_1 D_{12}\,[\bar{u}^*] - 2\eta_2\, D_{12}[\underline{\bar{v}}^*])\}\, dx$$

We next write

$$\int_0^L \delta''\bar{u}_2{}^*\, dx = \int_0^L \delta u_2{}^{*''}\, dx,$$

and set $\delta = k(u_1 - v_1)$, $u_2 = (\lambda+\hat{u}(h_o)\frac{\partial}{\partial x})\, k\,(u_1-v_1)$.

This leads to

$$0 = \int_0^L \{(-q+2\eta_2\, D_{22}\,[v])(v_2^*-\bar{u}_2^*)$$

$$+(u_1-v_1) < -Tk\,\bar{u}_2{}^{*''} + (\lambda-\hat{u}(h_o)\frac{\partial}{\partial x})\, k\,(\bar{q}^* - 2\eta_2\, D_{22}\,[\bar{v}^*]$$

$$- \bar{p}^* + 2\eta_1\, D_{22}[\underline{\bar{u}}^*]) + 2\eta_1\, D_{12}\,[\underline{\bar{u}}^*] >$$

$$+ (\bar{v}_1^* - \bar{u}_1^*)\cdot 2\eta_1\, D_{12}\,[\underline{u}]$$

$$+ v_1\,(2\eta_1\, D_{12}[\underline{\bar{u}}^*] - 2\eta_2\, D_{12}\,[\underline{\bar{v}}^*])\,\}\, dx.$$

This yields the following four conditions on $y = h_o$:

$$(6.13)\quad 2\eta_1\, D_{12}[\underline{\bar{u}}^*] = 2\eta_2\, D_{12}[\bar{v}^*],$$

(6.14) $\bar{u}_1^* = \bar{v}_1^*$,

(6.15) $\bar{u}_2^* = \bar{v}_2^*$,

(6.16) $$-Tk\, \bar{u}_2^{*''} + (\lambda - \hat{u}(h_o) \frac{\partial}{\partial x})\, k\, (\bar{q}^* - 2\eta_2\, D_{22}[\bar{v}^*]$$

$$- \vec{p} + 2\eta_1\, D_{22}\, [\underline{\bar{u}}^*]) + 2\eta_1\, D_{12}\, [\underline{\bar{u}}^*] = 0.$$

Thus the adjoint problem is given by the differential equations (6.9), (6.10), the Dirichlet conditions $\underline{u}^* = 0$, $\underline{\bar{v}}^* = 0$ on the walls and conditions (6.13)-(6.16) on the interface.

It is easy to establish a necessary and sufficient condition for the solvability of the inhomogeneous problem corresponding to (6.1)-(6.7). Suppose that the zeros on the right of the first equation in (6.1) and (6.2) and on the right of (6.4)-(6.7) are replaced by

(6.17) $\underline{g}_1\ (x,y),\ g_2(x,y),\ g_3\ (x),\ g_4(x),\ g_5(x),\ g_6(x)$,

respectively. This inhomogenous problem is solvable if and only if the data (6.17) are orthogonal to the kernel of the adjoint, that is, when (6.17) is such that

(6.18) $$\int_{\Omega_1} \underline{g}_1 \cdot \underline{\bar{u}}^*\, dxdy + \int_{\Omega_2} \underline{g}_2 \cdot \underline{\bar{v}}^*\, dxdy$$

$$= + \int_o^L \{g_3(\bar{q}^* - 2\eta_2\, D_{22}[\underline{\bar{v}}^*]) + g_4\, \bar{u}_1^* + g_5\, \bar{u}_2^*$$

$$+ g_6(\bar{q}^* - 2\eta_2\, D_{22}[\bar{v}^*] - \bar{p}^* + 2\eta_1\, D_{22}\, [\underline{\bar{u}}^*])\}\, dx.$$

We are interested in the neighborhood of a critical point, where a loss of stability occurs. There is a critical plate velocity $V_o = \hat{V}_o$ such that the real part of λ vanishes, and $\lambda = i\omega_o$. We put

$$V_o = \hat{V}_o(1+R),$$

so criticality is when $R = 0$. We get Hopf bifurcation when the loss of stability is strict

$$\mathrm{Re}\,\frac{\partial\lambda}{\partial R}\bigg|_{R=0} \neq 0.$$

Let

$$\underline{\zeta}_o = \begin{cases} \underline{u} & \text{in } \Omega_1 \quad (y < h_o)\ , \\ \underline{v} & \text{in } \Omega_2 \quad (y > h_o) \end{cases}$$

be a right eigenvector satisfying (6.1)-(6.7) at criticality. Then $\bar{\underline{\zeta}}_o$ is the right eigenvector belong to $-i\omega_o$. $\bar{\underline{\zeta}}_o{}^*$ and $\underline{\zeta}_o{}^*$ are the left eigenvectors belonging to $i\omega_o$ and $-i\omega_o$.

7. Domain perturbations and Hopf bifurcation

We have already demonstrated that Couette flow can bifurcate into a time-periodic solution in which we have travelling interfacial waves. To compute this solution we would, following Lindstedt, map the solution into a fixed frequency domain (2π periodic in s)

$$\omega dt = ds,$$

and replace

$$(7.1) \qquad (\dot{\underline{u}},\ \dot{\underline{v}},\ \dot{\delta}) \ \text{ with } \ \omega\left(\frac{\partial \underline{u}}{\partial s},\ \frac{\partial \underline{v}}{\partial s},\ \frac{\partial \delta}{\partial s}\right)$$

in (5.3), (5.4) and (5.10). We then map our problem into a fixed spatial domain, using a one to one linear mapping, which takes boundary points into boundary points

$$(7.2)_1 \qquad y = (h(x,t)-h_o)\,\frac{y_o-1}{h_o-1} + y_o \quad \begin{cases} h \leq y \leq 1, \\ h_o \leq y_o \leq 1, \end{cases}$$

and

$$(7.2)_2 \qquad y = \left(1 + \frac{h(x,t)-h_o}{h_o}\right) y_o \quad \begin{cases} 0 \leq y \leq h(x,t), \\ 0 \leq y_o \leq h_o\ . \end{cases}$$

We then change variables, putting $x = x_o$ and $y = \tilde{y}(x_o,y_o)$, where $\tilde{y}$ is defined by (7.2) in equations (5.3)-(5.10). The form of these equations changes under the change of variables. However, following ideas introduced by Joseph [18], [19] we find many simplifications. We shall now explain these simplifications.

First we introduce an amplitude parameter which is conveniently set as a projection

$$\varepsilon = [\, \underline{u},\ \underline{z}^*],$$

where

$$[\underline{a},\underline{b}] \overset{\text{def}}{=} \frac{1}{2\pi} \int_0^{2\pi} \langle\, \underline{a},\underline{b} \,\rangle \, ds,$$

and $\langle \underline{a},\ \underline{b} \rangle$ are intergrals over both regions of the type displayed in (6.8). We are working in the frame of § VIII.3 of Iooss-Joseph [17], and

$$(7.3) \qquad \underline{z}^* = e^{is}\, \underline{\zeta}_o^*,$$

where $\underline{\zeta}_o^*$ is the left eigenvector at criticality which was introduced at the end of the last section.

The bifurcating solution may be computed in a series of powers of ε. Thus

$$(7.4) \qquad \omega(\varepsilon) = \omega_o + \frac{\varepsilon^2}{2}\omega_2 + \frac{\varepsilon^4}{4!}\omega_4 + \ldots,$$

$$(7.5) \qquad V_o(\varepsilon) = \hat{V}_o(1+ \frac{\varepsilon^2}{2}\hat{V}_2 - \frac{\varepsilon^4}{4!}\hat{V}_4 + \ldots) .$$

It follows from the classical theory of Hopf bifurcation that ω and V_o are even functios of ε. we have assumed this in writing (7.4) and (7.5). Moreover,

$$(7.6) \qquad \begin{matrix} \underline{u}\,(x,y,s,\varepsilon) \\ \underline{v}\,(x,y,s,\varepsilon) \\ p\,(x,y,s,\varepsilon) \\ q\,(x,y,s,\varepsilon) \end{matrix} = \sum_{l=0}^{\infty} \frac{\varepsilon^{\ell+1}}{(\ell+1)!} \begin{matrix} \underline{u}^{[\ell]}(x_o,y_o,s) \\ \underline{v}^{[\ell]}(x_o,y_o,s) \\ p^{[\ell]}(x_o,y_o,s) \\ q^{[\ell]}(x\ ,y\ ,s) . \end{matrix}$$

The functions of x and y are defined in deformed domains with a wavy interface $h(x,s,\varepsilon) - h_o = \delta(x,s,\varepsilon)$. The functions of x_o and y_o was defined in the reference domain with a flat interface at $y_o = h$. The perturbation of the interface $\delta(x,s,\varepsilon)$ can be eliminated from (5.8); that is,

$$\delta(x,s,\varepsilon) = k\ (u_1 - v_1)(x,s,\varepsilon)$$

is an identity for all x,s,ε. The square brackets on the left of (7.6) indicate differentiation following the mapping evaluated at $\varepsilon = 0$. For example,

$$\underline{u}^{[n]}(x_o,y_o,s) = \frac{d^n}{d\varepsilon^n} u(x,y,s,\varepsilon)\Big|_{\varepsilon = 0}$$

(7.7)

$$= \frac{\partial^n}{\partial\varepsilon^n} u(x_o,y(x_o,y_o,s,\varepsilon),s,\varepsilon).$$

There is a simple differential calculus for domain perturbations. The partial derivatives, holding x, y fixed, at $\varepsilon = 0$ are also important. For these

$$\text{(7.8)} \quad \underline{u}^{\langle n\rangle}(x_o,y_o,s,\varepsilon) = \left[\frac{\partial^n}{\partial\varepsilon^n} u(x,y,s,\varepsilon)\right]_{\varepsilon=0} .$$

The two types of derivatives are related by the chain rule

$$\underline{u}^{[1]} = \underline{u}^{\langle 1\rangle} + y^{\langle 1\rangle}\frac{\partial \underline{u}^{\langle 0\rangle}}{\partial y} ,$$

$$\text{(7.9)} \quad \underline{u}^{[2]} = \underline{u}^{\langle 2\rangle} + 2y^{\langle 1\rangle}\frac{\partial \underline{u}^{\langle 1\rangle}}{\partial y} + (y^{\langle 1\rangle})^2 \frac{\partial^2 \underline{u}^{\langle 0\rangle}}{\partial y^2} + y^{\langle 2\rangle}\frac{\partial \underline{u}^{\langle 0\rangle}}{\partial y} ,$$

$$\underline{u}^{[n]}(x_o,y_o,s) = \underline{u}^{\langle n\rangle}(x_o,y_o,s) + \text{lower order terms},$$

where

$$\text{(7.10)} \quad y^{\langle n\rangle}(x_o,y_o,s) = \begin{cases} \delta^{\langle n\rangle}(x_o,s)\dfrac{y_o-1}{h_o-1} , & h_o < y_o < 1, \\[2ex] \delta^{\langle n\rangle}(x_o,s)\dfrac{y_o}{h_o} , & 0 < y_o < h_o . \end{cases}$$

On the free surface $y_o = h_o$ of the reference domain, we have

$$\text{(7.11)} \quad y^{\langle n\rangle}(x_o,h_o,s) = \delta^{\langle n\rangle}(x_o,s).$$

The equations governing the coefficients in (7.6) are very complicated because the differential operators with derivatives with respect to x and y in the field equations must be reexpressed by derivatives with respect to x_o, y_o under the change of variables

$x, y \to x_o, y_o$. Since this mapping is invertible we can continue (7.6) as

$$(7.12)\qquad \begin{matrix} \underline{u}(x,y,s,\varepsilon) \\ \underline{v}(x,y,s,\varepsilon) \\ \text{etc}. \end{matrix} \;=\; \sum_{\ell=o}^{\infty} \frac{\varepsilon^{\ell+1}}{(\ell+1)!} \; \begin{matrix} \underline{u}^{[\ell]}(x,y_o(x,y,s,\varepsilon),s) \\ \underline{v}^{[\ell]}(x,y_o(x,y,s,\varepsilon),s) \\ \text{etc}. \end{matrix}$$

Fortunately it is never necessary to solve the complicated equations which govern the derivatives $[\ell]$ on the right of (7.6). In fact we need only to do much simpler calculations for the partial derivatives $\langle\ell\rangle$. When the partial derivatives $\langle\ell\rangle$ are known the total derivatives $[\ell]$ may be computed by the chain rule (7.9). The point of simplicity of partial derivatives is that they don't perturb the operators which are defined on region Ω_1 and Ω_2, below and above the free surface, see [18] and [19]. For example,

$$(7.13)\qquad \operatorname{div} \underline{u}^{\langle n\rangle}(x_o,y_o,s) = \frac{\partial u_1^{\langle n\rangle}}{\partial x_o} + \frac{\partial u_2^{\langle n\rangle}}{\partial y_o} = 0,$$

whereas

$$\operatorname{div} \underline{u}^{[n]}(x_o,y_o,s) \neq 0 .$$

The same type of simplication holds for the perturbation equations which arise from (7.1), (5.3) and (5.4). For example,

$$(7.14)\qquad \rho\, [\omega_o \frac{\partial \underline{u}^{\langle 2\rangle}}{\partial s} + \hat{u}_1 \frac{\partial \underline{u}^{\langle 2\rangle}}{\partial x} + \underline{e}_x\, u_2^{\langle 2\rangle}\, \hat{u}_1' + \hat{u}_1^{\langle 2\rangle} \frac{\partial \underline{u}^{\langle 0\rangle}}{\partial x} + \underline{e}_x\, u_2^{\langle 0\rangle}\, \hat{u}_1'^{\langle 2\rangle} + \underline{u}^{\langle 1\rangle}\cdot\nabla\, \underline{u}^{\langle 1\rangle}$$
$$+ \omega_2 \frac{\partial \underline{u}^{\langle 0\rangle}}{\partial s}] = \eta_1\, \Delta \underline{u}^{\langle 2\rangle} - \nabla p^{\langle 2\rangle}.$$

The unknowns here are $\underline{u}^{\langle 2\rangle}$, $p^{\langle 2\rangle}$, ω_2 and $\hat{v}_2$.

It is not possible to avoid the total derivatives $[\ell]$ in the equations (5.6)-(5.10) because these are defined on a manifold, the interface, and not in a region. The interface conditions are of the form

(7.15) $\quad g(x, y = h(x,s,\varepsilon),\ s,\ \varepsilon) = 0$

and the perturbation of $y = h$ with ε cannot be avoided. In fact the interface conditions are identities on the interface so that tangential derivatives on them vanish (see [19]).

The following is a recipe for perturbations of the domain in bifurcation problems. First, we introduce the frequency $\omega(\varepsilon)$ into (5.3), (5.4) and (5.10) using (7.1). We then insert the series (7.4), (7.5) and the series

$$\begin{array}{lcl} \underline{u}(x,y,s,\varepsilon) & & \underline{u}^{\langle \ell \rangle}(x_o,y_o,s), \\ \underline{v}(x,y,s,\varepsilon) & & \underline{v}^{\langle \ell \rangle}(x_o,y_o,s), \\ p(x,y,s,\varepsilon) & = \sum\limits_{\ell=0} \dfrac{\varepsilon^{\ell+1}}{(\ell+1)!} & p^{\langle \ell \rangle}(x_o,y_o,s), \\ q(x,y,s,\varepsilon) & & q^{\langle \ell \rangle}(x_o,y_o,s), \\ \delta(x,s,\varepsilon) & & \delta^{\langle \ell \rangle}(x_o,s) \end{array} \tag{7.16}$$

into div $\underline{u}$ = div $\underline{v}$ = 0, (5.3), (5.4) and (5.5) and identify the perturbation equations. These equations hold in the reference domain. To get the equations which govern the interface conditions (5.6)-(5.10) we insert the series (7.6) and identify. Then we express the derivatives $[\ell]$ with partial derivatives $\langle \ell \rangle$, using the chain rule. The perturbation equations arising from interface conditions are thus defined on the flat interface $y = h_o$.

The series on the right of (7.16) may be expressed on the deformed domain by inverting the mapping, as in (7.12). In fact, the series on the right of (7.16) is equal to the series on the right of (7.6), though the partial sums of these two series are not equal (see equations of [18]).

8. Solvability of the perturbation equations

We must solve perturbation problems of the following form

(i) All functions of s are 2π periodic in s

(ii) All functions of $x = x_o$ are L periodic in x_o

(iii) In the region $0 < y_o < h_o$,

$$\frac{\partial u_1^{<n>}}{\partial x_o} + \frac{\partial u_2^{<n>}}{\partial y_o} = 0, \tag{8.1}$$

$$\rho[\omega_o \frac{\partial u_1^{<n>}}{\partial s} + \hat{u}\frac{\partial \underline{u}^{<n>}}{\partial x_o} + \underline{e}_x u_2^{<n>} \hat{u}'] - \eta_1 \Delta \underline{u}^{<n>} + \nabla p^{<n>} \tag{8.2}$$

$$= \theta_1(\omega_n, \hat{v}_n, x_o, s).$$

(iv) In the region $h_o < y_o < 1$ we have the same equations with $\underline{v}^{<n>}(x_o, y_o, s)$, $\hat{v}(y_o)$, $q^{<n>}$ and $\underline{\theta}_2$ replacing $\underline{u}^{<n>}$, $\hat{u}$, $p^{<n>}$ and $\underline{\theta}_1$.

(v) $u^{<n>} = 0$ at $y_o = 0$, $v^{<n>} = 0$ at $y_o = 1$.

(vi) On the interface $y = h_o + \delta$, we have by (5.8)

$$\delta = k [\![u_1]\!], \quad [\![u_1]\!] \stackrel{\text{def}}{=} u_1 - v_1. \tag{8.3}$$

We have eliminated δ from the interface equations (5.6), (5.7) and (5.10) with

Of course

$$\delta^{<n>} = k [\![u_1^{<n>}]\!] \quad \text{on} \quad y_o = h_o$$

(vii) The four interface conditions on $y_o = h_o$ are of the form

$$u_2^{<n>} - v_2^{<n>} = \theta_3(\hat{v}_n, x_o, s),$$

$$\eta_1 D_{12}[\underline{u}^{<n>}] - \eta_2 D_{12}[\underline{v}^{<n>}] = \theta_4(\hat{v}_n, x_o, s),$$

$$- p^{<n>} + \eta_1 D_{22}[\underline{u}^{<n>}] + q^{<n>} - \eta_2 D_{22}[\underline{v}^{<n>}]$$

$$- kT \frac{\partial^2}{\partial x_0^2} [\![u_1^{<n>}]\!] = \theta_5(\hat{v}_n, x_o, s),$$

$$k\left(\omega_o \frac{\partial}{\partial s} + \hat{u}_1(k_o) \frac{\partial}{\partial x_o}\right)[\![u_1^{\langle n\rangle}]\!] - u_2^{\langle n\rangle} = \theta_6(\omega_n, \hat{V}_n, x_o, s).$$

The inhomogeneous terms θ are linear in the unknown parameters ω_n and $\hat{V}_n$ and are otherwise known from computations at orders $\ell < n$.

These inhomogeneous problems can be solved uniquely in the space orthogonal to the null space of the adjoint operator introduced at the begining of §6. This null space is two dimensional and is spanned by

$$\underline{z}^* \quad \text{and} \quad \overline{\underline{z}}^*$$

defined by (7.3) and explained in §VIII. 3 of Iooss & Joseph. There are therefore two solvability conditions to be used in the determination of the parameters ω_n and $\hat{V}_n$.

REFERENCES

[1] S. Agmon, A. Douglis and L. Nirenberg, Estimates near the boundary for solutions of elliptic partial differential equations satisfying general boundary conditions, Comm. Pure Appl. Math. 12 (1959), 623-727 and 17 (1964), 35-92.

[2] J. T. Beale, Large time regularity of viscous surface waves, Arch. Rational Mech. Anal. 84 (1984), 307-352.

[3] J. T. Beale, The initial value problem for the Navier-Stokes equations with a free surface, Comm. Pure Appl. Math. 34 (1980), 359-392.

[4] M. G. Crandall and P. H. Rabinowitz, The Hopf bifurcation theorem in infinite dimensions, Arch. Rational Mech. Anal. 67 (1978), 53-72.

[5] C. E. Hickox, Instability due to viscosity and density stratification in axisymmetric pipe flow, Physics of Fluids 14 (1971), 251-262.

[6] A. P. Hooper and W. G. C. Boyd, Shear-flow instability at the interface between two viscous fluids, J. Fluid Mech. 128 (1983), 507-528.

[7] E. Hopf, Abzweigung einer periodischen Lösung eines Differentialsystems, Ber. Sächs. Akad. Wiss., Math-nat. Kl. 94 (1942), 1-22.

[8] G. Iooss, Existence et stabilité de la solution périodique secondaire intervenant dans les problèmes d'évolution du type Navier-Stokes, Arch. Rational Mech. Anal. 47 (1972), 301-329.

[9] V. I. Iudovich, The onset of auto-oscillations in a fluid, J. Appl. Math. Mech. 35 (1971), 587-603.

[10] D. D. Joseph and D. H. Sattinger, Bifurcating time periodic solutions and their stability, Arch. Rational Mech. Anal. 45 (1972), 75-109.

[11] D. D. Joseph, M. Renardy and Y. Renardy, Instability of the flow of two immiscible liquids with different viscosities in a pipe, J. Fluid Mech. 141 (1984), 309-317.

[12] T. Kato, Perturbation Theory for Linear Operators, Springer, 2nd ed. 1976.

[13] Y. Renardy and D. D. Joseph, Two-fluid Couette flow between concentric cylinders, submitted to J. Fluid Mech.

[14] D. H. Sattinger, Bifurcation of time periodic solution of the Navier-Stokes equations, Arch. Rational Mech. Anal. 41 (1971), 66-80.

[15] R. Temam, Navier-Stokes Equations, North Holland, 1979.

[16] C. S. Yih, Instability due to viscosity stratification, J. Fluid Mech. 27 (1967), 337-352.

[17] G. Iooss and D. D. Joseph, Elementary Stability and Bifurcation Theory, Springer 1980.

[18] D. D. Joseph, Domain perturbations. The higher order theory of infinitesimal water waves, Arch. Rational Mech. Anal. 51 (1973), 295-303.

[19] D. D. Joseph, Free surface problems in rheological fluid mechanics, Rheol. Acta 16(1977), 169-189.

MR/DDJ;jgb

SECURITY CLASSIFICATION OF THIS PAGE (When Data Entered)

REPORT DOCUMENTATION PAGE		READ INSTRUCTIONS BEFORE COMPLETING FORM
1. REPORT NUMBER 2689	2. GOVT ACCESSION NO. AD-A142	3. RECIPIENT'S CATALOG NUMBER 844
4. TITLE (and Subtitle) HOPF BIFURCATION IN TWO-COMPONENT FLOW		5. TYPE OF REPORT & PERIOD COVERED Summary Report - no specific reporting period
		6. PERFORMING ORG. REPORT NUMBER
7. AUTHOR(s) M. Renardy and D. D. Joseph		8. CONTRACT OR GRANT NUMBER(s) DAAG29-80-C-0041, MCS-7927062, Mod. 2, MCS-8215064, and DAAG29-82-K-0029
9. PERFORMING ORGANIZATION NAME AND ADDRESS Mathematics Research Center, University of Wisconsin, 610 Walnut Street, Madison, Wisconsin 53706		10. PROGRAM ELEMENT, PROJECT, TASK AREA & WORK UNIT NUMBERS Work Unit Number - 1 Applied Analysis
11. CONTROLLING OFFICE NAME AND ADDRESS See Item 18		12. REPORT DATE May 1984
		13. NUMBER OF PAGES 27
14. MONITORING AGENCY NAME & ADDRESS(if different from Controlling Office)		15. SECURITY CLASS. (of this report) UNCLASSIFIED
		15a. DECLASSIFICATION/DOWNGRADING SCHEDULE

16. DISTRIBUTION STATEMENT (of this Report)

Approved for public release; distribution unlimited.

17. DISTRIBUTION STATEMENT (of the abstract entered in Block 20, if different from Report)

18. SUPPLEMENTARY NOTES

U. S. Army Research Office
P. O. Box 12211
Research Triangle Park
North Carolina 27709

National Science Foundation
Washington, D.C. 20550

19. KEY WORDS (Continue on reverse side if necessary and identify by block number)

Two-Component Flow, Hopf bifurcation

20. ABSTRACT (Continue on reverse side if necessary and identify by block number)

The stability of viscosity-stratified bicomponent flow has been studied by long wave asymptotics, by short wave asymptotics and numerically. These studies have shown that interfacial instabilities arise from the viscosity difference between the two fluids. If the surface tension between the fluids is non-zero, then Hopf type bifurcations leading to traveling interfacial waves are expected. In this paper, we prove a rigorous theorem establishing the existence of bifurcating solutions of this nature.

DD FORM 1 JAN 73 1473 EDITION OF 1 NOV 65 IS OBSOLETE

UNCLASSIFIED

SECURITY CLASSIFICATION OF THIS PAGE (When Data Entered)

END

FILMED

8 - 84

DTIC