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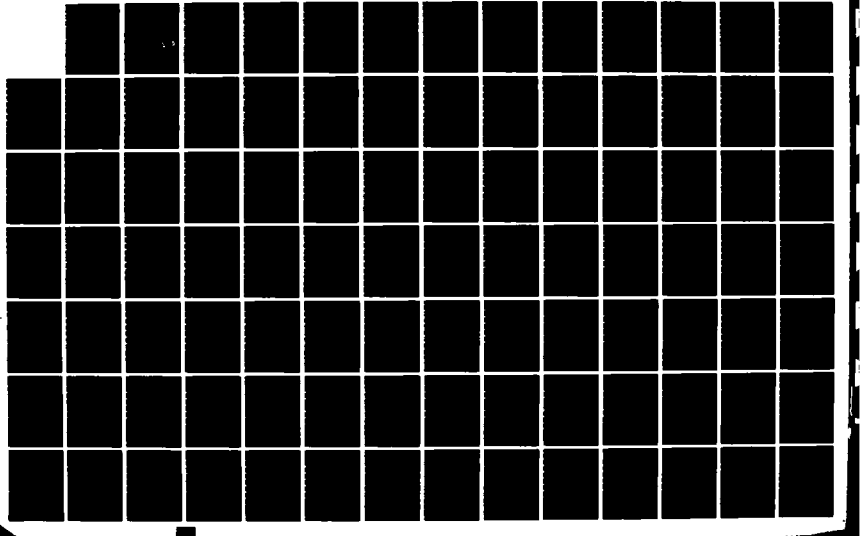

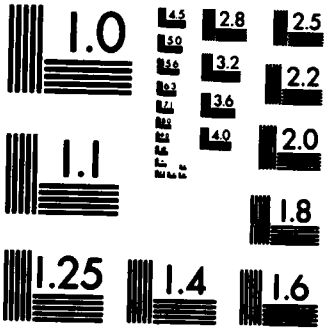

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AD-A142 831

FINAL REPORT ON

"HIGH-SPEED LOW-COST WAYS TO GET MESSAGES FROM A SENDER TO A RECEIVER WHEN SOME CHANNELS LINKING THEM BECOME INOPERATIVE."

REPORT DOCUMENTATION PAGE

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'HIGH-SPEED LOW-COST WAYS TO GET MESSAGES FROM A SENDER TO A RECEIVER WHEN /SOME CHANNELS SENDING THEM BECOME UNOPERATIVE'

Blakley, Bob

Final	FROM 83 Sep 30 TO 84 Mar 31	84 May 28	150

09 04 17 02	Science, Engineering, AFOSR SBIR, Electronic Technology, Electronic Component Reliability, Digital Communications, (See continuation on reverse side.)

This AFOSR SBIR Phase I Project produced explicitly the hyperfast pool/split/r[illegible] algorithms of the Bloom technique. These algorithms, once they obtain hardware implementation, will be used as follows. They will make it possible for a sender to get all desired digital information to a receiver by coding it for transmission over several parallel channels in such a way that decoding will recover everything sent even when up to a predetermined number of channels fail. This project developed a set of design principles for hardware implementation of such p/s/r processes by means of existing microprocessor chips of sizes 4, 8 and 16 bits.

C³I, Computer Science, Jamming, Antijam, Red Noise, Mathematics, Information Theory, Error Control Codes, Threshold Schemes, Hyperfast Bloom Processes, Pool/split/restitute Processes, Algorithms, Finite Fields, Galois Fields, Vandermonde Matrices, Reed-Solomon Codes, Vector Spaces, Linear Transformations, Gate Array, Programmable Logic Array, Parallel Processing, Microprocessors, Packet Switching, Chip Design, Fiber Optic Communications, Spread Spectrum, Message Gap, Distributed Processing, Netted Communications Systems, VLSI Design, Reliability, Survivability, Endurability, Responsiveness, Space Systems Technology, Distributed Communications, Radio Frequency Communications, High Bandwidth Communications, Channel Failure, Channel Fading, Sending Node, Receiving Node, Bandwidth Expansion, Novel Architecture, Fault Tolerance, Fail Safe Communications, Efficient Spectrum Utilization, Cost Effectiveness, Telecommunications Networks, Local Area Networks, Low Computational Complexity, Encode, Decode, Redundancy, Minimum Redundancy, Parallelism, Parallel Computations, Systolic Processors, Minimum Message Expansion.

BOX 11. TITLE (Include Security Classification)

UNCLASSIFIED: FINAL REPORT ON "High-speed low-cost ways to get messages from a sender to a receiver when some channels linking them become inoperative."

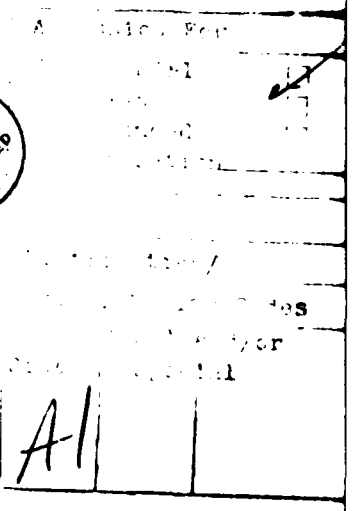
0. Introduction

This is the final report on USAF contract number F49620-83-C-0160, effective date 83 Sep 30, project number FQ 8671-8301504 3005/A1, duration 83 Sep 30 through 84 Mar 31.

The work described in the YLYK Ltd. proposal (see Appendix A) which led to this SBIR contract was completed on time, and within budget. Moreover the outcome was largely definitive and was at least as good as the target outcome of the proposal. This report is prepared to meet the 84 May 31 deadline for final report. In summary, the work was carried out on time, on target, within budget. This report is timely.

In the interests of readability this final report is organized as follows:

0. Introduction
1. Overview Narrative
2. Detailed Narrative
3. Summary of tasks, work, discoveries, recommendations and alternatives
4. Future
5. Appendices
   - A. The technical part of the YLYK Ltd. proposal which led to this contract
   - B. Tables of GF(2↑N) arithmetic
   - C. Selected tables of Vandermonde matrices
   - D. Tables of ENF (encode normal forms) produced by cold precomputations
   - E. Examples of the encode/decode process
   - F. Copy of Yeh/Reed/Truong paper on systolic multipliers for finite fields
   - G. Copy of Bloom paper on threshold schemes
   - H. Program for encoding procedure (including Stages 1, 2 and 4)
   - I. Program for decoding procedure (including Stages 1, 2, 3 and 4)



Section 3, "Summary of tasks, work, discoveries, recommendations and alternatives" is the heart of the report. It describes how YLYK Ltd. performed its agreed-upon Task 1 and Task 2. The reader may want to skim it before going through the report as a whole.

1. Overview Narrative

1.1 Red Noise

Appendix A contains a copy of the YLYK Ltd. proposal which led to the contract on which this is the final report. In the interests of readability we restate the idea behind the p/s/r processes, along with some realistic instances.

A sender S and a receiver R are linked by n channels of approximately equal capacity. All communications are digital, i.e. are strings of bits (0 or 1). The sender and the receiver anticipate traumas which will inactivate some of these channels. Nevertheless both the sender and the receiver expect at least k of the n channels to continue to function. Here, as everywhere, it is assumed that $k \leq n$.

They face the "red noise" problem. How does the sender S encode k channels worth of information for sending along n channels to the receiver R in such a way that R can recover all the information cheaply and quickly as long as any k of the n channels remain operative? The sender S must encode in ignorance of which k channels will survive the trauma and remain operative. Examples of the red noise problem are numerous. We sketch out a few here. We will return to them.

i. On-chip. Certain elements on a chip may fail permanently. The number n of channels is typically less than 100, often less than 10. Here k is usually almost as big as n, since chips with lots of hard failures are typically discarded. Perhaps $k = n - 1$ is especially important.

ii. Packet switching. Here the packets are the "channels". Occasionally a packet is destroyed or irrevocably misrouted. The number n of total packets for many practical examples would be less than 200, often less than 20. Most packets should arrive intact, so k would usually be near n. Perhaps $k = n - 1$ is especially important.

iii. Spread spectrum. Here a "channel" might be a frequency if the technique employed is frequency hopping. Perhaps quite a few frequencies are jammed. The number n of total frequencies should usually be less than 60,000 and often considerably less than 4,000. k can vary all over the lot. In battle conditions we might have $k < n/10$, e.g. only $k = 70$ "clean" frequencies among $n = 1,000$ frequencies being used. Those who feel that this is a pessimistic estimate should consult McEliece's recent paper on jamming in Longo's Springer-Verlag book, Secure Digital Communications.

iv. Hard wires or fibers. A control center on a weapons platform (plane, ship, etc.) might be connected by $n = 30$ parallel fibers to a propulsion unit, sensor, control surface, or weapons pod. It might be desirable to maintain full communication even after 20 fibers were cut. Here $k = 10 = n/3$. In such examples n less than 200 seems plausible. k can vary all over the lot.

v. Multiple channels between manned centers. A city might talk to a command post via a mixture of twisted pairs, fibers, microwave relay paths and satellite links. It would be desirable to keep up communication if half of the $n = 20$ channels joining them fail.

In all the foregoing examples the number n of total channels before failures occur would satisfy the inequality

$$2\uparrow 0 = 1 \leq n \leq 65,536 = 2\uparrow 16$$

(where we use the ALGOL arrow notation $2\uparrow 16$ instead of the older exponent notation 2^{16}). We will adopt the inequality above once for all as an explicit assumption :

At least one "channel";

At most 65,536 "channels".

The reader is asked to bear it in mind everywhere below. Another categorical assumption is:

Every signal is digital.

1.2 Bloom pool/split/restitute processes

A solution to the red noise problem is called a p/s/r process. We will discuss only Bloom p/s/r processes and their close relatives here. See Appendix G for the first exposition of the idea behind Bloom threshold schemes and p/s/r processes. They make use of many of the ideas which arise in Reed-Solomon error control codes. But we will not explicitly pursue any resemblances to the latter structure.

The idea behind a k-out-of-n Bloom p/s/r process is to enable a sender to use finite field arithmetic and linear algebra to smear k channels worth of information into n channels worth of transmission to a receiver R in such a way that all the original information can be quickly reclaimed from the outputs of any k of the n channels, even if n - k of them do not carry any information to the receiver (i.e. even if n - k of the n channels are inoperative).

Bloom's approach to building a k-out-of-n p/s/r process makes use of a field F containing at least n elements, and a k dimensional vector space V over F. It is easy to verify that there is at least one collection

$$B = \{B(1), B(2), \ldots, B(n)\} \subseteq V$$

of n vectors in general position in V (meaning that every k-member subset of B is a basis of V). Sender S and receiver R agree on one such B and refer everything to it. Given a list

$$I = (I(1), I(2), \ldots, I(k)) \subseteq F^k$$

of k pieces of information (i.e. k members of the field F) define a linear functional

$$L: V \to F$$

with the property that

$$L(B(j)) = I(j)$$

for each positive integer $j \leq k$. These k pieces of information provide a complete unique specification of the linear map L since

$$\{B(1), B(2), \ldots, B(k)\}$$

is a basis of V. But

$$\{B(w(1)), B(w(2)), \ldots, B(w(k))\}$$

is also a basis of V for any injection (one-to-one function)

$$w: \{1, 2, \ldots, k\} \to \{1, 2, \ldots, n\}$$

So you can reconstruct L, and therefore determine the list

$$I = (I(1), I(2), \ldots, I(k))$$

if you know the value of L at any k members

$$B(w(1)), B(w(2)), \ldots, B(w(k))$$

of the set V.

Now it is obvious how to encode and decode. To encode the list I, form L and send $L(B(j))$ down channel j for each positive integer $j \leq n$. To decode (i.e. to recover I from the signals received on any k of the n channels) form L and then determine

$$L(B(j)) = I(j)$$

for each positive integer $j \leq k$. This is possible since any $B(w(1)), B(w(2)), \ldots, B(w(k))$ make up a basis for V, and since a linear map L with domain V is determined by its values on a basis of V.

1.3 Making hyperfast Bloom p/s/r processes. Stages.

YLYK Ltd. set out to take this simple mathematical structure, the abstract Bloom p/s/r process, and produce an abstract design of a p/s/r process which would run very fast on very cheap hardware. In this Phase I SBIR effort no attempt was made to produce or design hardware. Rather, the purpose of the work was to produce an abstract design of a

system capable of operating at megabit per second rates and above. On the basis of this abstract design the hardware design should be possible with few or no further abstract considerations.

Roughly speaking, the problems to be overcome fall into 4 stages:

1. Cold precomputation. The cold precomputation must be done before the p/s/r equipment is built. These precomputations will not slow down system operation. It would be perfectly acceptable if they took months to perform. In fact they can be completed in a minute except in very large cases discussed below.

2. Cool precomputation. The cool precomputations take place each time sender S and receiver R agree on the k and the n for a session of communication using a k-out-of-n p/s/r process. The cool precomputations will involve a minor delay, probably causing no inconvenience. This delay will usually be less than a second in reasonable sized cases as noted below.

3. Hot precomputation. The hot precomputation takes place after some channels have gone down. The receiver determines which k channels are still operating. This amounts to finding out out which subset $B(w(1)), B(w(2)), \ldots, B(w(k))$ of B will be used. Since the communication session is ongoing, any delay here is undesirable. Either you lose information on the fly or you pay for a buffer to hold undecoded material until your decode goes on stream. Unfortunately the hot precomputations can take many milliseconds. It is doubtful that a significant further improvement over the scheme YLYK Ltd. has formulated is possible here.

4. Real-time on-line encode or decode. The real-time on-line encode or decode stage should be able to keep up with high bit rate inputs. In an "impedance matched" situation the computer clock should tick at least once per arriving bit. For example, consider a 5-out-of-9 p/s/r process. Suppose that each of the 5 operative channels carries a signal at 10 megabits per second and that the "matched computer clocks" in the decoding system therefore push the computer to perform 10 similar logical operations (such as XOR,

i.e. exclusive or, of 4-bit words) per microsecond. It would be desirable to produce decoded output on all 5 decoded plaintext channels at a rate of 10 megabits per second. It appears possible to achieve such throughput rates, but with a certain short lag time. For example, the 10 megabit per second decoded output might lag the received bit stream by 2 microseconds. In other words the decoded bit streams proceeds at the same rate as the received encoded bit streams. But the decoded streams lag the received encoded streams by a phase lag of 20 clock ticks, i.e. by 20 bits.

We must deal with each of these four computational stages separately. The first, the cold precomputation stage, is completely noncritical. Neither time nor memory is important as long as the needed output can be produced within months and does not consist of too many computer words. The second stage, the cool precomputation, is not very critical. Presumably it occurs in tranquil conditions while the sender S and the receiver R are agreeing on a k-out-of-n scheme. Days could elapse between the choice of k and n, and the time transmission starts. And almost always seconds will elapse. It is therefore unlikely that the procedure described below for cool precomputation will delay timely receipt of transmitted messages. Stage 3, the hot precompute, is usually the most critical. If it should take a second or more, one must decide whether to lose a lot of bits or spend money on buffers. Stage 3, therefore, requires extremely close attention. Stage 4, the real-time on-line decode, is crucial but not troublesome. There are ways to carry Stage 4 out at very high bit rates, given adequate hardware. There is a "phase lag" i.e. a lag of several bits between received input signal and decoded final signal. This lag can be reduced to a few microseconds in existing TTL logic. But reducing it to zero is an impossibility.

1.4. Making hyperfast Bloom p/s/r processes. Extreme cases of parameter settings.

So much for the four stages of computation. We turn now to parameter settings. How sensitive is a k-out-of-n p/s/r process to k and to n?

First let us dispense with the four extreme cases. These are the two trivial cases $k = 0$ or $k = n$ and the two easy but not completely trivial cases $k = 1$ or $k = n-1$. A 0-out-of-n p/s/r process is silly. No information sent on n channels produces no information received. No p/s/r coding is required. The n-out-of-n case is far from silly. It is the present state of affairs. Send a different message on each of n channels and hope they all get through. No p/s/r coding is required. The 1-out-of-n case is also easy to deal with without p/s/r coding. Send the same message on all channels and hope that at least one channel remains operative.

The (n-1)-out-of-n case is more interesting. It will also be important in some applications. Synchronize the channels. To p/s/r encode the information let the first $n-1$ channels transmit their messages unaltered. But at each time t, add (modulo 2) the bits on the first $n-1$ channels and send this sum (it, too, will be a bit) on the nth channel. To decode when one of the first $n-1$ channels, say the jth, fails you do as follows. If $i \in \{1, 2, \ldots, n\} \setminus \{j\}$ the decode transformation is the identity. The channel is carrying its message unaltered. But if $i = j$, just form the sum of the bits on channels $1, 2, \ldots, j-1, j+1, \ldots, n$. This will be what the jth channel would have carried if it were still operative. Note that the cold precomputation, cool precomputation and hot precomputation are nonexistent. The on line computation acts on a single bit from each channel. And, if implemented by fast hardware as indicated in Figure 1.4.1 below in the 7-out-of-8 case, the output bit rate is the same as the input bit rate, but with a lag of $3 = \log(8)$ bits (All logarithms in this report are the information theorist's logarithm to base 2).

For the first time we note a point which will be addressed more fully below. Encoding is a do-nothing operation on all plaintext channels (i.e. the first $n-1 = 7$ channels), and all plaintext channels remain synchronized. Encoding is a do-something operation on the 8th = nth channel. To keep all eight channels synchronized, the receiver must do something to <u>every</u> channel. In the 7-out-of 8 case this means 3 successive stages of adding 0 to what comes over every one of the first $n-1 = 7$ channels. A similar statement holds regarding the decode process.

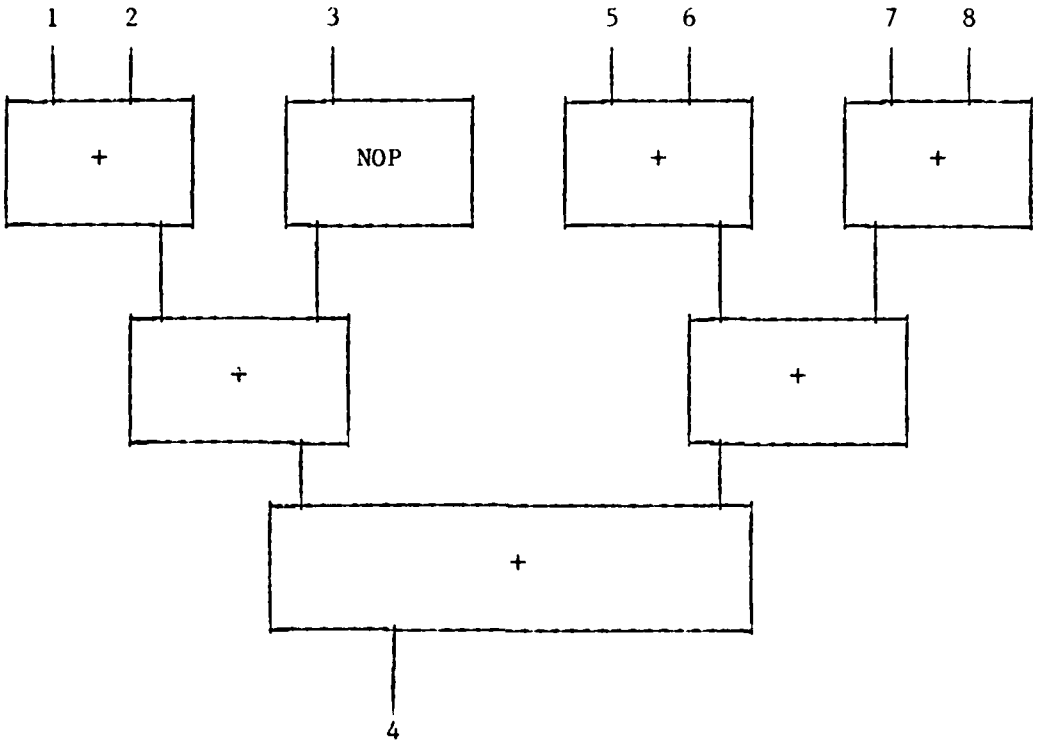


Figure 1.4.1.

The 7-out-of-8 p/s/r decode when channel 4 is inoperative. Assuming the modulo 2 adders (XOR) can operate as fast as bits are received the output bit stream will have the same speed as the input bit streams but will lag them by 3 bit positions. NOP means no operation. + stands for modulo 2 addition. Information flows downward.

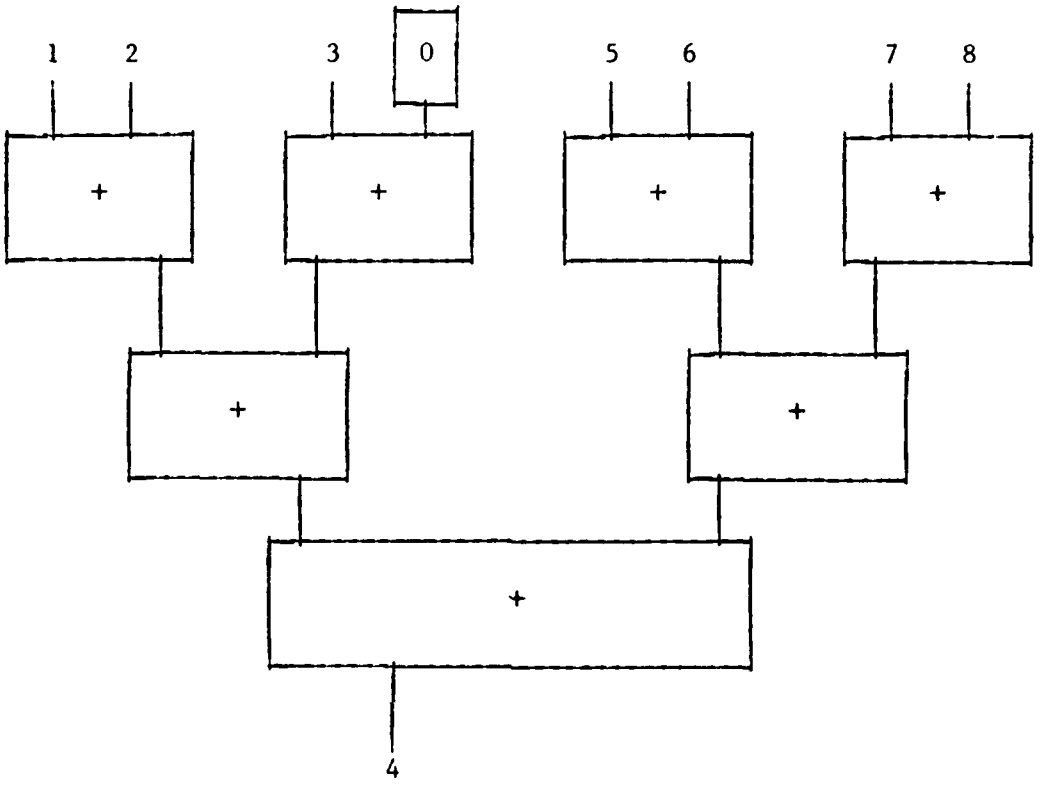


Figure 1.4.2

A variant of Figure 1.4.1. The receiver sends zeros into the decoder input corresponding to the missing channel 4. Information flows downward.

In each of the four extreme cases described in this subsection, the decode process could content itself with treating one bit at a time from each of the received channels. This is independent of n. Thus a very cheap programmable logic array (PLA) implementation of a hyperfast Bloom (n-1)-out-of-n p/s/r process is possible for very large n. The lag time would be about log(n).

1.5 Making hyperfast Bloom p/s/r processes. Mean parameter settings.

Turning now from the four extreme cases to all the other cases, which we shall call mean cases, we note that the p/s/r processes we are dealing with always satisfy the inequalities

$$2 \leq k \leq n - 2 \leq n \leq 65,536 .$$

No mean p/s/r encode or decode can deal with just one bit at a time. In fact one must deal with "words" of length at least log(n) from each channel. Recall that all logarithms are the information theorist's logarithm to base 2 in this report. As noted in the YLYK Ltd. proposal to Air Force for this Phase I SBIR proposal, encode and decode will be done using $GF(2^{\dagger}Q)$ arithmetic. As noted above, we will deal only with $Q \leq 16$. We have already discussed the extreme (n-1)-out-of-n case. This extreme case can be dealt with using GF(2) arithmetic. In dealing with mean cases we will usually make the following assumption:

$$Q \in \{4, 8, 12, 16\}.$$

Thus we will often deal just with the arithmetic of GF(16), GF(256), and GF(65,536). The reason for this is that 4, 8 and 16 bit words are natural objects to manipulate on standard hardware.

A case could be made for using only GF(65,536), i.e. for sticking to 16 bit words for standardization, since such an implementation can "do everything". But this size seems unwieldy at present. It may be better to try to get as much mileage as possible out of the GF(256) case, i.e. to try to get by with at most 256 transmitted channels. We will discuss some pros and cons later.

1.6 Making hyperfast Bloom p/s/r processes. Stage 4. Real-time on-line encode or decode.

In the mean cases of parameter settings one thing that does not change with parameter setting is the nature of the real-time on-line encode or decode in a superfast Bloom k-out-of-n p/s/r process. It is matrix multiplication. Encode is so like decode that we will concentrate on the latter in this section.

To each k-element subset

$$B^* = \{B(w(1)),\ B(w(2)),\ \ldots,\ B(w(k))\}$$

of

$$B = \{B(1),\ B(2),\ \ldots,\ B(n)\}$$

there corresponds a k by k matrix $DEC[B^*]$ such that

$$B(i) = \sum DEC[B^*](i,j)B(w(j))$$

for every positive integer $i \leq k$. The sum is over all positive integers $j \leq k$. As long as a given collection

$$\{w(1),\ w(2),\ \ldots,\ w(k)\}$$

of channels is operative this square matrix $DEC[B^*]$ is unchanging. So the block diagram for decoding the ith channel is contained in Figure 1.6.1 below. For illustrative purposes Figure 1.6.1 describes a 7-out-of-25 Bloom p/s/r process in which the receiver knows that channels 1, 2, 5, 7, 10, 12 and 19 are operative. Since $25 \leq 32 = 2\uparrow 5$ we can use 5-bit words, i.e. GF(32) arithmetic. So the 7 inputs to the decoder at time t are WORD1 on channel 1, WORD2 on channel 2, WORD5 on channel 5, ..., WORD19 on channel 19. Here of course

$$\begin{aligned} w(1) &= 1 \\ w(2) &= 2 \\ w(3) &= 5 \\ &\vdots \\ w(7) &= 19 \end{aligned}$$

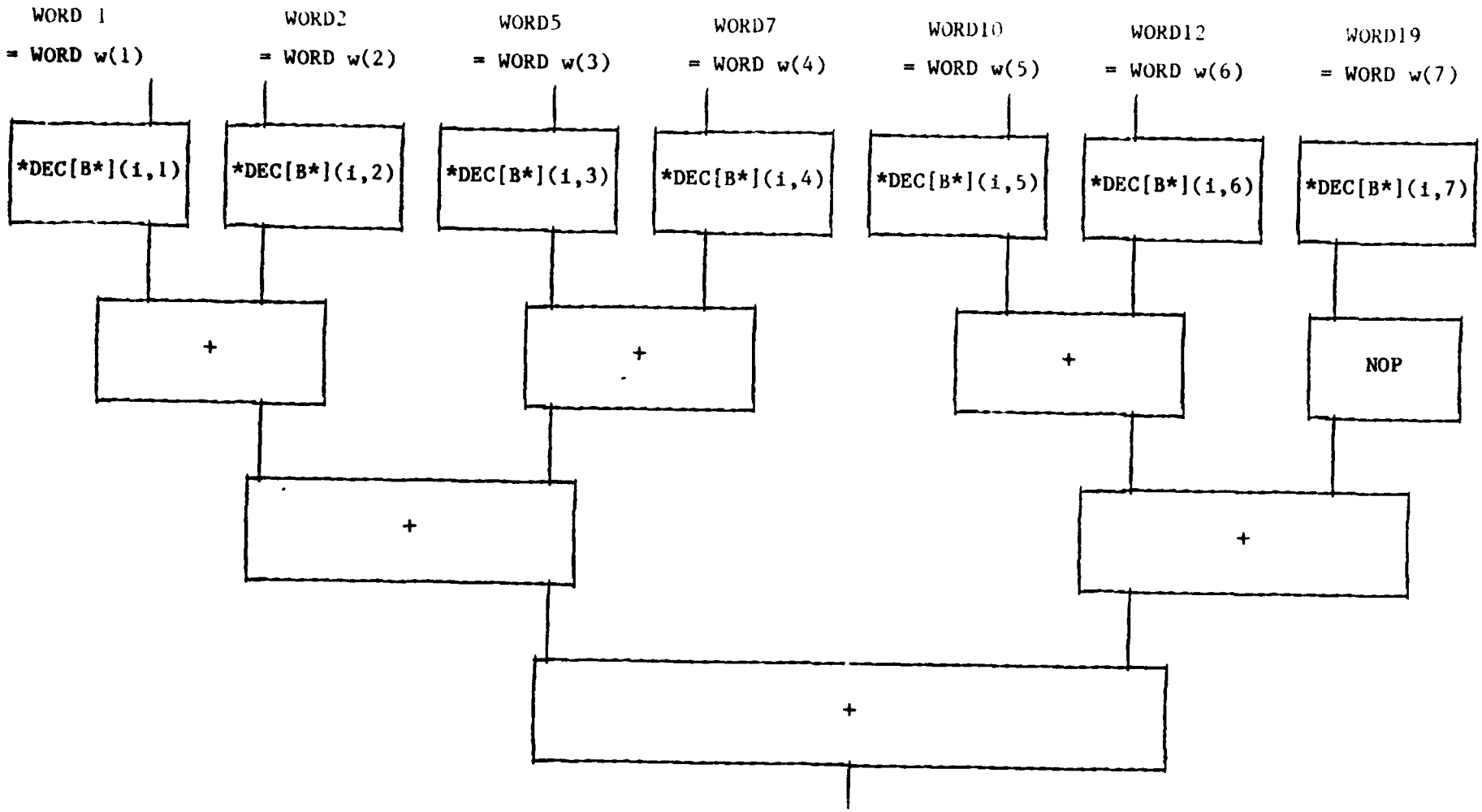


Figure 1.6.1

Example of decode for a
7-out-of-25 Bloom p/s/r.
Encode is similar.
Information flows downward.
NOP = no operation
+ = XOR of words, i.e. bitwise modulo 2 addition

On the face of things it would appear that one would have to use 5 cycles to fill in the (variable) 5-bit multiplicand WORD w(j) into the box which multiplies by the (fixed) 5-bit multiplier DEC[B*](i,j), then take more than 5 additional cycles to perform the GF(32) multiplication, then 3 more cycles to move through the adders (the add operation is XOR). This would involve an output stream slower than the one bit per cycle input stream. This, however, is not the case. We will show below how to produce a one bit per cycle output stream, using appropriate hardware. Of course the output will lag the input in phase. In the case above the lag will be about 18 cycles.

Again we note the need to keep parallel channels synchronized. This means that even the plaintext channels will be "encoded" (or "decoded"). This will be done by multiplying by 1, then adding 0, then adding another 0, and so on for the proper number of steps.

1.7 Making hyperfast Bloom p/s/r processes. Stage 3. Hot precomputation.

Recall that we are considering the mean cases of parameter settings. Turning now from Stage 4, real-time on-line decode, to Stage 3, hot precomputation, we come to an important problem. You want to shorten the hot precomputation because you must store or lose received bits while it takes place. It turns out that the hot precomputation should be done somewhat differently for different parameter settings in the mean cases of parameter settings.

In Section 2.5 below we take up this matter in more detail. If k or n-k is small, the hot precomputation proceeds quickly.

In summary, the only rub anywhere in the system occurs in the hot precomputation. And it is worst when k is close to n/2. In many applications, such as digital voice, where loss of one second's worth of transmission is tolerable, the rub can be ignored. In other applications, its presence may necessitate enough buffer memory to store several second's worth of received material.

Of course there is inevitably one other place where simple common sense dictates that expense is inevitable, not for memory to store signals but for memory and processing capability to do computations. In 16 bit applications in which $40{,}000 \leq k \leq n$ there are a lot of received channels and some very big (40,000 by 40,000) matrices to build. It is important to keep in mind the admonition that most systems with more than 256 channels are impractical. We return to this matter below.

1.8. Interfacing error control devices and cryptographic devices with p/s/r processes

p/s/r processes work best on channels which are virtually error-free while operative (like some optical fibers), but which can be rendered inoperative for long periods (e.g. by breaking the fibers). If the operative channels are also subject to intermittent errors then one should combine ordinary error control with p/s/r processes in the manner shown in Figure 1.8.1. First p/s/r encode, then error control encode, then transmit, then receive, then error control decode, then p/s/r decode. Doing an error control encode before the p/s/r encode would be silly. We will not belabor this point further.

Cryptographic encode should probably be placed before p/s/r encode and cryptographic decode after p/s/r decode, as in Figure 1.8.2. But this is a matter which will no doubt be determined by an appropriate branch of DOD, and we will therefore not treat it further.

Figure 1.8.3 shows the concatenation scheme for all three processes. All figures are to be understood as showing information flowing downward.

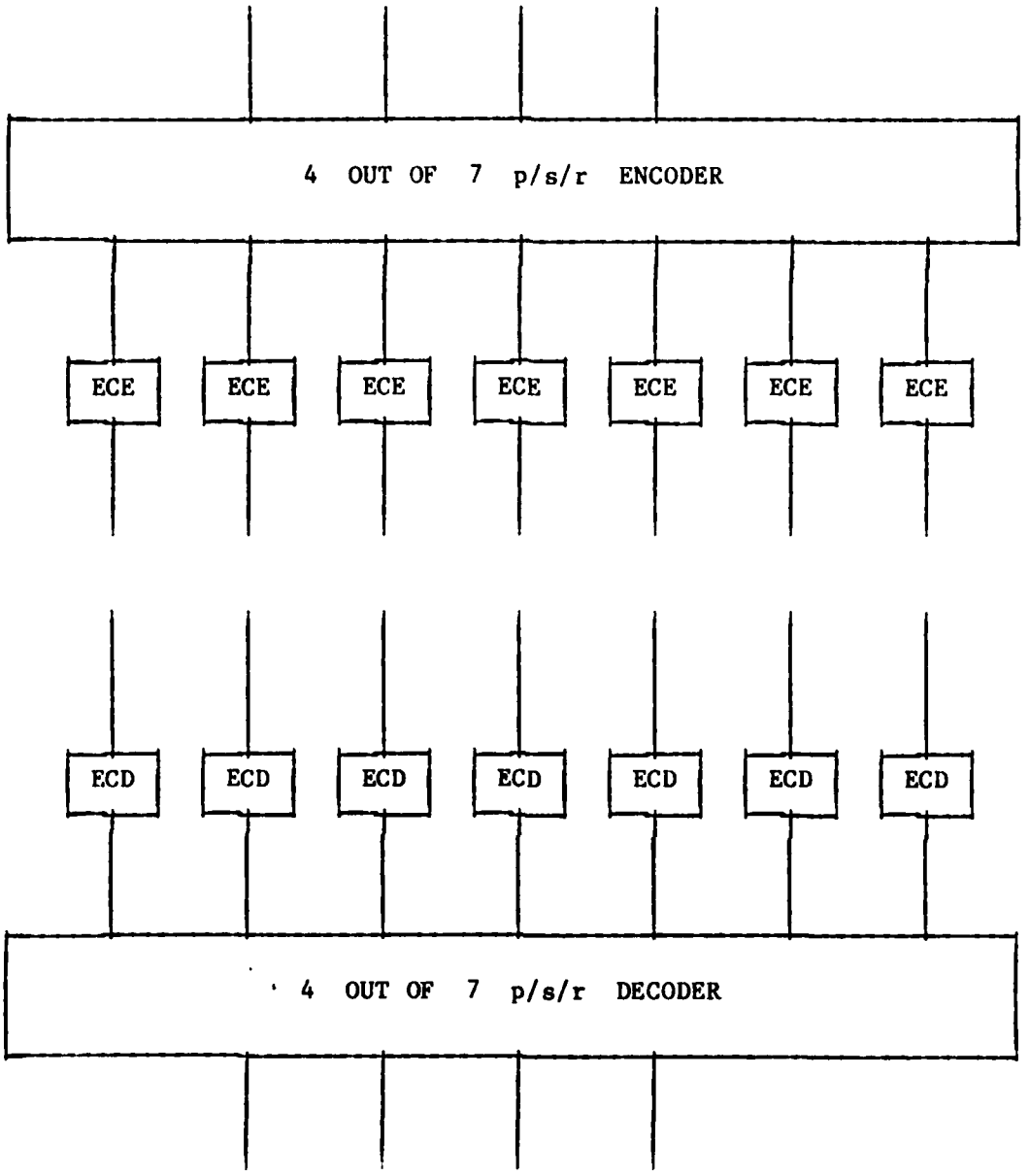


Figure 1.8.1

ECE = error control encode

ECD = error control decode

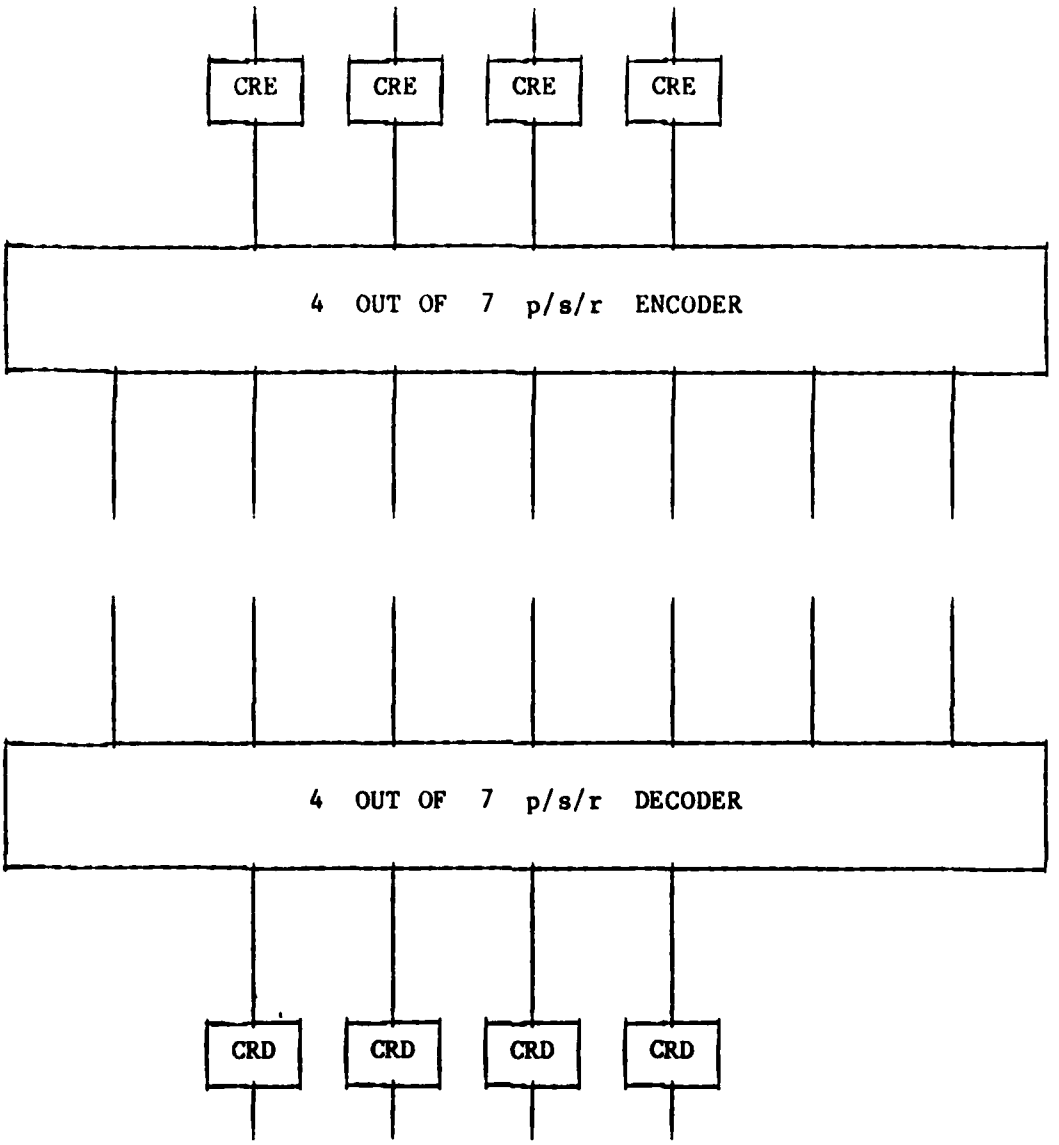


Figure 1.8.2

CRE = cryptographic encode

CRD = cryptographic decode

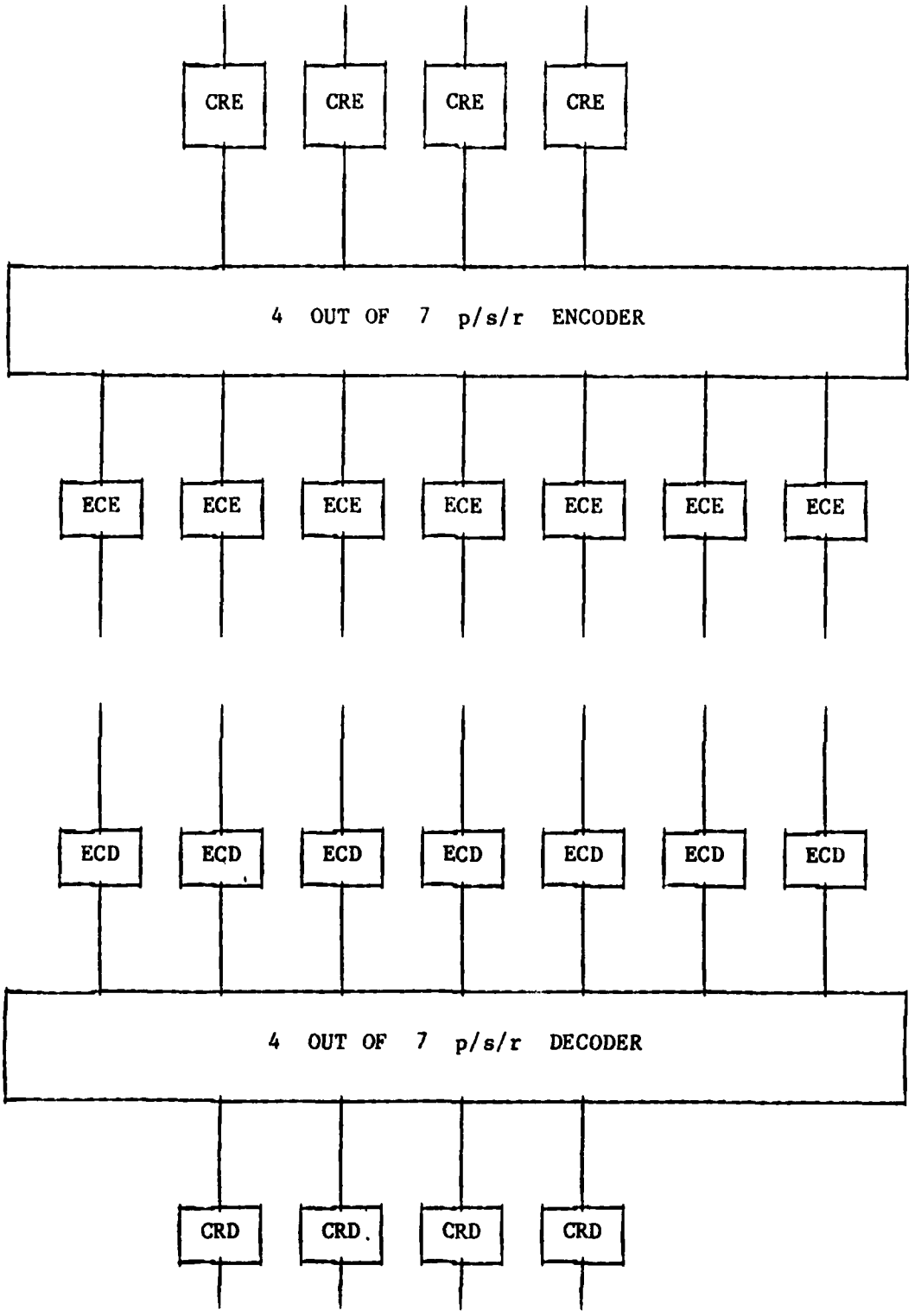

CRE
CRE
CRE
CRE
4 OUT OF 7 p/s/r ENCODER
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Figure 1.8.3

2. Detailed Narrative

2.1. Finite field arithmetic. Octal notation for polynomials and residue classes of polynomials.

It is no longer possible or desirable to avoid technicalities. We first make explicit the finite field arithmetic behind the Bloom p/s/r processes. GF(2) = Z/2Z is the field with two elements. Its arithmetic (i.e. its add, +, subtract, -, multiply, *, and divide, /) is summarized in the tables

+	0	1
0	0	1
1	1	0

-	0	1
0	0	1
1	1	0

*	0	1
0	0	0
1	0	1

/	0	1
0	undefined	0
1	undefined	1

Thus $x + y = x - y$ for every $x, y \in GF(2)$, the only nonzero product is $1 * 1 = 1$, and division by zero is impossible (undefined). You can put these things another way. +, -, and * are modulo 2 operations, and you cannot divide by zero. Alternatively, + and - are XOR of bits (exclusive or), * is AND of bits, and you cannot divide by zero.

Let p(x) be a polynomial over GF(2) which is irreducible (unfactorable) over GF(2). Examples of polynomials over GF(2) which are irreducible over GF(2) are:

x
x + 1
x↑2 + x + 1
x↑3 + x + 1
x↑4 + x + 1
x↑5 + x↑2 + 1
x↑6 + x + 1
x↑7 + x↑3 + 1
x↑8 + x↑4 + x↑3 + x↑2 + 1
x↑12 + x↑6 + x↑4 + x + 1
x↑16 + x↑12 + x↑3 + x + 1

Examples of polynomials over GF(2) which are reducible over GF(2) (i.e. polynomials which can be factored) are:

$$x\uparrow 2 = x * x$$

$$x\uparrow 2 + x = x * (x + 1)$$

$$x\uparrow 2 + 1 = (x + 1) * (x + 1)$$

$$x\uparrow 4 + x\uparrow 2 + 1 = (x\uparrow 2 + x + 1) * (x\uparrow 2 + x + 1)$$

$$x\uparrow 4 + x\uparrow 2 + x + 1 = (x + 1) * (x\uparrow 3 + x\uparrow 2 + 1)$$

Let n be a positive integer. The field GF(2↑n) is defined as follows. Let p(x) be an nth degree (monic) polynomial over GF(2) which is irreducible over GF(2). Let (p(x)) be the principal ideal generated by p(x) in the ring POL of polynomials over GF(2). Then GF(2↑n) is the quotient

$$GF(2\uparrow n) = POL/(p(x)).$$

of the ring POL modulo the principal ideal generated by p(x). For example if $p(x) = x\uparrow 3 + x + 1$ then the version of GF(8) = GF(2↑3) gotten by setting

$$GF(8) = POL/(p(x)) = POL/(x\uparrow 3 + x + 1)$$

consists of 8 residue classes modulo $p(x) = x\uparrow 3 + x + 1$, namely

$\underline{0} = \langle 0,0,0\rangle = CLASS\ (0) = \{0, x\uparrow 3 + x + 1, \ldots\}$

$\underline{1} = \langle 0,0,1\rangle = CLASS\ (1) = \{1, x\uparrow 3 + x, \ldots\}$

$\underline{2} = \langle 0,1,0\rangle = CLASS\ (x) = \{x, x\uparrow 3 + 1, \ldots\}$

$\underline{3} = \langle 0,1,1\rangle = CLASS\ (x+1) = \{x+1, x\uparrow 3, \ldots\}$

$\underline{4} = \langle 1,0,0\rangle = CLASS\ (x\uparrow 2) = \{x\uparrow 2, x\uparrow 3 + x\uparrow 2 + x + 1, \ldots\}$

$\underline{5} = \langle 1,0,1\rangle = CLASS\ (x\uparrow 2 + 1) = \{x\uparrow 2 + 1, x\uparrow 3 + x, \ldots\}$

$\underline{6} = \langle 1,1,0\rangle = CLASS\ (x\uparrow 2 + x) = \{x\uparrow 2 + x, x\uparrow 3 + x\uparrow 2 + 1, \ldots\}$

$\underline{7} = \langle 1,1,1\rangle = CLASS\ (x\uparrow 2 + x + 1) = \{x\uparrow 2 + x + 1, x\uparrow 3, \ldots\}$

It is too tedious to use a notation such as

$$\text{CLASS } (x\uparrow 2 + x)$$

or

$$\langle 1,0,1\rangle$$

or

$$\{x\uparrow 2 + x,\ x\uparrow 3 + x\uparrow 2 + 1,\ x\uparrow 4,\ \ldots\}$$

for a member of GF(8). Therefore we adopt the octal notation used in the MIT Press book of Peterson and Weldon on error correcting codes. An arabic numeral with neither overbar nor underbar is a whole number. Thus

$$7 = \text{VII} = \text{seven},$$

the number of days in the week. An arabic numeral with an overbar is a polynomial over GF(2). Thus

$$\overline{7} = \langle 1,1,1\rangle = x\uparrow 2 + x + 1.$$

And an arabic numeral with an underbar is a residue class (modulo some agreed upon irreducible polynomial p(x)) to which a polynomial q(x) belongs. Thus if $p(x) = x\uparrow 3 + x + 1$ is agreed upon in advance then

$$\begin{aligned}\underline{7} &= \{x\uparrow 2 + x + 1,\ x\uparrow 3 + x\uparrow 2,\ x\uparrow 4 + 1,\ x\uparrow 5 + x\uparrow 3 + x + 1,\ \ldots\}\\ &= \{\overline{7},\ \overline{14},\ \overline{21},\ \overline{53},\ \ldots\}\\ &= \text{CLASS } (\overline{7}) \text{ mod } (\overline{13})\end{aligned}$$

is the residue class modulo $x\uparrow 3 + x + 1$ whose lowest degree member is $\overline{7} = x\uparrow 2 + x + 1$.

We now agree on polynomials over GF(2) of degrees 2, 3, 4, 5, 6, 7, 8, 12 and 16. Each of them is irreducible over GF(2). In fact, each of them is a primitive irreducible polynomial over GF(2). There is no need to describe the notion of primitive here. Suffice it to say that it is a convenience, and is explained in Peterson and Weldon.

There are nine standard polynomials to be understood everywhere below. They are the polynomials on which our version of GF(4), GF(8), GF(16), GF(32), GF(64), GF(128), GF(256), GF(4,096) and GF(65,536) are based. It is, of course, well known that there is (up to isomorphism) only one Galois field of any given size.

The nine standard polynomials are

$$\overline{7} = x\uparrow 2 + x + 1$$

$$\overline{13} = x\uparrow 3 + x + 1$$

$$\overline{23} = x\uparrow 4 + x + 1$$

$$\overline{45} = x\uparrow 5 + x\uparrow 2 + 1$$

$$\overline{103} = x\uparrow 6 + x + 1$$

$$\overline{211} = x\uparrow 7 + x\uparrow 3 + 1$$

$$\overline{435} = x\uparrow 8 + x\uparrow 4 + x\uparrow 3 + x\uparrow 2 + 1$$

$$\overline{10123} = x\uparrow 12 + x\uparrow 6 + x\uparrow 4 + x + 1$$

$$\overline{210013} = x\uparrow 16 + x\uparrow 12 + x\uparrow 3 + x + 1$$

Members of $GF(2\uparrow 4) = GF(16)$ can thus be represented as 4-bit words, i.e. "numbers" expressible by two (underbarred) octal arabic numerals, neither of which is 8 or 9. Members of $GF(2\uparrow 8) = GF(256)$ "are" 8 bit words, i.e. "numbers" expressible by three (underbarred) octal arabic numerals (8 and 9 will not be used). For $GF(2\uparrow 12) = GF(4,096)$ we use 12 bit words, i.e. foursomes of underbarred arabic octal numerals (no 8 or 9 allowed). For $GF(2\uparrow 16) = GF(65,536)$ we use 16 bit words, underbarred 6 "digit" arabic numerals (with no occurrence of 8 or 9).

To exemplify the arithmetic of $GF(2\uparrow n)$ we will give tables for:

$$GF(4) \text{ as } POL/(x\uparrow 2 + x + 1) = POL/(\overline{7})$$

$$GF(8) \text{ as } POL/(x\uparrow 3 + x + 1) = POL/(\overline{13})$$

$$GF(16) \text{ as } POL/(x\uparrow 4 + x + 1) = POL/(\overline{23})$$

They are contained in Appendix B.

2.2 The linear algebra of Bloom p/s/r processes.

As noted above, the extreme parameter setting cases

$$(k,n) = (0,n)$$

$$(k,n) = (1,n)$$

$$(k,n) = (n,n)$$

require no coding. The extreme parameter setting case

$$(k,n) = (n-1,n)$$

can be very simply coded and decoded using only GF(2), and without cold, cool or hot precomputation. Thus we will consider the mean parameter setting cases, i.e. the cases involving (k,n) such that

$$2 \leq k \leq n-2 \leq n \leq 2\uparrow b = Q \leq 65,536$$

Here b is a parameter describing the size of (i.e. number of bits in) the computer word to be used in practical implementations. Its place in the scheme of things will be obvious below.

Let us begin with the $2\uparrow b$ by $2\uparrow b$ (i.e. Q by Q) Vandermonde matrix with entries in $GF(2\uparrow b) = GF(Q)$. This square matrix VAN is of the form

$$VAN = \begin{bmatrix} \underline{1} & \underline{0} & \underline{0} & \underline{0} & \cdots & \underline{0} & \underline{0} \\ \underline{1} & \underline{1} & \underline{1} & \underline{1} & \cdots & \underline{1} & \underline{1} \\ \underline{1} & \underline{2} & \underline{2}\uparrow 2 & \underline{2}\uparrow 3 & \cdots & \underline{2}\uparrow(b-2) & \underline{2}\uparrow(b-1) \\ \underline{1} & \underline{2}\uparrow 2 & \underline{2}\uparrow 4 & \underline{2}\uparrow 6 & \cdots & \underline{2}\uparrow 2(b-2) & \underline{2}\uparrow 2(b-1) \\ \vdots & \vdots & \vdots & \vdots & & \vdots & \vdots \\ \underline{1} & \underline{2}\uparrow(b-3) & \underline{2}\uparrow 2(b-3) & \underline{2}\uparrow 3(b-3) & \cdots & \underline{2}\uparrow(b-3)(b-2) & \underline{2}\uparrow(b-3)(b-1) \\ \underline{1} & \underline{2}\uparrow(b-2) & \underline{2}\uparrow 2(b-2) & \underline{2}\uparrow 3(b-2) & \cdots & \underline{2}\uparrow(b-2)(b-2) & \underline{2}\uparrow(b-2)(b-1) \end{bmatrix}$$

Note that the bases are (underbarred) members of GF(Q) and the exponents are (unbarred) integers. It is a fact that $\underline{2}$ is a primitive element in $GF(2\uparrow b)$ if $GF(2\uparrow b)$ is realized as $POL/(p(x))$ where

p(x) is a primitive polynomial. We will always use fields of this form. Examples of Vandermonde matrices are

$$VAN = \begin{bmatrix} \underline{1} & \underline{0} & \underline{0} & \underline{0} \\ \underline{1} & \underline{1} & \underline{1} & \underline{1} \\ \underline{1} & \underline{2} & \underline{3} & \underline{1} \\ \underline{1} & \underline{3} & \underline{2} & \underline{1} \end{bmatrix}$$

in GF(4) = POL/(x↑2 + x + 1) = POL/($\overline{7}$), and

$$VAN = \begin{bmatrix} \underline{1} & \underline{0} & \underline{0} & \underline{0} & \underline{0} & \underline{0} & \underline{0} & \underline{0} \\ \underline{1} & \underline{1} & \underline{1} & \underline{1} & \underline{1} & \underline{1} & \underline{1} & \underline{1} \\ \underline{1} & \underline{2} & \underline{4} & \underline{3} & \underline{6} & \underline{7} & \underline{5} & \underline{1} \\ \underline{1} & \underline{4} & \underline{6} & \underline{5} & \underline{2} & \underline{3} & \underline{7} & \underline{1} \\ \underline{1} & \underline{3} & \underline{5} & \underline{4} & \underline{7} & \underline{2} & \underline{6} & \underline{1} \\ \underline{1} & \underline{6} & \underline{2} & \underline{7} & \underline{4} & \underline{5} & \underline{3} & \underline{1} \\ \underline{1} & \underline{7} & \underline{3} & \underline{2} & \underline{5} & \underline{6} & \underline{4} & \underline{1} \\ \underline{1} & \underline{5} & \underline{7} & \underline{6} & \underline{3} & \underline{4} & \underline{2} & \underline{1} \end{bmatrix}$$

in GF(8) = POL/(x↑3 + x + 1) = POL/($\overline{13}$). See Appendix C for examples of Vandermonde matrices, for various fields.

Now let LEF[k] be a special Q by k submatrix of VAN. It consists of the first k columns of VAN. In our GF(8) example

$$\mathrm{LEF}[3] = \begin{bmatrix} \underline{1} & \underline{0} & \underline{0} \\ \underline{1} & \underline{1} & \underline{1} \\ \underline{1} & \underline{2} & \underline{4} \\ \underline{1} & \underline{4} & \underline{6} \\ \underline{1} & \underline{3} & \underline{5} \\ \underline{1} & \underline{6} & \underline{2} \\ \underline{1} & \underline{7} & \underline{3} \\ \underline{1} & \underline{5} & \underline{7} \end{bmatrix}$$

It is a well known property of Vandermonde matrices that every k by k submatrix of $\mathrm{LEF}[k]$ is nonsingular whenever k satisfies the inequalities

$$2 \leq k \leq Q = 2\uparrow b \ .$$

Thus the rows of $\mathrm{LEF}[k]$ can be regarded as a collection of $2\uparrow b = Q$ vectors in general position in a k dimensional vector space V over $\mathrm{GF}(2\uparrow b) = \mathrm{GF}(Q)$. But recall that $2 \leq k \leq n-2 \leq n \leq Q$. This means that we have the wherewithal to build a Bloom k-out-of-n p/s/r process. Consider any list $w(1), w(2), \ldots, w(k)$ of distinct row indices of $\mathrm{LEF}[k]$, i.e. any injection

$$w: \{1, 2, \ldots, k\} \to \{1, 2, \ldots, Q\}.$$

There is obviously a Q by k (coding) matrix COD_w corresponding to this w such that

$$\mathrm{ROW}[i] = \sum \mathrm{COD}_w(i,j)\mathrm{ROW}[w(j)]$$

for every positive integer $i \leq Q$. In particular

$$\mathrm{ROW}[i] = \sum \mathrm{COD}_e(i,j)\mathrm{ROW}[e(j)]$$

$$= \sum \mathrm{COD}_e(i,j)\mathrm{ROW}[j]$$

when e is the identity injection. All three sums above are over positive integer $j \leq k$.

The Bloom k-out-of-n p/s/r process now works as follows. Let I(1), I(2), ..., I(k) be the b-bit plaintext words the sender S has on source channels 1, 2, ..., k at time t. The sender encodes them to form b-bit words H(1), H(2), ..., H(n) for sending along broadcast channels 1, 2, ..., n as follows

$$H(i) = \sum COD_e(i,j)I(j)$$

for positive integer $i \leq n$, where the sum is over positive integer $j \leq k$. When the receiver R ascertains that channels w(1), w(2), ..., w(k) are operative, he decodes by finding

$$I(i) = \sum COD_w(i,j)H(w(j))$$

for positive integer $i \leq k$, where the sum is over positive integer $j \leq k$.

Before looking at implementation in the four stages we make a few comments. First, encoding is a process which depends only on k and Q, not on n (except in the trivial sense that you don't bother to encode *any messages* H(i) for channels n+1, n+2, ..., Q) and not on which channels are operative and which are inoperative. After all, the sender is not likely to know which channels are operative. Mathematically speaking, encoding makes use only of the (fixed) identity injection e.

Decoding, on the other hand, makes use of the (variable) injection w which embodies information known only to the receiver, namely which channels w(1), w(2), ..., w(k) are operative. So decoding depends on k, w and Q. Consequently decoding depends implicitly on n, since $1 \leq w(i) \leq n$ for every positive integer $i \leq k$.

If either sender S or receiver R can profit by taking n into account in a a more explicit fashion in their calculations, they are free to do so. But they don't have to. We will show below how to take advantage of a knowledge of n.

Comparing this description with the YLYK Ltd. proposal, the reader will note our assumption that

$$n \leq 2\uparrow b = Q .$$

That proposal held forth the possibility of the inequality

$$n \leq Q + 2$$

in many cases.

We abandoned this tack, fine tuning the field size, for four reasons:

1. It shortens word size by only one or two bits where it is possible;
2. It complicates coding and decoding where it is possible;
3. It is a very difficult problem to determine all the cases in which it is possible. See the MacWilliams and Sloane book on error correcting codes for more on this;
4. We now know how to achieve the desired goal of attaining hyperfast Bloom p/s/r processes without fine tuning the field size. The hyperfast real-time on-line decode is attained in a different way, by use of systolic multipliers, as we shall see below. Moreover, fine tuning field size is of no appreciable utility in attacking the other crucial problem, shortening the duration of Stage 3, the hot precomputation.

2.3. The first stage of computation in the mean cases of the Bloom p/s/r process, the manufacturer's cold precomputation

Recall that we have a field

$$GF(2\uparrow b) = GF(Q)$$

and that

$$2 \leq k \leq n-2 \leq n \leq 2\uparrow b = Q.$$

The entire problem of encoding and decoding in a k-out-of-n Bloom p/s/r process amounts to this. For each injection

$$w: \{1, 2, \ldots, k\} \rightarrow \{1, 2, \ldots, n\}$$

(including the identity injection $w = e$) find the k by k matrix COD_w such that

$$ROW[i] = \sum COD_w(i,j)ROW[w(j)]$$

where the sum is over positive integer $j \leq k$, and where $ROW[i]$ is the ith row of the Q by k matrix $LEF[k]$. Recall that $LEF[k]$ consists of the first k columns of the Q by Q Vandermonde matrix VAN over $GF(Q)$. Once COD_e is found, form

$$H(i) = \sum COD_e(i,j)I(j)$$

(where the sum is over every positive integer $j \leq k$) for every positive integer $i \leq n$ to encode. Once w is chosen and COD_w is found, form

$$I(i) = \sum COD_w(i,j)H(w(j))$$

(where the sum is over every positive integer $j \leq k$) for every positive integer $i \leq k$ to decode.

Obviously it is desirable to carry out computations as early as possible. We have agree to send the k plaintext messages $I(1), I(2), \ldots, I(k)$ (i.e. members of $GF(2\uparrow b) = GF(Q)$, i.e. b-bit words) down channels $1, 2, \ldots, k$ respectively. These words are unaltered. They are transmitted as is.

$$H(1) = I(1)$$

$$H(2) = I(2)$$

$$\vdots$$

$$H(k) = I(k)$$

What we need is the encoding for channels $k+1, k+2, \ldots, n$. In other words we need to express rows $k+1, k+2, \ldots, k+n$ of $LEF[k]$ in terms of rows $1, 2, \ldots, k$. To say we need dependences is to say we need vanishing linear combinations of the rows of $LEF[k]$. We need, therefore, a basis for the left kernel of $LEF[k]$ (The left kernel of a Q by k matrix L is the set of all length Q row vectors r such that rQ is the length k row vector with all zero entries). Let us take $GF(8)$ as an example.

$$\mathrm{LEF}[8] = \mathrm{VAN} = \begin{bmatrix} 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 \\ 1 & 2 & 4 & 3 & 6 & 7 & 5 & 1 \\ 1 & 4 & 6 & 5 & 2 & 3 & 7 & 1 \\ 1 & 3 & 5 & 4 & 7 & 2 & 6 & 1 \\ 1 & 6 & 2 & 7 & 4 & 5 & 3 & 1 \\ 1 & 7 & 3 & 2 & 5 & 6 & 4 & 1 \\ 1 & 5 & 7 & 6 & 3 & 4 & 2 & 1 \end{bmatrix}$$

is nonsingular.

$$\mathrm{LEF}[7] = \begin{bmatrix} 1 & 0 & 0 & 0 & 0 & 0 & 0 \\ 1 & 1 & 1 & 1 & 1 & 1 & 1 \\ 1 & 2 & 4 & 3 & 6 & 7 & 5 \\ 1 & 4 & 6 & 5 & 2 & 3 & 7 \\ 1 & 3 & 5 & 4 & 7 & 2 & 6 \\ 1 & 6 & 2 & 7 & 4 & 5 & 3 \\ 1 & 7 & 3 & 2 & 5 & 6 & 4 \\ 1 & 5 & 7 & 6 & 3 & 4 & 2 \end{bmatrix}$$

has rank 7. Therefore its kernel has dimension 1 and a calculation shows that it is spanned by the row vector

$$[\,1\ 1\ 1\ 1\ 1\ 1\ 1\ 1\,]$$

$$\mathrm{LEF}[6] = \begin{bmatrix} 1 & 0 & 0 & 0 & 0 & 0 \\ 1 & 1 & 1 & 1 & 1 & 1 \\ 1 & 2 & 4 & 3 & 6 & 7 \\ 1 & 4 & 6 & 5 & 2 & 3 \\ 1 & 3 & 5 & 4 & 7 & 2 \\ 1 & 6 & 2 & 7 & 4 & 5 \\ 1 & 7 & 3 & 2 & 5 & 6 \\ 1 & 5 & 7 & 6 & 3 & 4 \end{bmatrix}$$

has rank six. So its kernel contains the kernel of LEF[7]. A calculation shows that it is spanned by the row vectors

$$[\,\underline{1}\ \underline{1}\ \underline{1}\ \underline{1}\ \underline{1}\ \underline{1}\ \underline{1}\ \underline{1}\,]$$

and

$$[\,\underline{7}\ \underline{6}\ \underline{2}\ \underline{5}\ \underline{3}\ \underline{4}\ \underline{1}\ \underline{0}\,]$$

Similarly, one easily verifies that

$$\begin{bmatrix} \underline{1} & \underline{1} & \underline{1} & \underline{1} & \underline{1} & \underline{1} & \underline{1} & \underline{1} \\ \underline{7} & \underline{6} & \underline{2} & \underline{5} & \underline{3} & \underline{4} & \underline{1} & \underline{0} \\ \underline{2} & \underline{3} & \underline{2} & \underline{1} & \underline{3} & \underline{1} & \underline{0} & \underline{0} \end{bmatrix} \begin{bmatrix} \underline{1} & \underline{0} & \underline{0} & \underline{0} & \underline{0} \\ \underline{1} & \underline{1} & \underline{1} & \underline{1} & \underline{1} \\ \underline{1} & \underline{2} & \underline{4} & \underline{3} & \underline{6} \\ \underline{1} & \underline{4} & \underline{6} & \underline{5} & \underline{2} \\ \underline{1} & \underline{3} & \underline{5} & \underline{4} & \underline{7} \\ \underline{1} & \underline{6} & \underline{2} & \underline{7} & \underline{4} \\ \underline{1} & \underline{7} & \underline{3} & \underline{2} & \underline{5} \\ \underline{1} & \underline{5} & \underline{7} & \underline{6} & \underline{3} \end{bmatrix} = \begin{bmatrix} \underline{0} & \underline{0} & \underline{0} & \underline{0} & \underline{0} \\ \underline{0} & \underline{0} & \underline{0} & \underline{0} & \underline{0} \\ \underline{0} & \underline{0} & \underline{0} & \underline{0} & \underline{0} \end{bmatrix}$$

Going on in this way we arrive at the encode normal form ENF matrix over GF(8):

$$\begin{bmatrix} \underline{1} & \underline{1} & \underline{1} & \underline{1} & \underline{1} & \underline{1} & \underline{1} & \underline{1} \\ \underline{7} & \underline{2} & \underline{6} & \underline{5} & \underline{3} & \underline{4} & \underline{1} & \underline{0} \\ \underline{2} & \underline{3} & \underline{2} & \underline{1} & \underline{3} & \underline{1} & \underline{0} & \underline{0} \\ \underline{3} & \underline{5} & \underline{2} & \underline{5} & \underline{1} & \underline{0} & \underline{0} & \underline{0} \\ \underline{4} & \underline{3} & \underline{6} & \underline{1} & \underline{0} & \underline{0} & \underline{0} & \underline{0} \\ \underline{3} & \underline{2} & \underline{1} & \underline{0} & \underline{0} & \underline{0} & \underline{0} & \underline{0} \end{bmatrix}.$$

This matrix ENF is a Q-2 by Q matrix with 1 in the (j, Q-j+1)th entry and with 0 entries everywhere below these "antidiagonal" 1 s. In fact the matrix product ENF * VAN is a Q-2 by Q matrix with zeros above the "antidiagonal". Given any Q by Q Vandermonde matrix for GF(Q) it is elementary linear algebra to find the Q-2 by Q matrix ENF such that:

1. For each positive integer $j \leq Q-2$ the top j rows of ENF form a basis for the left kernel of LEF[Q-j]

2. The antidiagonal entries (i.e. ENF(j, Q-j+1) for every positive integer $j \leq Q-2$) are 1

3. The entries below the antidiagonal are 0.

This is the substance of the manufacturer's cold precomputation, Stage 1. A computer program incorporating this precomputation is contained in Appendix H. It could take months on an IBM 370 and still be perfectly satisfactory, since it will be done just once before the devices are fabricated. In fact the GF(16) computation takes seconds on an IBM PC. The GF(16) ENF is a 14 by 16 matrix whose entries are 4-bit words. See Appendix D for examples of ENF matrices for various fields.

The GF(256) cold precomputation, even without the shortcuts employed in Appendix H, takes far fewer than a billion machine cycles, i.e. a few minutes of mainframe time. To store its output requires 254*256 = 65,024 bytes of ROM. The GF(4,096) and GF(65,536) cold precomputations take longer.

Since finding a kernel basis and triangularizing its matrix takes a small constant times the cube of the dimension of the vector space, finding a 4094 by 4096 ENF matrix for GF(4,096 = GF(2↑12) could take as many 10↑13 single precision integer operations and single word logical operations on an IBM 370. This could take months. To store the output you would need more than 200 megabits of ROM.

To find a 65,534 by 65,536 ENF matrix over GF(65,536) = GF(2↑16) is a bigger task. Here we are talking about a fair sized multiple of 2↑48 operations, say 10↑17 to be on the safe side. Of course, this assumes no parallelism in the computer. But parallelism and vector structure are keynotes of the computation. However it looks like months of calculation on better adapted machines such as a CRAY I or the new MPP being put up at NASA, both of them well-suited to the sort of linear algebra computations required. It also means scrapping the PASCAL program in Appendix A and writing code which exploits the peculiarities of the machine it runs on. Also, storing its output is nontrivial. This output consists of 65,534 * 65,536 = 4,294,836,224 16-bit words. This means almost 9 gigabytes of ROM in the devices which implement such p/s/r processes.

What about larger fields? It seems doubtful that they can be exploited economically in the 1980s, or that they would be used even if computations were cheap. Some objections are:

i. 65,537 channels is a lot of channels. Is there a plausible application of k out of n p/s/r processes in a situation where n > 2↑16 = 65,536?

ii. Fields larger than GF(2↑16) cannot be handled on a 16 bit microprocessor without adopting unnatural expedients which slow things down.

iii. Stage 1, the cold precomputation stage in which ENF is formed, gets expensive and time consuming in GF(2↑b) for b > 16. For example production of an ENF for GF(2↑20) looks like a multiple of 2↑60 operations on a Von Neumann machine, say 10↑20 operations.

iv. Storing the ENF in fields bigger than GF(65,536) requires more than 9 gigabytes of ROM.

Summarizing the first stage, the manufacturer's cold precomputation stage, we see that the $2\uparrow b - 2$ by $2\uparrow b$ encode normal form matrix ENF has the following properties (pessimistic estimates):

Galois field	time to produce ENF	space to store ENF
GF(16) = GF(2↑4)	PC minutes	1 k bits
GF(256) = GF(2↑8)	mainframe minutes	600 k bits
GF(4,096) = GF(2↑12)	mainframe months	300 m bits
GF(65,536) = GF(2↑16)	supercomputer years	70 g bits

2.4 The second stage of computation in the mean cases of the Bloom p/s/r process, the sender's cool precomputation.

Recall that we have, once for all, chosen

$$GF(2\uparrow b) = GF(Q)$$

Thus the sender S must take k b-bit words at time t and encode this information into n b-bit words for transmission. Moreover

$$2 \leq k \leq n-2 \leq n \leq 2\uparrow b = Q$$

The ENF matrix is available to both sender and receiver. It contains information about VAN or, more specifically, about LEF[2], LEF[3], ..., LEF[Q-1]. The first row of ENF expresses the Qth row of LEF[Q-1] (and therefore of LEF[Q-2], ..., LEF[2]) as a linear combination of its first Q-1 rows. The second row of ENF expresses the (Q-1)st row of LEF[Q-2] (and therefore of LEF[Q-3], ..., LEF[2]) as a linear combination of its first Q-2 rows. And so on, to the bottom row (the (Q-2)nd row) of ENF. This row of ENF expresses the third row of LEF[2] in terms of the first two rows of LEF[2].

Once k and n are agreed upon, the sender S and receiver R fix their attention on LEF[k]. They can ignore its bottom Q - n rows. Thus they are looking at the upper left n by k submatrix UPLEF[n,k] of VAN. Clearly, the dependences they both need to know among the rows of LEF[k] (or, equivalently, of UPLEF[n,k]) are all contained (implicitly at least) in rows Q - n + 1, Q - n + 2, ..., Q - k of ENF.

For example a 3-out-of-7 p/s/r over GF(8) is based on knowing the dependences among the first seven rows of

$$\text{LEF}[3] = \begin{bmatrix} \underline{1} & \underline{0} & \underline{0} \\ \underline{1} & \underline{1} & \underline{1} \\ \underline{1} & \underline{2} & \underline{4} \\ \underline{1} & \underline{4} & \underline{6} \\ \underline{1} & \underline{3} & \underline{5} \\ \underline{1} & \underline{6} & \underline{2} \\ \underline{1} & \underline{7} & \underline{3} \\ \underline{1} & \underline{5} & \underline{7} \end{bmatrix}$$

and these are all expressed in rows

$$8 - 7 + 1 = 2$$
$$8 - 7 + 2 = 3$$
$$\vdots$$
$$8 - 3 = 5$$

of ENF, i.e. in the matrix

$$\text{MID}[2,5] = \begin{bmatrix} \underline{7} & \underline{2} & \underline{6} & \underline{5} & \underline{3} & \underline{4} & \underline{1} & \underline{0} \\ \underline{2} & \underline{3} & \underline{2} & \underline{1} & \underline{3} & \underline{1} & \underline{0} & \underline{0} \\ \underline{3} & \underline{5} & \underline{2} & \underline{5} & \underline{1} & \underline{0} & \underline{0} & \underline{0} \\ \underline{4} & \underline{3} & \underline{6} & \underline{1} & \underline{0} & \underline{0} & \underline{0} & \underline{0} \end{bmatrix}$$

Let MID[Q-n+1, Q-k] be the n - k by Q submatrix of ENF consisting of rows Q-n+1, Q-n+2, ..., Q-k of ENF. We now have the only matrix of interest to the sender S and the receiver R during this communication session using this k-out-of-n p/s/r. The last Q - n columns of MID[Q-n+1, Q-k] are, of course, zero. So they can, and will, be ignored in implementations. But a theoretical discussion proceeds more smoothly if we speak of all of MID[Q-n+1, Q-k]. The sender S sends channels 1, 2, ..., k in the clear (i.e. uncoded). But he needs to know how to encode channels k+1, k+2, ..., n. To do this the sender S can use elementary row operations to go from the already "triangularized" MID[Q-n+1, Q-k] to a "diagonalized" form SEN in which column k+1 has all zeros except for a 1 in the bottom row. Column k+2 has all zeros except for a 1 in the row above the bottom row, and column n has all zeros except for a 1 in the top row. This is a trivial variant of the process of reducing to Hermite normal form. Once he has produced the matrix SEN = SEN[k,n,Q], the sender S has finished his cool precomputation and he can start to encode and send. His encode amounts to

$$H(j) = \sum SEN(j,g)I(g) = \sum SEN[k,n,Q](j,g)I(g)$$

for each $j \in \{k+1, k+2, \ldots, n\}$, where the sum is over positive integers $g \leq k$.

The cool precomputation is linear algebraic, like the cold precomputation, but it is shorter. For a k-out-of-n p/s/r process it involves bringing an already triangularized matrix with n-k rows to a diagonalized form. This involves about (n-k)(n-k+1)/2 row operations. Therefore, approximately n(n-k)(n-k+1) arithmetical operations are involved, i.e. subtractions/additions (XORs) and Galois

field multiplication. It is only necessary to find n-k Galois field reciprocals if you do things carefully. This is helpful, since Galois field divisions require the Euclidean algorithm and are much slower than Galois field multiplications (unless we merely store arithmetical tables, an attractive expedient if $Q \leq 256$).

Consider a device built with Q synchronized parallel processors and a stored multiplication table they can all draw the same product from simultaneously. On such a device it would take about $c(n-k)\uparrow 2$ machine cycles for the computation, where c is around 10. Thus for GF(16) = GF(Q) (i.e. $2 \leq k \leq n-2 \leq n \leq 16$) we need 16 4-bit processors, a 16 by 16 table of 4-bit words (1 k bit ROM) and around $10 * 14\uparrow 2 = 1960$ machine cycles for parallel implementation on 16 processors. It would take about 50,000 cycles for implementation by one processor. This means a delay of several milliseconds before the sender S can send. For GF(256) we need 256 8-bit processors, a 256 by 256 table of 8-bit words (512 k bit ROM) and a delay of the order $10 * 254\uparrow 2 = 645,160$ machine cycles (i.e. about a second) before sending could start. With only one processor this delay could rise to $256 * 645,160$ which is approximately 200 million machine cycles. So it could take many seconds before transmission began. Of course the sender could send plaintext over the first k channels while waiting for the coding process for the last n - k channels to be found.

If some important (k,n) pairs were incorporated into firmware the sender's cool precomputation could be made part of the manufacturer's cold precomputation.

Turning to GF(65,536) a parallel implementation would need 65,536 16-bit processors, and $16 * 65,536\uparrow 2$ bits of ROM (i.e. 70 gigabits) The delay before sending could be as much as 40 billion machine cycles, an hour or so. Using just one 16-bit processor and doing the multiplications on the fly to dispense with the need for ROM could raise the delay before sending to years.

So, yet again, we see indications that 65,000 channels is a lot of channels to spread your messages among. But 256 channels once again looks very promising.

Let us summarize the second stage, the sender's cool precomputation stage. He extracts (from ENF) and row reduces (to a sort of Hermite normal form) the matrix MID[Q-n+1, Q-k] to produce an n - k by n matrix SEN[k,n,Q]. This matrix describes how to form the encoded words sent along channels k+1, k+2, ..., n at time t in terms of the "plaintext" words sent along channels 1, 2, ..., k at time t.

The work and memory required have upper bounds (since $n \leq Q$). These upper bounds are shown in the table below:

Field	Time to precompute by parallel implementation	Number and size of processors for parallel implementation	Storage required for SEN(k,n,Q)
GF(2↑4) = GF(16)	milliseconds	16 4-bit	1 k bit
GF(2↑8) = GF(256)	seconds	256 8-bit	600 k bit
GF(2↑12) = GF(4,096)	minutes	4,096 16-bit	300 m bit
GF(2↑16) = GF(65,536)	hours	65,536 16-bit	70 g bit

A computer program incorporating the cool precomputation is contained in Appendix H.

2.5 The third stage of computation in the mean cases of the Bloom p/s/r process, the hot precomputation.

The receiver R is moving right along, receiving all k plaintext channels from the sender S for a while, and then some channels fail. Using means which lie outside the scope of this Phase I SBIR effort, the receiver finds at least k channels which are still operative among the n channels the sender is using. He makes a choice of exactly k of these operative channels any way he chooses, perhaps by picking the first k of them but almost certainly in a predesigned automated manner. Such a choice amounts to an injection

$$w: \{1, 2, \ldots, k\} \to \{1, 2, \ldots, n\}.$$

Like the sender S, the receiver R has already singled out the matrix M[Q-n+1, Q-k]. In practice he has trimmed off all the zero columns on its right side.

On the face of things the receiver would have to use the information contained on the injection w to set up a way of using elementary row operations to do a reduction of MID[Q-n+1, Q-k] to a variant of Hermite normal form before real-time on-line decode could proceed.

This would appear to take as many as a small multiple of $n\uparrow 3$ operations in the small k case (since the relevant matrix is n-k by n). But there are artifices to reduce the computation time uniformly to yield a bound which is more like a small multiple of

$$P(n,k) = n * (n-k) * \min\{k, n-k\}$$

operations. Clearly $P(n,k) \leq (n\uparrow 3)/4 \leq (Q\uparrow 3)/4$, (the worst case being $k = n/2$).

Moreover P(n,k) is rather small (is less than $kn\uparrow 2$) if k is small, and is smaller still (is less than $n(n-k)\uparrow 2$) if n-k is small (i.e. if k is large).

The routines which achieve this improvement over straightforward linear algebraic row reductions are based on a trivial lemma, which is nevertheless worth stating.

Lemma: Let

$$\text{DATA} = \{1, 2, \ldots, n-k\} \cup \text{RANGE}(w)$$
$$\text{DESIDERATA} = \{1, 2, \ldots, k\} \setminus \text{RANGE}(w)$$
$$\text{DELENDA} = \{n-k+1, n-k+2, \ldots, n\} \setminus \text{RANGE}(w).$$

Then the sets DATA and DELENDA contain the same number of members. Moreover the set DATA is disjoint from both DESIDERATA and DELENDA.

Proof: Let A be the number of members of RANGE(w) which are no larger than n-k. In other words the set DATA contains A members. It follows that there are k - A members of RANGE(w) which exceed n - k. Hence the number of members of

$$\{n-k+1, n-k+2, \ldots, n\} \setminus \text{RANGE}(w)$$

is equal to

$$[n - (n-k)] - [k - A] = A.$$

Obviously DATA $\subseteq$ RANGE(w). On the other hand DESIDERATA $\cup$ DELENDA contains no member of RANGE(w). This ends the proof.

A computer program incorporating the hot precomputation is contained in Appendix I. The idea behind Stage 3, the receiver's cool precomputation in this program is to exploit the Lemma. It enables the receiver to avoid carrying out a complete row reduction of MID[Q-n+1, Q-k] to Hermite normal form. The DATA/DESIDERATA/DELENDA breakup of the set of column indices {1, 2, ..., n} has a partial reflection in the row indices of MID[Q-n+1, Q-k]. The result is that many rows are irrelevant to the production of the decode matrix COD_w described here. Moreover it is often possible to use this breakup to partition the rows of MID[Q-n+1, Q-k] into three sets, one of which can be ignored, and the second of which can be used to act on the third. A careful reading of the program will also explain the bound

$$P(n,k) = n * (n-k) * \min\{k, n-k\}$$

on the number of operations, a much smaller bound than the bound $n\uparrow 3$ which unimaginative use of standard linear algebraic techniques would suggest.

2.6 The fourth stage of computation in the mean cases of the Bloom p/s/r processes, the real-time on-line encode or decode

After finishing the third computational stage, the receiver's hot precomputation, the receiver R is ready to decode. He has a matrix REC whose rows are indexed by the set DESIDERATA, and which has n columns. Thus REC is no larger than a k by n matrix. Let $j \in$ DESIDERATA. Then REC(j,j) = 1. Moreover REC(j,k) = 0 for every $k \in$ DELENDA. Recall that + coincides with - in our Galois field GF(2↑b) = GF(Q)). It should be evident that the receiver can reclaim the word I(j) which has been sent along channel j at time t from the words H(w(g)) (where $1 \leq g \leq k$) according to the formula

$$I(j) = \sum REC(j,w(g))H(w(g))$$

for every positive integer j belonging to the set DESIDERATA. The sum above is over positive integer $g \leq k$.

Similarly the sender has used his cool precalculation to produce a matrix SEN such that

$$H(j) = \sum SEN(j,g)I(g)$$

for every integer $j \in \{k+1, k+2, \ldots, n\}$. The sum is over positive integer $g \leq k$.

The problem of the sender in encoding, and of the receiver in decoding, is to calculate quickly. This will be done as shown in Figure 1.6.1 above. So what remains is to multiply fast. And we can take advantage of the fact that in each of the top boxes in Figure 1.6.1 the multiplier remains fixed, though the multiplicands change with time. To carry out a multiply as fast as bits can be fed in is the goal. This can be done with systolic multipliers as shown in Appendix F.

To carry out a single GF(16) multiplication at maximum speed requires about 300 cells. To carry out 16 multiplications simultaneously requires about 4100 cells. The cells themselves consist of fewer than 10 active elements. So a very pessimistic estimate of the hardware required to carry out a GF(16) based p/s/r process is 100,000 active elements. This might require one or two programmable logic arrays.

The implementation of a GF(256) based p/s/r process would be larger. But, taking account of the fact that constant multipliers eliminate the need for flipflops in the basic cells in the implementation, we find that even GF(256) based p/s/r processes can be implemented using at most 256 PLAs. The chips for a p/s/r process involving at most 16 channels will cost less than $100 today. For a p/s/r process involving at most 256 channels the price would almost certainly be below $1000.

No pricing has been attempted, since no working prototypes exist. It seems likely that these cost estimates could be reduced substantially in a production mode. Other costs, such as boxes, wiring, etc. are easy to estimate.

There is an alternative approach which appears both faster and cheaper. The idea is to substitute memory for computations by storing tables of products and lists of reciprocals, perhaps even tables of quotients.

This is particularly attractive in the real-time on-line encode or decode since a single decoded channel (i.e. a single processor) keeps using the same multiplier. So each microprocessor can ask a common stored Q by Q multiplication table for a copy of the appropriate Q by 1 column corresponding to this multiply. Multiply thus becomes a memory fetch and the memory might even be resident on the microprocessor chip. A GF(16) based p/s/r process would need only 4*16 = 64 bits of memory for each microprocessor. A GF(256) based p/s/r process would require 8*256 = 2048 bits per microprocessor.

This sort of memory capacity goes for pennies. Of course, when there are $n = 65,536$ channels the picture changes. For each channel you need $16 * 65,536 = 400$ k bits of memory.

It is again worth stating explicitly that the decoding process and the encoding process are merely two variations on a theme. After cool precomputation the sender forms

$$H(j) = \sum SEN(j,g)I(g)$$

(where the sum is over positive integers $g \leq k$) to encode channel j for each $j \in \{k+1, k+2, \ldots, k+n\}$.

After hot precomputation the receiver forms

$$I(j) = \sum_{g} REC(j, w(g))H(w(g))$$

(where the sum is over positive integer $g \leq k$) to decode channel j for each $j \in$ DESIDERATA. So it suffices to describe the real-time on-line decode. The real-time on-line encode is more straightforward.

The abstract design shown in Figure 1.6.1 is the scheme which should be used. Once again we recall the need to maintain synchronization of channels in encode and decode. As in Section 1.4.1, it is easy to do.

2.7 Examples of Computations.

The programs contained in Appendices H and I have been used on examples, which are included. Appendix D gives tables of ENF for various fields GF(Q) produced by the cold precomputation program in Appendix H. Appendix E contains examples of the encoding and decoding processes as carried out in Stages 2, 3 and 4 by the programs in Appendices H and I.

3. Summary of tasks, work, discoveries, recommendations and alternatives.

The contract between AFOSR and YLYK Ltd. to perform the work reported on here describes two tasks.

Task 1: Implement the heuristic procedure described in Section 6 of the proposal by means of computer programs, in order to produce explicitly the hyperfast pool/split/restitute encode and decode algorithms of the Bloom technique. Analyze the results, putting the matrices in the most convenient form, using further computer programs if needed. Determine the explicit solutions of the cases of most practical importance.

Task 2: Develop a set of design principles for the implementation in hardware of such p/s/r processes by means of an existing 16-bit microprocessor.

Mathematically, hyperfast Bloom k-out-of-n p/s/r processes break up into cases and into stages. There are four "extreme" cases. The case $k = 0$ is silly. The cases $k = 1$ (send the same message on all channels) and $k = n$ (hope that all sent messages get to the receiver) are trivial within the present state of technology. The case $k = n-1$ is trivial from a mathematical and an engineering viewpoint. But it seems important and may not be currently in use. Its implementation should be separate from the remaining "mean" cases. This implementation involves $2n - 3$ bitwise XOR gates in the format shown in Figures 1.4.1 and 1.4.2. No precomputations are required. For $Q = 4000$ channels and $k = n-1 \leq Q$ this involves fewer than 8000 gates and a phase lag (as described above) of some 12 bits.

All other cases, i.e.

$$2 \leq k \leq n-2 \leq n \leq Q,$$

are called "mean" cases in contrast to "extreme" cases. In view of the facts turned up in the narrative above we make

Recommendation 1: Concentrate first on hyperfast Bloom p/s/r processes over GF(2), GF(16) and GF(256). Over GF(2) you can implement an (n-1)-out-of-n p/s/r process for any reasonable size n. It will act on 1-bit words. Over GF(16) you can implement a k-out-of-n p/s/r process whenever

$$2 \leq k \leq n-2 \leq n \leq 16.$$

It will act on 4-bit words. Over GF(256) you can implement a k-out-of-n process whenever

$$2 \leq k \leq n-2 \leq n \leq 256.$$

It will act on 8-bit words. These three implementations will be general purpose (i.e. the boxes will allow the user to vary k and n).

Recommendation 2: Somebody who intends to use more than 256 channels should consider dedicated (i.e. k and n fixed in firmware or hardware) boxs and should try strenuously to keep the number of channels small. 4000 channels seems to be at or above the technically feasible upper bound.

For the mean cases

$$2 \leq k \leq n-2 \leq n \leq 2\uparrow b = Q$$

of a Bloom p/s/r process there are four stages of computation. The first two are noncritical straightforward linear algebraic reductions and we will not consider them further here, except to make

Recommendation 3: If a particular parameter setting (k,n) will be widely used (e.g. all F16s will always communicate with base by means of a 77-out-of-92 p/s/r process) then the second stage of precomputation, the sender's cool precomputation, can be dispensed with (more exactly, can be incorporated into the cold precomputation performed before the boxes are manufactured) in boxes dedicated to 77-out-of-92. The third stage of computation, the receiver's hot precomputation can be performed expeditiously.

A dedicated box can also free a participant in a battle from unnecessary attention to details. It will usually be cheaper than a general purpose box.

Recommendation 4. Maintain synchronization of parallel channels in encode and in decode. Do this by "doing something" trivial to the plaintext channels so that they acquire the phase lag associated with the channels which are encoded (or decoded) nontrivially. We have already discussed the obvious, and inexpensive, expedients which suffice to maintain such synchronization.

Recommendation 5: If it is desirable to combine p/s/r processes with cryptography or conventional error control then the following architecture should be employed. Encryption should precede p/s/r encoding which should, in turn, precede conventional error control encoding on the sender's end. By the same token conventional error control decoding should precede p/s/r process decoding which should, in turn precede decryption.

If it were built today a memory intensive ultraparallel prototype of a general purpose k-out-of-n send/receive box for $2 \leq k \leq n-2 \leq 254$ would be configured as follows. It would have about 1 mbit of ROM, broken up into 512 kbits to store a GF(256) multiplication table, 510 kbits to store ENF and 2 kbits to store a list of reciprocals of the nonzero elements of GF(256). For these purposes four 256 kbit (= 2↑18 bit) ROM chips will suffice. The box would employ 256 8-bit processors, perhaps Z80s, to do the cool precomputations (when switched on send mode) as well as the hot precomputation (when switched on receive mode). There would be no logical harm, and only a small time penalty if n is over 200, in having the precomputations done as if n = 256, the maximum number of channels. Cool and hot precomputations would take about a second.

The real-time on-line decode would be done by 65,536 = 256↑2 dedicated "dumb" processors. The processors will be arranged in 256 clusters of 256 processors. There might be as many as 256 dumb processors on one PLA chip. During a given session (i.e. for given k and n in send mode, and for given k, n and w in receive mode.)

Each processor would use a 2048 bit RAM which stored an appropriate column of the GF(256) multiplication table in ROM. This RAM will have been filled by the Z80s during precomputation. The 8-bit word arriving on channel i will be split into two copies eight times so that a copy of each arriving word goes into each cluster of 256 processors on its ith channel. When a word arrives at a dumb processor the processor multiplies that word by its session constant, (i.e. treats the word as an address and outputs the contents of that address). After that the outputs from each cluster are XORed together through 8 layers as in a deeper version of Figure 1.6.1. This yields decodes or encodes for each channel. This requires 128 mbits of RAM and 65,000 (extremely) dumb units capable only of outputting the contents of an address. This configuration would require 512 RAM chips with 256 kbit capacity each. We have noted that one mbit of ROM will also be needed, as well as 256 Z80s. The dumb procesors can be parts of a PLA. Presumably some 256 PLA chips would be capable of holding the needed 65,536 processors.

The system would require shift register storage devices (perhaps 1000 cells per register) and would have to verify synchronization of inputs and impose synchronization of outputs. This would require some sort of synchronization pulses in the bit streams entering and leaving the box. A promising method is to use two voltage levels for bits and a third for synch pulses, as is standard in television transmission in the U. S.

These estimates are all on the highly pessimistic side, since detailed hardware design has not yet been undertaken.

A smaller device in which $2 \leq k \leq n-2 \leq 14$ would require sixteen 4-bit microprocessors, less than 3 kbits of ROM, 256 dumb processors and 16 kbits of RAM. Phase lag would be about 10 bits.

The splitting scheme in Figure 3.1.1 looks forbidding in two dimensions. But in three dimensions it is very simple, no matter how many channels there are. Figure 3.1.2 is a different rendering of the same process. It suggests regularity of the architecture more directly.

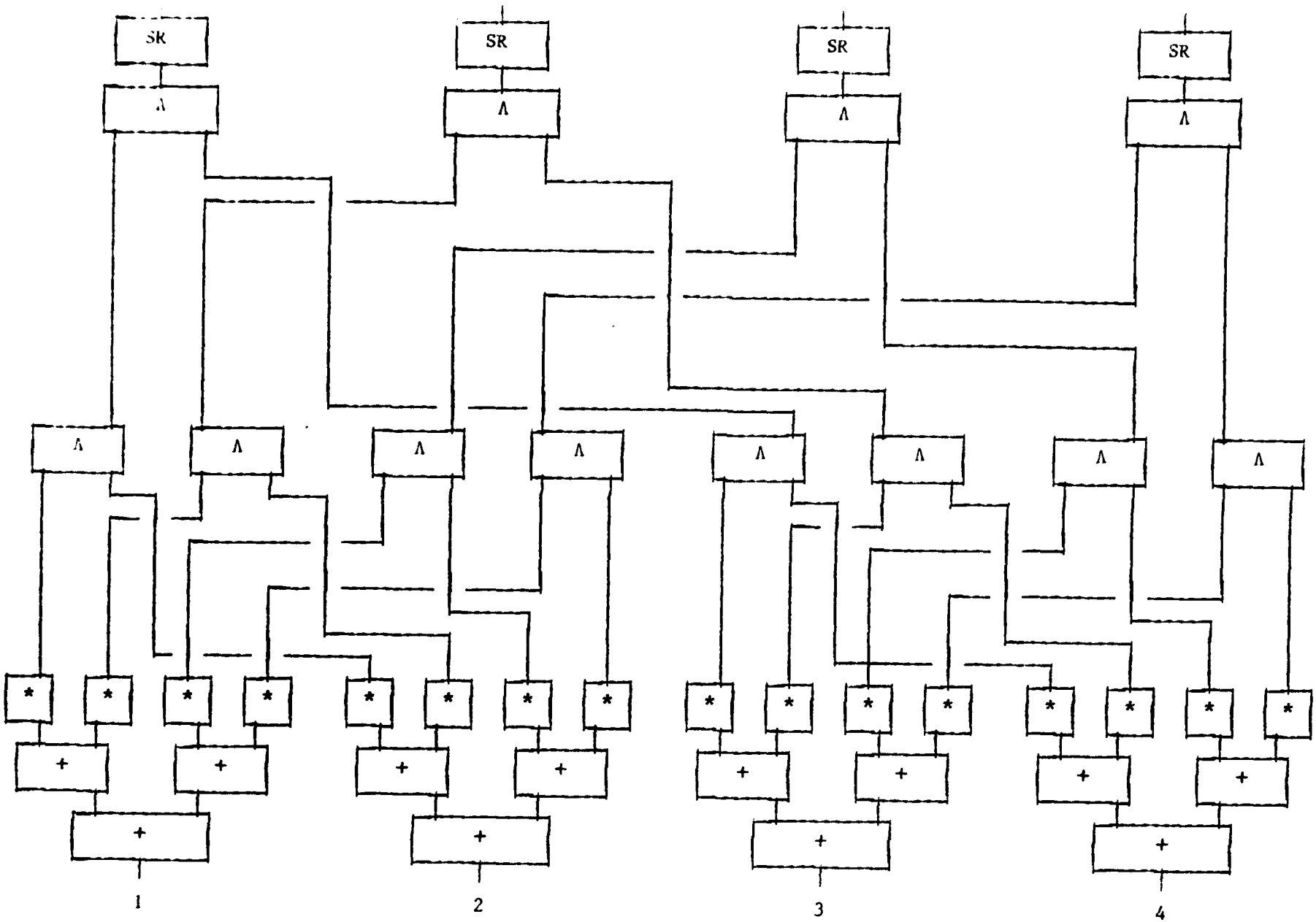

Figure 3.1.1. Abstract schematic for decode of four channels. Information flows downward.

SR = shift register * = multiply successive inputs by a constant multipler

+ = XOR of words Λ = take input, make two copies of it, and output one copy on each of the two outgoing lines

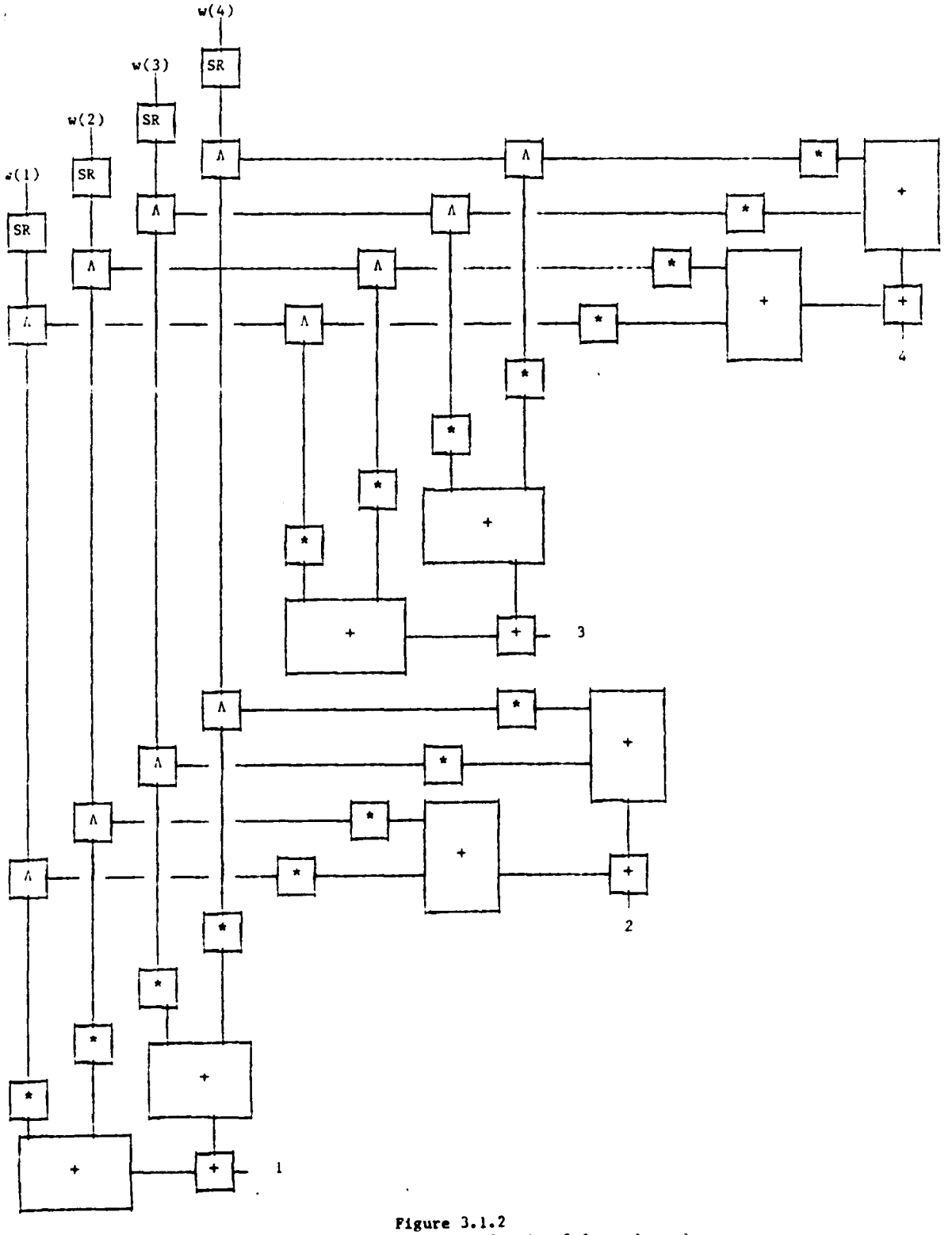

Figure 3.1.2
Alternate abstract schematic for decode of four channels

There are a number of choices facing somebody who designs hardware implementations of p/s/r processes.

YLYK Ltd. has found a very large number of ways to decode. We finally fixed on the DATA/DESIDERATA/DELENDA approach to minimize the number of row operations at the receiver's hot precomputation stage. But other more pedestrian approaches sometimes use less computer code. In subsequent efforts, these alternative approaches should be borne in mind. Which one is used depends on what aspect of the decoding process is most important. Our approach was to minimize the time interval between discovery of what channels were inoperative, and beginning of real-time on-line decode.

There is one alternative which should be resolved as late as possible in an SBIR Phase II effort to produce a prototype. The reason for delaying a decision is the continual shift in relative costs and speeds of hardware in the marketplace. The alternative in question is whether to use computation or memory to do Galois field multiplies and divides. One the one hand there are systolic multipliers. On the other, a table of GF(16) products requires only 16*16*4 = 1024 bits of memory. The table below tells the story for various fields.

Field	Number of bits to store table of products	Number of bits to store table of quotients	Number of bits to store list or reciprocals
GF(16)	16*16*4 = 1024	16*15*4 = 960	15*4 = 60
GF(256)	256*256*8 = 512k	256*255*8 = 510k	255*8 = 2k
GF(4,096)	202 m	202 m	50 k
GF(65,536)	69 g	69 g	1.1 m

Memory is cheap. The problem is speed. If words can be accessed quickly enough, the use of lookup for multiplication and division is attractive. XOR of words will, of course, be used for addition and subtraction.

Consider a GF(16) based p/s/r process. If each of 16 4-bit microprocessors has 64 bits of memory "on-chip" the receiver's hot precompute can load the appropriate column of the multiplication table into these 64 locations on each microprocessor. This will reduce multiplication to a lookup of a 4-bit word on a list of 16 words. A GF(256) based p/s/r process would need 2048 bits of memory "on-chip" available to each of the 256 processors used in real-time on-line decode. Multiply would be lookup of an 8-bit word on a list of 256 words after the appropriate column of the multiplication table had been loaded into a given processor. What we have said about real-time on-line decode applies also to real-time on-line encode, of course.

The relative merits of this approach, as opposed to a systolic system for computing products algorithmically, could change drastically as new products came onto the market or the prices of old products fell.

Another unresolved alternative concerns all three stages of precomputation. Should we use many "smart" existing processors for the precomputations or smarten up the custom designed processors used for real-time on-line encode or decode so that they can carry out the precomputations as well as the encode/decode?

Many of the cheapest old 4-bit and 8-bit processors operate below 1 mhz, whereas newer more expensive PLA can be driven faster. It would take development time to configure smart PLA to perform precomputations, whereas existing processors can be quickly programmed. It seems prudent to delay this decision as long as possible, with a view to the state of the components market the day it is made. Other choices seem more straightforward. It hardly seems worthwhile to try to fine tune field size so as to get, for example, a 17-out-of-34 p/s/r process over GF(32). The simplicity of assuming that n is no larger than the field size is worth seeking. Possible exceptions to this approach can be made on an individual basis, and will likely lead to a dedicated single purpose box, such as 3-out-of-6 p/s/r process over GF(4).

4. Future.

At this point, what remains is to cast the p/s/r processes into hardware. Three obvious general purpose (i.e. variable k and n) implementations would be:

1. (n-1)-out-of-n, for $n \leq 1000$ using GF(2) arithmetic on 1-bit words and requiring no precomputation;

2. k-out-of-n, for $2 \leq k \leq n-2 \leq 14$ using GF(16) arithmetic on 4-bit words and requiring precomputations of a few milliseconds in (cool) Stage 2 and (hot) Stage 3;

3. k-out-of-n for $2 \leq k \leq n-2 \leq 254$ using GF(256) arithmetic on 8-bit words and requiring precomputations lasting about a second in Stage 2 and Stage 3.

It would be interesting to produce a few dedicated (i.e. fixed k and n) implementations such as:

4. 3900-out-of-4000 using GF(4,096) arithmetic on 12-bit words (in practice they would probably be the last 12 bits of 16-bit words) no Stage 2 precomputation, and a several second Stage 3 precomputation.

5. 100-out-of-4000 using GF(4,096) arithmetic, no Stage 2 precomputation and a several second Stage 3 precomputation.

6. Some half-and-half implementation, i.e. a k-out-of 2k for the largest value of k which would yield a tolerably short Stage 3 (hot) precomputation. Possibly a 500-out-of 1000 implementation using GF(1,024) arithmetic on 10-bit words could hold the Stage 3 precomputation down to just a few seconds.

One mathematical topic which was not targeted for the Phase I SBIR effort is dynamic reconfiguration. Suppose a sender and a receiver start out using a 200-out-of-250 p/s/r process to communicate over 250 channels which are all operative at the start. Suppose that a new

channel goes down every few seconds. It is probably possible to do the necessary reconfiguration precomputations one at a time after each failure so as to keep communications going with negligible interruptions as the receiver migrates from one set of 200 channels to another "nearby" set of 200, to another, and so on.

Careful analysis might be able to reduce the Stage 3 hot precomputation times, given that only one channel at a time goes down. The viewpont of this proposal is that the receiver deals with n-k channel failures at once.

An engineering/ergonomics consideration which will have to be tackled in Phase II, or shortly after, is the question of how the receiver will ascertain which channels have gone down. Will it be by human decision that a channel carries nothing or carries garbage? Or will it be by some automated means of sensing when a channel goes sour statistically, and is therefore presumed to be down? Or will it be by sending periodic check sequences on each channel, the idea being that their absence on the receiving end signifies channel failure? Or will still some other system be used? There are many existing protocols and algorithms to sense when a channel is or is not operational. If possible a p/s/r process box should be a module in a larger system. This architecture would enable the user in the field to decide which method of sensing inoperative channels is appropriate to the system in use.

Such considerations may or may not influence the p/s/r hardware directly, but will certainly be important in the context in which a p/s/r process is imbedded. Matters of this sort will be taken up in more detail in YLYK Ltd's SBIR Phase II Proposal to AFOSR. Up to now speed has been the dominant consideration. In Phase II cost will come more to the fore.

Appendix A

The technical part of the YLYK Ltd. proposal
which led to this contract

U.S. DEPARTMENT OF DEFENSE

SMALL BUSINESS INNOVATION RESEARCH PROGRAM
PHASE I—FY 1983
PROJECT SUMMARY

FOR DOD USE ONLY

Program Office	Proposal No.	Topic No.

TO BE COMPLETED BY PROPOSER

Name and Address of Proposer

YLYK Ltd.
PO Box 7966
Ann Arbor, Michigan 48107

Name and Title of Principal Investigator

Mr. Bob Blakley
President, YLYK Ltd.

Title of Project

High-speed low-cost ways to get messages from a sender to a receiver when some channels linking them become inoperative.

Technical Abstract (Limit to two hundred words)

Military communications systems are subject to trauma. Certain channels fail for protracted periods of time. The red noise problem arises when some, but not all, of the channels linking a sender to a receiver become inoperative. The solutions to this problem are called pool/split/restitute processes. P/s/r processes amount to ways to encode digital messages at a sending node so as to make sure that all transmitted information gets through and is decoded correctly at the receiving node whenever at least k out of the n channels linking those two nodes remain operative. P/s/r processes are designed to work even though the sending node has no way to tell which of the channels it is using are inoperative.

It has been known for at least two years that the encode and the decode operations in a p/s/r process are faster and simpler than those in any but the weakest and most trivial error correcting codes. Moreover the bandwidth expansion is typically smaller in a p/s/r process than in an error correcting code adapted to do the same job. This project is aimed at producing a further orders-of-magnitude improvement in the theory of p/s/r processes. This carries over into a comparable improvement in implementing them.

Anticipated Benefits/Potential Commercial Applications of the Research or Development

The availability of best-possible p/s/r processes to solve the red noise problem will make it cheap and easy to design fault-tolerant or fail-safe communications systems at all levels of complexity, from the microscopic to the global. The ability to overcome the unpredictable permanent failure of a certain specified proportion of the channels of communication in a system may have major consequences in chip layout, design of wiring within military platforms, commercial vehicles, telecommunications networks, and global C^3I structures. The speed and simplicity of the implementation of p/s/r processes gives promise of widespread cheap channel-failure insurance in gigabit per second communications.

3. Identification and significance of the problem/opportunity.

This proposal deals with research and development work on the red noise problem [AS82].* It is one facet of the message gap [BR81; AN83]. It is associated with the difficulty experienced by two or more centers in communication with one another when a catastrophic long-lasting failure of some of the communication channels linking them occurs.

More specifically, the red noise problem concerns a sending node and a receiving node linked by several parallel channels over which information is moving in digital form. The problem is this. Suppose you are prepared to accept the failure of n-k out of the n channels which are initially functioning. How do you encode the information at the sending node so that all of it gets through as long as any k channels remain operative? How do you decode this information at the receiving node? Ways of doing this are called pool/split/restitute processes.

Examples of systems faced with the red noise problem are numerous. A few of them are:

I. Within a single vehicle or platform -- such as a missile, an aircraft, a ship, a tank or a spacecraft -- there might be eight separate wires or fibers carrying information from an area containing power supplies, engines, control devices and weapons to an area containing human or electronic controllers. It is imperative that the controllers continue to receive all of the highest priority types of information even though three wires (nobody knows in advance which three) or fibers are cut by accident or trauma. This guaranteed 5 out of 8 reliability may have to be cheap in the sense that it must be provided by tiny inexpensive circuitry;

I. At the global level or the theater level, consider communications between commanders and subordinates, or between separate command centers (whether these are vehicles or cities or redoubts or satellites is irrelevant mathematically) connected by ten communications channels. Several of these channels might be optical fibers, several might be microwave relay tower chains, and a few might be satellite relay links. In the event of emergency it might be imperative for all high level communications to get through continuously after six of these ten channels fail, even when the sender does not know which four of his outgoing channels are successfully carrying their information to the intended receiver. It might be imperative to provide this guaranteed 4 out of 10 reliability to communications systems working at very high bit rates;

III. On the microscopic scale, VLSI and VHSIC are forcing more active elements and more pathways onto a chip. It is increasingly important to assure the safe arrival of every bit at the proper place in timely fashion even though certain circuit elements fail. This must be done in an extremely simple way so as not to gobble up too much of the chip just for this assurance of reliability. Perhaps it would be desirable to use an 8 out of 10 p/s/r process to move a 16-bit word from memory along ten 2-bit channels so that the whole word gets through despite the failure of any two of those ten channels.

IV. The word "channel" should not be allowed to obscure the abstract possibilities. Separate packets in a local area network can be treated as separate channels since each packet can be 500, 1000, 2000 or some such large number [ST83] of bits. The bits in a single packet get through all together

*Footnote: All entries in square brackets refer to the bibliographic citations list beginning on page 17.

or not at all, according as the packet reaches its destination, or else is destroyed in a collision or otherwise goes astray [BL83a, p. 5; P082, pp. 76-101]. If collisions and misroutings are present, but rare, a 63 out of 64 p/s/r process applied to successive batches of 63 packets from a given sender to a single receiver might provide cheap insurance at a bandwidth expansion of 1/64 = 1.5%.

Obviously, comparable examples could be produced in many other contexts. But abstractly they all point up the same need. It is important to find extremely simple encode/decode schemes to provide cheap ways of assuring very high bit rate solutions to the problem of getting all the important information from sender to receiver whatever channels remain -- in the absence of prior (or even concurrent) knowledge of which channels are the lucky survivors -- as long as there are enough channels still operative to come up to the initial specifications.

This might sound reminiscent of the use of error correcting codes to correct burst errors, and in a way it is. However, during the two years since the red noise problem was recognized [AS82] as important in its own right, tailor-made solutions have been advanced which are much cheaper (algorithmically, but this entails a comparable dollar saving in implementation) than, and much faster than, the use of standard error correcting code techniques [BL83a, pp. 367-389; MC77, pp. 181-186, 212-213; BE68, pp. 393-394; VI79, pp. 227-300] to solve it.

A moment's reflection shows why this might be so. Error correcting codes are designed to deal with errors occuring anywhere in the transmitted data stream (as long as these errors are not too numerous) [VI79, p. 34]. These errors can be very irregularly spaced. In a mathematical sense which should become clearer below, red noise errors can be viewed as occurring with a definite periodicity in the received bit stream. Such a well behaved type of error, of course, constitutes a subproblem of the general error correction problem. So it seems plausible (and turns out actually to be the case) that the solution might be conceptually simple, as well as easy to implement in a cheap fast way. The recent literature [AS82] and some as yet unpublished work, bears this out. But in 1983 a further remarkable simplification and speedup of both the encoding and decoding processes used to solve the red noise problem has been suggested by current research. Several important instances of this further orders-of-magnitude improvement have been discovered and verified as the result of a powerful heuristic principle. The research on this project will attempt to turn this heuristic principle into a rigorous tool for producing this orders-of-magnitude improvement of both the speed and the cost of the encoding/decoding schemes for combatting red noise in many or all cases of the problem. It aims to produce a complete taxonomy of best possible (or, more properly speaking, almost best possible) solutions of the red noise problem. Time permitting, it will make a preliminary abstract analysis of how to design electronic implementation of these coding/decoding processes using cheap off-the-shelf components to attain bit rates well above a megabit per second.

4. Background, technical approach and anticipated benefits.

4a. Background. An understanding of the red noise problem and the objects which solve it, namely pool/split/restitute processes, is best acquired by looking at the history of the last five years. In a 1978 NSF proposal, Blakley invented a new cryptographic object, the threshold scheme (He called it a key safeguarding scheme, but Denning's well known cryptography and data security textbook [DE82] has made threshold scheme the standard terminology). His paper describing the notion, and giving the first example was presented at NCC '79 and published [BL79] in the proceedings of that meeting.

A k out of n threshold scheme is a mathematical way of utilizing a source of random bits to take an important piece of digital information, called a substance

(there isn't much harm in thinking of a substance as just being a plaintext message) and produce n output pieces of information called shadows of the original substance. A shadow can, without too much inaccuracy, be thought of as being part of a ciphertext message. Every shadow is about the same size as the substance and, collectively, the shadows securely carry the full import of the substance in the following sense. There is, on the one hand, a trivial algorithm which can reproduce the substance if any k of the n shdows are inputted to it. But, on the other hand, it is impossible to gain any inkling of the value of the substance on the basis of knowledge of only k-1 or fewer of the shadows. The justification of this latter statement is somewhat technical. Nevertheless the basic idea can be expressed fairly briefly in terms of what Konheim [KO81, p. 31] calls the Bayesian opponent. Just as it is possible to prove [BL81a] the one-time pad [DI79 pp. 399-400, DE82 pp. 86-87] perfectly secure in the Shannon [SH49] sense, so it is possible to prove that a k out of n threshold scheme is (Shannon) perfectly secure up to threshold k. This means that the Bayesian opponent cannot modify a (perhaps shrewd) initial guess regarding the substance on the basis of knowledge of only k-1 shadows. Somewhat more formally:

> A posteriori probability that the substance has a value equal to S (given that the objects h(1), h(2), ..., h(k-1) are known to be shadows of that substance) = A priori probability that the substance has a value equal to S.

To be more concrete, suppose there is a roll of magnetic tape (the substance) which contains the full inventory of payloads, locations and targets of all missiles belonging to A on day D. Somebody might think this information important enough to merit protection by a 4 out of 9 threshold scheme. This will involve use of a trivial algorithm which takes this original roll of tape, together with 4 tape rolls worth of random bits, and produces 9 rolls of mag tape (the 9 shadows of the original substance) as outputs. Now an opponent of A, let us call it R, might quite correctly suspect at the outset that several of these missiles are targeted on some important spot, call it M. But if R can only obtain 3 of the (shadow) rolls of mag tape it cannot shed any new light on this initial conjecture. It started out with a good bet that its conjecture is correct. It winds up with exactly the same odds. If R can get 4 of the 9 rolls, of course, the game is over. It has crossed the threshold of information and can reconstruct the entire original roll of mag tape. So it knows everything A does.

Shamir, by the way, introduced the threshold terminology in a paper [SH79] which independently invented the idea of threshold scheme a few months after [BL79], and gave a better example of how to implement the notion. After the Blakley [BL79] and Shamir [SH79] papers appeared, several people interested in information theory and computer science took up the topic. Asmuth and Bloom [AS81] produced a huge family of threshold schemes, of which Shamir's was a special case. They also gave the only way known to date for "spoofproofing" a threshold scheme, a notion we won't consider further here. But they paid a price for this extra feature, a small departure from Shannon perfect security. Then Bloom [BL81b] generalized the one-time pad (really the 2 out of 2 case of a threshold scheme, rather than a true [BL80; DE82, p. 152] cryptosystem) so as to produce essentially the fastest possible threshold scheme. He also noted that it is possible to reduce message expansion in a threshold scheme, but only at the cost of reducing security.

Blakley [BL79], Shamir [SH79], Asmuth and Bloom [AS81], and Bloom [BL81b] independently discovered that any k out of n threshold scheme which made use of a finite field [JA64, pp. 58-62; PL82, pp. 44-58; BI83a, pp. 65-92] required that the field contain at least n elements. Bloom gave a persuasive argument [BL81b] to the effect that this was necessary in order to attain Shannon perfect security. Davida, DeMillo and Lipton [DA80] produced another threshold scheme. Hellman, in company with his students Karnin and Green [KA81], produced schemes without sharp thresholds and

showed that adding certain desirable features to threshold schemes necessarily impairs Shannon perfect security, thus explaining what Asmuth and Bloom [AS81] had observed regarding spoolproofing. McEliece and Sarwate [MC81] produced yet another threshold scheme, and drew the theories of threshold schemes and of error correcting codes into a single compass by exhibiting an explicit relationship between Shamir's [SH79] scheme and Reed-Solomon codes [RE60; BE74, pp. 70-71].

Two aspects of threshold schemes worth noting explicitly are:

I. Threshold schemes are related to error correcting codes. But the "decode" in a threshold scheme is trivial, whereas decode can be a formidable [BE78; NT81] problem, even an NP-complete [GA78] problem, in an error correcting code.

II. As of 1982, most k out of n threshold schemes made use of finite fields (Galois fields) [JA64, pp. 58-62; MA77, pp. 93-124; PE72, pp. 144-169]. All [AS83; BL79; BL81b; SH79] such schemes required a field with at least n elements.

Last year, Asmuth and Blakley [AS82] explicitly enunciated the red noise problem and solved it by means of a p/s/r process based on the Chinese Remainder Theorem. This p/s/r process could be viewed as being just "an Asmuth-Bloom threshold scheme completely lacking in cryptographic security". Its great advantage was its flexibility in dealing with information sources with very different bit rates. Years ago Stone [ST63] had used much the same approach to solve a problem in the theory of error correcting codes.

4b. Technical approach. With this background it is now possible to give the general framework of the present research. The principal investigator, Bob Blakley, has already taken a Bloom threshold scheme and produced from it the corresponding p/s/r process. It will be called, simply, a Bloom p/s/r process below. He has simulated its operation on a high speed digital computer.

The k out of n case of this Bloom p/s/r process works as follows. Suppose that b is a whole number (positive integer [MA67, p. 47]) so big that $2^b \geq n$. Then any ancestral list (a(1), a(2), ..., a(k)) of k words [MA67, p. 43] (each of which is a b-bit word) is turned into a descendant list (d(1), d(2), ..., d(n)) of n b-bit words. This is the encode (i.e. the pool/split) process. It is done in such a way that any k-word sublist [MA67, p. 228] (d(j(1)), d(j(2)), ..., d(j(k)) of the descendant list (d(1), d(2), ..., d(n)) contains enough information to reclaim the ancestral list (a(1), a(2), ..., a(k)) in its entirety. This is done by a decode (i.e. restitute) process which uses no more than trivial linear algebra over the finite field $GF(2^b)$. By comparison with threshold schemes and error correcting codes this Bloom-style p/s/r process has the following features.

I. Its k out of n case effects only (n/k)-fold message expansion. Thus its 8 out of 10 case effects a 25% message expansion (from 1 unit to 10/8 = 1.25 units). This expansion is quite obviously best possible for a scheme which can recover eight b-bit ancestral words from any eight of ten b-bit descendant words.

II. The Bloom p/s/r is, to all intents and purposes, the p/s/r process which uses the smallest possible number of arithmetic operations in the finite field it utilizes. Its "encode" (i.e. pool/split) and "decode" (i.e. restitute) processes are both trivial, exhibiting much less computational complexity than the decodes in any error correcting code which might be adapted to do the same job. The reason for this is that the error correcting code exhibits overkill because it is a general purpose tool. It is invented to deal with many more types [HA80, p. 24] of "errors" than one encounters when dealing

with red noise. This p/s/r process is a special-purpose tool for dealing with red noise.

III. P/s/r processes are not cryptographic objects in any sense of the word. They do not involve any type of cryptosecurity. They do nothing more than guard against loss of signal, and therefore fall within the general area of error control.

4c. Anticipated benefits. The linear algebra of large finite fields can take many machine cycles per multiply or divide. It can also, in the worst circumstances, make considerable demands on memory. During 1983 a heuristic principle has come to light which massively reduces this aspect of the computation in numerous cases. From Bloom p/s/r processes it produces hyperfast p/s/r processes which encode and decode bytes or larger words in less than ten machine cycles (on highly parallel processors) for almost all practical choices of k and n. This heuristic suggests the possibility of comparable reductions in many other cases. Consider an example which at first blush seems extreme. In April, 1983 we have reduced the memory requirements for one implementation of a 60 out of 62 scheme by orders of magnitude. As regards the parameters, 60 out of 62, one cannot readily conceive of so many fibers joining two nodes. But, returning to the packet-switching example above, it is easy to imagine one or two packets out of sixty going astray. Also, recently developed continuously reconfiguring multimicroprocessor control systems [EL83] appear to have many virtual channels.

At any rate it appears that this heuristic principle -- already successful in making a k out of k+1 or a k out of k+2 Bloom p/s/r process capable of decoding in something like 3k machine cycles on an ordinary microprocessor, and in about log(k)+2 cycles on a parallel processor -- will lead to ways to reduce the run time of hardware implementation of all k out of n schemes by comparable amounts. This should make them able to run on gate arrays, programmable logic arrays or other standard cell [NE83, pp. 470-471] hardware, or even other cheap off-the-shelf devices, at rates well above the megabit per second range.

The ability to code and decode at such bit rates becomes increasingly desirable with the emergence of tiny cheap cleaved coupled-cavity lasers [TH83]. They make it possible to use a 73 mile fiber without a repeater [AB83] to communicate at 420 megabits per second with an error rate of 10^{-9} [TH83; LI83, p. 363]. It seems likely [GO83] that terabit per second communication systems are in the offing now that 30 femtosecond light pulses are available. Theoretically, further orders-of-magnitude improvements in processing gains because of exploitation of photonic efficiency of detectors [GA83, p. 526] as well as by means of preservation of polarization [RA83] are possible even after that. Until optical computers are developed we will need code/decode schemes of minuscule computational complexity to deal with such bit rates.

The hyperfast p/s/r processes have a further advantage, in addition to low computational complexity (which amounts to high-speed low-cost implementability on simple hardware). They can also be implemented in a highly parallel way, so that separate devices can do concurrent decoding for separate channels, and each device can do many operations in parallel.

It is now clear how to move digital information with minimum redundancy and maximum speed (an unusual plus, best possible in two ways) at a modest dollar cost (which does, however, rise with desired data throughput rate) so as to overcome a predetermined level of threat of channel failure.

Presumably the existence of such a capability could affect the design of everything from chips to the fiber "wiring" of missiles, ships, tanks, planes and the

design of C^3I systems in the future. It certainly helps implement the military digital switching systems criteria [R083, p. 19] of survivability, endurability, distributed communications, responsiveness, efficient spectrum utilization, and cost effectiveness. Thus this proposal arguably addresses 6 of the 9 military operational system requirements detailed in [R083, p. 19].

4d. Foundations for Phase II. By the end of Phase I all the algorithmic principles needed to encode and decode in a hyperfast p/s/r should be known for every n and k. Basic principles of design for hardware implementation of these algorithms should also be available. The design principle will lead one way if minimum cost is the ultimate goal, another way (high parallelism and custom design) if ultrahigh speed is the overriding aim, and still a third way if a single unit is to be used for various different values of k and n.

But in any case the abstract basis on which to proceed to build bench systems, and then prototypes of field systems, will be firmly in place. The actual design and testing program can begin as soon as Phase I is complete.

5. Phase I technical objectives.

In Phase I we hope to exploit the heuristic described in Section 6 below to use the Bloom approach to suggest a new collection of p/s/r processes, the hyperfast p/s/r processes. The k out of n case can be expected to restitute (i.e. decode) 15 bits of the information contained in one input channel using fewer than 3k machine cycles on an existing 16-bit microprocessor if $n - k$ is smaller than 16. The memory requirement for table lookup implementation will be well under 1000 bytes.

The various channels can be recovered concurrently on separate machines if so desired. Since the decode process is simply a linear combination $\Sigma\, c(i)d(i)$ of received words $d(1),\ldots,d(k)$ with fixed coefficients $c(1),\ldots,c(k)$ it is even possible to design a vector microprocessor machine which can move 2k words concurrently, then perform k products by table lookup concurrently, then add (which is just XOR in $GF(2^b)$, and thus has no carry propagation) k summands in a single operation. The vector fetch and the vector dot product $\Sigma\, c(i)d(i)$ can, in theory, be done in 1 cycle each. The XOR of k summands can be done in $\log(k)$ cycles or fewer (the log being to base 2). It is even possible to use VLSI to produce a cheap ultraparallel implementation in terms of hardwired functions with more than two inputs if n is not too large.

Examples of times to restitute 15 bits on one channel in implementing the k out of n case of such a hyperfast p/s/r process appear to be:

k	n	number of cycles (ordinary microprocessor implementation)	number of cycles (k-vector microprocessor implementation)	number of cycles (ultraparallel implementation)
1	10	3	3	4
2	10	6	3	4
4	10	12	4	4
8	10	24	5	4
16	30	48	6	4
32	40	96	7	4
64	70	192	8	4
128	140	384	9	4
256	270	768	10	4
512	520	1536	11	4
1000	1019	3072	12	4

The expected form of the encode algorithm is so similar to the expected form of the decode that we will not discuss it here. See Section 6 below.

The first technical objective of Phase I, then, is production of the encode algorithm and the decode algorithm for the k out of n case of a hyperfast p/s/r process. Each of these is in the form of a bunch of separate and independent dot products in k or n dimensional vector spaces over $GF(2^b)$ for some positive integer b near log(k).

The second technical objective is a portfolio of abstract design principles for implementation of such a p/s/r process. YLYK Ltd. plans to sketch the abstract principles behind implementing such a k out of n p/s/r process by means of an existing 16-bit microprocessor, an existing programmable logic array or gate array, and a hypothetical vector microprocessor with a 16-bit word size, and vectors of up to 1024 words.

It is to be emphasized that the plan for Phase I is to deliver the encode and decode algorithms in definitive and final form. But YLYK Ltd. will only sketch, as time allows, the basic abstract features of hardware implementation. YLYK Ltd. will not produce hardware, or even the final design of hardware, in Phase I.

6. Phase I work plan.

It is no longer possible to avoid technicalities. Before we describe the heuristic device for producing these cases of hyperfast p/s/r processes and, thereafter, finding the general hyperfast p/s/r process it is necessary to look more deeply into the geometry of Bloom p/s/r processes. The collection

$$V(k,F) = \{(1,m,m^2,m^3,\ldots,m^{k-1}) \in F^k: m \in F\}$$

is in general position [YA68, p. 164; MA77, p. 326] in the k-dimensional vector space F^k over any field F. In other words, suppose that k lies between 2 and the cardinality [MA67, p. 53] of F. Then every k by k matrix of the form

$$\begin{bmatrix} 1 & m(1) & m(1)^2 & \ldots & m(1)^{k-1} \\ 1 & m(2) & m(2)^2 & \ldots & m(2)^{k-1} \\ \cdot & \cdot & \cdot & \cdot & \cdot \\ \cdot & \cdot & \cdot & \cdot & \cdot \\ \cdot & \cdot & \cdot & \cdot & \cdot \\ 1 & m(k) & m(k)^2 & \ldots & m(k)^{k-1} \end{bmatrix}$$

(where the m(i) are pairwise distinct) is nonsingular because it is a Vandermonde matrix [HO71, p. 125]. The formal definition, then, is that a set of vectors is in general position in a k-dimensional vector space W if every one of its k-member subsets is a basis for the space W. More important than what we said about V(k,F), but far less trivial, is the fact that

$$V^*(k,F) = V(k,F) \cup \{(0,0,\ldots,0,1,0,\ldots,0)\} = V(k,F) \cup \{e\}$$

(where the 1 is in any position) is also in general position. This requires use of the theory of symmetric polynomials [RE67, pp. 457-458]. So getting just one more vector into the set takes a lot more doing. But so far the extra effort seems essential to what we propose to do. The way a Bloom k out of n p/s/r process works is to take a fairly large set of vectors (at least n of them) in general position in the k dimensional vector space $GF(q)^k$ over the field GF(q) of q

elements. There's no harm in taking $V^*(k, GF(q))$ if $q \geq n$. Suppose that q is a power of 2, i.e. that $q = 2^b$. Suppose, also, that the p/s/r process is meant to work by accepting one b-bit word after another from each of k input channels (ancestral channels) at the source. It should then send one b-bit word after another down each of n descendant channels to the receiver. Each one of these descendant channels is identified with a vector belonging to $V^*(k, GF(2^b))$. Once some channels fail, and a decoding scheme is employed on k of the channels which still work, it acts the same way on every successive b bits in each channel. So it suffices to look at a single time slice through the system. In such a slice encoding is done by defining a linear map [H071, p. 67] $L : GF(2^b)^k \to GF(2^b)$ by setting

$$L(w(i)) = \text{the } i\text{th b-bit ancestral message}$$

for the vectors $w(1), w(2), \ldots, w(k)$, in some ordering of $V^*(k, GF(2^b))$, which corresponds to the k ancestral inputs. These are assumed to be sent unaltered down the first k descendant channels. In addition to that, the sender solves for any other member y of $V^*(k, GF(2^b))$ in the form

$$y = c(y,1)w(1) + \ldots + c(y,k)w(k)$$

as a linear combination of the $w(i)$ with coefficients $c(y,i)$ drawn from $GF(2^b)$. Down the channel corresponding to y is sent the message

$$Ly = L(c(y,1)w(1) + \ldots + c(y,k)w(k)) = c(y,1)Lw(1) + \ldots + c(y,k)Lw(k).$$

Addition is $GF(2^b)$ addition (i.e. exclusive or, XOR, of b-bit words) and multiplication is $GF(2^b)$ multiplication, since both $c(y,i)$ and $Lw(i)$ are members of $GF(2^b)$. All the linear algebra is a precomputation, of course. Hence the $c(y,i)$ are available before encoding starts. Decoding involves a once-for-all solution (another precomputation) of linear equations to find the $\{w(1),w(2),\ldots,w(k)\}$ in terms of a collection of any k of the y's. This gives the $Lw(i)$'s (the ancestral b-bit messages) in terms of the Ly's (the descendant b-bit messages). The whole thing works because any k members of $V^*(k, GF(2^b))$ are a basis for the vector space $GF(2^b)^k$, i.e. because of the general position assumption.

This sounds abstract, for the usual reason. It was written to fit into a small compass, without too many numbers and subscripts littering the printed page. But all the objects are explicitly given. For example, a 3 out of 7 p/s/r process could make use of the field $GF(8)$, the 3-dimensional vector space $GF(8)^3$, and the 9-member set

$$V^*(3, GF(8)) = \{(1,m,m^2): m \in GF(8)\} \cup \{\varepsilon\},$$

where ε is either $(0,1,0)$ or $(0,0,1)$. For a Bloom p/s/r process it doesn't matter which. For our purposes, building hyperfast p/s/r processes, the choice of ε seems to be crucially important. It appears to require an amount of trial and error tedious for humans, but trivial on a computer.

A k out of n Bloom threshold scheme would require use of $GF(2^t)$ where $2^t \geq n$. Thus a 990 out of 1000 scheme would require $GF(1024)$ multiplications. In table lookup mode this would require a table of over one million 10-bit words.

Obviously one would trade time off against memory. But then each multiplication would involve dozens of machine cycles, and each division could require hundreds. The simple heuristic we describe below says that the threshold scheme analogy is hopelessly pessimistic. A 990 out of 1000 hyperfast p/s/r process should require only GF(16) multiplications. This uses only a table of 256 four-bit words.

The heuristic for producing hyperfast k out of 2^b+1 p/s/r processes which use linear algebra over extremely small fields of characteristic two [JA64, p. 61; PL82, p. 46; BL83a, p. 80] goes as follows. Do not use just any collection of 2^b+1 vectors in general position over $GF(2^b)^k$. Use $V^*(k, GF(2^b))$, where the vector $\varepsilon = (0,0,\ldots,0,1,0,\ldots,0)$ is chosen by trial and error from among the k possible unit coordinate vectors [NO69, pp. 473-474] in $GF(2^b)^k$ to satisfy the following condition.

Heuristic: A k out of k+j hyperfast p/s/r process can be formed, in the Bloom manner, over $GF(2^b)$ if $j < 2^b$. Form a Bloom p/s/r process using $V^*(k, GF(2^b))$ for each possible choice of $\varepsilon = (0,0,\ldots,0,1,0,\ldots,0)$ and examine the corresponding coefficients c(y,i). There is a minimal ε, in the sense that all the c(y,i) for this ε belong to a smallest subfield of $GF(2^u)$, where $2^u \geq k+j$. This minimal ε may have the property that all c(y,i) belong to $GF(2^e)$, where $j < 2^e$, and where e is the smallest integer exponent for which this is true.

In the following paragraphs we will give some motivation for the heuristic. Here is a summary of the known cases of a hyperfast p/s/r process it has suggested, directly or indirectly:

4 out of 5 over GF(2), followed by general k out of k+1 over GF(2);
3 out of 6 over GF(4), and 4 out of 6 over GF(4);
7 out of 14 over GF(8).

This last was made possible, with limited computer power, by adroit use of Zech's logs [MA, p. 91-92]. It might lead to a more general k out of k+7 hyperfast p/s/r process over GF(8) soon. Conceivably the cases 8 out of 14, 9 out of 14, and 10 out of 14 can also be produced over GF(8) and made to give rise to more general cases involving k out of k+7, k out of k+6, k out of k+5 and k out of k+4 over GF(8). But to get such things as a k out of k+8 p/s/r process using only the arithmetic of GF(16) will likely require the effort and the computer power of an IBM PC programmed in assembly language running for hours.

We recall that GF(2) = Z/(2) is [BL83a, pp. 69,75] the field of two elements, i.e. the integers modulo 2, i.e. the set {0,1} under the addition and multiplication tables

+	0	1
0	0	1
1	1	0

*	0	1
0	0	0
1	0	1

.

The following encode and decode rules obviously work for a 4 out of 5 p/s/r, where all arithmetic is done in GF(2). To encode (i.e. to pool/split) an ancestral list (a(1), a(2), a(3), a(4)) of four 1-bit words, let

d(1) = a(1); d(2) = a(2); d(3) = a(3); d(4) = a(4);

d(5) = a(1) + a(2) + a(3) + a(4).

To decode if d(5) is missing set:

a(1) = d(1); a(2) = d(2); a(3) = d(3); a(4) = d(4).

If d(1) is missing set;

$$a(1) = d(2) + d(3) + d(4) + d(5); \quad a(2) = d(2); \quad a(3) = d(3); \quad a(4) = d(4).$$

If d(2) or d(3) or d(4) is missing the obvious analog of the case immediately above decodes successfully. This can be more readily seen in terms of matrices over [H071, p. 6] the field GF(2)

$$A=\begin{bmatrix} a(1) \\ a(2) \\ a(3) \\ a(4) \end{bmatrix}, \quad D=\begin{bmatrix} d(1) \\ d(2) \\ d(3) \\ d(4) \\ d(5) \end{bmatrix}, \quad E=\begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \\ 1 & 1 & 1 & 1 \end{bmatrix}, \quad M[1]=\begin{bmatrix} 0 & 1 & 1 & 1 & 1 \\ 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 \end{bmatrix}, \quad M[2]=\begin{bmatrix} 1 & 0 & 0 & 0 & 0 \\ 1 & 0 & 1 & 1 & 1 \\ 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 \end{bmatrix},$$

$$M[3] = \begin{bmatrix} 1 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 \\ 1 & 1 & 0 & 1 & 1 \\ 0 & 0 & 0 & 1 & 0 \end{bmatrix}, \quad M[4] = \begin{bmatrix} 1 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 \\ 1 & 1 & 1 & 0 & 1 \end{bmatrix}, \quad M[5] = \begin{bmatrix} 1 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 \end{bmatrix}.$$

Then encoding is the rule D = EA. And decoding is the rule A = M[i]D when d(i) is missing. This works because

$$M[i]D = M[i](EA) = (M[i]E)A = IA = A$$

when the ith entry of D is absent. This is because every M[i] is a left inverse [N069, p. 11] of the nonsquare matrix E, and because M[i]D is independent of d(i) (since the ith column of M[i] contains only zeros). Clearly [N069, pp. 11-17, 132-135] E cannot have a right inverse [N069, p. 11].

Instead of a 4 out of 5 p/s/r we could as easily have defined a k out of k+1 p/s/r process using only the arithmetic of GF(2). This is quite unlike what happens when threshold schemes are involved. To implement a k out of k+1 threshold scheme you must use the arithmetic of the much larger field GF(Q), where $Q \geq k+1$.

The extreme simplicity of this k out of k+1 p/s/r process (its use of only GF(2) arithmetic) is not a fluke. Moving up the scale, it is possible to implement a k out of k+3 hyperfast p/s/r process using only the arithmetic of GF(4). This is, one recalls [MA77, p. 101; BL83a, p. 75], the set {0,1,r,s} under the addition and multiplication tables

+	0	1	r	s
0	0	1	r	s
1	1	0	s	r
r	r	s	0	1
s	s	r	1	0

,

*	0	1	r	s
0	0	0	0	0
1	0	1	r	s
r	0	r	s	1
s	0	s	1	r

It is commonplace to represent these four "numbers" as 2-bit words:

$$0 = (0,0); \qquad 1 = (0,1); \qquad r = (1,0); \qquad s = (1,1).$$

Evidently, then, + is just the 2-bit word exclusive or operation, XOR. And * can be implemented by means of a table with sixteen 2-bit entries.

For brevity we merely give the matrix form of a 3 out of 6 hyperfast p/s/r process in terms of matrices over [H071, p. 6] the field GF(4).

$$A=\begin{bmatrix} a(1) \\ a(2) \\ a(3) \end{bmatrix},\quad D=\begin{bmatrix} d(1) \\ d(2) \\ d(3) \\ d(4) \\ d(5) \\ d(6) \end{bmatrix},\quad E=\begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \\ 1 & 1 & 1 \\ 1 & r & s \\ 1 & s & r \end{bmatrix},\quad M[1,2,3]=\begin{bmatrix} 0 & 0 & 0 & 1 & 1 & 1 \\ 0 & 0 & 0 & 1 & s & r \\ 0 & 0 & 0 & 1 & r & s \end{bmatrix},\quad M[1,2,4]=\begin{bmatrix} 0 & 0 & 1 & 0 & s & r \\ 0 & 0 & 1 & 0 & 1 & 1 \\ 0 & 0 & 1 & 0 & 0 & 0 \end{bmatrix},$$

$$M[1,2,5]=\begin{bmatrix} 0 & 0 & s & r & 0 & s \\ 0 & 0 & r & s & 0 & s \\ 0 & 0 & 1 & 0 & 0 & 0 \end{bmatrix},\quad M[1,2,6]=\begin{bmatrix} 0 & 0 & r & s & r & 0 \\ 0 & 0 & s & r & r & 0 \\ 0 & 0 & 1 & 0 & 0 & 0 \end{bmatrix},\quad M[1,3,4]=\begin{bmatrix} 0 & 1 & 0 & 0 & r & s \\ 0 & 1 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 1 & 1 \end{bmatrix},$$

$$M[1,3,5]=\begin{bmatrix} 0 & r & 0 & s & 0 & r \\ 0 & 1 & 0 & 0 & 0 & 0 \\ 0 & s & 0 & r & 0 & r \end{bmatrix},\quad M[1,3,6]=\begin{bmatrix} 0 & s & 0 & r & s & 0 \\ 0 & 1 & 0 & 0 & 0 & 0 \\ 0 & r & 0 & s & s & 0 \end{bmatrix},\quad M[1,4,5]=\begin{bmatrix} 0 & s & r & 0 & 0 & 1 \\ 0 & 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 & 0 \end{bmatrix},$$

$$M[1,4,6]=\begin{bmatrix} 0 & r & s & 0 & 1 & 0 \\ 0 & 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 & 0 \end{bmatrix},\quad M[1,5,6]=\begin{bmatrix} 0 & 1 & 1 & 1 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 & 0 \end{bmatrix},\quad M[2,3,4]=\begin{bmatrix} 1 & 0 & 0 & 0 & 0 & 0 \\ 1 & 0 & 0 & 0 & r & s \\ 1 & 0 & 0 & 0 & s & r \end{bmatrix},$$

$$M[2,3,5]=\begin{bmatrix} 1 & 0 & 0 & 0 & 0 & 0 \\ s & 0 & 0 & r & 0 & 1 \\ r & 0 & 0 & s & 0 & 1 \end{bmatrix},\quad M[2,3,6]=\begin{bmatrix} 1 & 0 & 0 & 0 & 0 & 0 \\ r & 0 & 0 & s & 1 & 0 \\ s & 0 & 0 & r & 1 & 0 \end{bmatrix},\quad M[2,4,5]=\begin{bmatrix} 1 & 0 & 0 & 0 & 0 & 0 \\ r & 0 & s & 0 & 0 & r \\ 0 & 0 & 1 & 0 & 0 & 0 \end{bmatrix},$$

$$M[2,4,6]=\begin{bmatrix} 1 & 0 & 0 & 0 & 0 & 0 \\ s & 0 & r & 0 & s & 0 \\ 0 & 0 & 1 & 0 & 0 & 0 \end{bmatrix},\quad M[2,5,6]=\begin{bmatrix} 1 & 0 & 0 & 0 & 0 & 0 \\ 1 & 0 & 1 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 & 0 \end{bmatrix},\quad M[3,4,5]=\begin{bmatrix} 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 & 0 \\ s & r & 0 & 0 & 0 & s \end{bmatrix},$$

$$M[3,4,6]=\begin{bmatrix} 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 & 0 \\ r & s & 0 & 0 & r & 0 \end{bmatrix},\quad M[3,5,6]=\begin{bmatrix} 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 & 0 \\ 1 & 1 & 0 & 1 & 0 & 0 \end{bmatrix},\quad M[4,5,6]=\begin{bmatrix} 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 & 0 \end{bmatrix}.$$

To encode, set D = EA. To decode, set A = M[w,x,y]D when d(w), d(x) and d(y) are missing. This works because

$$M[w,x,y]D = M[w,x,y](EA) = (M[w,x,y]E)A = IA = A$$

even though the wth, xth and yth entries (d(w), d(x) and d(y)) of D are unknown (the product M[w,x,y]D is independent of them because the wth, xth and yth columns of M[w,x,y] are zero). It is easy to verify that, in the arithmetic of GF(4), every one of the twenty matrices M[w,x,y] is a left inverse of E. Finally, it is a straightforward matter to produce k out of k+3 generalizations of this hyperfast p/s/r process, using only the arithmetic of GF(4).

The major part of the work plan is to write, to run, and to analyze the output of, computer programs for using the heuristic principle to find the encode and decode algorithms for successively larger cases of hyperfast p/s/r processes. This will involve a great deal of run time. Hence it will be necessary to obtain an IBM PC and

use it throughout the project. See Section 8 below. First we propose to find the form of general:

k out of k+4; k out of k+5; and k out of k+6

p/s/r processes using only GF(8) arithmetic, then general

k out of k+7, ..., k out of k+14

p/s/r processes using only GF(16) arithmetic, then general

k out of k+15, ..., k out of k+30

p/s/r processes using only GF(32) arithmetic, and so on. These results, which already contain the larger part of the foreseeable practical use of cheap hyperfast p/s/r processes, can be expected to lead to the form of the general k out of k+j p/s/r process using only the arithmetic of $GF(2^b)$, where $j < 2^b$.

Once this is done, the rest of the work plan is to do an abstract design of hardware implementation of p/s/r processes. For packet switching [SL81] and other sequential-arrival-of-words type applications, low cost and minimum parallelism may be the overriding design consideration. For other applications, perhaps involving physically parallel channels transmitting concurrently, cost and use of off-the-shelf components may take a back seat to speed. In this case it may be necessary to provide abstract designs of encode and decode processes utilizing parallel processing, or even the ultimate ultraparallel implementation so as to approach the four-machine-cycle ideal of encoding and decoding speed mentioned in Section 5 above.

The last part of the work plan, also an abstract design task, is to sacrifice speed or economy or both so as to produce general purpose decoders. In other words we want to classify the pairs ((k,n), (k*,n*)) with the property that an encoder (resp. decoder) for a k out of n p/s/r process will encode (resp. decode) for a k* out of n* process as well.

Cases of this are known. It is easy to turn the implementation of a 3 out of 6 hyperfast p/s/r process using only GF(4) arithmetic into the implementation of a 2 out of 4 hyperfast p/s/r process using only GF(4) arithmetic by "tying some channels to ground", i.e. by sending only zeros over them (or having the receiver pretend that only zeros are sent over them). We omit details, which a reader can easily work out. Obviously you pay a price in bandwidth. In this example a 3 megabit per second throughput is reduced to 2 megabits per second. It is reasonable to conjecture that a k out of n implementation can be trivially turned into a k* out of n* implementation this way if $k^* \leq k$, $n^* \leq n$, and $n^*-k^* \leq n-k$. It would be desirable to verify this conjecture and, if possible, extend it. The advantage of having a few versatile boxes (general purpose communication tools) can sometimes outweigh the panoply of unique advantages peculiar to each of a large number of dedicated boxes (precision single purpose tools) in a military context.

Actual hardware design is not part of Phase I. It will be left to Phase II.

7. Phase I statement of work.

The work will start with the production, and numerous runs, of a program to implement the heuristic device described in Section 6 above. It is strongly indicated by much evidence in the cases n = k, n = k+1, n = k+2, n = k+3, and n = k+7 that a properly chosen Bloom p/s/r gives rise to an appropriate hyperfast p/s/r for any choice of k and n. The program will produce the list of matrices which embody this hyperfast k out of n case, for each choice of k and n. By the end of two months the first of these results (the cases $4 \leq k \leq 7$, n around 60) will be available. Within the following month or two, the other cases most important to the general solution of the problem of building all hyperfast p/s/r processes should be

available. They may not be in the best form. If not, an interactive matrix manipulation program will be produced to format them in the manner most conducive to reading off the general structure of the matrices which embody a hyperfast p/s/r process. The last two months will be devoted to discovering, and then proving correct, the form of the general hyperfast p/s/r process. Even if the general solution is not found, most cases with any conceivable practical importance will have been settled.

The abstract design principles for implementation can proceed concurrently with the discovery process over the last 3 of the 6 months of the project. The reason for this is that the general form of the solution is known. Both encode and decode are dot products between vectors in an n dimensional or a k dimensional vector space. What is not yet conclusively demonstrated, though we gave a well motivated conjecture in Section 6, is the size of the fields underlying these vector spaces for a given choice of n and k. And the number of occurrences of each member of that field is quite mysterious. But, as these pieces fall into place case by case, the abstract design principles can evolve iteratively.

At the end of the sixth month YLYK Ltd. will deliver a report. The report will contain a catalog of k out of n cases of hyperfast p/s/r processes embodied in lists of matrices for various important values of k and n. If the work meets with complete success it will in fact give the form of the list of matrices embodying the general k out of n hyperfast p/s/r. Finally, it will describe the abstract design principles of implementing such p/s/r processes on currently available off-the-shelf hardware, as well as on a hypothetical vector machine or even a hypothetical ultraparallel processor.

8. Facilities/equipment.

So far the hyperfast k out of k+1, 4 out of 6, 3 out of 6, and 7 out of 14 cases of a p/s/r process have been produced with no more computer power than an HP 41C. This is because the fields in question are quite small. Hence no matrix larger than 14 by 14 is needed to turn the heuristic principle described in Section 6 above into an infinite collection of encode/decode rules. But in order to go beyond this it will be necessary to at least double the size of the Galois field F in question. It will also be necessary to do linear algebra with matrices larger than 30 by 30. And by the time the general form of the encode/decode procedure for k out of n processes emerges we will probably be dealing with something like a k out of k+100 case. This will involve fields with more than 100 elements, and (extremely sparse) matrices of size approximately 20,000 by 20,000 over such fields.

The calculations involved will require a computer capable of supporting FORTRAN, as well as being easily programmable in its own assembly language, and with sizable memory. The IBM Personal Computer is just about the smallest of the machines capable of carrying out this program. But with 64K RAM, and assuming adroit programming and use of disk memory, it will be possible to explore the consequences of the heuristic principle mentioned in Section 6 above within the size ranges aforementioned. No other special equipment will be required to complete the project.

The 2-room facilities available to YLYK Ltd. at Ann Arbor are adequate to the task at hand. They can accommodate the IBM PC and provide the principal investigator with a work area and necessary library and drafting facilities. Other personnel can be accommodated there, or else assigned duties to be performed on their own premises in consultant fashion.

9. Consultants.

Charles Asmuth (Ph.D., Mathematics, University of Chicago, 1976) did postdoctoral work at the Institute for Advanced Study in Princeton, New Jersey. He taught in the

department of mathematics at Texas A&M before taking his present position as assistant professor in the department of mathematics and computer science at Rutgers University (Newark). He is author or coauthor of some ten papers in mathematics and its applications, especially information theory and cryptography. He will be a consultant on the proposed research. His combination of knowledge in electrical engineering, computer science and abstract algebra will be useful in going from the classification of hyperfast p/s/r processes to implementation.

G. R. Blakley (Ph.D. Mathematics, University of Maryland, 1960) did postdoctoral work at Cornell and Harvard. He has been on the mathematics department faculty of the University of Illinois (Urbana), SUNY at Buffalo, and Texas A&M (where he was department head for many years, and where he is currently a professor). He is author or coauthor of some 30 papers in mathematics and its applications, especially information theory and cryptography. He will be a consultant on the proposed research. His expertise in linear algebra will be useful in finding a general scheme under which the anticipated abundance of hyperfast p/s/r processes can be classified.

John Bloom (Ph.D., Mathematics, CalTech, 1977) taught at the department of mathematics, Texas A&M University, before taking his present research and development position at Chevron, La Habra, California. He is author or coauthor of some ten papers and technical reports in mathematics and its applications, including information theory. He will be a consultant on the proposed research. His expertise in algebraic number theory and algebraic geometry will be especially useful in the very first phase, formulating the programs which implement the heuristic based on the Bloom p/s/r processes and produce examples of hyperfast p/s/r processes for various choices of k and n.

10. Related work. Bibliographic citations list.

Bob Blakley served as a draftsman for the City of Bryan, Texas, in the summer of 1978. He is an expert scientific programmer, having been employed at various times over the last three years in software production and maintenance by research contracts and grants in the Mathematics, Mechanical Engineering, Statistics, Chemistry, Biochemistry and Biophysics departments of Texas A&M University, the Geophysical Fluid Dynamics Laboratory at Princeton University and the University of Michigan Computer Center, as well as for YLYK Ltd. of Ann Arbor, Michigan. He has had extensive experience in algebraic scientific software production, some of it in collaboration with G. R. Blakley. He has produced sizable module [HE74] theoretic generalizations of linear algebraic programs for chemical applications. He has produced programs for the arithmetic of g-adic rings [MA81] and the arithmetic of finite fields of characteristic 2. He has implemented computer simulations of both the Asmuth-Blakley [AS82] p/s/r analog of the Asmuth-Bloom threshold scheme [AS83] and the Bloom-style p/s/r analog of the Bloom threshold scheme [BL81b]. He has a substantial academic background in logic, computer science and natural languages. He is conversant with a dozen computer languages, several of which are assembly languages.

C. A. Asmuth is one of the leading practitioners in the theory of threshold schemes [AS83], p/s/r processes [AS82] and their applications [AS81]. He has a practical familiarity with digital electronics extending back many years. His grasp of abstract algebra and abstract harmonic analysis is highly sophisticated.

G. R. Blakley invented [BL79] threshold schemes, and is a major contributor [BL80; BL81a; BL82] to their theory. With Asmuth, he first explicitly identified the red noise problem [AS82] and solved it (though Bloom certainly [BL81b] foreshadowed this solution). He works actively [BL83b] on minimal computational complexity algorithms for scientific and mathematical computations. His interest in linear algebra, and its applications outside mathematics, goes back twenty years, and has issued in numerous publications not cited here because they are not directly relevant

to the topic at hand. The term linear algebra is used here in an expansive sense which includes matrix analysis on the analytic side, and integer matrices -- and, more generally, module theory -- on the abstract algebraic side. He is currently principal investigator on a National Security Agency grant to do unclassified research in information theory, some aspects of which are related to the theory and practice of p/s/r processes.

J. Bloom is the inventor of the Bloom threshold scheme [BL81b], the fastest known. His work prefigured the development of the Bloom-style p/s/r processes and the hyperfast p/s/r processes. His influence is major and his insight into every aspect of the subject is incisive. His grasp of geometry, including algebraic geometry, is powerful. He has devoted the last two years to sophisticated programming efforts on computers near the edge of the envelope.

C. Asmuth, Bob Blakley, G. R. Blakley and J. Bloom have all known each other for more than five years. They communicate effortlessly with each other on technical matters. The requested travel funds will be used to get two or more of them together for periods of several days at several points during the work.

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11. Key Personnel.

YLYK Ltd. was incorporated in Delaware on 4 June 1979. It is currently headquartered in Ann Arbor, Michigan. It has produced software, designed algorithms, designed systems in the area of coding, communications and cryptography, and has conducted studies.

Bob Blakley, born 13 July 1960 in Washington D.C., is a citizen of the U.S.A. and a 1982 honors graduate of Princeton University. He married Karen Hejtmancik of College Station, Texas, on 7 August 1982. His previous technical employment history can be found in Section 10 above. He is currently involved in part time teaching and graduate study in computer science at the University of Michigan. He is coauthor of three papers on cryptography and information theory in Cryptologia, Volume 2 (1978), pp. 305-321, Volume 3 (1979), pp. 29-42, and Volume 3 (1979), pp. 105-118. He is president of YLYK Ltd., and will be principal investigator on the proposed research. His Social Security Number is 460-06-2353.

12. Current and pending support.

SBIR proposals very similar to this proposal, all bearing the title

> High-speed low-cost ways to get messages from a sender to a receiver when some channels linking them become inoperative,

and all having Bob Blakley, President, YLYK Ltd., as principal investigator are being submitted in May 1983 to the following DOD components under DOD Program Solicitation Number 83.1, Small Business Research Program, Closing date 31 May 1983:

Appendix B

Tables of GF(2↑N) arithmetic

Addition Table for GF 2**(2) Mod(7)

+	0	1	2	3
0	0	1	2	3
1	1	0	3	2
2	2	3	0	1
3	3	2	1	0

Multiplication Table for GF 2**(2) Mod(7)

+	0	1	2	3
0	0	0	0	0
1	0	1	2	3
2	0	2	3	1
3	0	3	1	2

Addition Table for GF 2**(3) Mod(13)

+	0	1	2	3	4	5	6	7
0	0	1	2	3	4	5	6	7
1	1	0	3	2	5	4	7	6
2	2	3	0	1	6	7	4	5
3	3	2	1	0	7	6	5	4
4	4	5	6	7	0	1	2	3
5	5	4	7	6	1	0	3	2
6	6	7	4	5	2	3	0	1
7	7	6	5	4	3	2	1	0

Multiplication Table for GF 2**(3) Mod(13)

+	0	1	2	3	4	5	6	7
0	0	0	0	0	0	0	0	0
1	0	1	2	3	4	5	6	7
2	0	2	4	6	3	1	7	5
3	0	3	6	5	7	4	1	2
4	0	4	3	7	6	2	5	1
5	0	5	1	4	2	7	3	6
6	0	6	7	1	5	3	2	4
7	0	7	5	2	1	6	4	3

Addition Table for GF 2**(4) Mod(23)

+	00	01	02	03	04	05	06	07	10	11	12	13	14	15	16	17
00	00	01	02	03	04	05	06	07	10	11	12	13	14	15	16	17
01	01	00	03	02	05	04	07	06	11	10	13	12	15	14	17	16
02	02	03	00	01	06	07	04	05	12	13	10	11	16	17	14	15
03	03	02	01	00	07	06	05	04	13	12	11	10	17	16	15	14
04	04	05	06	07	00	01	02	03	14	15	16	17	10	11	12	13
05	05	04	07	06	01	00	03	02	15	14	17	16	11	10	13	12
06	06	07	04	05	02	03	00	01	16	17	14	15	12	13	10	11
07	07	06	05	04	03	02	01	00	17	16	15	14	13	12	11	10
10	10	11	12	13	14	15	16	17	00	01	02	03	04	05	06	07
11	11	10	13	12	15	14	17	16	01	00	03	02	05	04	07	06
12	12	13	10	11	16	17	14	15	02	03	00	01	06	07	04	05
13	13	12	11	10	17	16	15	14	03	02	01	00	07	06	05	04
14	14	15	16	17	10	11	12	13	04	05	06	07	00	01	02	03
15	15	14	17	16	11	10	13	12	05	04	07	06	01	00	03	02
16	16	17	14	15	12	13	10	11	06	07	04	05	02	03	00	01
17	17	16	15	14	13	12	11	10	07	06	05	04	03	02	01	00

Multiplication Table for GF 2**(4) Mod(23)

+	00	01	02	03	04	05	06	07	10	11	12	13	14	15	16	17
00	00	00	00	00	00	00	00	00	00	00	00	00	00	00	00	00
01	00	01	02	03	04	05	06	07	10	11	12	13	14	15	16	17
02	00	02	04	06	10	12	14	16	03	01	07	05	13	11	17	15
03	00	03	06	05	14	17	12	11	13	10	15	16	07	04	01	02
04	00	04	10	14	03	07	13	17	06	02	16	12	05	01	15	11
05	00	05	12	17	07	02	15	10	16	13	04	01	11	14	03	06
06	00	06	14	12	13	15	07	01	05	03	11	17	16	10	02	04
07	00	07	16	11	17	10	01	06	15	12	03	04	02	05	14	13
10	00	10	03	13	06	16	05	15	14	04	17	07	12	02	11	01
11	00	11	01	10	02	13	03	12	04	15	05	14	06	17	07	16
12	00	12	07	15	16	04	11	03	17	05	10	02	01	13	06	14
13	00	13	05	16	12	01	17	04	07	14	02	11	15	06	10	03
14	00	14	13	07	05	11	16	02	12	06	01	15	17	03	04	10
15	00	15	11	04	01	14	10	05	02	17	13	06	03	16	12	07
16	00	16	17	01	15	03	02	14	11	07	06	10	04	12	13	05
17	00	17	15	02	11	06	04	13	01	16	14	03	10	07	05	12

Addition Table for GF 2**(5) Mod(45)

+	00	01	02	03	04	05	06	07	10	11	12	13	14	15	16	17	20	21	22	23	24	25	26	27	30	31	32	33	34	35	36	37
00	00	01	02	03	04	05	06	07	10	11	12	13	14	15	16	17	20	21	22	23	24	25	26	27	30	31	32	33	34	35	36	37
01	01	00	03	02	05	04	07	06	11	10	13	12	15	14	17	16	21	20	23	22	25	24	27	26	31	30	33	32	35	34	37	36
02	02	03	00	01	06	07	04	05	12	13	10	11	16	17	14	15	22	23	20	21	26	27	24	25	32	33	30	31	36	37	34	35
03	03	02	01	00	07	06	05	04	13	12	11	10	17	16	15	14	23	22	21	20	27	26	25	24	33	32	31	30	37	36	35	34
04	04	05	06	07	00	01	02	03	14	15	16	17	10	11	12	13	24	25	26	27	20	21	22	23	34	35	36	37	30	31	32	33
05	05	04	07	06	01	00	03	02	15	14	17	16	11	10	13	12	25	24	27	26	21	20	23	22	35	34	37	36	31	30	33	32
06	06	07	04	05	02	03	00	01	16	17	14	15	12	13	10	11	26	27	24	25	22	23	20	21	36	37	34	35	32	33	30	31
07	07	06	05	04	03	02	01	00	17	16	15	14	13	12	11	10	27	26	25	24	23	22	21	20	37	36	35	34	33	32	31	30
10	10	11	12	13	14	15	16	17	00	01	02	03	04	05	06	07	30	31	32	33	34	35	36	37	20	21	22	23	24	25	26	27
11	11	10	13	12	15	14	17	16	01	00	03	02	05	04	07	06	31	30	33	32	35	34	37	36	21	20	23	22	25	24	27	26
12	12	13	10	11	16	17	14	15	02	03	00	01	06	07	04	05	32	33	30	31	36	37	34	35	22	23	20	21	26	27	24	25
13	13	12	11	10	17	16	15	14	03	02	01	00	07	06	05	04	33	32	31	30	37	36	35	34	23	22	21	20	27	26	25	24
14	14	15	16	17	10	11	12	13	04	05	06	07	00	01	02	03	34	35	36	37	30	31	32	33	24	25	26	27	20	21	22	23
15	15	14	17	16	11	10	13	12	05	04	07	06	01	00	03	02	35	34	37	36	31	30	33	32	25	24	27	26	21	20	23	22
16	16	17	14	15	12	13	10	11	06	07	04	05	02	03	00	01	36	37	34	35	32	33	30	31	26	27	24	25	22	23	20	21
17	17	16	15	14	13	12	11	10	07	06	05	04	03	02	01	00	37	36	35	34	33	32	31	30	27	26	25	24	23	22	21	20
20	20	21	22	23	24	25	26	27	30	31	32	33	34	35	36	37	00	01	02	03	04	05	06	07	10	11	12	13	14	15	16	17
21	21	20	23	22	25	24	27	26	31	30	33	32	35	34	37	36	01	00	03	02	05	04	07	06	11	10	13	12	15	14	17	16
22	22	23	20	21	26	27	24	25	32	33	30	31	36	37	34	35	02	03	00	01	06	07	04	05	12	13	10	11	16	17	14	15
23	23	22	21	20	27	26	25	24	33	32	31	30	37	36	35	34	03	02	01	00	07	06	05	04	13	12	11	10	17	16	15	14
24	24	25	26	27	20	21	22	23	34	35	36	37	30	31	32	33	04	05	06	07	00	01	02	03	14	15	16	17	10	11	12	13
25	25	24	27	26	21	20	23	22	35	34	37	36	31	30	33	32	05	04	07	06	01	00	03	02	15	14	17	16	11	10	13	12
26	26	27	24	25	22	23	20	21	36	37	34	35	32	33	30	31	06	07	04	05	02	03	00	01	16	17	14	15	12	13	10	11
27	27	26	25	24	23	22	21	20	37	36	35	34	33	32	31	30	07	06	05	04	03	02	01	00	17	16	15	14	13	12	11	10
30	30	31	32	33	34	35	36	37	20	21	22	23	24	25	26	27	10	11	12	13	14	15	16	17	00	01	02	03	04	05	06	07
31	31	30	33	32	35	34	37	36	21	20	23	22	25	24	27	26	11	10	13	12	15	14	17	16	01	00	03	02	05	04	07	06
32	32	33	30	31	36	37	34	35	22	23	20	21	26	27	24	25	12	13	10	11	16	17	14	15	02	03	00	01	06	07	04	05
33	33	32	31	30	37	36	35	34	23	22	21	20	27	26	25	24	13	12	11	10	17	16	15	14	03	02	01	00	07	06	05	04
34	34	35	36	37	30	31	32	33	24	25	26	27	20	21	22	23	14	15	16	17	10	11	12	13	04	05	06	07	00	01	02	03
35	35	34	37	36	31	30	33	32	25	24	27	26	21	20	23	22	15	14	17	16	11	10	13	12	05	04	07	06	01	00	03	02
36	36	37	34	35	32	33	30	31	26	27	24	25	22	23	20	21	16	17	14	15	12	13	10	11	06	07	04	05	02	03	00	01
37	37	36	35	34	33	32	31	30	27	26	25	24	23	22	21	20	17	16	15	14	13	12	11	10	07	06	05	04	03	02	01	00

Multiplication Table for GF 2**(5) Mod(45)

*	00	01	02	03	04	05	06	07	10	11	12	13	14	15	16	17	20	21	22	23	24	25	26	27	30	31	32	33	34	35	36	37
00	00	00	00	00	00	00	00	00	00	00	00	00	00	00	00	00	00	00	00	00	00	00	00	00	00	00	00	00	00	00	00	00
01	00	01	02	03	04	05	06	07	10	11	12	13	14	15	16	17	20	21	22	23	24	25	26	27	30	31	32	33	34	35	36	37
02	00	02	04	06	10	12	14	16	20	22	24	26	30	32	34	36	05	07	01	03	15	17	11	13	25	27	21	23	35	37	31	33
03	00	03	06	05	14	17	12	11	30	33	36	35	24	27	22	21	25	26	23	20	31	32	37	34	15	16	13	10	01	02	07	04
04	00	04	10	14	20	24	30	34	05	01	15	11	25	21	35	31	12	16	02	06	32	36	22	26	17	13	07	03	37	33	27	23
05	00	05	12	17	24	21	26	33	15	10	07	02	31	34	23	26	32	37	20	25	16	13	04	01	27	22	35	30	03	06	11	14
06	00	06	14	12	30	36	24	22	25	23	31	37	15	13	01	07	17	11	03	05	27	21	33	35	32	34	26	20	02	04	16	10
07	00	07	16	11	34	33	22	25	35	32	23	24	01	06	17	10	37	30	21	26	03	04	15	12	02	05	14	13	36	31	20	27
10	00	10	20	30	05	15	25	35	12	02	32	22	17	07	37	27	24	34	04	14	21	31	01	11	36	26	16	06	33	23	13	03
11	00	11	22	33	01	10	23	32	02	13	20	31	03	12	21	30	04	15	26	37	05	14	27	36	06	17	24	35	07	16	25	34
12	00	12	24	36	15	07	31	23	32	20	16	04	27	35	03	11	21	33	05	17	34	26	10	02	13	01	37	25	06	14	22	30
13	00	13	26	35	11	02	37	24	22	31	04	17	33	20	15	06	01	12	27	34	10	03	36	25	23	30	05	16	32	21	14	07
14	00	14	30	24	25	31	15	01	17	03	27	33	32	26	02	16	36	22	06	12	13	07	23	37	21	35	11	05	04	10	34	20
15	00	15	32	27	21	34	13	06	07	12	35	20	26	33	14	01	16	03	24	31	37	22	05	10	11	04	23	36	30	25	02	17
16	00	16	34	22	35	23	01	17	37	21	03	15	02	14	36	20	33	25	07	11	06	10	32	24	04	12	30	26	31	27	05	13
17	00	17	36	21	31	26	07	10	27	30	11	06	16	01	20	37	13	04	25	32	22	35	14	03	34	23	02	15	05	12	33	24
20	00	20	05	25	12	32	17	37	24	04	21	01	36	16	33	13	15	35	10	30	07	27	02	22	31	11	34	14	23	03	26	06
21	00	21	07	26	16	37	11	30	34	15	33	12	22	03	25	04	35	14	32	13	23	02	24	05	01	20	06	27	17	36	10	31
22	00	22	01	23	02	20	03	21	04	26	05	27	06	24	07	25	10	32	11	33	12	30	13	31	14	36	15	37	16	34	17	35
23	00	23	03	20	06	25	05	26	14	27	17	34	12	31	11	32	30	13	33	10	36	15	35	16	24	07	27	04	22	01	21	02
24	00	24	15	31	32	16	27	03	21	05	34	10	13	37	06	22	07	23	12	36	35	11	20	04	26	02	33	17	14	30	01	25
25	00	25	17	32	36	13	21	04	31	14	26	03	07	22	10	35	27	02	30	15	11	34	06	23	16	33	01	24	20	05	37	12
26	00	26	11	37	22	04	33	15	01	27	10	36	23	05	32	14	02	24	13	35	20	06	31	17	03	25	12	34	21	07	30	16
27	00	27	13	34	26	01	35	12	11	36	02	25	37	10	24	03	22	05	31	16	04	23	17	30	33	14	20	07	15	32	06	21
30	00	30	25	15	17	27	32	02	36	06	13	23	21	11	04	34	31	01	14	24	26	16	03	33	07	37	22	12	10	20	35	05
31	00	31	27	16	13	22	34	05	26	17	01	30	35	04	12	23	11	20	36	07	02	33	25	14	37	06	10	21	24	15	03	32
32	00	32	21	13	07	35	26	14	16	24	37	05	11	23	30	02	34	06	15	27	33	01	12	20	22	10	03	31	25	17	04	36
33	00	33	23	10	03	30	20	13	06	35	25	16	05	36	26	15	14	27	37	04	17	24	34	07	12	21	31	02	11	22	32	01
34	00	34	35	01	37	03	02	36	33	07	06	32	04	30	31	05	23	17	16	22	14	20	21	15	10	24	25	11	27	13	12	26
35	00	35	37	02	33	06	04	31	23	16	14	21	10	25	27	12	03	36	34	01	30	05	07	32	20	15	17	22	13	26	24	11
36	00	36	31	07	27	11	16	20	13	25	22	14	34	02	05	33	26	10	17	21	01	37	30	06	35	03	04	32	12	24	23	15
37	00	37	33	04	23	14	10	27	03	34	30	07	20	17	13	24	06	31	35	02	25	12	16	21	05	32	36	01	26	11	15	22

Appendix C

Selected tables of Vandermonde matrices

Vandermonde Matrix for GF 2**(2) mod 7 is:

```
1 0 0 0
1 1 1 1
1 2 4 1
1 3 2 1
```

Vandermonde Matrix for GF 2**(3) mod 13 is:

```
1 0 0 0 0 0 0 0
1 1 1 1 1 1 1 1
1 2 4 8 3 6 5 1
1 4 3 9 12 10 1 1
1 8 12 3 9 4 6 1
1 3 9 1 3 9 1 1
1 6 10 9 3 5 4 1
1 5 12 10 4 7 9 1
```

Vandermonde Matrix for GF 2**(4) mod 23 is:

```
01 00 00 00 00 00 00 00 00 00 00 00 00 00 00 00
01 01 01 01 01 01 01 01 01 01 01 01 01 01 01 01
01 02 04 10 03 06 14 17 05 12 07 16 17 15 11 01
01 04 03 14 05 07 17 11 02 10 06 13 12 16 15 01
01 10 14 12 17 01 10 14 12 17 01 10 14 12 17 01
01 03 05 17 02 06 12 15 04 14 07 11 10 13 16 01
01 06 07 01 06 07 01 06 07 01 06 07 01 06 07 01
01 14 17 10 12 01 14 17 10 12 01 14 17 10 12 01
01 17 11 14 15 06 17 03 16 10 07 04 12 02 05 01
01 05 02 12 04 07 10 16 03 17 06 15 14 11 13 01
01 12 10 17 14 01 12 10 17 14 01 12 10 17 14 01
01 07 06 01 07 06 01 07 06 01 07 06 01 07 06 01
01 16 13 10 11 07 14 04 15 12 06 02 17 05 03 01
01 17 12 14 10 01 17 12 14 10 01 17 12 14 10 01
01 15 16 12 13 06 10 02 11 17 07 05 14 03 04 01
01 11 15 17 16 07 12 05 13 14 06 03 10 04 02 01
```

Vandermonde Matrix for GF Dekt 5: mod 45 is:

```
01 00 00 00 00 00 00 00 00 00 00 00 00 00 00 00 00 00 00 00 00 00 00 00 00 00 00 00 00 00 00 00
01 01 01 01 01 01 01 01 01 01 01 01 01 01 01 01 01 01 01 01 01 01 01 01 01 01 01 01 01 01 01 01
01 02 04 10 20 05 12 24 15 32 21 07 16 34 35 37 33 23 03 06 14 30 25 17 36 31 27 13 26 11 22 01
01 04 20 12 15 21 16 35 33 03 14 25 36 27 26 22 32 10 05 24 32 07 34 37 23 06 30 12 31 16 11 01
01 10 12 32 16 37 03 30 36 13 22 04 05 15 07 35 23 14 17 27 11 02 20 24 21 34 33 06 25 31 26 01
01 20 15 16 33 14 36 26 02 05 32 34 23 30 31 11 04 12 21 35 03 25 27 22 10 24 07 37 06 17 13 01
01 05 21 37 14 31 22 29 32 35 06 36 11 10 15 34 03 17 26 04 24 16 23 25 13 02 12 07 33 30 27 01
01 12 16 03 36 22 05 07 23 17 11 20 21 33 25 26 10 32 37 30 13 04 15 35 14 27 02 24 34 06 31 01
01 24 25 30 26 29 07 03 31 02 15 37 25 11 05 16 06 27 04 32 33 17 22 12 34 14 13 10 21 23 36 01
01 15 33 36 02 32 17 01 04 21 03 27 10 07 06 13 20 16 14 26 05 34 30 11 12 35 25 22 24 37 17 01
01 30 03 13 05 25 17 02 21 06 26 12 37 36 04 07 14 11 24 33 31 10 16 30 22 15 23 27 20 34 25 01
01 21 14 22 32 06 11 15 03 26 24 23 13 12 33 27 05 37 31 20 35 36 10 34 17 04 16 25 02 07 30 01
01 17 25 04 34 36 20 37 27 12 23 26 15 06 22 21 30 02 16 17 10 35 31 05 33 13 24 03 11 32 14 01
01 16 26 05 23 11 21 25 13 37 13 15 14 02 34 31 12 03 22 07 17 20 33 26 32 30 04 35 27 24 06 01
01 24 37 15 30 10 33 11 07 26 12 06 02 35 17 32 25 20 23 22 16 31 24 14 04 37 26 21 17 05 02 01
01 35 26 07 31 15 25 06 05 04 33 22 34 13 21 36 24 30 20 03 02 37 11 16 27 32 17 12 14 10 23 01
01 17 22 35 11 24 26 16 13 17 27 21 31 32 26 15 17 24 25 12 30 05 14 20 05 16 03 04 21 02 37 01
01 37 05 22 34 25 10 06 01 14 05 30 12 25 24 17 15 36 22 31 21 27 07 13 16 26 34 11 35 22 37 01
01 12 19 14 13 17 32 27 16 11 37 02 03 20 30 24 36 21 17 34 22 33 04 06 05 25 15 31 07 26 35 01
01 17 06 17 21 26 37 04 14 24 31 16 02 23 30 25 32 13 33 02 06 12 36 07 11 33 10 30 15 27 24 01
01 16 24 27 35 04 30 32 26 30 20 17 07 22 02 12 21 34 02 14 15 13 37 10 25 21 11 23 05 36 16 01
01 14 32 11 03 24 13 22 05 21 35 10 17 16 02 30 21 22 06 15 26 23 12 27 37 20 36 34 04 25 07 01
01 30 07 02 25 16 04 17 34 10 26 35 20 31 37 05 27 33 12 13 23 24 26 03 15 11 06 32 22 14 21 01
01 25 24 20 27 27 15 22 30 16 10 31 37 24 11 14 07 04 36 37 12 26 06 21 02 17 35 05 13 03 32 01
01 17 27 24 22 25 35 12 11 20 34 05 26 14 16 20 13 06 07 10 27 03 21 04 31 23 32 02 36 33 15 01
01 26 23 21 10 13 14 34 12 22 17 33 32 04 27 06 16 05 11 25 37 15 02 31 03 07 20 26 30 35 24 01
01 31 06 34 24 02 27 14 35 15 04 13 30 37 32 10 26 25 33 21 20 11 17 23 07 05 22 36 03 16 12 01
01 27 30 33 07 12 02 13 25 23 16 24 04 26 17 03 34 15 10 11 36 06 35 32 20 22 31 14 37 21 05 01
01 13 17 06 37 07 24 10 22 27 25 03 35 21 12 04 11 31 30 23 34 32 05 02 26 36 14 33 16 15 20 01
01 26 31 25 06 33 34 21 24 20 02 11 27 17 14 23 35 07 15 05 04 22 13 36 30 03 37 16 32 12 10 01
01 11 13 31 17 30 06 23 37 34 07 32 24 05 10 02 22 26 27 36 25 14 03 33 35 16 21 15 12 20 04 01
01 22 11 26 13 27 31 36 17 25 30 14 06 03 23 33 37 35 34 16 07 21 32 15 24 12 05 20 10 04 02 01
```

Appendix D

Tables of ENF (encode normal forms) produced by cold precomputations

RNF Matrix for GF 2**(2) mod 7 is:

1 1 1 1
1 2 1 0

RNF Matrix for GF 2**(3) mod 13 is:

1 1 1 1 1 1 1 1
7 2 6 5 3 4 1 0
2 3 2 1 3 1 0 0
3 5 2 5 1 0 0 0
4 3 6 1 0 0 0 0
3 2 1 0 0 0 0 0

RNF Matrix for GF 2**(4) mod 23 is:

01 01 01 01 01 01 01 01 01 01 01 01 01 01 01 01
17 02 0e 16 15 13 07 14 11 03 04 12 05 10 01 00
14 10 15 14 04 11 15 04 01 10 05 11 05 01 00 00
07 14 10 0e 03 17 03 15 12 03 05 12 01 00 00 00
15 07 02 14 02 04 11 14 12 10 04 01 00 00 00 00
15 01 12 07 15 03 11 15 01 03 01 00 00 00 00 00
02 14 16 0e 15 04 02 04 11 01 00 00 00 00 00 00
11 15 16 07 02 17 15 14 01 00 00 00 00 00 00 00
1e 14 12 14 03 11 07 01 00 00 00 00 00 00 00 00
13 01 02 06 04 13 01 00 00 00 00 00 00 00 00 00
17 07 10 14 15 01 00 00 00 00 00 00 00 00 00 00
1e 04 15 1e 01 00 00 00 00 00 00 00 00 00 00 00
17 11 0e 01 00 00 00 00 00 00 00 00 00 00 00 00
13 12 01 00 00 00 00 00 00 00 00 00 00 00 00 00

RNF Matrix for GF 2**(5) and 45 is:

```
01 01 01 01 01 01 01 01 01 01 01 01 01 01 01 01 01 01 01 01 01 01 01 01 01 01 01 01 01 01 01
35 02 06 1a 3a 32 21 05 10 22 03 04 12 2e 13 24 17 34 27 21 25 15 30 27 11 20 07 14 32 23 01 00
07 10 35 06 17 32 11 25 33 01 10 25 27 24 23 34 06 07 17 22 34 1e 22 24 33 11 35 16 32 01 00 00
33 12 11 27 36 05 17 1a 31 36 01 21 20 35 01 23 31 12 05 11 31 11 06 04 03 32 35 14 01 00 00 00
21 21 31 32 32 06 04 21 24 21 15 23 25 15 36 1e 22 11 36 22 01 24 10 36 03 11 07 01 00 00 00 00
34 27 36 21 02 24 32 02 04 13 36 34 10 21 26 27 21 27 10 35 03 01 10 04 33 20 01 00 00 00 00 00
17 30 31 32 13 30 24 10 27 0e 13 02 1e 30 03 34 35 16 31 15 03 24 06 24 11 01 00 00 00 00 00 00
15 26 22 21 33 31 37 04 36 26 30 33 11 23 07 17 12 32 31 35 01 11 22 27 01 00 00 00 00 00 00 00
13 05 10 04 17 05 23 10 05 31 15 05 26 34 36 03 12 16 10 22 31 1e 30 01 00 00 00 00 00 00 00 00
14 35 01 35 15 16 17 14 24 23 04 32 11 01 36 17 35 27 36 11 34 15 01 00 00 00 00 00 00 00 00 00
27 36 31 1e 26 01 25 20 1e 36 06 17 11 34 07 34 21 11 05 22 25 01 00 00 00 00 00 00 00 00 00 00
13 20 32 03 04 33 17 27 05 13 0e 32 2e 23 07 27 22 13 17 31 01 00 00 00 00 00 00 00 00 00 00 00
24 33 16 14 37 35 07 37 05 36 04 05 11 30 26 1e 31 07 37 01 00 00 00 00 00 00 00 00 00 00 00 00
12 11 3e 1e 17 1e 07 27 16 37 15 37 1e 21 36 23 06 34 01 00 00 00 00 00 00 00 00 00 00 00 00 00
24 1e 02 37 17 35 17 23 24 31 20 02 10 15 01 34 17 01 00 00 00 00 00 00 00 00 00 00 00 00 00 00
11 17 02 1e 37 37 25 14 05 25 17 34 25 05 23 24 01 00 00 00 00 00 00 00 00 00 00 00 00 00 00 00
05 16 3e 14 04 01 17 10 3e 0e 36 37 2e 24 13 01 00 00 00 00 00 00 00 00 00 00 00 00 00 00 00 00
27 11 1e 37 1e 1e 13 04 27 11 1e 31 23 26 01 00 00 00 00 00 00 00 00 00 00 00 00 00 00 00 00 00
32 33 32 1e 05 3 37 10 04 21 31 25 12 01 30 00 00 00 00 00 00 00 00 00 00 00 00 00 00 00 00 00
23 20 31 35 17 31 24 32 24 36 10 04 01 00 00 00 00 00 00 00 00 00 00 00 00 00 00 00 00 00 00 00
11 36 01 04 33 23 32 21 31 01 03 01 00 00 00 00 00 00 00 00 00 00 00 00 00 00 00 00 00 00 00 00
31 35 10 21 13 24 04 16 33 32 01 00 00 00 00 00 00 00 00 00 00 00 00 00 00 00 00 00 00 00 00 00
32 05 22 32 02 0e 17 25 10 01 00 00 00 00 00 00 00 00 00 00 00 00 00 00 00 00 00 00 00 00 00 00
14 26 31 21 32 05 11 05 01 00 00 00 00 00 00 00 00 00 00 00 00 00 00 00 00 00 00 00 00 00 00 00
11 30 36 32 36 32 21 01 00 00 00 00 00 00 00 00 00 00 00 00 00 00 00 00 00 00 00 00 00 00 00 00
14 37 31 27 17 33 01 00 00 00 00 00 00 00 00 00 00 00 00 00 00 00 00 00 00 00 00 00 00 00 00 00
01 21 11 06 36 01 00 00 00 00 00 00 00 00 00 00 00 00 00 00 00 00 00 00 00 00 00 00 00 00 00 00
30 12 35 16 01 00 00 00 00 00 00 00 00 00 00 00 00 00 00 00 00 00 00 00 00 00 00 00 00 00 00 00
17 10 06 01 00 00 00 00 00 00 00 00 00 00 00 00 00 00 00 00 00 00 00 00 00 00 00 00 00 00 00 00
03 02 01 00 00 00 00 00 00 00 00 00 00 00 00 00 00 00 00 00 00 00 00 00 00 00 00 00 00 00 00 00
```

Appendix E

Examples of the encode/decode process

```
Please enter, on one line and separated by a blank,
the field-base and modulus to be used.  The
field-base should be a decimal number and the
modulus should be an octal number.
4 23
Please enter the number of channels to be sent
by the transmitting node.  This should be a
decimal number.
10
please enter the number of channels active at
the receivers node; this should be a decimal number.
8

DNF matrix for  8 out of 10
channels over GF 2**(4) mod 23 is:


01 00 11 17 16 04 15 11 13 02
00 01 14 02 07 01 10 04 01 04

Please enter, on one line and separated by blanks,
the numbers of the  8 channels active
at the receiving node.  These numbers should be decimal.
3 4 5 6 7 8 9 10
Decoder matrix for the active channels listed above is:

01 00 11 17 16 04 15 11 13 02
00 01 14 02 07 01 10 04 01 04
00 00 01 00 00 00 00 00 00 00
00 00 00 01 00 00 00 00 00 00
00 00 00 00 01 00 00 00 00 00
00 00 00 00 00 01 00 00 00 00
00 00 00 00 00 00 01 00 00 00
00 00 00 00 00 00 00 01 00 00

please enter, on one line and separated by blanks,
the data received on each of the channels active at
the receivers node.  The data should be in the form
of octal numbers, and should be entered in order of
increasing channel number.
12 13 14 15 16 17 07 14

the  8 transmitted cleartext words were
(octal numbers expressed in channel order):

10 11 12 13 14 15 16 17
do you want to decode another  8 words?
(type y or n).
n
Please enter, on one line and separated by a blank,
the field-base and modulus to be used.  The
field-base should be a decimal number and the
modulus should be an octal number.
4 23
Please enter the number of channels to be sent
by the transmitting node.  This should be a
decimal number.
10
please enter the number of channels active at
the receivers node; this should be a decimal number.
8
```

```
GMF matrix for  6 out of 10
channels over GF 2**(4) mod 23 is:


01 00 11 17 16 04 15 11 13 02
00 01 14 02 07 01 10 04 01 04

please enter, on one line and separated by blanks,
the numbers of the  3 channels active
at the receiving node.  These numbers should be decimal.
2 3 4 6 7 8 9 10
Decoder matrix for the active channels listed above is:

01 02 02 13 00 06 16 01 11 12
00 01 00 00 00 00 00 00 00 00
00 00 01 00 00 00 00 00 00 00
00 00 00 01 00 00 00 00 00 00
00 06 16 14 01 06 05 13 06 13
00 00 00 00 00 01 00 00 00 00
00 00 00 00 00 00 01 00 00 00
00 00 00 00 00 00 00 01 00 00

please enter, on one line and separated by blanks,
the data received on each of the channels active at
the receivers node.  The data should be in the form
of octal numbers, and should be entered in order of
increasing channel number.
11 12 13 15 16 17 07 14

the  8 transmitted cleartext words were
(octal numbers expressed in channel order):

10 11 12 13 14 15 16 17
do you want to decode another  8 words?
(type y or n).
n
```

```
Please enter, on one line and separated by a blank,
the field-base and modulus to be used.  The
field-base should be a decimal number and the
modulus should be an octal number.
4 23
Please enter the number of channels to be sent
by the transmitting node.  This should be a
decimal number.
10
please enter the number of channels active at
the receivers node; this should be a decimal number.
8

Enf matrix for  8 out of 10
channels over GF 2**(4) mod 23 is:


01 00 11 17 16 04 15 11 13 02
00 01 14 02 07 01 10 04 01 04

please enter, on one line and separated by blanks,
the numbers of the  8 channels active
at the receiving node.  These numbers should be decimal.
2 3 4 5 6 7 8 10
Decoder matrix for the active channels listed above is:

01 13 04 12 12 17 12 03 00 10
00 01 00 00 00 00 00 00 00 00
00 00 01 00 00 00 00 00 00 00
00 00 00 01 00 00 00 00 00 00
00 00 00 00 01 00 00 00 00 00
00 00 00 00 00 01 00 00 00 00
00 00 00 00 00 00 01 00 00 00
00 00 00 00 00 00 00 01 00 00

please enter, on one line and separated by blanks,
the data received on each of the channels active at
the receivers node.  The data should be in the form
of octal numbers, and should be entered in order of
increasing channel number.
11 12 13 14 15 16 17 14

the  8 transmitted cleartext words were
(octal numbers expressed in channel order):

10 11 12 13 14 15 16 17
do you want to decode another  8 words?
(type y or n).
n
Please enter, on one line and separated by a blank,
the field-base and modulus to be used.  The
field-base should be a decimal number and the
modulus should be an octal number.
4 23
Please enter the number of channels to be sent
by the transmitting node.  This should be a
decimal number.
10
please enter the number of channels active at
the receivers node; this should be a decimal number.
8
```

```
DNF matrix for  8 out of 10
channels over GF 2** 4) mod 23 is:


01 00 11 17 16 04 15 11 13 02
00 01 14 02 07 01 10 04 01 04

please enter, on one line and separated by blanks,
the numbers of the  8 channels active
at the receiving node.  These numbers should be decimal.
1 2 3 4 6 7 8 10
Decoder matrix for the active channels listed above is:

01 00 00 00 00 00 00 00 00 00
00 01 00 00 00 00 00 00 00 00
00 00 01 00 00 00 00 00 00 00
00 00 00 01 00 00 00 00 00 00
14 15 05 01 01 10 01 07 00 12
00 00 00 00 00 01 00 00 00 00
00 00 00 00 00 00 01 00 00 00
00 00 00 00 00 00 00 01 00 00

please enter, on one line and separated by blanks,
the data received on each of the channels active at
the receivers node.  The data should be in the form
of octal numbers, and should be entered in order of
increasing channel number.
10 11 12 13 15 16 17 14

the  8 transmitted cleartext words were
(octal numbers expressed in channel order):

10 11 12 13 14 15 16 17
do you want to decode another  8 words?
(type y or n).
n
Please enter, on one line and separated by a blank,
the field-base and modulus to be used.  The
field-base should be a decimal number and the
modulus should be an octal number.
4 23
Please enter the number of channels to be sent
by the transmitting node.  This should be a
decimal number.
10
[illegible] the number of channels
the receivers node; this should be a decimal number.
8

DNF matrix for  8 out of 10
channels over GF 2**(4) mod 23 is:


01 00 11 17 16 04 15 11 13 02
00 01 14 02 07 01 10 04 01 04

please enter, on one line and separated by blanks,
the numbers of the  8 channels active
at the receiving node.  These numbers should be decimal.
1 2 3 4 5 6 7 8
Decoder matrix for the active channels listed above is:

01 00 00 00 00 00 00 00 00 00
00 01 00 00 00 00 00 00 00 00
00 00 01 00 00 00 00 00 00 00
```

```
00 00 00 00 01 00 00 00 00 00
00 00 00 00 00 01 00 00 00 00
00 00 00 00 00 00 01 00 00 00
00 00 00 00 00 00 00 01 00 00

please enter, on one line and separated by blanks,
the data received on each of the channels active at
the receivers node.  The data should be in the form
of octal numbers, and should be entered in order of
increasing channel number.
10 11 12 13 14 15 16 17

the  8 transmitted cleartext words were
(octal numbers expressed in channel order):

10 11 12 13 14 15 16 17
do you want to decode another  8 words?
(type y or n).
n
```

please enter, on one line and separated by blanks,
the field-base, modulus, number of channels to be sent,
and number of channels to be received. The modulus
should be an octal number; all other numbers should
be decimal.

4 23 10 3

thank you...please wait

ENF MATRIX----------

01 17 14 07 15 05 02 11 06 13 17 16 17 03
01 02 10 14 07 01 14 15 14 01 07 14 10 02
01 06 15 10 02 12 16 16 12 02 10 15 06 01
01 16 14 06 14 07 06 07 14 06 14 16 01 00
01 15 04 03 02 15 15 02 03 04 15 01 00 00
01 13 11 17 04 02 04 17 11 13 01 00 00 00
01 07 15 02 11 11 03 15 07 01 00 00 00 00
01 14 04 15 14 15 04 14 01 00 00 00 00 00
01 11 01 12 12 01 11 01 00 00 00 00 00 00
01 03 10 03 10 03 01 00 00 00 00 00 00 00
01 04 05 05 04 01 00 00 00 00 00 00 00 00
01 12 11 12 01 00 00 00 00 00 00 00 00 00
01 05 05 01 00 00 00 00 00 00 00 00 00 00
01 10 01 00 00 00 00 00 00 00 00 00 00 00
01 01 00 00 00 00 00 00 00 00 00 00 00 00
01 00 00 00 00 00 00 00 00 00 00 00 00 00

SUBMATRIX---------->

02 14 1a 06 15 04 03 04 11 01
11 15 1e 07 02 17 15 14 01 00
0e 14 12 14 03 11 07 01 00 00
12 01 02 06 04 13 01 00 00 00
17 07 10 14 15 01 00 00 00 00
1a 14 15 1e 01 00 00 00 00 00
17 10 06 01 00 00 00 00 00 00

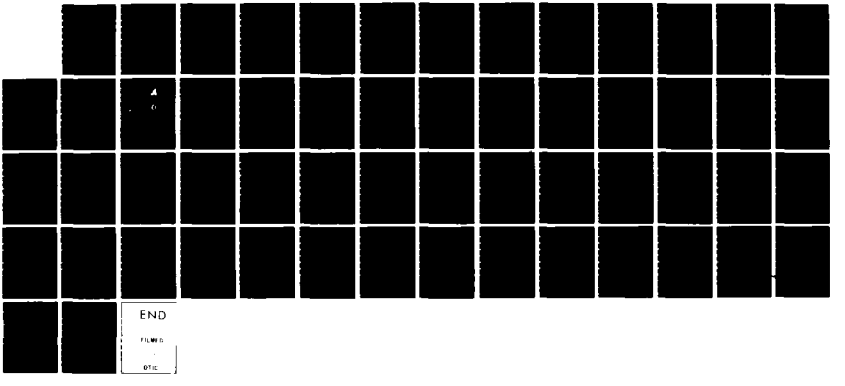
ENCODE KEY---------->

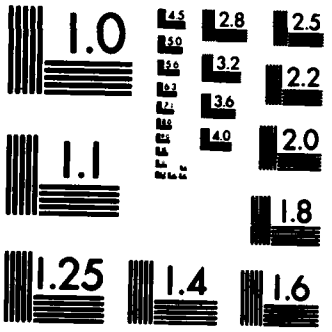
16 11 06 00 00 00 00 00 00 01
13 04 16 00 00 00 00 00 01 00
05 15 11 00 00 00 00 01 00 00
16 10 07 00 00 00 01 00 00 00
01 01 01 00 00 01 00 00 00 00
13 05 17 00 01 00 00 00 00 00
17 19 06 01 00 00 00 00 00 00

```
please enter, on one line, in octal and separated
by blanks, the values to be transmitted over the
transmitters  3 channels
0 1 2
words transmitted are (in channel order):
00 01 02 04 10 03 06 14 13 05

do you want to send another  3 words?
(type y or n)
y
please enter, on one line, in octal and separated
by blanks, the values to be transmitted over the
transmitters  3 channels
3 4 5
words transmitted are (in channel order):
03 04 05 11 17 02 17 05 16 16

do you want to send another  3 words?
(type y or n)
y
please enter, on one line, in octal and separated
by blanks, the values to be transmitted over the
transmitters  3 channels
6 7 10
words transmitted are (in channel order):
16 07 10 14 06 11 02 14 11 15

do you want to send another  3 words?
(type y or n)
y
please enter, on one line, in octal and separated
by blanks, the values to be transmitted over the
transmitters  3 channels
11 12 13
words transmitted are (in channel order):
11 12 13 16 13 10 14 14 12 15

do you want to send another  3 words?
(type y or n)
y
please enter, on one line, in octal and separated
by blanks, the values to be transmitted over the
transmitters  3 channels
14 15 16
words transmitted are (in channel order):
14 15 16 10 04 17 12 00 07 11

do you want to send another  3 words?
(type y or n)
y
please enter, on one line, in octal and separated
by blanks, the values to be transmitted over the
transmitters  3 channels
17 10 4
words transmitted are (in channel order):
17 10 04 15 04 03 05 06 10 12

do you want to send another  3 words?
(type y or n)
n
```

```
please enter, on one line and separated by blanks,
the field-base, modulus, number of channels to be sent,
and number of channels to be received.  The modulus
should be an octal number; all other numbers should
be decimal.

: 23 10 5

thank you...please wait


INF MATRIX---------------------->

01 17 14 07 15 05 02 11 06 13 17 16 17 03
11 02 10 14 07 01 14 15 14 01 07 14 10 02
01 05 15 10 02 12 16 16 12 02 10 15 06 01
01 16 14 06 14 07 06 07 14 06 14 16 01 00
01 15 04 03 02 15 15 02 03 04 15 01 00 00
01 13 11 17 04 03 04 17 11 13 01 00 00 00
01 07 15 03 11 11 03 15 07 01 00 00 00 00
01 14 04 15 14 15 04 14 01 00 00 00 00 00
01 11 01 12 12 01 11 01 00 00 00 00 00 00
01 03 10 03 10 03 01 00 00 00 00 00 00 00
01 04 05 05 04 01 00 00 00 00 00 00 00 00
01 12 11 12 01 00 00 00 00 00 00 00 00 00
01 05 05 01 00 00 00 00 00 00 00 00 00 00
01 10 01 00 00 00 00 00 00 00 00 00 00 00
01 01 00 00 00 00 00 00 00 00 00 00 00 00
01 00 00 00 00 00 00 00 00 00 00 00 00 00
```

AD-A142 831 HIGH SPEED LOW-COST WAYS TO GET MESSAGES FROM A SENDER 2/2
TO A RECEIVER WHEN.. (U) YLYK LTD ANN ARBOR MI
B BLAKLEY 28 MAY 84 YLYK/AFOSR/SBIRI/83-84/001
UNCLASSIFIED AFOSR-TR-84-0528 F49620-83-C-0160 F/G 17/2.1 NL





MICROCOPY RESOLUTION TEST CHART
NATIONAL BUREAU OF STANDARDS-1963-A

SUBMATRIX---------->

02 14 16 06 15 04 03 04 11 01
11 15 16 07 02 17 15 14 01 00
06 14 12 14 03 11 07 01 00 00
13 01 02 06 04 13 01 00 00 00
17 07 10 14 15 01 00 00 00 00

```
ENCODE KEY---------------------->

03 06 03 17 10 00 00 00 00 01
10 16 15 15 07 00 00 00 01 00
05 16 06 16 02 00 00 01 00 00
10 05 05 13 02 00 01 00 00 00
17 07 10 14 15 01 00 00 00 00
```

```
please enter, on one line, in octal and separated
by blanks, the values to be transmitted over the
transmitters  5 channels
0 1 2 3 4
words transmitted are (in channel order):
00 01 02 03 04 02 11 13 14 04

do you want to send another  5 words?
(type y or n)
y
please enter, on one line, in octal and separated
by blanks, the values to be transmitted over the
transmitters  5 channels
5 6 7 10 11
words transmitted are (in channel order):
05 06 07 10 11 17 15 11 01 04

do you want to send another  5 words?
(type y or n)
y
please enter, on one line, in octal and separated
by blanks, the values to be transmitted over the
transmitters  5 channels
12 13 14 15 16
words transmitted are (in channel order):
12 13 14 15 16 13 16 07 06 13

do you want to send another  5 words?
(type y or n)
y
please enter, on one line, in octal and separated
by blanks, the values to be transmitted over the
transmitters  5 channels
17 10 4 14 0
words transmitted are (in channel order):
17 10 04 14 00 16 05 00 12 03

do you want to send another  5 words?
(type y or n)
n
```

```
please enter, on one line and separated by blanks,
the field-base, modulus, number of channels to be sent,
and number of channels to be received.  The modulus
should be an octal number; all other numbers should
be decimal.

4 23 10 8

thank you...please wait


ENF MATRIX---------------------->

01 17 14 07 15 05 02 11 06 13 17 16 17 03
01 02 10 14 07 01 14 15 14 01 07 14 10 02
01 06 15 10 02 12 16 16 12 02 10 15 06 01
01 16 14 06 14 07 06 07 14 06 14 16 01 00
01 15 04 03 02 15 15 02 03 04 15 01 00 00
01 13 11 17 04 03 04 17 11 13 01 00 00 00
01 07 15 03 11 11 03 15 07 01 00 00 00 00
01 14 04 15 14 15 04 14 01 00 00 00 00 00
01 11 01 12 12 01 11 01 00 00 00 00 00 00
01 03 10 03 10 03 01 00 00 00 00 00 00 00
01 04 05 05 04 01 00 00 00 00 00 00 00 00
01 12 11 12 01 00 00 00 00 00 00 00 00 00
01 05 05 01 00 00 00 00 00 00 00 00 00 00
01 10 01 00 00 00 00 00 00 00 00 00 00 00
01 01 00 00 00 00 00 00 00 00 00 00 00 00
01 00 00 00 00 00 00 00 00 00 00 00 00 00
```

SUBMATRIX---------->

02 14 16 06 15 04 03 04 11 01
11 15 16 07 02 17 15 14 01 00

ENCODE KEY---------->

17 03 11 14 14 12 14 02 00 01
11 15 16 07 02 17 15 14 01 00

```
please enter, on one line, in octal and separated
by blanks, the values to be transmitted over the
transmitters  8 channels
0 1 2 3 4 5 6 7
words transmitted are (in channel order):
00 01 02 03 04 05 06 07 17 04

do you want to send another  8 words?
(type y or n)
y
please enter, on one line, in octal and separated
by blanks, the values to be transmitted over the
transmitters  8 channels
10 11 12 13 14 15 16 17
words transmitted are (in channel order):
10 11 12 13 14 15 16 17 07 14

do you want to send another  8 words?
(type y or n)
n
```

```
Please enter, on one line and separated by a blank,
the field-base and modulus to be used.  The
field-base should be a decimal number and the
modulus should be an octal number.
4 23
Please enter the number of channels to be sent
by the transmitting node.  This should be a
decimal number.
10
please enter the number of channels active at
the receivers node; this should be a decimal number.
3

DNF matrix for  3 out of 10
channels over GF 2**(4) mod 23 is:


01 00 00 00 00 00 00 14 05 10
00 01 00 00 00 00 00 04 12 17
00 00 01 00 00 00 00 01 01 01
00 00 00 01 00 00 00 10 17 06
00 00 00 00 01 00 00 05 12 16
00 00 00 00 00 01 00 11 16 06
00 00 00 00 00 00 01 05 13 17

please enter, on one line and separated by blanks,
the numbers of the  3 channels active
at the receiving node.  These numbers should be decimal.
1 5 8
Decoder matrix for the active channels listed above is:

01 00 00 00 00 00 00 00 00 00
10 01 00 00 05 00 00 14 00 00
16 00 01 00 13 00 00 04 00 00

please enter, on one line and separated by blanks,
the data received on each of the channels active at
the receivers node.  The data should be in the form
of octal numbers, and should be entered in order of
increasing channel number.
6 6 14

the  3 transmitted cleartext words were
(octal numbers expressed in channel order):

06 07 10
do you want to decode another  3 words?
(type y or n).
y
please enter, on one line and separated by blanks,
the data received on each of the channels active at
the receivers node.  The data should be in the form
of octal numbers, and should be entered in order of
increasing channel number.
3 17 5

the  3 transmitted cleartext words were
(octal numbers expressed in channel order):

03 04 05
do you want to decode another  3 words?
```

```
Please enter, on one line and separated by a blank,
the field-base and modulus to be used.  The
field-base should be a decimal number and the
modulus should be an octal number.
4 23
Please enter the number of channels to be sent
by the transmitting node.  This should be a
decimal number.
10
Please enter the number of channels active at
the receivers node; this should be a decimal number.
5

INF matrix for  5 out of 10
channels over GF 2**(4) mod 23 is:


 01 00 00 00 00 15 14 12 02 10
 00 01 00 00 00 14 14 13 03 11
 00 00 01 00 00 12 02 16 02 05
 00 00 00 01 00 10 14 05 13 13
 00 00 00 00 01 04 13 06 16 06

Please enter, on one line and separated by blanks,
the numbers of the  5 channels active
at the receiving node.  These numbers should be decimal.
5 7 8 9 10
Decoder matrix for the active channels listed above is:

01 00 00 00 00 15 14 12 02 10
00 01 00 00 00 14 14 13 03 11
00 00 01 00 00 12 02 16 02 05
00 00 00 01 00 10 14 05 13 13
00 00 00 00 01 04 13 06 16 06

Please enter, on one line and separated by blanks,
the data received on each of the channels active at
the receivers node.  The data should be in the form
of octal numbers, and should be entered in order of
increasing channel number.
13 16 07 06 13

The  5 transmitted cleartext words were
(octal numbers expressed in channel order):

12 13 14 15 16
Do you want to decode anotner  5 words?
(type y or n).
y
Please enter, on one line and separated by a blank,
the field-base and modulus to be used.  The
field-base should be a decimal number and the
modulus should be an octal number.
4 23
Please enter the number of channels to be sent
by the transmitting node.  This should be a
decimal number.
10
please enter the number of channels active at
the receivers node; this should be a decimal number.
8

INF matrix for  8 out of 10
channels over GF 2**(4) mod 23 is:
```

```
01 00 11 17 16 04 15 11 13 02
00 01 14 02 07 01 10 04 01 04

please enter, on one line and separated by blanks,
the numbers of the  8 channels active
at the receiving node.  These numbers should be decimal.
1 2 3 4 6 7 8 10
Decoder matrix for the active channels listed above is:

01 00 00 00 00 00 00 00 00 00
00 01 00 00 00 00 00 00 00 00
00 00 01 00 00 00 00 00 00 00
00 00 00 01 00 00 00 00 00 00
14 13 05 01 01 10 01 07 00 12
00 00 00 00 00 01 00 00 00 00
00 00 00 00 00 00 01 00 00 00
00 00 00 00 00 00 00 01 00 00

please enter, on one line and separated by blanks,
the data received on each of the channels active at
the receivers node.  The data should be in the form
of octal numbers, and should be entered in order of
increasing channel number.
10 11 12 13 15 16 17 14

the  8 transmitted cleartext words were
(octal numbers expressed in channel order):

10 11 12 13 14 15 16 17
do you want to decode another  8 words?
(type , or n).
```

Appendix F

Copy of Yeh/Reed/Truong paper
on systolic multipliers for finite fields

Systolic Multipliers for Finite Fields $GF(2^m)$

C.-S. YEH, STUDENT MEMBER, IEEE, IRVING S. REED, FELLOW, IEEE, AND T. K. TRUONG, MEMBER, IEEE

Abstract — **Two systolic architectures are developed for performing the product-sum computation $AB + C$ in the finite field $GF(2^m)$ of 2^m elements, where A, B, and C are arbitrary elements of $GF(2^m)$. The first multiplier is a serial-in, serial-out one-dimensional systolic array, while the second multiplier is a parallel-in, parallel-out two-dimensional systolic array. The first multiplier requires a smaller number of basic cells than the second multiplier. The second multiplier needs less average time per computation than the first multiplier if a number of computations are performed consecutively. To perform single computations both multipliers require the same computational time. In both cases the architectures are simple and regular and possess the properties of concurrency and modularity. As a consequence they are well suited for use in VLSI systems.**

Index Terms — **Finite field, logic design, primitive element, systolic array.**

I. Introduction

FINITE or Galois fields have many important and practical applications. Finite fields can be applied to error-correcting codes [1]–[3], switching theory [4], and digital signal processing [5]. For example, finite fields are used in the construction of many error-correcting codes. Reed-Solomon (RS) codes utilize the finite field $GF(2^m)$ of 2^m elements, where m is a positive integer. The encoding and decoding algorithm of a binary RS code require algebraic operations in some field $GF(2^m)$, rather than the usual binary arithmetic operations.

The operations of addition and multiplication in a finite field are quite different from the usual binary arithmetic operations. Because of their simplicity and practical usefulness, only the finite fields $GF(2^m)$ are considered in this paper. Addition in $GF(2^m)$ is bit independent and straightforward. Clearly it is easier than the usual binary addition. On the contrary, multiplication in $GF(2^m)$ is more complex and difficult than binary integer multiplication.

Several circuits have been proposed [1]–[3], [6]–[8] to realize multiplication in $GF(2^m)$. Unfortunately, these circuits are not suited for use in VLSI systems, due to irregular wire routing and complicated control problems as well as a nonmodular structure or lack of concurrency [9].

In this paper two parallel architectures are designed to perform multiplication in $GF(2^m)$. In Section II an algorithm is derived for multiplication in $GF(2^m)$. This algorithm is mapped into the hardware design in Sections III and IV. In Section III a one-dimensional systolic multiplier for $GF(2^m)$ is designed. This multiplier is serial-in, serial-out. In Section IV, a parallel-in, parallel-out multiplier in $GF(2^m)$ is developed. The latter multiplier has a two-dimensional array structure.

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II. Multiplication in $GF(2^m)$

It is assumed that the reader is familiar with the basic concepts of finite fields. The properties of finite fields are covered in detail in [1]–[3]. In the following the properties of finite fields are reviewed briefly as required.

A finite field must contain p^m elements, where p is a prime integer and m is a positive integer. The finite field $GF(2^m)$ contains 2^m elements. $GF(2^m)$ is an extension field of the ground field $GF(2)$ of 2 elements, i.e., $GF(2) = \{0, 1\}$. All arithmetic operations in $GF(2^m)$ are performed by taking the results modulo 2.

The nonzero elements of $GF(2^m)$ are generated by a primitive element α, where α is a root of a primitive irreducible polynomial $F(x) = x^m + f_{m-1}x^{m-1} + \cdots + f_1x + f_0$ over $GF(2)$. For example $F(x) = x^4 + x + 1$ is one such primitive irreducible polynomial for $GF(2^4)$.

The nonzero elements of $GF(2^m)$ can be represented as the powers of α, i.e., $GF(2^m) = \{0, \alpha^1, \alpha^2, \cdots, \alpha^{2^m-2} = 1\}$. Since $F(\alpha) = 0$, $\alpha^m = f_{m-1}\alpha^{m-1} + \cdots + f_1\alpha + f_0$. Therefore, an element of $GF(2^m)$ can be also expressed as a polynomial of α with degree less than m. That is, $GF(2^m) = \{a_{m-1}\alpha^{m-1} + \cdots + a_1\alpha + a_0 \mid a_i \in GF(2)$ for $0 \le i \le m-1\}$. In the following discussion, the polynomial representation is used to represent the finite field $GF(2^m)$.

Let $A = a_{m-1}\alpha^{m-1} + \cdots + a_1\alpha + a_0$ and $B = b_{m-1}\alpha^{m-1} + \cdots + b_1\alpha + b_0$ be two elements in $GF(2^m)$. Then $A + B = S_{m-1}\alpha^{m-1} + \cdots + S_1\alpha + S_0$, where $S_i = a_i + b_i \pmod 2$ for $0 \le i \le m-1$. Therefore, addition in $GF(2^4)$ is realized easily by m independent EXCLUSIVE-OR gates.

Suppose $P = p_{m-1}\alpha^{m-1} + \cdots + p_1\alpha + p_0$ is the product of A and B, i.e., $P = AB$. P can be written as follows [1]–[3]:

$$P = \sum_{k=0}^{m-1} (A\alpha^k)b_k = \sum_{k=0}^{m-1}\left(\sum_{n=0}^{m-1} a_n^{(k)}\alpha^n\right)b_k = \sum_{n=0}^{m-1}\left(\sum_{k=0}^{m-1} a_n^{(k)}b_k\right)\alpha^n \tag{1}$$

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where $a_n^{(k)}$ is the coefficient of α^n in $A\alpha^k$, i.e., $A\alpha^k = a_{m-1}^{(k)}\alpha^{m-1} + \cdots + a_1^{(k)}\alpha + a_0^{(k)}$ for $0 \le k \le m-1$. From (1), one obtains $p_n = a_n^{(0)}b_0 + a_n^{(1)}b_1 + \cdots + a_n^{(m-2)}b_{m-2} + a_n^{(m-1)}b_{m-1}$.

The computation of $A\alpha^k$ can be performed recursively on k for $0 \le k \le m-1$. Initially, for $k = 0$, $A\alpha^0 = A$, i.e., $a_n^{(0)} = a_n$ for $0 \le n \le m-1$. For $1 \le k \le m-1$,

$$A\alpha^k = (A\alpha^{k-1})\alpha = \sum_{n=0}^{m-1} a_n^{(k-1)}\alpha^{n+1} = a_{m-1}^{(k-1)}\alpha^m + \sum_{n=1}^{m-1} a_{n-1}^{(k-1)}\alpha^n. \tag{2}$$

Substituting $\alpha^m = f_{m-1}\alpha^{m-1} + \cdots + f_1\alpha + f_0$ into (2) yields

$$A\alpha^k = \sum_{n=1}^{m-1} (a_{n-1}^{(k-1)} + a_{m-1}^{(k-1)}f_n)\alpha^n + a_{m-1}^{(k-1)}f_0. \tag{3}$$

From (3), one obtains

$$a_n^{(k)} = a_{n-1}^{(k-1)} + a_{m-1}^{(k-1)}f_n \qquad \text{for } 1 \le n \le m-1$$
$$a_0^{(k)} = a_{m-1}^{(k-1)}f_0. \tag{4}$$

Fig. 1 illustrates the step-by-step operations of a procedure for performing $P = AB + C$ in $GF(2^4)$. In Fig. 1 $a_n^{(k)}$, b_n, c_n, f_n, and p_n are the nth bits of $A\alpha^k$, B, C, F, and P, respectively, where F is the primitive irreducible polynomial. $p_n^{(i)}$ is the partial sum of p_n.

In the following sections this procedure is mapped into two systolic architectures. The above symbols (e.g., F, P, $a_n^{(k)}$) are still used in the following sections.

STEP NUMBER	OPERATIONS	
1	$p_3^{(0)} = c_3$	$a_3^{(0)} = a_3$
2	$p_3^{(1)} = p_3^{(0)} + a_3^{(0)}b_0$, $p_2^{(0)} = c_2$	$a_2^{(0)} = a_2$
3	$p_2^{(1)} = p_2^{(0)} + a_2^{(0)}b_0$, $p_1^{(0)} = c_1$	$a_1^{(0)} = a_1$, $a_3^{(1)} = a_2^{(0)} + a_3^{(0)}f_3$
4	$p_3^{(2)} = p_3^{(1)} + a_3^{(1)}b_1$, $p_1^{(1)} = p_2^{(0)} + a_1^{(0)}b_0$, $p_0^{(0)} = c_0$	$a_0^{(0)} = a_0$, $a_2^{(1)} = a_3^{(0)} + a_2^{(0)}f_2$
5	$p_2^{(2)} = p_2^{(1)} + a_2^{(1)}b_1$, $p_0^{(1)} = p_0^{(0)} + a_0^{(0)}b_0$,	$a_1^{(1)} = a_2^{(0)} + a_1^{(0)}f_1$ $a_3^{(2)} = a_2^{(1)} + a_3^{(1)}f_3$
6	$p_3^{(3)} = p_3^{(2)} + a_3^{(2)}b_2$, $p_1^{(2)} = p_1^{(1)} + a_1^{(1)}b_1$,	$a_0^{(1)} = a_1^{(0)} + a_0^{(0)}f_0$ $a_2^{(2)} = a_2^{(1)} + a_3^{(1)}f_2$
7	$p_2^{(3)} = p_2^{(2)} + a_2^{(2)}b_2$, $p_0^{(2)} = p_0^{(1)} + a_0^{(1)}b_1$,	$a_1^{(2)} = a_2^{(1)} + a_3^{(1)}f_1$ $a_3^{(3)} = a_2^{(2)} + a_3^{(2)}f_3$
8	$p_3 = p_3^{(4)} = p_3^{(3)} + a_3^{(3)}b_3$, $p_1^{(3)} = p_1^{(2)} + a_1^{(2)}b_2$,	$a_0^{(2)} = a_1^{(1)}f_0$ $a_2^{(3)} = a_2^{(2)} + a_3^{(2)}f_2$
9	$p_2 = p_2^{(4)} = p_2^{(3)} + a_2^{(3)}b_3$, $p_0^{(3)} = p_0^{(2)} + a_0^{(2)}b_2$,	$a_1^{(3)} = a_2^{(2)} + a_1^{(2)}f_1$
10	$p_1 = p_1^{(4)} = p_1^{(3)} + a_1^{(3)}b_3$,	$a_0^{(3)} = a_1^{(2)}f_0$
11	$p_0 = p_0^{(4)} = p_0^{(3)} + a_0^{(3)}b_3$	

Fig. 1. A procedure for computing $P = AB + C$ in the finite field $GF(2^4)$, where A, B, C, and P are elements of $GF(2^4)$.

III. A Serial-In, Serial-Out Systolic Multiplier for $GF(2^m)$

In this section a one-dimensional systolic array is developed to compute $P = AB + C$ in $GF(2^m)$. A similar structure was proposed to multiply the usual two's complement binary numbers [10]. For simplicity in description the ensuing discussion is limited to the particular finite field $GF(2^4)$. In Fig. 2 this architecture is shown for $GF(2^4)$. The primitive irreducible polynomial is $F = f_3\alpha^3 + f_2\alpha^2 + f_1\alpha + f_0$. Input d_n receives the b_n of B. The nth bits c_n, a_n, and f_n of C, A, and F are received serially at inputs e_0, g_0, and h_0, respectively. Two control signals, START and END, are used in the design. Inputs r_0 and t_0 receive START and END control signals, respectively.

Output e_4 serially transmits the nth bit p_n of the result P out of the system. The order of the inputs and outputs are also shown in Fig. 2. The flip-flops associated with inputs t_0 and h_0 are used for the purpose of synchronization.

The circuit of cell L_i is shown in Fig. 3. The operation of flip-flops in this system is synchronized implicitly by a clock signal. In Fig. 3, when $r_i^* = 1$, $u_i = g_i^*$ at the next time unit (through switch SW). When $r_i^* = 0$, u_i sustains its value. Two principle operations of the system are the following:

$$e_{i+1} \leftarrow (g_i^* d_i) \oplus e_i^*$$
$$g_{i+1}^* \leftarrow (u_i h_i^*) \oplus (g_i^* t_i^*) \tag{5}$$

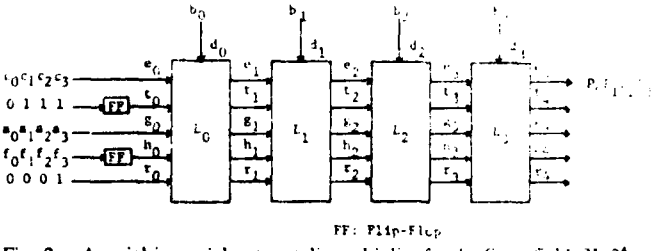


Fig. 2. A serial-in, serial-out systolic multiplier for the finite field $GF(2^4)$.

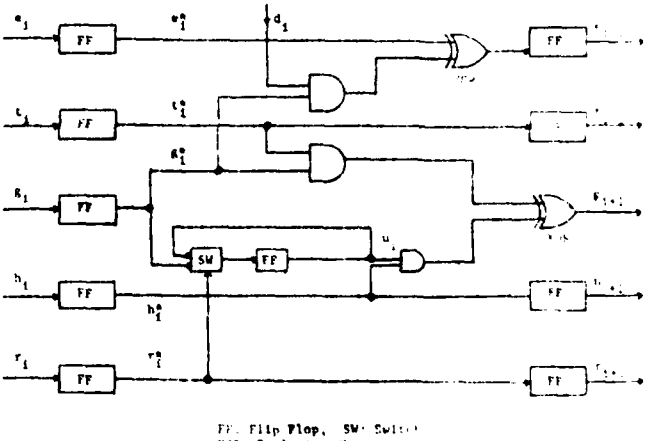


Fig. 3. The circuit of the cell L_i used in the multiplier shown in Fig. 2.

TABLE I
SOME PROPERTIES OF TWO SYSTOLIC MULTIPLIERS FOR $GF(2^m)$

Multiplier /	The multiplier in Fig. 2	The multiplier in Fig. 4
Number of basic cells	m	m^2
Input/output format	serial	parallel
Minimum average time per computation (time units)	m	1
Delay between first input and first output of a computation (time units)	$2m$	$2m$ ($3m$ if input/output delay is also counted)
Number of control signals	2	0

where $0 \le i \le 3$, $\oplus$ denotes EXCLUSIVE-OR operation, and the backwards arrow denotes the substitution operation.

A comparison of the procedure in Fig. 1 and the structure in Figs. 2 and 3 yields the following facts. The signal u_i in L_i is equal to $a_i^{(i)}$ in $A\alpha^i$. The signal g_i^* is equal to $a_n^{(i)}$ in $A\alpha^i$ for some n. The signal e_i^* is equal to the partial sum of $AB + C$.

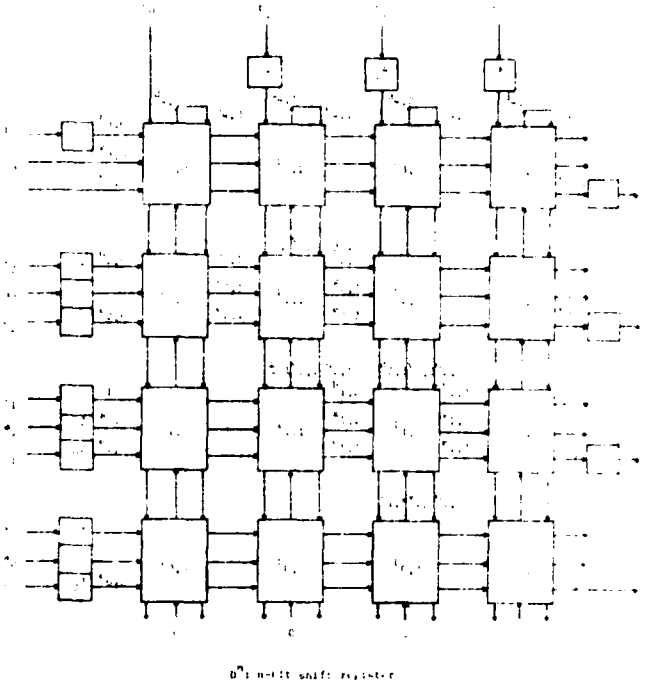
The multiplier in Fig. 2 can be generalized to the finite field $GF(2^m)$ by simply concatenating m identical cells. Extra registers and control signals are required if the b_i's are inputted serially into the system in the same order as the a_i's. Some properties of this multiplier are listed in Table I.

IV. A PARALLEL-IN, PARALLEL-OUT MULTIPLIER FOR $GF(2^m)$

In this section a parallel-in, parallel-out, two-dimensional systolic array is designed for performing $P = AB + C$ in $GF(2^m)$. A similar structure was designed [11] to perform multiplications in standard binary arithmetic. The discussion in this section is again limited to the finite field $GF(2^4)$. An analogous development can be constructed for any other finite field $GF(2^m)$. Fig. 4 shows this multiplier for $GF(2^4)$. In Fig. 4 D^n denotes an n-bit shift register or delay device. Inputs $d_{4,n}$'s, $c_{n,0}$'s, $g_{n,0}$'s, and $h_{n,0}$'s receive in parallel the b_n's of B, c_n's of C, a_n's of A, and f_n's of F, respectively, for $0 \le n \le 3$. The p_n's of the result P are transmitted out the system in parallel from outputs $e_{n,4}$'s for $0 \le n \le 3$.

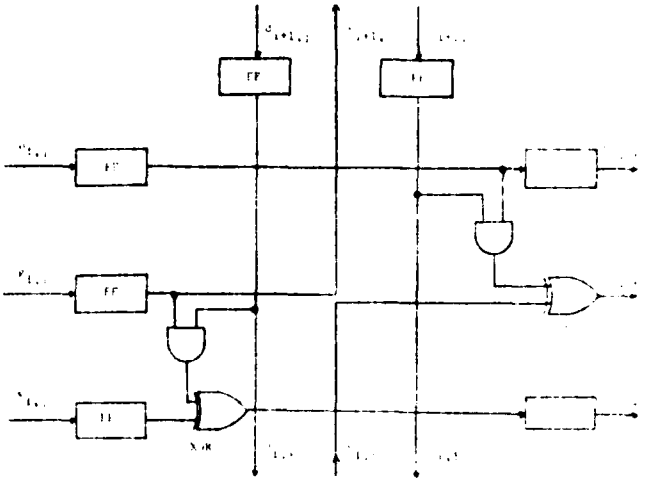
The circuit of a basic cell $L_{i,j}$ is shown in Fig. 5. This circuit is similar to the circuit shown in Fig. 3. Two of the primary operations of a basic cell are the same as the operations given in (5). One may use degenerative versions of the circuit in Fig. 5 for the cells in the bottom row and the rightmost column of the array structure in Fig. 4 since some inputs and outputs of these cells are not used. Note that the signal $g_{n,k}$ is equal to the $a_n^{(k)}$ of $A\alpha^k$. The signal $u_{n,k}$ is equal to $a_n^{(k)}$ of $A\alpha^k$ for $0 \le n \le 3$.

Some properties of the multiplier in Fig. 4 are also listed in Table I. The multiplier in Fig. 4 is "programmable" since F is changeable. If F is fixed or seldom changed then the design can be simplified by eliminating all flip-flops associated with $h_{i,0}$. For such a case buffers and long wires may be required.



D^n = n-bit shift register

Fig. 4. A two-dimensional parallel-in, parallel-out systolic multiplier for the finite field $GF(2^4)$.



FF = Flip-flop, XOR = Exclusive-OR

Fig. 5. The circuit of basic cell $L_{i,j}$ used in the multiplier shown in Fig. 4.

V. CONCLUSION

Two parallel architectures are designed for performing multiplication in the finite field $GF(2^m)$ of 2^m elements. A comparison between these two multipliers is listed in Table I. The multiplier in Fig. 4 can be viewed as a "time expansion" of the multiplier in Fig. 2. Both multipliers are suited well for VLSI systems because of the simple control, the regular interconnection pattern, the modular structure, and finally the complete concurrency of their operations.

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Appendix G

Copy of Bloom paper on threshold schemes

A Note on Superfast Threshold Schemes

John R. Bloom

Abstract: Threshold schemes, or key safeguarding schemes, are innovative new approaches to cryptokey transfer or secure data storage problems. This note outlines a class of schemes which approach optimality of speed and simplicity. The schemes are based on linear maps over finite fields. These schemes are the proper generalization of Vernam pads.

Key words and phrases: privacy, security, cryptography, message passing

CR Categories: 3.81, 5.6, 5.25

A threshold scheme is a method for producing, from a message x, n "shadows" $y_1, \ldots, y_n$, with the properties that:

1. Any r shadows suffice to determine x.

2. No $r-1$ shadows give any information about x.

Threshold schemes have been discussed in papers by Blakley [2], Shamir [4], where many applications are discussed, and Asmuth and Bloom [1], where a class of schemes including Shamir's is discussed, and further cross checking capabilities are also introduced. This paper introduces a class of schemes of optimum speed and simplicity when the message length is large compared to r. These schemes are the generalization of Vernam pads.

To generate such a threshold scheme, pick $v_0, v_1, \ldots, v_n$ vectors in $\mathbb{F}_q^r$ so that no r are linearly dependent. This can be done if $q > n$, and conjecturally for no smaller q. (See [3] pp. 323-328).

Considering the message x and the shadows y_i to be elements of $\mathbb{F}_q$, construct a linear map L from $\mathbb{F}_q^r$ to $\mathbb{F}_q$ with $Lv_0 = x$

and $Lv_1, \ldots, Lv_{r-1}$ random. Letting $y_i = Lv_i$, $i = 1,\ldots,n$ one has produced a threshold scheme. Property 1 is satisfied since any r v_i's span $\mathbb{F}_q^r$, and property 2 is satisfied since v_0 is not dependent on any $r-1$ v_i's.

In practice one picks q as small as possible and reduces x to a sequence m messages of size q.

Proposition: To produce a sequence of m shadows for fixed i requires at most mr additions and mr multiplications. To reconstitute the sequence of m x's requires at most $\frac{r^3}{3} + (m+1)r$ additions and $\frac{r^3}{3} + (m+1)r$ multiplications. The algorithm meeting these requirements is described below.

For fixed i, there is a vector $w \in \mathbb{F}_q^r$ with $v_i = \sum_{j=0}^{r-1} w_j v_j$. One can construct $y_i = Lv_i$ from the relation $Lv_i = \sum_{j=0}^{r-1} w_j Lv_j$. To reconstitute x from $y_{i1}, \ldots, y_{ir}$, one solves $\sum_{j=1}^{r} u_j v_{ij} = v_0$ for the vector u by Gaussian elimination and forms $x = Lv_0$ from the relation $Lv_0 = \sum_{j=1}^{r} u_j y_{ij}$. This algorithm clearly satisfies the op counts given above.

Since q can be chosen extremely small in many applications, two savings are possible. If the u's are stored, no Gaussian elimination is necessary. If a table of Zech's logarithms is stored ([3], p. 91) the encoding and decoding algorithms reduce to r additions and r table look-ups.

For large m, these threshold schemes take $(2+\varepsilon)r$ operations. Conjecturally, one cannot have threshold schemes requiring $(2-\varepsilon)r$

operations for large r, n. An elementary result is the following.

Proposition: A threshold scheme cannot be decoded in fewer than r operations.

Proof. Since a threshold scheme requires that no r-1 y_i's determine x, all r shadows must be used.

The definition of a threshold scheme requires that each shadow carry as much information as the message x. This message expansion can be overcome by using a pseudo-threshold scheme. All existing schemes have such variants, only the variant of this paper's superfast scheme is outlined.

Pick v_{-k}, ..., v_0, ..., v_n in $\mathbb{F}_q^r$ so that no r are linearly dependent. Form a linear map L with $Lv_{-j} = x_j$ for j = 0, ..., K where K < r and x_0, ..., x_K are messages or parts of a message x. Let $y_i = Lv_i$ i = 1,...,n. All other details are as before.

For these schemes one has, for each i that no r-1 y_i's give any information about x_i, and this may suffice for many applications, but r-1 y_i's do give information about the tuple $(x_0, ..., x_k)$. In essense, given r-s y_i's, if one correctly guesses s of the x_i's the rest follow.

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Appendix H

Program for encoding procedure
(including Stages 1, 2 and 4)

```
                                                                Page   1
                                                                05-24-84
                                                                00:17:17
:] IC  Line#   Source Line       IBM Personal Computer Pascal Compiler V1.00
   20      1   program enf (input,output);
           2
   10      3   const  WORDLENGTH = 16;
   10      4          MAXINDEX   = 32;
           5
   10      6   type   mat_row       = array[1..MAXINDEX] of integer;
   10      7          matrix        = array[1..MAXINDEX] of mat_row;
   10      8          channel_array = array[1..MAXINDEX] of integer;
           9
          10
          11   { the TWO_TO_THE function makes up for the lack of a generalize
          11   d
          12     exponentiation operator in standard Pascal.  It returns two
          13     raised to the power of its caller-supplied argument.
          13    }
          14
   20     15   function TWO_TO_THE (argument : integer): integer;
          16
   20     17   var  accumulator : integer;
   20     18        index       : integer;
          19
   20     20   begin
   21     21     accumulator := 1;
   21     22     for index := 1 to argument do
   21     23       accumulator := accumulator * 2;
 = 21     24     TWO_TO_THE := accumulator;
   10     25   end;

Symtab    25   Offset Length   Variable - TWO_TO_THE
               -    2     14   Return offset, Frame length
               -    6      2   (function return)  :              Integer
               -    0      2   ARGUMENT                        :Integer ValueP
               -   10      2   INDEX                           :Integer
               -    8      2   ACCUMULATOR                     :Integer

          26
          27
          28   { the READ_OCTAL routine; this routine allows the user of
          29     the program to input his values in octal rather than in
          30     decimal; it replaces the Pascal standard "read" routine. }
          31
   20     32   procedure READ_OCTAL (var total : integer);
          33
   20     34   const BLANK  = ' ';
          35
   20     36   var   inchar : char;

READ_OCTAL
```

```
JG IC  Line#   Source Line        IBM Personal Computer Pascal Compiler V1.00
           37
           38
    20     39   begin
    21     40     read (inchar);
    21     41     total  := 0;
           42
    21     43     while (inchar = BLANK) do
    21     44       read (inchar);
           45
    21     46     while not (inchar = BLANK) do begin
    22     47       total := total * 8 + (ord(inchar) - ord('0'));
    22     48       read (inchar)
    22     49     end
           50
    10     51   end;

RD  READ   51   Offset Length   Variable - READ_OCTAL
                -    2      8   Return offset, Frame length
                -    0      2   TOTAL                          :Integer VarP
                -    6      1   INCHAR                         :Char

           52
           53
           54   {the WRITE_OCTAL routine; it replaces the Pascal standard
           55    "write" routine and allows the program to report its
           56    output values in octal rather than in decimal.          }
           57
    20     58   procedure WRITE_OCTAL (number     : integer;
    20     59                          field_base : integer );
           60
    20     61   var outbuf : array [1..WORDLENGTH] of char;
    20     62       temp   : integer;
    20     63       index  : integer;
           64
           65
    20     66   begin
    21     67     for index := 1 to WORDLENGTH do outbuf[index] := '0';
    21     68     index := 1;
           69
    21     70     while (number > 0) do begin
    22     71       temp := number mod 8;
    22     72       outbuf[index] := chr (ord('0') + temp);
    22     73       index := index + 1;
    22     74       number := number div 8
    21     75     end;
           76

WRITE_OCTAL
```

```
Nest IC  Line#   Source Line        IBM Personal Computer Pascal Compiler V1.00
   21      77      temp := ((field_base + 2) div 3);
   21      78      if (temp < 1) then temp := 1;
   21      79      if (temp < (index - 1)) then temp := index - 1;
   21      80      for index := temp downto 1 do write(outbuf[index]);
   21      81      write(' ')
           82
   10      83    end;

Symtab     83    Offset Length   Variable - WRITE_OCTAL
                 -     4     30   Return offset, Frame length
                 -     0      2   NUMBER                          :Integer ValueP
                 -    24      2   TEMP                            :Integer
                 -    26      2   INDEX                           :Integer
                 -     2      2   FIELD_BASE                      :Integer ValueP
                 -    22     16   OUTBUF                          :Array

           84
           85
           86
           87    { The ADD function returns the logical xor of its two caller-
           88      supplied arguments.  This is addition over GF(n) for any n. }
           89
   20      90    function ADD (term1 : integer;
   20      91                  term2 : integer ): integer;
           92
   20      93    begin
 = 21      94      ADD := ( (term1 or term2) and (not(term1 and term2)) )
   10      95    end;

Symtab     95    Offset Length   Variable - ADD
                 -     4     10   Return offset, Frame length
                 -     8      2   (function return)  :            Integer
                 -     0      2   TERM1                          :Integer ValueP
                 -     2      2   TERM2                          :Integer ValueP

           96
           97
           98    { The MULTIPLY function performs multiplication over GF(n)
           99      modulo the caller-supplied modulus and returns the result
          100      of the multiplication.                                      }
          101
   20     102    function MULTIPLY (factor1          : integer;
   20     103                       factor2          : integer;
   20     104                       modulus          : integer;
   20     105                       field_base       : integer ): integer;
          106

MULTIPLY
```

```
VS IC  Line#   Source Line        IBM Personal Computer Pascal Compiler V1.00
   20    107   var  index   : integer;
   20    108        answer  : integer;
         109
   20    110   begin
         111
   21    112     answer := 0;
         113
   21    114     for index := 0 to (field_base - 1) do
   21    115       begin
   22    116         answer := answer * 2;
         117
   22    118         if (( (factor1 mod TWO_TO_THE (field_base - index))
   22    119               div TWO_TO_THE (field_base - (index+1)) ) > 0)
   22    120           then answer := ADD (answer, factor2);
         121
   22    122         if ( (answer div TWO_TO_THE (field_base)) > 0)
   22    123           then answer := ADD (answer,
   22    124                                TWO_TO_THE (field_base) + modulus)
   21    125       end;
 = 21    126     MULTIPLY := answer
   10    127   end;

Symtab   127   Offset Length   Variable - MULTIPLY
               -    8     20   Return offset, Frame length
               -   12      2   (function return)  :          Integer
               -    0      2   FACTOR1                     :Integer ValueP
               -   14      2   INDEX                       :Integer
               -   16      2   ANSWER                      :Integer
               -    2      2   FACTOR2                     :Integer ValueP
               -    4      2   MODULUS                     :Integer ValueP
               -    6      2   FIELD_BASE                  :Integer ValueP

         128
         129
         130   { The INVERSE function.  It accepts a field element and
         131     returns the element's multiplicative inverse.  This
         132     implementation is very slow & primitive-- it should be
         133     replaced by Davida's inverse routine or some other fast
         134     implementation at the first opportunity.                  }
         135
   20    136   function INVERSE ( element       : integer;
   20    137                      field_base    : integer;
   20    138                      modulus       : integer  ): integer;
         139
   20    140   var   index   : integer;
   20    141         answer  : integer;

INVERSE
```

```
JG IC  Line#   Source Line        IBM Personal Computer Pascal Compiler V1.00
    20     142         squares : integer;
           143
    20     144   begin
           145
    21     146     answer := 1;
    21     147     squares := element;
           148
    21     149     for index := 1 to (field_base - 1) do
    21     150       begin
    22     151         squares := MULTIPLY (squares,squares,modulus,field_base);
    22     152         answer  := MULTIPLY (answer,squares,modulus,field_base)
    21     153       end;
           154
  - 21     155     INVERSE := answer
           156
    10     157   end;

Symtab     157   Offset Length   Variable - INVERSE
                 -     6     20   Return offset, Frame length
                 -    10      2   (function return)  :              Integer
                 -     0      2   ELEMENT                          :Integer ValueP
                 -    12      2   INDEX                            :Integer
                 -    14      2   ANSWER                           :Integer
                 -     2      2   FIELD_BASE                       :Integer ValueP
                 -     4      2   MODULUS                          :Integer ValueP
                 -    16      2   SQUARES                          :Integer

           158
           159   { The DIVIDE function performs Galois-field division.
           160     it accepts dividend, divisor, modulus, and field-base
           161     (in that order), takes the inverse of the divisor, and
           162     multiplies the result by the dividend.                  }
           163
    20     164   function DIVIDE ( dividend      : integer;
    20     165                      divisor       : integer;
    20     166                      modulus       : integer;
    21     167                      field_base    : integer  ): integer;
           168
    20     169   var divisor_inverse : integer;
           170
    20     171   begin
           172
    21     173     divisor_inverse := INVERSE (divisor, field_base, modulus);
  - 21     174     DIVIDE := MULTIPLY (dividend, divisor_inverse,
    21     175                         modulus, field_base       )
           176

    DIVIDE
```

```
[S IC  Line#   Source Line        IBM Personal Computer Pascal Compiler V1.00
   10    177   end;

Symtab   177   Offset Length   Variable - DIVIDE
               -    8     16   Return offset, Frame length
               -   12      2   (function return)  :              Integer
               -    0      2   DIVIDEND                        :Integer ValueP
               -    2      2   DIVISOR                         :Integer ValueP
               -    4      2   MODULUS                         :Integer ValueP
               -    6      2   FIELD_BASE                      :Integer ValueP
               -   14      2   DIVISOR_INVERSE                 :Integer

          178
          179
          180   { The CONSTRUCT_VAN routine.  This procedure constructs a
          181     square vandermonde matrix with the dimension supplied by the
          182     calling routine.                                              }
          183
   20     184   procedure CONSTRUCT_VAN (var van           : matrix;
   20     185                            n               : integer;
   20     186                            field_base      : integer;
   20     187                            modulus         : integer );
          188
   20     189   var   row       : integer;
   20     190         column    : integer;
   20     191         exponent  : integer;
   20     192         index     : integer;
   20     193         temp      : integer;
          194
   20     195   begin
          196
   21     197     if (n < 3) then writeln ('van dimension < 3: error')
   21     198     else
   21     199       begin
          200
          201   {build first row of van.  }
          202
   22     203         van [1][1] := 1;
   22     204         for column := 2 to n do
   22     205           van [1][column] := 0;
          206
          207   {build second row of van. }
          208
   22     209         for column := 1 to n do
   22     210           van [2][column] := 1;
          211
          212   {build third row of van. }

CONSTRUCT_VAN
```

```
)G IC  Line#   Source Line        IBM Personal Computer Pascal Compiler V1.00
          213
    22    214          van [3][1] := 1;
    22    215          van [3][2] := 2;
    22    216          for column := 3 to n do
    22    217            van [3][column] :=
    22    218                MULTIPLY (van [3][column - 1], 2,
    22    219                          modulus, field_base              );
          220
          221    {build remaining rows of van.}
          222
    22    223          if (n > 3) then
    22    224            for row := 4 to n do begin
    23    225              van [row][1] := 1;
    23    226              for column := 2 to n do
    23    227                van [row][column] :=
    23    228                    MULTIPLY (van [row - 1][column], van [3][column
    23    228    ],
    23    229                              modulus, field_base
    23    229      )
    23    230            end
          231
          232
    22    233        end
    10    234    end;

Symtab    234    Offset Length   Variable - CONSTRUCT_VAN
                 -    8     32   Return offset, Frame length
                 -    0      2   VAN                          :Array   VarP
                 -    2      2   N                            :Integer ValueP
                 -   12      2   ROW                          :Integer
                 -    4      2   FIELD_BASE                   :Integer ValueP
                 -    6      2   MODULUS                      :Integer ValueP
                 -   14      2   COLUMN                       :Integer
                 -   18      2   INDEX                        :Integer
                 -   20      2   TEMP                         :Integer
                 -   16      2   EXPONENT                     :Integer

          235
          236
          237    { the BUILD_ENF routine.  It accepts the modulus and field-base
          238      desired by the user and the number of channels to be
          239      transmitted and produces a CODING-NORMAL-FORM matrix (enf).
          240      This matrix is (transmitted) X (transmitted - 2), and is
          241      gotten by column-reducing the first (transmitted - 2) columns
          242      of a (transmitted) X (transmitted) Vandermonde matrix so
          243      that the resulting matrix is upper-right triangular.

CONSTRUCT_VAN
```

```
JB IC  Line#   Source Line        IBM Personal Computer Pascal Compiler V1.00
          243   }
          244
    20    245   procedure BUILD_ENF (var enf          : matrix;
    20    246                             transmitted : integer;
    20    247                             modulus     : integer;
    20    248                             field_base  : integer );
          249
    20    250   var  columns          : integer;
    20    251        rows             : integer;
    20    252        reducing_col     : integer;
    20    253        reduced_elt      : integer;
    20    254        row              : integer;
    20    255        column           : integer;
    20    256        dimension        : integer;
          257
          258
    21    259   begin
          260
    21    261     dimension := TWO_TO_THE (field_base);
          262
    21    263     CONSTRUCT_VAN (enf, dimension, field_base, modulus);
          264
    21    265     rows := dimension;
    21    266     columns := dimension - 2;
    21    267     for reducing_col := 1 to columns do begin
          268
          269   { divide reducing-col through by its lead element (we want
          270     ones along the diagonal.)                                  }
          271
    22    272       for row := 1 to (rows - reducing_col) do
    22    273         enf[row][reducing_col] :=
    22    274              DIVIDE (enf[row][reducing_col],
    22    275                      enf[(rows-reducing_col)+1][reducing_col],
    22    276                      modulus, field_base);
    22    277       enf[(rows-reducing_col)+1][reducing_col] := 1;
          278
          279   { column-reduce to clear the row containing the lead element of
          280     reducing-col (that lead element is now a 1).
          281   }
    22    282       if (reducing_col < columns) then
    22    283         for column := (reducing_col + 1) to columns do begin
    23    284           reduced_elt := enf[(rows-reducing_col)+1][column];
    23    285           for row := 1 to (rows - reducing_col) do
    23    286             enf[row][column] :=
    23    287                  ADD (enf[row][column],

BUILD_ENF
```

```
LC  Line#   Source Line        IBM Personal Computer Pascal Compiler V1.00
    268                          MULTIPLY ( reduced_elt, enf[row][reducing_c
    288   ol],
    289                                    modulus, field_base );
    289       );
    290             enf[(rows-reducing_col)+1][column] := 0
    291         end
    292
    293     end
    294   end;

Symtab  294   Offset Length   Variable - BUILD_ENF
              -    8     36   Return offset, Frame length
              -    0      2   ENF                          :Array   VarP
              -    2      2   TRANSMITTED                  :Integer ValueP
              -    4      2   MODULUS                      :Integer ValueP
              -   12      2   COLUMNS                      :Integer
              -   14      2   ROWS                         :Integer
              -   20      2   ROW                          :Integer
              -   22      2   COLUMN                       :Integer
              -    6      2   FIELD_BASE                   :Integer ValueP
              -   24      2   DIMENSION                    :Integer
              -   18      2   REDUCED_ELT                  :Integer
              -   16      2   REDUCING_COL                 :Integer

        295
        296
        297   { the TRANSPOSE routine accepts a matrix and its dimensions
        298     and produces the transpose of the matrix.                  }
        299
   20   300   procedure TRANSPOSE (     m       : matrix;
   20   301                          var m_prime : matrix;
   20   302                              m_rows  : integer;
   20   303                              m_cols  : integer );
        304
   20   305   var  row : integer;
   20   306        col : integer;
        307
   20   308   begin
   21   309     for row := 1 to m_rows do
   21   310       for col := 1 to m_cols do
   21   311         m_prime[col][row] := m[row][col];
   10   312   end;

Symtab  312   Offset Length   Variable - TRANSPOSE
              - 2054   2066   Return offset, Frame length
              - 2046   2048   M                            :Array   ValueP

TRANSPOSE
```

```
IE IC  Line#   Source Line        IBM Personal Computer Pascal Compiler V1.00
               - 2048      2   M_PRIME                          :Array   VarP
               - 2050      2   M_ROWS                           :Integer ValueP
               - 2052      2   M_COLS                           :Integer ValueP
               - 2058      2   ROW                              :Integer
               - 2060      2   COL                              :Integer

          313
          314
          315   {the EXTRACT_SUBMATRIX routine accepts the enf matrix, the
          316    number of channels to be transmitted, and the number of
          317    channels to be received.  It produces a smaller matrix
          318    which will be used to construct the encode and decode keys
          319    for this particular configuration of transmitted and
          320    received channels.                                          }
          321
    20    322   procedure EXTRACT_SUBMATRIX (var submatrix      : matrix;
    20    323                                    enf            : matrix;
    20    324                                    transmitted    : integer;
    20    325                                    received       : integer;
    20    326                                    field_base     : integer );
          327
    20    328   var  enf_prime     : matrix;
    20    329        row            : integer;
    20    330        column         : integer;
    20    331        dimension      : integer;
    20    332        index          : integer;
          333
          334
    20    335   begin
          336
    21    337     dimension := TWO_TO_THE (field_base);
          338
    21    339     TRANSPOSE (enf, enf_prime, dimension, dimension-2);
          340
    21    341     index := 0;
          342
    21    343     for row := (dimension - (transmitted - 1)) to
    21    344                (dimension - received) do begin
    22    345       index := index + 1;
    22    346       for column := 1 to transmitted do
    22    347         submatrix[index][column] := enf_prime[row][column]
    22    348     end
          349
    10    350   end;

$/mtab    350   Offset Length   Variable - EXTRACT_SUBMATRIX

EXTRACT_SUBMATRIX
```

IBM Personal Computer Pascal Compiler V1.00

```
 IC  Line#   Source Line      Return offset, Frame length
             - 2056   4122    Return offset, Frame length
             -    0      2    SUBMATRIX                    :Array   VarP
             - 2048   2048    ENF                          :Array   ValueP
             - 2052      2    RECEIVED                     :Integer ValueP
             - 4108      2    ROW                          :Integer
             - 4110      2    COLUMN                       :Integer
             - 4114      2    INDEX                        :Integer
             - 4106   2048    ENF_PRIME                    :Array
             - 4112      2    DIMENSION                    :Integer
             - 2050      2    TRANSMITTED                  :Integer ValueP
             - 2054      2    FIELD_BASE                   :Integer ValueP

         351
         352
         353   { the BUILD_ENCODE_KEY builds the matrix which will be used
         354     to produce the (transmitted - received) coded channels for
         355     transmission.  The first (received) channels are sent in
         356     the clear.  }
         357
   20    358   procedure BUILD_ENCODE_KEY (var encode_key     : matrix;
   20    359                                  submatrix      : matrix;
   20    360                                  transmitted    : integer;
   20    361                                  received       : integer;
   20    362                                  modulus        : integer;
   20    363                                  field_base     : integer );
         364
   20    365   var columns       : integer;
   20    366       rows          : integer;
   20    367       col           : integer;
   20    368       row           : integer;
   20    369       reducing_row  : integer;
   20    370       reduced_elt   : integer;
         371
         372
   20    373   begin
   21    374     rows := transmitted - received;
   21    375     columns := transmitted;
         376
   21    377     for reducing_row := rows downto 2 do
   21    378       for row := (reducing_row - 1) downto 1 do begin
         379
   22    380         reduced_elt := submatrix[row]
   22    381                                [received+(rows-reducing_row)+1];
   22    382         for col := 1 to (received+(rows-reducing_row)) do
   22    383           submatrix [row][col] :=
   22    384             ADD (submatrix[row][col],

BUILD_ENCODE_KEY
```

```
00 IC  Line#   Source Line        IBM Personal Computer Pascal Compiler V1.00
    22    385                    MULTIPLY (reduced_elt, submatrix[reducing_row][c
    22    385   col],
    22    386                                modulus, field_base)
    22    386       );
    22    387         submatrix[row][received+(rows-reducing_row)+1] := 0
          388
    21    389       end;
          390
    21    391       for row := 1 to rows do
    21    392         for col := 1 to columns do
    21    393           encode_key[row][col] := submatrix[row][col]
          394
    10    395   end;

0 bmteb   395   Offset Length   Variable - BUILD_ENCODE_KEY
                - 2058   2084   Return offset, Frame length
                -    0      2   ENCODE_KEY                    :Array   VarP
                - 2048   2048   SUBMATRIX                    :Array   ValueP
                - 2052      2   RECEIVED                     :Integer ValueP
                - 2054      2   MODULUS                      :Integer ValueP
                - 2062      2   COLUMNS                      :Integer
                - 2064      2   ROWS                         :Integer
                - 2066      2   COL                          :Integer
                - 2068      2   ROW                          :Integer
                - 2050      2   TRANSMITTED                  :Integer ValueP
                - 2056      2   FIELD_BASE                   :Integer ValueP
                - 2072      2   REDUCED_ELT                  :Integer
                - 2070      2   REDUCING_ROW                 :Integer

          396
          397
          398
          399
          400
          401   { the ENCODE procedure.  It accepts the number of channels
          402     transmitted, the number of channels to be received, the
          403     modulus, and the field_base.  It then generates an encoding
          404     key and begins reading plaintext words.  It encodes the
          405     plaintext words and prints them out (transmits them)
          406     until it encounters an end-of-file flag.                    }
          407
    20    408   procedure ENCODE ( transmitted   : integer;
    20    409                      received      : integer;
    20    410                      modulus       : integer;
    20    411                      field_base    : integer;
    20    412                      output_channel : channel_array );

ENCODE
```

```
SG IC  Line#   Source Line        IBM Personal Computer Pascal Compiler V1.00
          413
    20    414   var enf              : matrix;
    20    415       enf_prime        : matrix;
    20    416       submatrix        : matrix;
    20    417       encode_key       : matrix;
    20    418       decode_key       : matrix;
    20    419       cool_decoder     : matrix;
    20    420       index            : integer;
    20    421       key_column       : integer;
    20    422       row              : integer;
    20    423       column           : integer;
    20    424       dimension        : integer;
    20    425       EOT              : boolean;
    20    426       response         : char;
          427
          428
    20    429   begin
          430
    21    431     dimension := TWO_TO_THE (field_base);
          432
    21    433     BUILD_ENF (enf, transmitted, modulus, field_base);
    21    434           TRANSPOSE(enf,enf_prime,dimension,(dimension - 2));
    21    435           writeln;writeln('ENF MATRIX---------------------->');
    21    436           for row := 1 to (dimension - 2) do begin
    22    437             writeln;
    22    438             for column := 1 to dimension do
    22    439               WRITE_OCTAL (enf_prime[row][column], field_base)
    21    440           end;
    21    441           page;
    21    442     EXTRACT_SUBMATRIX (submatrix, enf, transmitted, received,fiel
    21    442   d_base);
    21    443           writeln;writeln('SUBMATRIX---------------------->');
    21    444           for row := 1 to (transmitted - received) do begin
    22    445             writeln;
    22    446             for column := 1 to transmitted do
    22    447               WRITE_OCTAL (submatrix[row][column], field_base)
    21    448           end;
    21    449           page;
    21    450     BUILD_ENCODE_KEY (encode_key, submatrix, transmitted, receiv
    21    450   d,
    21    451                             modulus, field_base);
    21    452           writeln;writeln('ENCODE KEY--------------------->');
    21    453           for row := 1 to (transmitted - received) do begin
    22    454             writeln;
    22    455             for column := 1 to transmitted do
    22    456               WRITE_OCTAL (encode_key[row][column], field_base)

ENCODE
```

```
JG IC  Line#   Source Line        IBM Personal Computer Pascal Compiler V1.00
   21    457           end;
   21    458           page;
         459
         460   { the encode routine now reads in "received" cleartext words,
         461     generates "transmitted" - "received" coded words, and sends
         462     all "transmitted" words out.                                 }
         463
   21    464     EOT := FALSE;
   21    465     repeat
   22    466       begin
   23    467         writeln('please enter, on one line, in octal and separate
   23    467   d');
   23    468         writeln('by blanks, the values to be transmitted over the
   23    468   ');
   23    469         writeln('transmitters ',received:2,' channels');
   23    470         for index := 1 to received do begin
   24    471           READ_OCTAL (output_channel[index]);
   23    472         end;
         473
   23    474         writeln('words transmitted are (in channel order):');
         475
   23    476         for index := (received+1) to transmitted do begin
   24    477           output_channel[index] := 0;
   24    478           for key_column := 1 to received do
   24    479             output_channel[index] :=
   24    480                 ADD ( output_channel[index],
   24    481                       MULTIPLY (output_channel[key_column],
   24    482                                 encode_key[(transmitted-index)+1]
   24    483                                           [key_column],
   24    484                                 modulus, field_base)
   24    484    )
   23    485         end;
         486
   23    487         for index := 1 to transmitted do
   23    488           WRITE_OCTAL (output_channel[index], field_base);
   23    489         writeln;writeln;
         490
   23    491         writeln('do you want to send another ',received:2,' words
   23    491   ?');
   23    492         writeln('(type y or n)');
   23    493         readln(response);
   23    494         if (response = 'n') then EOT := TRUE;
         495
   23    496       end
   21    497     until (EOT);
   21    498     page

ENCODE
```

```
JG IC  Line#   Source Line        IBM Personal Computer Pascal Compiler V1.00
          499
    10    500   end;

Symtab    500   Offset Length    Variable - ENCODE
                -   72  12400    Return offset, Frame length
                -    0      2    TRANSMITTED                    :Integer ValueP
                -    2      2    RECEIVED                       :Integer ValueP
                -    4      2    MODULUS                        :Integer ValueP
                - 2122   2048    ENF                            :Array
                -12364      2    INDEX                          :Integer
                -12368      2    ROW                            :Integer
                -12374      1    EOT                            :Boolean
                -12370      2    COLUMN                         :Integer
                -    6      2    FIELD_BASE                     :Integer ValueP
                - 4170   2048    ENF_PRIME                      :Array
                -12372      2    DIMENSION                      :Integer
                -12376      1    RESPONSE                       :Char
                - 6218   2048    SUBMATRIX                      :Array
                - 8266   2048    ENCODE_KEY                     :Array
                -10314   2048    DECODE_KEY                     :Array
                -12366      2    KEY_COLUMN                     :Integer
                -   70     64    OUTPUT_CHANNEL                 :Array   ValueP
                -12362   2048    COOL_DECODER                   :Array

          501
          502
          503   { THE MAIN ROUTINE.  THIS CODE READS IN THE NUMBER OF
          504     CHANNELS SENT AND THE NUMBER OF CHANNELS WHICH NEED
          505     TO BE RECIEVED, AND GENERATES AN ENCODE-NORMAL FORM
          506     MATRIX FOR THAT CHOICE OF 'TRANSMITTED' AND 'RECIEVED'.  }
          507
          508
    10    509   var field_base      : integer;
    10    510       modulus         : integer;
    10    511       transmitted     : integer;
    10    512       received        : integer;
          513       channels        : channel_array;
    10    514       index           : integer;
          515
          516
          517
    10    518   begin
          519
    11    520     writeln(chr(27),'M');        {enable elite type on printer}
          521
    11    522     writeln('please enter, on one line and separated by blanks,')

ENF
```

```
JG IC  Line#   Source Line        IBM Personal Computer Pascal Compiler V1.00
   11    522   ;
   11    523     writeln('the field-base, modulus, number of channels to be se
   11    523   nt,');
   11    524     writeln('and number of channels to be received.  The modulus'
   11    524   );
   11    525     writeln('should be an octal number; all other numbers should'
   11    525   );
   11    526     writeln('be decimal.');
   11    527     writeln;
         528
   11    529     read (field_base);
   11    530     READ_OCTAL (modulus);
   11    531     modulus := modulus - TWO_TO_THE(field_base);
   11    532     read (transmitted);
   11    533     readln (received);
   11    534     writeln;
         535
   11    536     writeln('thank you...please wait');writeln;
         537
   11    538     ENCODE (transmitted, received, modulus, field_base, channels)
   11    538   ;
         539
   11    540     writeln(chr(27),chr(64));        {disable special print modes
   11    540   }
         541
   00    542   end.

Symtab   542   Offset Length   Variable
                    0     76   Return offset, Frame length
                   74      2   INDEX                          :Integer Static
                    4      2   MODULUS                        :Integer Static
                   10     64   CHANNELS                       :Array   Static
                    8      2   RECEIVED                       :Integer Static
                    2      2   FIELD_BASE                     :Integer Static
                    6      2   TRANSMITTED                    :Integer Static


              Errors  Warns  In Pass One
                   0      0
```

Appendix I

Program for decoding procedure
(including Stages 1, 2, 3 and 4)

```
                                                                  Page   1
                                                                  05-23-84
                                                                  17:43:56
SC IC  Line#   Source Line        IBM Personal Computer Pascal Compiler V1.00
   20      1   program dnf (input,output);
           2
   10      3   const  zero          = 0;
   10      4          one           = 1;
   10      5          maxindex      = 32;
   10      6          WORDLENGTH    = 16;
           7
   10      8   type   mat_row       = array [1..maxindex] of integer;
   10      9          matrix        = array [1..maxindex] of mat_row;
          10
   10     11   var    index                    : integer;
   10     12          van                      : matrix;
   10     13          dnf                      : matrix;
   10     14          dnf_prime                : matrix;
   10     15          received                 : integer;
   10     16          transmitted              : integer;
   10     17          field_base               : integer;
   10     18          modulus                  : integer;
   10     19          row                      : integer;
   10     20          col                      : integer;
   10     21          rows                     : integer;
   10     22          datarow                  : integer;
   10     23          datarows                 : integer;
   10     24          extra_desid              : integer;
   10     25          columns                  : integer;
   10     26          dimension                : integer;
   10     27          temp                     : integer;
   10     28          channel                  : integer;
   10     29          desired_channels         : integer;
   10     30          reducing_elt             : integer;
   10     31          reduced_elt              : integer;
   10     32          dead_channels            : integer;
   10     33          dead_channel             : mat_row;
   10     34          decoder                  : matrix;
   10     35          desired_channel          : mat_row;
   10     36          data                     : matrix;
   10     37          desiderata               : matrix;
   10     38          active_channel           : mat_row;
   10     39          codeword                 : mat_row;
   10     40          clearword                : integer;
   10     41          continue                 : char;
   10     42          active                   : boolean;
   10     43          EOT                      : boolean;      {End Of Transmission
   10     43   }
          44
          45

0 Errors
```

```
                                                                    ,Page   2
                                                                    05-23-84
                                                                    17:43:58
 ?? [C  Line#   Source Line        IBM Personal Computer Pascal Compiler V1.00
          45    { the TWO_TO_THE function makes up for the lack of a generalize
          46    d
          47      exponentiation operator in standard Pascal.  It returns two
          48      raised to the power of its caller-supplied argument.
          49     }
          49
    20    50    function TWO_TO_THE (argument : integer): integer;
          51
    20    52    var  accumulator : integer;
    20    53         index       : integer;
          54
    20    55    begin
    21    56      accumulator := 1;
    21    57      for index := 1 to argument do
    21    58        accumulator := accumulator * 2;
  * 21    59      TWO_TO_THE := accumulator;
    10    60    end;

Symtab      50    Offset Length   Variable - TWO_TO_THE
                  -     2     14  Return offset, Frame length
                  -     6      2  (function return)  :              Integer
                  -     0      2  ARGUMENT                         :Integer ValueP
                  -    10      2  INDEX                            :Integer
                  -     8      2  ACCUMULATOR                      :Integer

          61
          62
          63
          64    { the READ_OCTAL routine; this routine allows the user of
          65      the program to input his values in octal rather than in
          66      decimal; it replaces the Pascal standard "read" routine. }
          67
    20    68    procedure READ_OCTAL (var total : integer);
          69
    21    70    const BLANK  = ' ';
          71
          72    var   inchar : char;
          73
          74
    20    75    begin
    21    76      read (inchar);
    21    77      total  := 0;
          78
    21    79      while (inchar = BLANK) do
    21    80        read (inchar);
          81

END OCTAL
```

```
                                                                  Page   3
                                                                  05-23-84
                                                                  17:44:02
 IC  Line#   Source Line        IBM Personal Computer Pascal Compiler V1.00
 21      82     while not (inchar = BLANK) do begin
 22      83       total := total * 8 + (ord(inchar) - ord('0'));
 22      84       read (inchar)
 22      85     end
         86
 10      87   end;

mtab    87   Offset Length   Variable - READ_OCTAL
             -    2      8   Return offset, Frame length
             -    0      2   TOTAL                          :Integer VarP
             -    6      1   INCHAR                         :Char

         88
         89
         90   {the WRITE_OCTAL routine; it replaces the Pascal standard
         91    "write" routine and allows the program to report its
         92    output values in octal rather than in decimal.          }
         93
 20      94   procedure WRITE_OCTAL (number     : integer;
 20      95                          field_base : integer );
         96
 20      97   var outbuf : array [1..WORDLENGTH] of char;
 20      98       temp   : integer;
 20      99       index  : integer;
        100
        101
 20     102   begin
 21     103     for index := 1 to WORDLENGTH do outbuf[index] := '0';
 21     104     index := 1;
        105
 21     106     while (number > 0) do begin
 22     107       temp := number mod 8;
 22     108       outbuf[index] := chr (ord('0') + temp);
 22     109       index := index + 1;
 22     110       number := number div 8
 21     111     end;
        112
 21     113     temp := ((field_base + 2) div 3);
 21     114     if (temp < 1) then temp := 1;
 21     115     if (temp < (index - 1)) then temp := index - 1;
 21     116     for index := temp downto 1 do write(outbuf[index]);
 21     117     write(' ')
        118
 10     119   end;

mtab   119   Offset Length   Variable - WRITE_OCTAL

ITE_OCTAL
```

```
JG IC  Line#   Source Line        IBM Personal Computer Pascal Compiler V1.00
               -    4     30   Return offset, Frame length
               -    0      2   NUMBER                          :Integer ValueP
               -   24      2   TEMP                            :Integer
               -   26      2   INDEX                           :Integer
               -    2      2   FIELD_BASE                      :Integer ValueP
               -   22     16   OUTBUF                          :Array

          120
          121
          122
          123   { The ADD function returns the logical xor of its two caller-
          124     supplied arguments.  This is addition over GF(n) for any n. }
          125
    20    126   function ADD (term1 : integer;
    20    127                 term2 : integer ): integer;
          128
    20    129   begin
  = 21    130     ADD := ( (term1 or term2) and (not(term1 and term2)) )
    10    131   end;

Symtab    131   Offset Length   Variable - ADD
               -    4     10   Return offset, Frame length
               -    8      2   (function return)  :             Integer
               -    0      2   TERM1                          :Integer ValueP
               -    2      2   TERM2                          :Integer ValueP

          132
          133
          134   { The MULTIPLY function performs multiplication over GF(n)
          135     modulo the caller-supplied modulus and returns the result
          136     of the multiplication.                                    }
          137
    20    138   function MULTIPLY (factor1          : integer;
    20    139                      factor2          : integer;
    20    140                      modulus          : integer;
    20    141                      field_base       : integer ): integer;
          142
    20    143   var  index    : integer;
    20    144        answer   : integer;
          145
    20    146   begin
          147
    21    148     answer := 0;
          149
    21    150     for index := 0 to (field_base - 1) do
    21    151       begin

MULTIPLY
```

```
16 IC  Line#   Source Line        IBM Personal Computer Pascal Compiler V1.00
   22     152         answer := answer * 2;
          153
   22     154         if (( (factor1 mod TWO_TO_THE (field_base - index))
   22     155               div TWO_TO_THE (field_base - (index+1)) ) > 0)
   22     156           then answer := ADD (answer, factor2);
          157
   22     158         if ( (answer div TWO_TO_THE (field_base)) > 0)
   22     159           then answer := ADD (answer,
   22     160                                TWO_TO_THE (field_base) + modulus)
   21     161       end;
 = 21     162     MULTIPLY := answer
    0     163   end;

Symtab   163   Offset Length   Variable - MULTIPLY
               -    6     20   Return offset, Frame length
               -   12      2   (function return)  :              Integer
               -    0      2   FACTOR1                          :Integer ValueP
               -   14      2   INDEX                            :Integer
               -   16      2   ANSWER                           :Integer
               -    2      2   FACTOR2                          :Integer ValueP
               -    4      2   MODULUS                          :Integer ValueP
               -    6      2   FIELD_BASE                       :Integer ValueP

          164
          165
          166   { The INVERSE function.  It accepts a field element and
          167     returns the element's multiplicative inverse.  This
          168     implementation is very slow & primitive-- it should be
          169     replaced by Davida's inverse routine or some other fast
          170     implementation at the first opportunity.                  }
          171
   20     172   function INVERSE ( element        : integer;
   20     173                      field_base     : integer;
   20     174                      modulus        : integer  ): integer;
          175
   20     176   var   index   : integer;
   20     177         answer  : integer;
   20     178         squares : integer;
          179
   20     180   begin
          181
   21     182     answer := 1;
   21     183     squares := element;
          184
   21     185     for index := 1 to (field_base - 1) do
   21     186       begin

VERSE
```

```
JG IC  Line#   Source Line        IBM Personal Computer Pascal Compiler V1.00
   22    187         squares := MULTIPLY (squares,squares,modulus,field_base):
   22    188         answer  := MULTIPLY (answer,squares,modulus,field_base)
   21    189       end;
         190
 = 21    191     INVERSE := answer
         192
   10    193   end;

Symtab   193   Offset Length   Variable - INVERSE
                 -     6      20   Return offset, Frame length
                 -    10       2   (function return)  :              Integer
                 -     0       2   ELEMENT                          :Integer ValueP
                 -    12       2   INDEX                            :Integer
                 -    14       2   ANSWER                           :Integer
                 -     2       2   FIELD_BASE                       :Integer ValueP
                 -     4       2   MODULUS                          :Integer ValueP
                 -    16       2   SQUARES                          :Integer

         194
         195   { The DIVIDE function performs Galois-field division.
         196     it accepts dividend, divisor, modulus, and field-base
         197     (in that order), takes the inverse of the divisor, and
         198     multiplies the result by the dividend.                  }
         199
   20    200   function DIVIDE ( dividend      : integer;
   20    201                     divisor       : integer;
   20    202                     modulus       : integer;
   20    203                     field_base    : integer  ): integer;
         204
   20    205   var divisor_inverse : integer;
         206
   20    207   begin
         208
   21    209     divisor_inverse := INVERSE (divisor, field_base, modulus);
 = 21    210     DIVIDE := MULTIPLY (dividend, divisor_inverse,
   21    211                         modulus, field_base       )
         212
   10    213   end;

Symtab   213   Offset Length   Variable - DIVIDE
                 -     8      16   Return offset, Frame length
                 -    12       2   (function return)  :              Integer
                 -     0       2   DIVIDEND                         :Integer ValueP
                 -     2       2   DIVISOR                          :Integer ValueP
                 -     4       2   MODULUS                          :Integer ValueP
                 -     6       2   FIELD_BASE                       :Integer ValueP

DIVIDE
```

```
JG IC  Line#   Source Line        IBM Personal Computer Pascal Compiler V1.00
               -   14      2   DIVISOR_INVERSE                  :Integer

          214
          215
          216   { The HERMITE_NORMALIZE routine takes a matrix which is
          217     at least two columns wide and which is also at least
          218     as tall as it is wide and reduces it to Hermite normal
          219     form (i.e. to a form with an identity matrix at the top.) }
          220
    20    221   procedure HERMITE_NORMALIZE  (var m           : matrix;
    20    222                                     rows        : integer;
    20    223                                     cols        : integer;
    20    224                                     modl        : integer;
    20    225                                     f_base      : integer  );
          226
    20    227   var row              : integer;
    20    228       col              : integer;
    20    229       reducing_col     : integer;
    20    230       reducing_elt     : integer;
    20    231       reduced_elt      : integer;
    20    232       index            : integer;
    20    233       temp             : integer;
          234
    20    235   begin
          236
    21    237     if (cols < 2) then
    21    238       writeln ('stripped matrix has <2 cols: error')
    21    239     else begin
          240
    22    241       for reducing_col := 1 to cols do begin
    23    242         index := reducing_col;
    23    243         while ( (m [reducing_col][index] = 0) and
    23    244                 (index < reducing_col) ) do
    23    245           index := index + 1;
          246
    23    247         if (not(index = reducing_col)) then              {switch c
    23    247   ols}
    23    248           for row := 1 to rows do begin
    24    249             temp := m [row][reducing_col];
    24    250             m [row][reducing_col] := m [row][index];
    24    251             m [row][index] := temp
    23    252           end;
          253
    23    254         reducing_elt := m [reducing_col][reducing_col];
          255
          256   {set leading elts. of columns to 1 by dividing cols by constant

HERMITE_NORMALIZE
```

```
Lvl IC  Line#   Source Line        IBM Personal Computer Pascal Compiler V1.00
          256   .}
          257
    23    258         temp := reducing_elt;
    23    259         if (not (temp = 1) ) then begin
    24    260           m[reducing_col][reducing_col] := 1;
    24    261           for row := (reducing_col + 1) to rows do
    24    262             m [row][reducing_col] := DIVIDE (m[row][reducing_col]
    24    262   , temp,
    24    263                                             modl, f_base       )
    23    264         end;
          265
          266   {column-reduce by clearing row 'reducing-col' using entry
          267    m[reducing_col][reducing_col].                             }
          268
    23    269         for col := 1 to cols do
          270
    23    271           if (not(col = reducing_col)) then begin
    24    272             reduced_elt := m [reducing_col][col];
    24    273             if (not(reduced_elt = 0)) then
    24    274               for row := reducing_col to rows do
    24    275                 m [row][col] :=
    24    276                   ADD (m [row][col],
    24    277                        MULTIPLY (m [row][reducing_col],
    24    278                                  reduced_elt, modl, f_base  )  )
    24    279           end
          280
    23    281       end
    22    282     end
          283
    10    284   end;

Symtab  284   Offset Length   Variable - HERMITE_NORMALIZE
              -   10     42   Return offset, Frame length
              -    0      2   M                              :Array   VarP
              -    2      2   ROWS                           :Integer ValueP
              -    4      2   COLS                           :Integer ValueP
              -   14      2   ROW                            :Integer
              -   16      2   COL                            :Integer
              -    6      2   MODL                           :Integer ValueP
              -    8      2   F_BASE                         :Integer ValueP
              -   24      2   INDEX                          :Integer
              -   26      2   TEMP                           :Integer
              -   18      2   REDUCING_COL                   :Integer
              -   22      2   REDUCED_ELT                    :Integer
              -   20      2   REDUCING_ELT                   :Integer

HERMITE_NORMALIZE
```

```
   IC  Line#   Source Line        IBM Personal Computer Pascal Compiler V1.00
               285
               286
               287
               288
               289   { The CONSTRUCT_VAN routine.  This procedure constructs a
               290     square vandermonde matrix with the dimension supplied by the
               291     calling routine.                                          }
               292
    2C         293   procedure CONSTRUCT_VAN (var van          : matrix;
    20         294                                n            : integer;
    20         295                                field_base   : integer;
    20         296                                modulus      : integer );
               297
    20         298   var   row      : integer;
    20         299         column   : integer;
    20         300         exponent : integer;
    20         301         index    : integer;
    20         302         temp     : integer;
               303
    20         304   begin
               305
    21         306     if (n < 3) then writeln ('van dimension < 3: error')
    21         307     else
    21         308       begin
               309
               310   {build first row of van.  }
               311
    22         312         van [1][1] := 1;
    22         313         for column := 2 to n do
    22         314           van [1][column] := 0;
               315
               316   {build second row of van. }
               317
    22         318         for column := 1 to n do
    22         319           van [2][column] := 1;
               320
               321   {build third row of van. }
               322
    22         323         van [3][1] := 1;
    22         324         van [3][2] := 2;
    22         325         for column := 3 to n do
    22         326           van [3][column] :=
    22         327               MULTIPLY (van [3][column - 1], 2,
    22         328                         modulus, field_base           );
               329
               330   {build remaining rows of van.}

ONSTRUCT_VAN
```

```
 IO IO  Line#   Source Line        IBM Personal Computer Pascal Compiler V1.00
          331
    22    332          if (n > 3) then
    22    333            for row := 4 to n do begin
    23    334              van [row][1] := 1;
    23    335              for column := 2 to n do
    23    336                van [row][column] :=
    23    337                    MULTIPLY (van [row - 1][column], van [1][column
    23    337   ],
    23    338                                  modulus, field_base
    23    338     )
    23    339            end
          340
          341
    22    342        end
    10    343   end;

 Level    343   Offset Length   Variable - CONSTRUCT_VAN
                -    8     32   Return offset, Frame length
                -    0      2   VAN                          :Array   VarP
                -    2      2   N                            :Integer ValueP
                -   12      2   ROW                          :Integer
                -    4      2   FIELD_BASE                   :Integer ValueP
                -    6      2   MODULUS                      :Integer ValueP
                -   14      2   COLUMN                       :Integer
                -   18      2   INDEX                        :Integer
                -   20      2   TEMP                         :Integer
                -   16      2   EXPONENT                     :Integer

          344
          345
          346
          347   { the TRANSPOSE routine accepts a matrix and its dimensions
          348     and produces the transpose of the matrix.                  }
          349
    20    350   procedure TRANSPOSE (     m       : matrix;
    20    351                         var m_prime : matrix;
                                          m_rows  : integer;
    20    353                               m_cols  : integer );
          354
    20    355   var  row : integer;
    20    356        col : integer;
          357
    20    358   begin
    21    359     for row := 1 to m_rows do
    21    360       for col := 1 to m_cols do
    21    361         m_prime[col][row] := m[row][col];

TRANSPOSE
```

```
JG IC  Line#   Source Line        IBM Personal Computer Pascal Compiler V1.00
   10    362   end;

Symtab   362   Offset Length   Variable - TRANSPOSE
               - 2054   2066   Return offset, Frame length
               - 2046   2048   M                          :Array   ValueP
               - 2048      2   M_PRIME                    :Array   VarP
               - 2050      2   M_ROWS                     :Integer ValueP
               - 2052      2   M_COLS                     :Integer ValueP
               - 2058      2   ROW                        :Integer
               - 2060      2   COL                        :Integer

          363
          364
          365
          366   { THE MAIN ROUTINE.  THIS CODE GOES THROUGH THE ENTIRE
          367     DECODING PROCESS, WHICH IS BROKEN INTO COLD, COOL AND
          368     HOT PRECOMPUTE STAGES AND ONLINE DECODE STAGE.          }
          369
    10    370   begin
          371
          372   { COLD PRECOMPUTE STAGE BEGINS HERE.                      }
          373
          374   { First, we read in the modulus and field-base for the
          375     Galois field to be used in our calculations.            }
          376
    11    377     writeln('Please enter, on one line and separated by a blank,'
    11    377   );
    11    378     writeln('the field-base and modulus to be used.  The');
    11    379     writeln('field-base should be a decimal number and the');
    11    380     writeln('modulus should be an octal number.');
          381
    11    382     read(field_base);
    11    383     READ_OCTAL(modulus);
          384
    11    385     modulus := modulus - TWO_TO_THE(field_base);
          386
          387
          388   { Next, we construct a Vandermonde matrix called VAN.     }
          389
    11    390     dimension := TWO_TO_THE(field_base);
          391
    11    392     CONSTRUCT_VAN (van, dimension, field_base, modulus);
          393
          394
          395   { COOL PRECOMPUTE STAGE BEGINS HERE.                      }
          396

EOF
```

```
JG IC  Line#   Source Line        IBM Personal Computer Pascal Compiler V1.00
          397   { Next, we read in the number of channels to be sent
          398     by the transmitting node.                                 }
          399
    11    400     writeln('Please enter the number of channels to be sent');
    11    401     writeln('by the transmitting node.  This should be a');
    11    402     writeln('decimal number.');
          403
    11    404     readln(transmitted);
          405
          406
          407   { Next, we read the number of channels to be received by
          408     the receiving node.                                       }
          409
    11    410     writeln('please enter the number of channels active at');
    11    411     writeln('the receivers node; this should be a decimal number.
    11    411   ');
          412
    11    413     readln(received);
          414
          415
          416   { Now we strip away the extraneous rows and columns of the
          417     vandermonde matrix.  We leave only the topmost n rows and
          418     the leftmost k columns of VAN.                            }
          419
    11    420     rows := transmitted;
    11    421     columns := received;
          422
          423
          424   { Next, we hermite-normalize van to give us a tall, thin matrix
          425     with an identity at the top.                              }
          426
    11    427     HERMITE_NORMALIZE (van, rows, columns, modulus, field_base);
          428
          429
          430   { Finally, we construct our "special" left-kernel for the
          431     stripped, col-reduced VAN.  This matrix is short and
          432     fat, with an identity at the left, and it is our DNF
          433     (Decode-Normal-Form) matrix.                              }
          434
    11    435     for row := 1 to (transmitted - received) do
    11    436       for col := 1 to received do
    11    437         dnf [row][col] := van [row + received][col];
          438
    11    439     for row := 1 to (transmitted - received) do
    11    440       for col := (received + 1) to transmitted do
    11    441         if ( (col - received) = row) then

DNF
```

```
   IC  Line#   Source Line        IBM Personal Computer Pascal Compiler V1.00
    11    442           dnf [row][col] := 1
    11    443         else
    11    444           dnf [row][col] := 0;
          445
    11    446     TRANSPOSE (dnf, dnf_prime, (transmitted - received), transmit
    11    446   ted);
          447
    11    448     HERMITE_NORMALIZE (dnf_prime, transmitted, (transmitted - rec
    11    448   eived),
    11    449                              modulus, field_base);
          450
    11    451     TRANSPOSE (dnf_prime, dnf, transmitted, (transmitted - receiv
    11    451   ed));
          452
          453
    11    454     writeln;
    11    455     writeln ('DNF matrix for ',received:2,' out of ',transmitted:
    11    455   2);
    11    456     write ('channels over GF 2**(',field_base:1,') mod ');
    11    457     temp := modulus + TWO_TO_THE (field_base);
    11    458     WRITE_OCTAL(temp, field_base);
    11    459     writeln('is: ');writeln;
    11    460     for row := 1 to (transmitted - received) do begin
    12    461       writeln;
    12    462       for col := 1 to transmitted do
    12    463         WRITE_OCTAL (dnf [row][col], field_base);
    11    464     end;
    11    465     writeln;writeln;
          466
          467   { HOT PRECOMPUTE STAGE BEGINS HERE.                          }
          468
    11    469     writeln('please enter, on one line and separated by blanks,')
    11    469   ;
    11    470     writeln('the numbers of the ',received:2,' channels active');
    11    471     writeln('at the receiving node.  These numbers should be deci
    11    471   mal.');
          472
    11    473     for index := 1 to (received - 1) do read (active_channel [ind
    11    473   ex]);
    11    474     readln (active_channel [received]);
          475
          476
          477   { Here we fill up the data matrix.                             }
          478
    11    479     row := 1;
    11    480     for index := 1 to received do

DNF
```

```
JG IC  Line#   Source Line        IBM Personal Computer Pascal Compiler V1.00
   11    481       if (active_channel [index] <= (transmitted - received)) the
   11    481   n begin
   12    482         for col := 1 to transmitted do
   12    483           data [row][col] :=
   12    484                 dnf [ active_channel [index] ][col];
   12    485         row := row + 1
   11    486       end;
   11    487     datarows := row - 1;
         488
         489
         490   { Here we fill up the desiderata matrix and those rows of the
         491     decoder matrix corresponding to channels which we are recevin
         491   g.}
         492
   11    493     desired_channels := 0;
   11    494     dead_channels := 0;
   11    495     extra_desid := 1;
   11    496     for channel := 1 to transmitted do begin
   12    497       active := FALSE;
   12    498       for index := 1 to received do
   12    499         if (active_channel [index] = channel) then active := TRUE
   12    499   ;
   12    500       if ( (not active) and (channel <= received) ) then begin
   13    501        desired_channels := desired_channels + 1;
   13    502        desired_channel [desired_channels] := channel;
   13    503        if (channel > (transmitted - received)) then begin
   14    504         dead_channels := dead_channels + 1;
   14    505         dead_channel [dead_channels] := channel;
   14    506         for col := 1 to transmitted do
   14    507           desiderata [channel][col] := data [extra_desid][col];
   14    508         extra_desid := extra_desid + 1
   14    509        end
   13    510        else
   13    511         for col := 1 to transmitted do
   13    512           desiderata [channel][col] := dnf [channel][col]
   13    513       end
   12    514       else if (not active) then begin
   13    515         dead_channels := dead_channels + 1;
   13    516         dead_channel [dead_channels] := channel
   13    517       end
   12    518       else if (channel <= received) then
   12    519         for col := 1 to transmitted do
   12    520           if (col = channel) then decoder [channel][col] := 1
   12    521           else decoder [channel][col] := 0
   11    522     end;
         523

END
```

```
;6 IC  Line#   Source Line        IBM Personal Computer Pascal Compiler V1.00
          524
          525   { Here we clear the columns in the desiderata matrix correspond
          525   ing
          526     to channels which we are not receiving.
          526      }
          527
    11    528     row := 1;
    11    529     for index := 1 to dead_channels do begin
    12    530       if (dead_channel [index] > (transmitted - received)) then b
    12    530   egin
    13    531         for channel := 1 to desired_channels do
    13    532         if(not(desired_channel[channel] = dead_channel[index])) th
    13    532   en begin
    14    533           reducing_elt := data [row][dead_channel[index]];
    14    534           reduced_elt := desiderata [ desired_channel[channel] ]
    14    535                                     [dead_channel[index]];
    14    536           for col := 1 to transmitted do
    14    537             desiderata [ desired_channel[channel] ] [col] :=
    14    538               ADD ( MULTIPLY(desiderata [desired_channel[channel]][
    14    538   col],
    14    539                              reducing_elt,modulus, field_base),
    14    540                     MULTIPLY(data[row][col],reduced_elt,
    14    541                              modulus, field_base))
    13    542         end;
    13    543           for datarow := (row + 1) to datarows do begin
    14    544           reducing_elt := data[row][dead_channel[index]];
    14    545           reduced_elt := data [datarow][dead_channel[index]];
    14    546           for col := 1 to transmitted do
    14    547             data [datarow][col] :=
    14    548               ADD ( MULTIPLY(data[datarow][col],reducing_elt,
    14    549                     modulus, field_base),
    14    550                     MULTIPLY(data[row][col],reduced_elt,
    14    551                              modulus, field_base))
    13    552           end;
    13    553           row := row + 1
    13    554         end
    11    555     end;
          556
          557
          558   { Here we obtain ones in the "lead" columns of the desiderata
          559     rows by dividing through by the values previously in those
          560     columns.                                                    }
          561
    11    562     for channel := 1 to desired_channels do begin
    12    563       reducing_elt := desiderata [desired_channel[channel]]
    12    564                                  [desired_channel[channel]];

DNF
```

```
JG IC  Line#   Source Line        IBM Personal Computer Pascal Compiler V1.00
   12    565       for col := 1 to transmitted do
   12    566         desiderata [desired_channel[channel]][col] :=
   12    567           DIVIDE (desiderata [desired_channel[channel]][col],
   12    568                   reducing_elt, modulus, field_base);
   11    569     end;
         570
         571
         572   { Now fill up the rows of the decoder matrix corresponding
         573     to channels which were desired but not active.            }
         574
   11    575     for channel := 1 to desired_channels do
   11    576       for col := 1 to transmitted do
   11    577         decoder [desired_channel[channel]][col] :=
   11    578             desiderata [desired_channel[channel]][col];
         579
         580
         581   { Now we print out the decoder matrix.                      }
         582
   11    583     writeln('Decoder matrix for the active channels listed above
   11    583   is:');
         584
   11    585     writeln;
   11    586     for row := 1 to received do begin
   12    587       for col := 1 to transmitted do
   12    588         WRITE_OCTAL ( decoder [row][col], field_base);
   12    589       writeln
   11    590     end;
   11    591     writeln;
         592
         593
         594   { DECODING BEGINS HERE.                                     }
         595
   11    596     EOT := FALSE;
   11    597     while (not EOT) do begin
         598
   12    599     writeln('please enter, on one line and separated by blanks,')
   12    599   ;
   12    600     writeln('the data received on each of the channels active at'
   12    600   );
   12    601     writeln('the receivers node.  The data should be in the form'
   12    601   );
   12    602     writeln('of octal numbers, and should be entered in order of'
   12    602   );
   12    603     writeln('increasing channel number.');
         604
   12    605     for index := 1 to transmitted do

EOF
```

```
JS IC  Line#   Source Line        IBM Personal Computer Pascal Compiler V1.00
   12    606       codeword [index] := 0;
         607
   12    608     for index := 1 to received do
   12    609       READ_OCTAL ( codeword [ active_channel [index] ] );
         610
   12    611     writeln;
   12    612     writeln('the ',received:2,' transmitted cleartext words were'
   12    612   );
   12    613     writeln('(octal numbers expressed in channel order):');
   12    614     writeln;
         615
   12    616     for index := 1 to received do begin
   13    617       clearword := 0;
   13    618       for col := 1 to transmitted do
   13    619         clearword := ADD (clearword,
   13    620                           MULTIPLY (decoder[index][col],
   13    621                                     codeword[col],
   13    622                                     modulus, field_base) );
   13    623       WRITE_OCTAL (clearword, field_base)
   12    624     end;
   12    625     writeln;
         626
   12    627     writeln('do you want to decode another ',received:2,' words?'
   12    627   );
   12    628     writeln('(type y or n).');
         629
   12    630     readln(continue);
   12    631     if (continue = 'n') then EOT := TRUE
         632
   12    633     end
         634
   00    635   end.

Symtab   635   Offset Length   Variable
                    0  12592   Return offset, Frame length
                 2052   2048   DNF                          :Array   Static
                 6158      2   COL                          :Integer Static
                12590      1   EOT                          :Boolean Static
                    4   2048   VAR                          :Array   Static
                 6156      2   ROW                          :Integer Static
                 6160      2   ROWS                         :Integer Static
                 8360   2048   DATA                         :Array   Static
                 6172      2   TEMP                         :Integer Static
                    2      2   INDEX                        :Integer Static
                12588      1   ACTIVE                       :Boolean Static
                 6162      2   DATAROW                      :Integer Static

No errors
```

JG IC Line# Source Line IBM Personal Computer Pascal Compiler V1.00

	6168	2	COLUMNS	:Integer Static
	6174	2	CHANNEL	:Integer Static
	6248	2048	DECODER	:Array Static
	6154	2	MODULUS	:Integer Static
	6164	2	DATAROWS	:Integer Static
	12520	64	CODEWORD	:Array Static
	12586	1	CONTINUE	:Char Static
	4100	2048	DNF_PRIME	:Array Static
	6148	2	RECEIVED	:Integer Static
	6170	2	DIMENSION	:Integer Static
	12584	2	CLEARWORD	:Integer Static
	6152	2	FIELD_BASE	:Integer Static
	10408	2048	DESIDERATA	:Array Static
	6150	2	TRANSMITTED	:Integer Static
	6166	2	EXTRA_DESID	:Integer Static
	6180	2	REDUCED_ELT	:Integer Static
	6176	2	DESIRED_CHANNELS	:Integer Static
	6178	2	REDUCING_ELT	:Integer Static
	6184	64	DEAD_CHANNEL	:Array Static
	6182	2	DEAD_CHANNELS	:Integer Static
	8296	64	DESIRED_CHANNEL	:Array Static
	12456	64	ACTIVE_CHANNEL	:Array Static

Errors	Warns	In Pass One
0	0	