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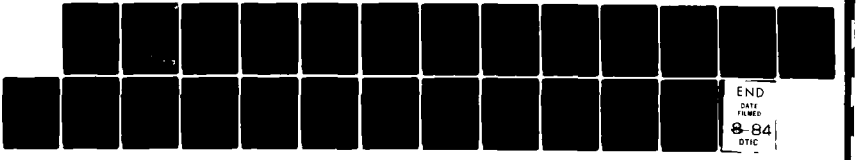

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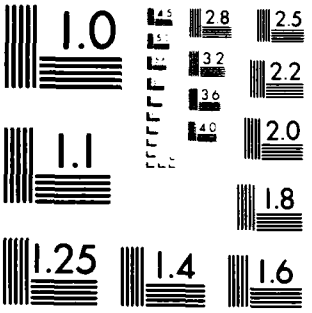

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Longitudinal and Transverse Instabilities in a High Current Modified Betatron Electron Accelerator

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Plasma Physics Division

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Jaycor, Inc.

May 17, 1984

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In this paper we derive a general dispersion relation which describes the longitudinal and transverse instabilities on an intense non-neutral electron beam in a modified betatron field configuration. In a modified betatron, the usual betatron fields are supplemented by a strong toroidal stabilizing magnetic field. The negative mass/kink, longitudinal resistive wall and transverse resistive wall instabilities have been analyzed in this new field configuration for an intense, non-neutral, confined electron ring. Our model includes self field effects as well as induced field effects due to the

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20. ABSTRACT (Continued)

image charges and currents on the toroidal chamber walls. The electron ring, which is assumed to have an energy spread, is taken to be located near the center of a toroidal chamber of finite conductivity. All the instabilities examined here are shown to be significantly reduced by the presence of the strong toroidal magnetic field. In the limit of low beam current and no toroidal magnetic field our results reduce to the usual expressions. We find an unstable hybrid mode of oscillation which is a coupled azimuthal and longitudinal mode and exists at beam energies below the transition energy. The negative mass/kink instability associated with an ultra high current ($I \approx 10$ kA) modified betatron electron accelerator is analyzed and evaluated in some detail. Growth rate curves are given for various ranges of system parameters, including electron beam energy spread. It is found that with a moderate energy spread on a 10 kA injected electron beam, the negative mass/kink instability can be stabilized.

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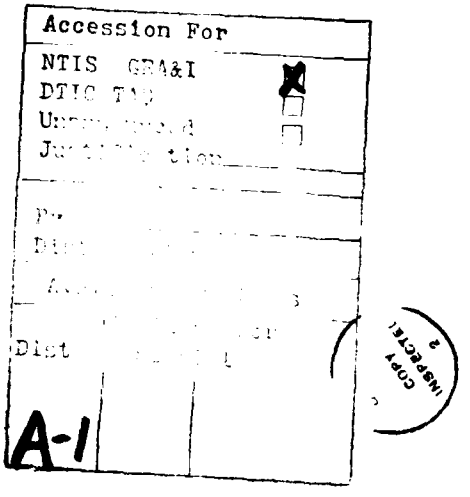

LONGITUDINAL AND TRANSVERSE INSTABILITIES IN A HIGH CURRENT MODIFIED BETATRON ELECTRON ACCELERATOR

1. INTRODUCTION

In this paper we examine a number of potentially destructive instabilities which can arise in a high current modified betatron electron accelerator. One of our main results is that the negative mass/kink type of instability can in an ultra high current modified betatron, be stabilized by a combination of a strong toroidal magnetic field and energy spread on the injected electron beam.[1] The modified betatron accelerator consists of an external toroidal magnetic field in addition to the usual external betatron field components (see Fig. 1). The strong toroidal magnetic field B_θ is the salient feature of the modified betatron accelerator. We will show that the addition of this field component greatly improves the stability characteristics of the intense electron ring. In order for the toroidal magnetic field to be effective in stabilizing the various instabilities, it is necessary that $|B_\theta| >> |B_z|$ where B_z is the vertical magnetic field. Although the toroidal field is beneficial in so far as stability is concerned, it requires a somewhat more involved electron beam injection scheme.[2,3]

From simple space charge considerations alone, it has been shown that the total number of electrons (current) that can be contained in the modified betatron field greatly exceeds the number that can be contained in a conventional betatron configuration.[4,5] Taking N_{mb} to be the maximum number of electrons that can be stably confined in a modified betatron and N_{cb} to be the corresponding number for a conventional betatron, it has been found that solely from space charge considerations

$$N_{mb} = \frac{1}{2} (B_\theta / B_z)^2 N_{cb},$$

where $|B_z| << |B_\theta|$. For a relativistic electron beam the maximum injection current that can be confined in a modified betatron accelerator is

$$I_{b,\max} = 2.1 (r_b / r_0)^2 \gamma^3 (B_\theta / B_z)^2 \ [kA],$$

where r_b and r_0 are the minor and major electron ring radii respectively and γ is the usual relativistic factor. It is clear from the expression for $I_{b,\max}$ that for injection energies in the MeV range extremely high currents (in the tens of kiloampere range) can be confined in the modified betatron for very modest values of r_b, r_0 and B_θ.

There are, however, numerous beam instabilities which may also place limits on the beam current, especially at the early stages of the acceleration process, when the beam energy is lowest. Some of the instabilities that have been found to limit the current in conventional low current betatrons are the negative mass, longitudinal resistive wall, transverse resistive wall and resonant instabilities. These instabilities, associated with conventional cyclic accelerators, have been examined in great detail and are well understood. A large body of literature exists on this subject and a number of excellent papers and review articles discuss these and other instabilities for conventional tenuous beam cyclic accelerators.[6-23] Far less is known, however, of the corresponding instabilities for intense electron rings

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in a modified betatron accelerator. In this paper we derive a general dispersion relation which describes longitudinal and transverse instabilities associated with an intense relativistic electron ring in a modified betatron field configuration. A recent analysis of longitudinal instabilities, i.e., negative mass and resistive wall, in the modified betatron has been performed.[23] The theoretical model used in Ref. (23) does not provide for transverse oscillations of the beam center and, therefore, a comparison with our results cannot be made. The present analysis of instabilities in high current modified betatrons does not consider resonant instabilities due to errors in the external fields. This general class of instabilities has been addressed in a separate paper.[24] In the present analysis the intense electron ring is non-neutralized in both charge and current and, hence, self field effects are included in our analysis. Self field effects, as measured by the self field index n_s, are shown to play an important role in so far as the strength of the various instabilities are concerned. Since the electron ring is confined in a toroidal chamber of finite conductivity, induced fields from the image charges and currents are included in our model. We also allow the beam center to undergo self-consistent transverse oscillations. In all parameter regimes, this extra degree of freedom for the beam dynamics is very important in the negative mass type of instability in the modified betatron.

Although we are considering high current electron rings (multi-kiloampere range) we will limit ourselves to low ν/γ beams where $\nu = |e|^2 N/(2\pi m_0 c^2 r_0)$ is Budker's parameter (N is the total number of electrons in the ring). The low ν/γ beam assumption places only weak limitations on the analysis since it does not necessarily imply low beam currents. There are however a number of physical implications associated with low ν/γ beams. The fractional change in beam energy in traversing the minor radius of the beam is proportional to ν/γ. Hence, by employing the low ν/γ approximation we are neglecting this energy shear. Due to the particle drifts associated with the self forces of the beam, i.e., electric and magnetic forces, and the external toroidal magnetic field B_θ, the beam electrons rotate in the poloidal direction. This poloidal motion generates a toroidal diamagnetic field which opposes the external B_θ field. However, for beams with $\nu/\gamma \ll 1$ this diamagnetic field may be neglected. Because of these and other simplifying assumptions our analysis assumes that $\nu/\gamma \ll 1$.

The derivation of the linear dispersion relation is performed in Section 2. Here, the fields are expressed as the sum of external and perturbed fields. Azimuthal density perturbations as well as transverse beam displacements are considered in our analysis since both are coupled in the modified betatron field configuration. The resulting dispersion relation describing longitudinal (azimuthal) and transverse beam instabilities in a modified betatron is analyzed in various limits in the subsequent sections. In Section 3 the emphasis is on the various limiting regimes associated with the dispersion relation. In Section 3a the kink (purely transverse mode) is analyzed and the growth rate for the transverse resistive wall instability obtained. In the following sections the coupled azimuthal and transverse modes (negative mass/kink) modes are analyzed. In Section 3b we consider the special case of an intense electron beam in a conventional betatron ($B_\theta = 0$) and in Section 3c a tenuous electron beam in a modified betatron is considered. The most interesting and perhaps relevant limit is that of an intense beam in a modified betatron. This special case is studied in Section 3d in some detail. In this section we first evaluate the growth rate associated with a nonthermal beam in a toroidal chamber of infinite conductivity. The resulting instability is shown to be a hybrid mode associated with the coupling of transverse and azimuthal beam oscillations. In this limit the effects of a finite wall resistivity are shown to be in general negligible. We also find that a modest amount of beam energy spread will stabilize the instability even for electron beam currents as high as 10 kA. Finally in Section 4 we discuss our results and assess the potential realization of an ultra high current modified betatron accelerator.

2. DERIVATION OF DISPERSION RELATION

Our model consists of an intense non-neutral electron ring confined within a conducting toroidal chamber as shown in Fig. 2. The electron ring is assumed to have a circular cross section with minor radius r_b and *center* at $r = r_0 + \Delta r(\theta,t)$ and $z = \Delta z(\theta,t)$. The beam is enclosed in a toroidal chamber

of finite conductivity, σ, with minor radius $a >> r_b$ and major radius $r_0 >> a$. The beam center is initially centered at $(r_0, 0)$ but can undergo small displacements, hence $|\Delta r|, |\Delta z| << a$.

The containment of the beam in a conventional betatron is accomplished by an applied axial magnetic field B_z with a local variation proportional to r^{-n}, where $n = -(r/B_z)\, dB_z/dr$ is the external field index. A radial component B_r is present, to account for the nonuniformity in B_z. The modified betatron utilizes an additional azimuthal magnetic field B_θ, with a free-space spatial variation proportional to r^{-1}. Expanded about $(r_0, 0)$, these fields are

$$B_z = B_{0z}(1 - n(r - r_0)/r_0), \tag{1a}$$

$$B_r = -B_{0z} nz/r_0, \tag{1b}$$

$$B_\theta = B_{0\theta}(1 - (r - r_0)/r_0), \tag{1c}$$

where B_{0z}, $B_{0\theta}$ are constant.

It is convenient at this point to define a reference particle in perfect unperturbed circular motion about the axis with $r = r_0$ and $z = 0$. Under the condition that the electron beam itself is also centered at $(r_0, 0)$, the orbit of the reference particle is governed solely by the external fields (neglecting toroidal effects). The azimuthal velocity of this particle is $v_0 = r_0\, \Omega_{0z}/\gamma_0$, where $\gamma_0 = (1 - v_0^2/c^2)^{-1/2}$ and $\Omega_{0z} = |e|B_{0z}/m_0 c$. Furthermore, since the canonical angular momentum P_θ is defined to within an additive constant, we take it to be zero for the reference particle. Note that Gaussian units are used throughout this paper, and $|e|$ and m_0 are the elementary charge and the electron rest mass while c is the speed of light.

The total unperturbed fields acting on the particles consist of the external fields, Eqs. (1), as well as the self and induced fields. The induced field contribution results from a displacement of the beam from the center of the chamber. In the absence of toroidal effects, image charges and currents are induced on the conducting chamber walls only when the beam is displaced off center. These image charges and currents result in the induced field contribution. The self fields, on the other hand, are simply the self electric and magnetic fields associated with an unconfined beam. Assuming a constant profile for both the beam charge and current density it is straightforward to show that within the beam, the combined expressions for the self and induced fields $E_r^{(1)}$, $E_z^{(1)}$, $B_r^{(1)}$, $B_z^{(1)}$ are to lowest order

$$\begin{aligned}
E_r^{(1)} &= -2\pi|e|n_0[r - r_0 - (1 - r_b^2/a^2)\Delta r], \\
E_z^{(1)} &= -2\pi|e|n_0[z - (1 - r_b^2/a^2)\Delta z], \\
B_r^{(1)} &= -2\pi|e|\beta_0 n_0[z - (1 - r_b^2/a^2)(1 + \xi)\Delta z], \\
B_z^{(1)} &= 2\pi|e|\beta_0 n_0[r - r_0 - (1 - r_b^2/a^2)(1 + \xi)\Delta r],
\end{aligned} \tag{2}$$

where n_0 is the uniform equilibrium number density, $\beta_0 = v_0/c$ is the uniform normalized azimuthal velocity, $\xi = (1 + i)(r_b^2/(a^2 - r_b^2))\, \sqrt{|\omega|/\omega}\ (\delta/a)$, $\delta = c/\sqrt{2\pi\sigma|\omega|}$ is the skin depth associated with the finite conductivity chamber and ω is the frequency associated with the perturbed beam displacement (for example in the case of a pure negative mass instability, $\omega \approx l\Omega_{0z}/\gamma_0$ where l is the harmonic number).

The perturbing fields excited by the perturbed charge and current density of the beam within the conducting chamber can be found in the usual way by solving Maxwell's equations. It will become clear later that it is necessary to express only the perturbed azimuthal electric field $E_\theta^{(1)} = \hat{E}_\theta \exp[i(l\theta - \omega t)]$ in terms of the perturbing line charge $\lambda^{(1)} = \hat{\lambda} \exp[i(l\theta - \omega t)]$ where l is an integer and ω is the frequency of the perturbation. Since the perturbed field $E_\theta^{(1)}$ varies slightly across the beam when $r_b << a$, we will use its value at the center of the beam in our analysis. For an electron beam centered at $(r_0, 0)$ and having a uniform density and current profile, the relationship between $\hat{E}_\theta$ and $\hat{\lambda}$ can be shown to be

$$\hat{E}_\theta = -i \frac{l}{r_0} \frac{(1 + 2 \ln a/r_b)}{\gamma_0^2} (1 - i\epsilon_{\parallel})\hat{\lambda} \tag{3}$$

where $\epsilon_{\parallel} = \gamma_0^2 \beta_0^2 \sqrt{|\omega|/\omega}(\delta/a)/(1 + 2 \ln a/r_b)$. The γ_0^2 term in (3) is a electromagnetic contribution. In obtaining Eqs. (2) and (3) we have assumed that the wavelength of the perturbation is large compared to the torus minor radius, i.e., $2\pi r_0/l >> a$ and that toroidal effects can be neglected, i.e., $\nu/\gamma_0 << 1$. It has also been assumed that the azimuthal phase velocity of the perturbation is very close to the beam velocity, i.e., $\omega \approx l\nu_0/r_0 = l\Omega_{0z}/\gamma_0$. It should be noted that Eq. (3) is valid for a beam centered at $(r_0, 0)$ whereas our analysis will be applicable for a beam with a general displacement centered at $(r_0 + \Delta r, \Delta z)$. We will show later that the beam displacement from the center of the torus is itself proportional to $E_\theta^{(1)}$, if the beam is initially centered at $(r_0, 0)$. Hence, for a beam displaced off center by $(\Delta r, \Delta z)$, the correction to Eq. (3) is of higher order in the perturbing field and therefore will be neglected. The external fields in Eq. (1), the self and induced fields in Eq. (2) as well as the perturbing azimuthal field in Eq. (3) will be used in the particle orbit equations to obtain the linearized particle trajectories. Knowledge of the linearized particle trajectories will permit the evaluation of the perturbed line charge as a linear function of E_θ. The remainder of our analysis deals with deriving the self consistent linear relationship between the perturbed line charge and perturbing azimuthal electric field. Coupling this relationship with Eq. (3) will result in a dispersion relation for the various modes which include the negative mass, longitudinal resistive wall and transverse resistive wall instabilities.

Employing a Lagrangian representation for the particle dynamics, the line charge of the beam is $\lambda = -|e| \int n_b(r,z,\theta,t)dS$, where $\int dS$ is the integral over the beam cross sectional area and $n_b(r,z,\theta,t)$ is the beam density. The line charge can be expressed in the form

$$\lambda = -|e| n_0 r_0 \int_0^{r_b} \rho_0 d\rho_0 \int_0^{2\pi} d\alpha_0 \eta(\rho_0, \alpha_0) \int_{-\infty}^{\infty} d\Delta P g(\Delta P) \int_0^{2\pi} d\theta_0 \, \delta(\theta - \tilde{\theta})/\tilde{r} \tag{4}$$

where n_0 is the ambient beam density, (ρ_0, α_0) are the particle's initial poloidal radial distance and angle measured from the center of the beam, $\eta(\rho_0, \alpha_0)$ denotes the initial distribution of ρ_0 and α_0, ΔP is the particle's initial canonical angular momentum, $g(\Delta P)$ describes the initial distribution of ΔP and is normalized such that $\int d\Delta P g(\Delta P) = 1$, θ_0 is the particle's initial azimuthal angle, $\tilde{\theta} = \tilde{\theta}(\rho_0, \alpha_0, \Delta P, \theta_0, t)$ is the particle's azimuthal angle at time t and $\tilde{r} = \tilde{r}(\rho_0, \alpha_0, \Delta P, \theta_0, t)$ is the particle's radial position at time t. To find a linear expression for the perturbed line charge $\lambda^{(1)}$, we expand the particle trajectories in terms of the perturbing field, i.e., $\tilde{r} = r^{(0)} + r^{(1)}$ and $\tilde{\theta} = \theta_0 + \theta^{(0)} + \theta^{(1)}$ where $r^{(1)}$ and $\theta^{(1)}$ are linear in $E_\theta^{(1)}$. Performing this expansion on Eq. (4) we find that

$$\lambda^{(1)} = |e| n_0 \int_0^{r_b} \rho_0 d\rho_0 \int_0^{2\pi} d\alpha_0 \eta \int_{-\infty}^{+\infty} g d\Delta P (\partial\theta^{(1)}/\partial\theta_0 + r^{(1)}/r^{(0)})\Big|_{\theta_0 = \tilde{\theta} - \theta^{(0)}}. \tag{5}$$

It now remains to evaluate the linear part of the trajectories $r^{(1)}$ and $\theta^{(1)}$, in order to obtain the linear perturbed line charge. Using the external as well as the self and induced fields in Eqs. (1) and (2) we find that the linearized particle equations, correct to first order in Δr, Δz and E_θ, are

$$\ddot{r} + \omega_r^2(r - r_0) - \frac{\Omega_{0z}}{\gamma_0}\frac{\Delta P}{\gamma_0 m_0 r_0} - \frac{\Omega_{0\theta}}{\gamma_0}\dot{z} = -\frac{\Omega_{0z}^2}{\gamma_0^2} n_s\left[1 - \frac{r_b^2}{a^2}\right](1 - \beta_0^2\gamma_0^2\xi)\Delta r + \frac{\Omega_{0z}}{\gamma_0}\frac{P_\theta^{(1)}}{\gamma_0 m_0 r_0}, \quad (6a)$$

$$\ddot{z} + \omega_z^2 z + \frac{\Omega_{0\theta}}{\gamma_0}\dot{r} = -\frac{\Omega_{0z}^2}{\gamma_0^2} n_s(1 - r_b^2/a^2)(1 - \beta_0^2\gamma_0^2\xi)\Delta z, \quad (6b)$$

$$\dot{\theta} = \frac{\Omega_{0z}}{\gamma_0}\left[1 - \frac{r - r_0}{r_0}\right] + \frac{\Delta P + P_\theta^{(1)}}{\gamma_0^3 m_0 r_0^2}, \quad (6c)$$

where $\omega_r^2 = (1 - n - n_s)(\Omega_{0z}/\gamma_0)^2$, $\omega_z^2 = (n - n_s)(\Omega_{0z}/\gamma_0)^2$, n is the external field index, $n_s = \omega_b^2/(2\gamma_0\Omega_{0z}^2)$ is the self field index, $\omega_b^2 = 4\pi|e|^2 n_0/m_0$, $\Omega_{0\theta}/\gamma_0 = |e|B_{0\theta}/\gamma_0 m_0$ is the toroidal cyclotron frequency and $P_\theta^{(1)}$ is the first order perturbing canonical angular momentum which is shown later to be proportional to $E_\theta^{(1)}$. The self field index n_s is a measure of the electron beam self fields and is related to Budker's parameter by the relation $n_s = 2(\nu/\gamma_0)(c/\Omega_{0z}r_b)^2$. Note that for a low beam current, and no toroidal field, i.e., $n_s \to 0$ and $B_{0\theta} \to 0$, the terms ω_r and ω_z reduce to the usual radial and axial betatron frequencies. In obtaining Eqs. (6) we made use of the following linearized relations

$$\gamma = \gamma_0 + (\Delta P + P_\theta^{(1)})\Omega_{0z}/(\gamma_0 m_0 c^2), \quad (7a)$$

$$\nu_\theta = \nu_0 + (\Delta P + P_\theta^{(1)})/(\gamma_0^3 m_0 r_0), \quad (7b)$$

where $\gamma = (1 - v_\theta^2/c^2)^{-1/2}$ and v_θ is the azimuthal particle velocity correct to first order. As a reminder, we note that the Δr and Δz terms in Eqs. (6a,b) are to be considered first order terms.

The significance of the toroidal magnetic field on the stability of the particle orbits, under the influence of only self field forces (space charge forces), can be seen from an examination of Eqs. (6a,b). To study the orbit stability due to only self field forces we set Δz, Δr and $P_\theta^{(1)}$ equal to zero in Eqs. (6a,b). The frequencies ω_r and ω_z may be imaginary for large values of n_s (which is the present case of interest). However, unlike the conventional betatron, particle orbit stability is achieved by having a large toroidal magnetic field. It can be shown[4] from Eqs. (6a,b) that under the influence of only self fields, particle orbit stability is obtained if $n_s \leqslant (B_{0\theta}/2B_{0z})^2$ when $B_{0\theta} >> B_{0z}$. In conventional betatrons, on the other hand, orbit stability due to self field forces require that $n_s \leqslant 1/2$. Since n_s is proportional to the beam current the conditions for stability against self field forces show that the modified betatron field configuration can confine far higher beam currents than the conventional betatron.

The zero order solution to Eqs. (6) are

$$r^{(0)} = r_0 + \frac{\Omega_{0z}}{\gamma_0}\frac{\Delta P}{\gamma_0 m_0 r_0 \omega_r^2}, \quad (8a)$$

$$z^{(0)} = 0, \quad (8b)$$

$$\theta^{(0)} = (\Omega_{0z}/\gamma_0 + k\Delta P)t, \quad (8c)$$

where $k = [1/\gamma_0^2 - 1/(1 - n - n_s)]/\gamma_0 m_0 r_0^2$. In obtaining Eqs. (8) we have neglected the homogeneous part of the solution and will, henceforth, not consider the effects due to finite amplitude betatron oscillations in our model. These effects, however, can be included in a straightforward way. To solve for the first order solutions to Eqs. (6) we first note that the perturbed canonical momentum satisfies $\dot{P}_\theta^{(1)} = -|e|\beta_0\,\partial(A_\theta^{(1)} - \beta_0^{-1}\phi^{(1)})/\partial\theta$, where $A_\theta^{(1)}$ and $\phi^{(1)}$ are the perturbed vector and scalar potentials associated with the perturbed charge and current density of the beam. Since the perturbing azimuthal electric field is

$$E_\theta^{(1)} = \hat{E}_\theta \exp\,[i(l\theta - \omega t)] = -c^{-1}\partial A_\theta^{(1)}/\partial t - r_0^{-1}\partial\phi^{(1)}/\partial\theta$$

we find that to lowest order

$$P_\theta^{(1)} = -i\Delta\omega_l^{-1}|e|r_0\hat{E}_\theta\, e^{i(l\theta_0 - \Delta\omega_l t)}, \tag{9}$$

where again we are considering frequencies near harmonics of the cyclotron frequency, i.e., $\omega \approx l\Omega_{0z}/\gamma_0$ and $\Delta\omega_l = \omega - l(\Omega_{0z}/\gamma_0 + k\Delta P)$. To solve for the first order trajectories we first represent the center of the beam displacement and the first order solution as

$$(\Delta r^{(1)}, \Delta z^{(1)}) = (\Delta\hat{r}, \Delta\hat{z})e^{i(l\theta - \omega t)}, \tag{10a}$$

$$(r^{(1)}, z^{(1)}) = (\hat{r},\hat{z})e^{i(l\theta - \omega t)} \approx (\hat{r},\hat{z})e^{i(l\theta_0 - \Delta\omega_l t)}. \tag{10b}$$

Using this representation together with Eq. (9) we find that the first order solution to Eqs. (6) is given by

$$\hat{r} = D^{-1}\left[-(\omega_z^2 - \Delta\omega_l^2)\Omega_s^2\Delta\hat{r} + i\Delta\omega_l(\Omega_{0\theta}/\gamma_0)\Omega_s^2\Delta\hat{z} - \frac{i|e|}{\gamma_0 m_0}(\omega_z^2 - \Delta\omega_l^2)\frac{(\Omega_{0z}/\gamma_0)}{\Delta\omega_l}\hat{E}_\theta\right], \tag{11a}$$

$$\hat{z} = D^{-1}\left[-i\Delta\omega_l(\Omega_{0\theta}/\gamma_0)\Omega_s^2\Delta\hat{r} - (\omega_r^2 - \Delta\omega_l^2)\Omega_s^2\Delta\hat{z} + \frac{|e|}{\gamma_0 m_0}\,\frac{\Omega_{0\theta}}{\gamma_0}\,\frac{\Omega_{0z}}{\gamma_0}\hat{E}_\theta\right], \tag{11b}$$

where $D = (\omega_r^2 - \Delta\omega_l^2)(\omega_z^2 - \Delta\omega_l^2) - \Delta\omega_l^2\Omega_{0\theta}^2/\gamma_0^2$ and $\Omega_s^2 = n_s(1 - r_b^2/a^2)(1 - \beta_0^2\gamma_0^2\xi)\Omega_{0z}^2/\gamma_0^2$. Equations (11) state the first order solutions in terms of $\Delta\hat{r}$, $\Delta\hat{z}$ and $\hat{E}_\theta$. To obtain $\hat{r}$ and $\hat{z}$ in terms of only $\hat{E}_\theta$ we note that the average value of $(\hat{r},\hat{z})$ over all the particles is just $(\Delta\hat{r}, \Delta\hat{z})$. Since we are disregarding the effects of finite amplitude betatron oscillations, the average of $(\hat{r},\hat{z})$ over all particles in any given beam cross section, denoted by $(<\hat{r}>, <\hat{z}>)$, is

$$(<\hat{r}>, <\hat{z}>) \equiv \int d\Delta P g(\Delta P)(\hat{r},\hat{z}) = (\Delta\hat{r}, \Delta\hat{z}). \tag{12}$$

Performing this particle average over Eq. (11) we obtain

$$\Delta\hat{r} = \frac{i|e|}{\gamma_0 m_0}\,\frac{C_r}{R}\hat{E}_\theta, \tag{13a}$$

$$\Delta\hat{z} = \frac{|e|}{\gamma_0 m_0}\,\frac{C_z}{R}\hat{E}_\theta, \tag{13b}$$

where

$$R = <Q_z><Q_r> - \Omega_s^4 <\Delta\omega_l D^{-1}>^2 \Omega_{0\theta}^2/\gamma_0^2,$$

$$Q_r = 1 + \Omega_s^2(\omega_r^2 - \Delta\omega_l^2)D^{-1}, \quad Q_z = 1 + \Omega_s^2(\omega_z^2 - \Delta\omega_l^2)D^{-1},$$

$$C_r = \frac{\Omega_{0z}}{\gamma_0}\left[\Omega_s^2 \frac{\Omega_{0\theta}^2}{\gamma_0^2}\left\langle\frac{1}{D}\right\rangle\left\langle\frac{\Delta\omega_l}{D}\right\rangle - \left\langle\frac{\omega_z^2 - \Delta\omega_l^2}{\Delta\omega_l D}\right\rangle <Q_r>\right].$$

$$C_z = \frac{\Omega_{0z}}{\gamma_0}\left[\frac{\Omega_{0\theta}}{\gamma_0}\ \frac{1}{D}\ <Q_z> - \Omega_s^2\ \frac{\Omega_{0\theta}}{\gamma_0}\left\langle\frac{\Delta\omega_l}{D}\right\rangle\left\langle\frac{\omega_z^2 - \Delta\omega_l^2}{\Delta\omega_l D}\right\rangle\right].$$

Combining Eqs. (12) and (13) we find

$$\hat{r} = \frac{i|e|}{\gamma_0 m_0} R^{-1} D^{-1}\left[-\Omega_s^2(\omega_z^2 - \Delta\omega_l^2)C_r + \Omega_s^2 \Delta\omega_l \frac{\Omega_{0\theta}}{\gamma_0} C_z - R(\omega_z^2 - \Delta\omega_l^2)\Delta\omega_l^{-1}\frac{\Omega_{0z}}{\gamma_0}\right]\hat{E}_\theta, \tag{14a}$$

$$\hat{z} = \frac{|e|}{\gamma_0 m_0} R^{-1} D^{-1}\left[\Omega_s^2 \Delta\omega_l \frac{\Omega_{0\theta}}{\gamma_0} C_r - \Omega_s^2(\omega_r^2 - \Delta\omega_l^2)C_z + R\ \frac{\Omega_{0z}}{\gamma_0}\ \frac{\Omega_{0\theta}}{\gamma_0}\right]\hat{E}_\theta. \tag{14b}$$

We can now evaluate the perturbed line charge given in Eq. (5). From Eq. (6c) we find that

$$\dot{\theta}^{(1)} = -\frac{\Omega_{0z}}{\gamma_0}\ \frac{r^{(1)}}{r_0} + \frac{P_\theta^{(1)}}{\gamma_0^3 m_0 r_0^2},$$

where $r^{(1)}$ and $P_\theta^{(1)}$ are given by Eqs. (14a) and (9). Substituting $\theta^{(1)}$ into Eq. (5) and using Eq. (14a) together with Eqs. (9) we find that the perturbed line charge is

$$\lambda^{(1)} = il\ \frac{\omega_b^2}{4\gamma_0}\ \frac{r_b^2}{r_0} <\Delta\omega_l^{-2}(1/\gamma_0^2 - D^{-1}(\Omega_{0z}/\gamma_0)^2(\omega_z^2 - \Delta\omega_l^2))$$

$$- \Omega_s^2(\Delta\omega_l DR)^{-1}(\Omega_{0z}/\gamma_0)[(\omega_z^2 - \Delta\omega_l^2)C_r - \Delta\omega_l(\Omega_{0\theta}/\gamma_0)C_z]> \hat{E}_\theta e^{i(l\theta - \omega t)}. \tag{15}$$

In obtaining Eq. (15) we have assumed a constant density profile beam, i.e., $\eta(\rho_0, \alpha_0) = 1$ and have neglected finite amplitude betatron oscillations.

Combining Eqs. (3) and (15) we obtain the following dispersion relation

$$1 = \frac{\omega_b^2}{4\gamma_0^3}\frac{r_b^2}{r_0^2}I^2(1 + 2\ln a/r_b)(1 - i\epsilon_{\parallel})\left\{ \frac{1}{\Delta\omega_l^2}\left[\frac{1}{\gamma_0^2} - \frac{(\omega_z^2 - \Delta\omega_l^2)\Omega_{0z}^2/\gamma_0^2}{D}\right]\right.$$

$$\left. - \frac{\Omega_s^2\Omega_{0z}/\gamma_0}{\Delta\omega_l DR}\left[(\omega_z^2 - \Delta\omega_l^2)C_r - \Delta\omega_l\frac{\Omega_{0\theta}}{\gamma_0}C_z\right]\right\}. \tag{16}$$

Equation (16) is the complete linear dispersion relation for an intense electron ring in a modified betatron confined in a conducting chamber of finite conductivity. The azimuthal and transverse modes of oscillation are coupled via the presence of the external azimuthal magnetic field. Thus, for example, the negative mass instability in the modified betatron excites both azimuthal and transverse oscillations. In a conventional betatron field configuration longitudinal oscillations are excited by the negative mass instability. As is the case in a conventional betatron a pure kink type mode can exist in a modified betatron (Section 3a). In this case the radial and axial beam displacements are, however, coupled through the presence of the external azimuthal magnetic field. This is unlike the situation in a conventional betatron where the transverse oscillations are uncoupled. In the following section we examine the general dispersion relation given by Eq. (10) in various limiting situations.

3. LIMITING REGIMES OF DISPERSION RELATION

a. Kink instability (transverse resistive wall).

Due to the finite wall conductivity a kink instability can develop on the electron ring. In the case of finite wall conductivity this instability is referred to as the transverse resistive wall instability. If the dissipative mechanism were collisions with say a background plasma we would refer to the instability as the hose instability. The transverse resistive wall instability is characterized by a uniform density in the azimuthal direction, hence $E_\theta^{(1)} = 0$. Since the beam undergoes a transverse displacement we see from Eqs. (13a,b) that the dispersion relation for this mode is $R = 0$. When $B_\theta = 0$, this reproduces the dispersion relations of Ref. (11). For $B_{0\theta} \neq 0$ and an external field index equal to 1/2, i.e., $n = 1/2$, the dispersion relation for the transverse resistive wall instability becomes

$$1 + \Omega_s^2\int \frac{g(\Delta P)\,d(\Delta P)}{\omega_r^2 - \Delta\omega_l^2 \pm \Delta\omega_l\Omega_{0\theta}/\gamma_0} = 0, \tag{17}$$

where ω_r^2 has been used for both ω_r^2 and ω_z^2 (since $n = 1/2$).

For a highly conducting chamber, the skin depth is small compared to the minor radius a, i.e., $\delta/a << 1$. Then, for a cold beam, $g(\Delta P_0) = \delta(\Delta P_0)$, the growth is

$$\Gamma = \frac{B_{0z}}{B_{0\theta}}\frac{\Omega_{0z}}{\gamma_0}\frac{n_s\beta_0^2\gamma_0^2(\delta/a)r_b^2/a^2}{\sqrt{1 + 4(B_{0z}/B_{0\theta})^2(1/2 - n_s r_b^2/a^2)}}. \tag{18}$$

This relation shows that for $B_{0z}/B_{0\theta} << 1$, the growth rate is proportional to the vanishingly small ratio $B_{0z}/B_{0\theta}$. On the other hand, if $B_{0\theta} = 0$, then Eq. (18) reproduces the result of Ref. (11), for $n = 1/2$.

In addition if the conductivity of the wall is taken to be infinite, the dispersion relation in Eq. (17) gives, for a cold beam and for $n = 1/2$, the following four stable transverse beam modes:

$$\Delta\omega_l = \pm \frac{1}{2} \frac{\Omega_{0\theta}}{\gamma_0} \left[1 \ (\pm) \sqrt{1 + 4 \left| \frac{B_{0z}}{B_{0\theta}} \right|^2 \left| \frac{1}{2} - n_s \left| \frac{r_b}{a} \right|^2 \right|} \right] \tag{19}$$

Since $1/2 - n_s(r_b/a)^2$ is necessarily much less than $(B_{0\theta}/2\,B_{0z})^2$, there are two pairs of stable transverse modes of oscillation, viz. with $\Delta\omega_l = \pm\,(\Omega_{0\theta}/\gamma_0)$, i.e., the fast oscillation, and $\Delta\omega_l = \pm\omega_B$, i.e., the slow oscillation, where

$$\omega_B = (1/2 - n_s(r_b/a)^2)(B_{0z}/B_{0\theta})\Omega_{0z}/\gamma_0$$

is the bounce frequency. The slow oscillation at the bounce frequency is the result of the $\mathbf{E} \times \mathbf{B}$ drift, caused by the transverse induced image fields and the applied azimuthal magnetic field.

In the following subsections the instabilities associated with the azimuthal and transverse beam oscillations are analyzed for a number of limiting cases.

b. Negative Mass and Longitudinal Resistive Wall Instability in a High Current Conventional Betatron ($B_{0\theta} = 0$ and $n_s \neq 0$).

In this subsection we consider the negative mass and longitudinal resistive wall instability in a conventional betatron, $B_{0\theta} = 0$, with self field effects included, $n_s \neq 0$. In this limit the dispersion relation in Eq. (16) simplifies substantially since the coupling term which is proportional to $B_{0\theta}$ vanishes. For a cold electron beam, i.e., $g(\Delta P) = \delta(\Delta P)$ the dispersion relation takes the form

$$\omega = l \frac{\Omega_{0z}}{\gamma_0} + \left[l\eta \ (1 - i\epsilon_{\parallel})(1 - \alpha)/\gamma_0^2) \right]^{1/2}, \tag{20}$$

where

$$\eta^2 = (\omega_b^2/4\gamma_0)(r_b/r_0)^2(1 + 2 \ln a/r_b)/\gamma_0^2 = (c/r_0)^2(\nu/\gamma_0)(1 + 2 \ln a/r_b)/\gamma_0^2,$$

and $\alpha = \gamma_0^2/(1 - n - n_s r_b^2/a^2)$. Except for the self field term $n_s r_b^2/a^2$ in the expression for α, Eq. (19) is identical to the usual dispersion relation describing the negative mass and longitudinal resistive wall instability in a conventional betatron.[8,9,13,14] In a conventional betatron the self field index n_s must be less than $1/2$ to insure particle orbit stability, i.e., ω_r^2 and $\omega_z^2 > 0$. Since $(r_b/a)^2 << 1$ the term $n_s(r_b/a)^2$ can be omitted. Thus in a conventional betatron self field effects play a substantially more important role in particle orbit stability than in the negative mass/longitudinal resistive wall instability.

c. Negative Mass and Longitudinal Resistive Wall Instability in a Low Current Modified Betatron ($B_{0\theta} \neq 0$, $n_s \gtrsim 0$).

In this limit we consider the instabilities in a low current, $n_s \approx 0$, modified betratron, $B_{0\theta} \neq 0$, accelerator. The dispersion relation in Eq. (16) again simplifies substantially since $\Omega_s = 0$ and $Q_r = Q_z = R = 1$. The resulting expression is

$$1 = l^2\eta^2(1 - i\epsilon_{\parallel})\ \gamma_0^{-2} \int \frac{d(\Delta P) g(\Delta P)}{\Delta\omega_l^2} \left[1 - \frac{\gamma_0^2}{1 - n} \frac{\omega_r^2\omega_z^2 - \Delta\omega_l^4}{D} \right] \tag{21}$$

where $\Delta\omega_l = \omega - l\Omega_{0z}/\gamma_0$ and $D = (\omega_r^2 - \Delta\omega_l^2)\ (\omega_z^2 - \Delta\omega_l^2) - \Delta\omega_l^2\Omega_{0\theta}^2/\gamma^2$. For perfectly conducting walls, i.e., $\epsilon_{\parallel} = 0(\sigma = \infty)$, the dispersion relation in Eq. (21) describes the negative mass instability in a low current, i.e., $n_s = 0$, modified betatron and is identical to the result in Ref. (16) for the case of zero betatron amplitudes. In the limit that $B_{0\theta} >> B_{0z}$ the term $\Delta\omega_l^2$ can be neglected compared to ω_r^2

and ω_z^2 but not compared to $\omega_r^2\omega_z^2/(\Omega_{0\theta}/\gamma_0)^2$. For a cold electron beam, i.e., $g(\Delta P) = \delta(\Delta P)$, the leading term of the growth rate in this limit is

$$\Gamma = \Omega_{0z} \frac{B_{0z}}{B_{0\theta}} n^{1/2} \left[1 - \frac{1-n}{\gamma_0^2}\right]^{1/2}. \tag{22}$$

The growth rate in Eq. (22) is independent of beam density and is proportional to the vanishingly small ratio $B_{0z}/B_{0\theta}$.

d. Negative Mass/Kink and Longitudinal Resistive Wall Instability in a High Current Modified Betatron ($B_{0\theta} >> B_{0z}$ and $n_s >> 1$).

From space charge considerations alone we have seen that the modified betatron configuration can confine ultra high current electron beams if the azimuthal magnetic field is sufficiently strong. In fact for space charge confinement in the modified betatron we require that n_s be less than $(B_{0\theta}/2B_{0z})^2$. Since $B_{0\theta} >> B_{0z}$, the self field index n_s may be much greater than unity. In conventional betatrons, on the other hand, n_s is limited to values less than 1/2. We will now examine in some detail the dispersion relation in Eq. (16) in the high current regime, i.e., $n_s >> 1$ but less than $(B_{0\theta}/2B_{0z})^2$. In this regime the quantities $|\omega_r^2|$, $|\omega_z^2|$, $|\Omega_s^2|$ and $(\Omega_{0\theta}/\gamma_0)^2$ are much greater than $(\Omega_{0z}/\gamma_0)^2$, while $|\omega_r^2 + \Omega_s^2|$, $|\omega_z^2 + \Omega_s^2|$ and $|\omega_r^2\omega_z^2/(\Omega_{0\theta}/\gamma_0)^2|$ are comparable to $(\Omega_{0z}/\gamma_0)^2$. We also note that $|\Delta\omega_{0l}| << \Omega_{0z}/\gamma_0$, where $\Delta\omega_{0l} = \omega - l\Omega_{0z}/\gamma_0$. In view of these observations and inequalities the dispersion relation in Eq. (16) for a cold electron beam, i.e., $g(\Delta P) = \delta(\Delta P)$ becomes

$$1 = l^2 \frac{\eta^2}{\gamma_0^2} (1 - i\epsilon_{\parallel}) \left[\frac{1-\alpha}{\Delta\omega_{0l}^2} + \frac{\alpha}{\Delta\omega_{0l}^2 - \omega_B^2(1 + i\,\epsilon_{\perp})}\right] \tag{23}$$

where $\epsilon_{\parallel} = \gamma_0^2 \beta_0^2 \sqrt{|\omega|/\omega}\ (\delta/a)/(1 + 2\ln a/r_b)$ and $\epsilon_{\perp} = -2\ \gamma_0^2\beta_0^2\ n_s(r_b/a)^2(\delta/a)\ (1/2 - n_s r_b/a)^{-1}$. For a highly conducting chamber $\epsilon_{\parallel} << 1$ and $\epsilon_{\perp} << 1$. To investigate the nature of the roots of the dispersion relation in Eq. (23) for a perfectly conducting wall, i.e., $\epsilon_{\parallel} = \epsilon_{\perp} = 0$, we define

$$F(\Delta\omega_{0l}^2) = \frac{1-\alpha}{\Delta\omega_{0l}^2} + \frac{\alpha}{\Delta\omega_{0l}^2 - \omega_B^2}. \tag{24}$$

The dispersion relation is then simply $F(\Delta\omega_{0l}^2) = (\gamma_0/l\eta)^2$. The function $F(\Delta\omega_{0l}^2)$ is depicted in Fig. 3 and has singularities at $\Delta\omega_{0l}^2 = 0$ and $\Delta\omega_{0l}^2 = \omega_B^2$. These singularities correspond respectively to the azimuthal and transverse (kink) modes of oscillations on the beam. The nature of the roots of the dispersion relation $F(\Delta\omega_{0l}^2) = (\gamma_0/l\eta)^2$, depend upon the sign of α as can be seen in Fig. 3.

We first consider the situation where $\alpha < 0$, Fig. 3a. For $(\gamma_0/l\eta)^2$ sufficiently large, i.e., low beam density and/or high beam energy, the dispersion relation has two real positive roots for $\Delta\omega_{0l}^2$. Hence, the roots of the primarily azimuthal mode, $\Delta\omega_{0l} \approx 0$, and of the primarily transverse mode, $\Delta\omega_{0l} \approx \pm\,\omega_B$ are stable. If on the other hand $(\gamma_0/l\eta)^2$ is significantly small, i.e., high beam density and/or low beam energy, there are again two real positive roots for $\Delta\omega_{0l}^2$. These roots are stable and represent primarily the transverse modes of the beam since $\Delta\omega_{0l}^2 > (1-\alpha)\omega_B^2$. For intermediate values of $(\gamma_0/l\eta)^2$ the roots $\Delta\omega_{0l}^2$ are complex and the azimuthal and transverse modes couple, resulting in an unstable hybrid mode.

We now consider the situation where $\alpha > 0$. If α is positive it must be greater than $2\gamma_0^2 >> 1$. In this case the function $F(\Delta\omega_{0l}^2)$ takes the form shown in Fig. 3b. Here there are two real roots for $\Delta\omega_{0l}^2$, the positive root satisfying $\Delta\omega_{0l}^2 > \omega_B^2$ and the negative satisfying $(1-\alpha)\omega_B^2 < \Delta\omega_{0l}^2 < 0$. The positive root is of course stable while the negative root is unstable regardless of the value of $(\gamma_0/l\eta)^2$. Since this unstable root satisfies $|\Delta\omega_{0l}^2| < (\alpha - 1)\omega_B^2 \approx \alpha\omega_B^2$, the growth rate is bounded by $\sqrt{\alpha}\omega_B$. Therefore, the maximum growth rate decreases as the azimuthal magnetic field $B_{0\theta}$ is increased.

The two possible regimes, i.e., $\alpha < 0$ and $\alpha > 2\gamma_0^2 >> 1$, correspond to values of γ_0 less than γ_{tran} and γ_0 greater than γ_{tran} respectively, where

$$\gamma_{tran} = 1 + \left[2\,\frac{r_0}{a}\right]^{4/3} (\beta_0 \nu)^{2/3}. \tag{25}$$

A typical curve of the real and imaginary part of $\Delta\omega_l$ for a cold beam and infinite wall conductivity is shown in Fig. 4. The growth rate as a function of γ_0 is double-peaked. The peak located at $\gamma_0 < \gamma_{tran}$ corresponds to the regime where $\alpha < 0$ and is the unstable hybrid mode, *i.e.*, strongly coupled azimuthal and longitudinal beam mode. The peak in the growth rate occurring at $\gamma_0 > \gamma_{tran}$ corresponds to the $\alpha > 2\gamma_0^2$ regime and is again a hybrid mode resulting from the coupling of the azimuthal and transverse modes.

From the dispersion relation in Eq. (23) the growth rate in the regime $\gamma_0 > \gamma_{tran}$ is given by

$$\Gamma = \mathrm{Im}(\omega) = \frac{1}{\sqrt{2}} \left\{ \left[4\alpha\omega_B^2 \left[l\frac{\eta}{\gamma_0} \right]^2 + \left[\omega_B^2 - l^2 \frac{\eta^2}{\gamma_0^2} \right]^2 \right]^{1/2} - \left[\omega_B^2 + l^2 \frac{\eta^2}{\gamma_0^2} \right] \right\}^{1/2}. \tag{26}$$

For γ_0 slightly greater than γ_{tran} the growth rate in Eq. (26) reduces to

$$\Gamma = \omega_B \alpha^{1/2}, \quad \gamma_0 \gtrsim \gamma_{tran}. \tag{27}$$

For γ_0 much greater than γ_{tran} the growth rate in Eq. (26) reduces to

$$\Gamma = l(\eta/\gamma_0)\alpha^{1/2}, \quad \gamma_0 >> \gamma_{tran}. \tag{28}$$

It should be noted that the growth in Eq. (28), i.e., for $\gamma_0 >> \gamma_{tran}$, is identical to the growth rate in the absence of the applied toroidal magnetic field, see Eq. (20). The upper bound for the growth rate in the regime where $\gamma_0 > \gamma_{tran}$ is

$$\Gamma < (\omega_B l \eta/\gamma_0)^{1/2} \alpha^{1/4} = 0.62(1 + 2\ln a/r_b)^{1/2} (c/r_0) \nu^{1/6} \sqrt{lc/\Omega_{0\theta} r_0}\, (a/r_0)^{1/6}. \tag{29}$$

For $\gamma_0 < \gamma_{tran}$ ($\alpha < 0$) the dispersion relation in Eq. (23) yields either two real and positive solutions for $\Delta\omega_{0l}^2$ or a pair of complex conjugate roots for $\Delta\omega_{0l}^2$. If

$$-\alpha > \frac{1}{4} \left[\frac{l\eta}{\gamma_0 \omega_B} - \frac{\gamma_0 \omega_B}{l\eta} \right]^2 \tag{30}$$

the roots are complex and the growth rate is given by

$$\Gamma = \frac{1}{2} \left[\left[l\frac{\eta}{\gamma_0} \sqrt{-\alpha} \right]^2 - \left[\omega_B - l\frac{\eta}{\gamma_0} \sqrt{1-\alpha} \right]^2 \right]^{1/2}. \tag{31}$$

If the inequality in Eq. (30) is not satisfied the hybrid mode is stable.

In Figs. 5 through 8 the cold beam growth rate is shown as a function of beam energy for various system parameters. In all these figures the growth rate is a double peaked function of energy, $\mathscr{E} = (\gamma_0 - 1) m_0 c^2$. As has been discussed earlier, the first peak ($\gamma_0 < \gamma_{tran}$) is the unstable mode hybrid mode while the second peak ($\gamma_0 > \gamma_{tran}$) is primarily the azimuthal oscillation mode. For $\gamma_0 >> \gamma_{tran}$ the unstable mode becomes the usual negative mass instability and the growth rate is independent of the applied toroidal magnetic field. The transition between these two regions occur at $\mathscr{E} = \mathscr{E}_{tran} = (\gamma_{tran} - 1)\, m_0 c^2$ where γ_{tran} is given by Eq. (25).

In Fig. 5 the growth rate is given for various values of beam current. For $\mathscr{E} > \mathscr{E}_{tran}$ the peak growth rate scales as $I_b^{1/6}$ in agreement with Eq. (29). The growth rate for various values of applied toroidal magnetic field is given in Fig. 6. Here in accordance with Eq. (29) the peak growth rate is proportional to $B_{0\theta}^{1/2}$ for $\mathscr{E} > \mathscr{E}_{tran}$. Figure 7 gives the growth rate for various values of the major beam radius. In this figure the peak growth rate is proportional to $r_0^{-5/3}$ again in agreement with Eq. (29).

Finally, in Fig. 8 the growth rate is plotted for various values of the azimuthal harmonic number l as a function of beam energy. For $\mathscr{E} > \mathscr{E}_{tran}$ the peak growth rate is proportional to $l^{1/2}$ as given in Eq. (29).

The effects of finite wall conductivity are represented in the dispersion relation, Eq. (23), by the terms $\epsilon_{\parallel}$ and $\epsilon_{\perp}$. These terms describe the effects of wall resistivity on the azimuthal and transverse beam oscillations respectively. For a good but not perfect conductor both $\epsilon_{\parallel}$ and $\epsilon_{\perp}$ are small compared to unity. The presence of resistivity primarily affects those modes that would have been stable, if the chamber walls were perfectly conducting. For those modes which are unstable for perfectly conducting walls, resistivity introduces only slight changes in the real and imaginary parts of the frequency.

The discussion so far has been limited to monoenergetic beam distributions. As the growth rates in Figs. 5 through 8 show the instabilities for ultra high current beams are unacceptably high. However, actual beams have a spread in the canonical angular momentum (energy spread) which can substantially reduce the instability growth rate. For purposes of showing the stabilizing effects of temperature we will use a Lorentzian beam distribution,

$$g(\Delta P) = \frac{P_{th}/\pi}{\Delta P^2 + P_{th}^2}, \tag{32}$$

where P_{th} denotes the thermal spread in canonical angular momentum. The convenient feature of the Lorentzian distribution function is that

$$\frac{1}{(\Delta\omega_l)^j} = \int_{-\infty}^{\infty} \frac{d\Delta P g(\Delta P)}{(\Delta\omega_l)^j} = \frac{1}{(\Delta\omega_{0l} + i|lkP_{th}|)^j} \tag{33}$$

for any integer $j \geq 0$. With the relationship in Eq. (33) it can be seen that in the general dispersion relation of Eq. (16), or any simplified version such as Eqs. (23), the growth rate for a thermal beam is simply given by

$$\Gamma_{th} = \Gamma - lP_{th}|k| \tag{34}$$

where Γ is the corresponding growth rate for a cold beam, $P_{th} = 0$.

Using Eq. (7a) we find that the thermal spread in the canonical angular momentum P_{th} is related to the thermal spread in the beam energy $\mathscr{E}_{th} = \gamma_{th} m_0 c^2$ by

$$\gamma_{th} = P_{th}/m_0 c r_0. \tag{35}$$

Setting Γ_{th} equal to zero and using Eq. (35) we find that the thermal energy spread necessary to stabilize the instability is

$$\gamma_{th} \geq \Gamma/(lm_0 c r_0 |k|) \tag{36}$$

where Γ is the cold beam growth rate.

Figures (9) and (10) illustrate the stabilizing effects of a thermal energy spread on the negative mass/kink instability. In these figures the parameter values are $r_0 = 100$ cm, $a = 10$ cm, $r_b = 2$ cm $B_{0\theta} = 10$ kG and $I = 10$ kA and 5 kA. The fundamental azimuthal harmonic number, $l = 1$, was used in both figures. If the fundamental mode is stabilized by an energy spread so are the higher order modes. In Fig. 9 the entire growth rate spectrum is stabilized for $\gamma_{th} \approx 0.55$. For an injection energy of $\mathscr{E} = (\gamma_0 - 1)\, m_0 c^2 = 3$ MeV, a 10% energy spread is needed to insure stability against the negative mass/kink instability. Whereas in Fig. 10 we see that for a 5 kA injected electron beam of $\mathscr{E} = 3$ MeV, a 5% energy spread would be necessary to insure stability. Unlike conventional betatrons, the electron beam of an ultra high current modified betatron can tolerate a substantial energy spread without significant expansion of the beam's minor radius.

4. SUMMARY

In this paper, we have investigated the transverse and longitudinal instabilities on an intense, confined electron ring in a modified betatron field configuration. The analysis in Sec. 2 resulted in a general dispersion relation for these instabilities. Of paramount importance in the analysis was the freedom of the beam center to oscillate in a self-consistent manner within the torus. We found that the transverse beam oscillation is not only relevant for the transverse resistive wall instability but is an important aspect of the negative mass type of instability.

The dispersion relation was evaluated for various limiting cases in Sec. 3. We first investigated the transverse resistive wall instability in the modified betatron. The applied azimuthal magnetic field couples both transverse oscillation directions and substantially reduces the growth rate of the instability.

In Sec. 3b we considered the limiting case of a conventional betatron ($B_{0\theta} = 0$) and found that self field effects could be neglected provided $n_s < 1/2$. The condition that $n_s < 1/2$ is also the requirement for space charge confinement.

In Sec. 3c we considered the low current, i.e., $n_s << 1/2$, modified betatron, i.e., $B_{0\theta} \neq 0$. Our results in this limit are in agreement with those in Ref. 16.

Finally in Sec. 3d the instabilities associated with an ultra high current, i.e., $n_s >> 1$, modified betatron with $B_{0\theta} >> B_{oz}$ were studied. We found that in the modified betatron the conducting chamber walls coupled the longitudinal and transverse oscillations, resulting in a hybrid mode. The negative mass instability, which in the conventional betatron is primarily a longitudinal mode of oscillation, becomes a hybrid mode of oscillation in the modified betatron. In general, for a cold beam the hybrid mode is found to be unstable. However, stability can be achieved by a moderate amount of beam energy spread. Numerous illustrations of the functional dependence of the growth rate on the system parameters are given for a 10 kA injected electron ring.

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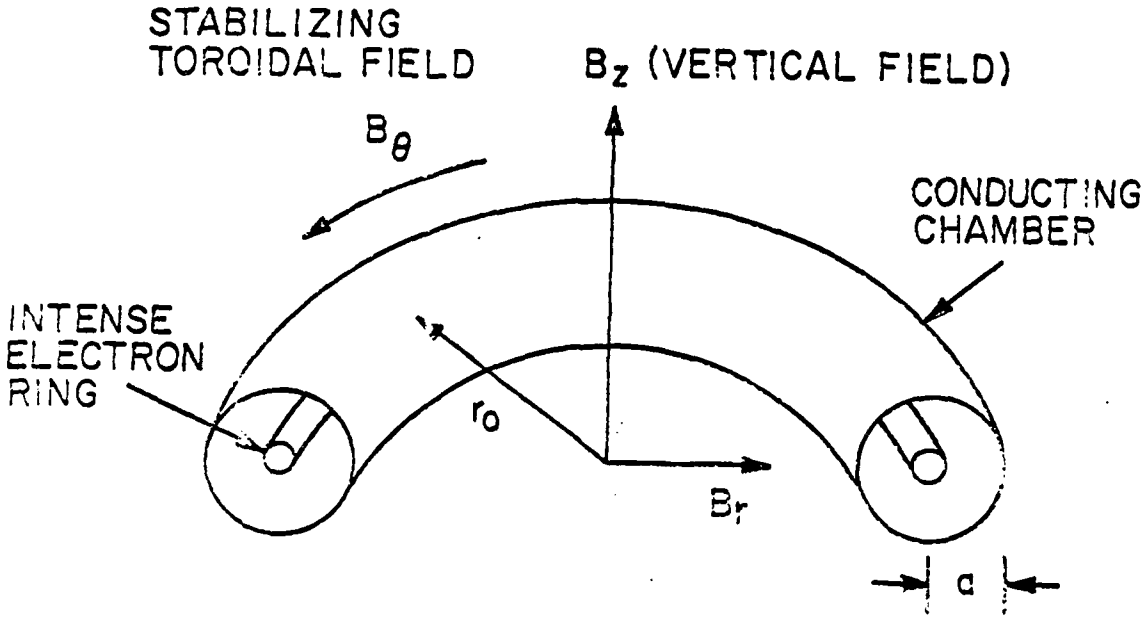


Fig. 1 — Configuration of the modified betatron electron accelerator. An intense electron ring is confined within a toroidal conducting chamber of major radius r_0 and minor radius a. The external magnetic fields consist of the nonuniform axial field B_z, the associated radial field B_r and the stabilizing azimuthal field B_θ.

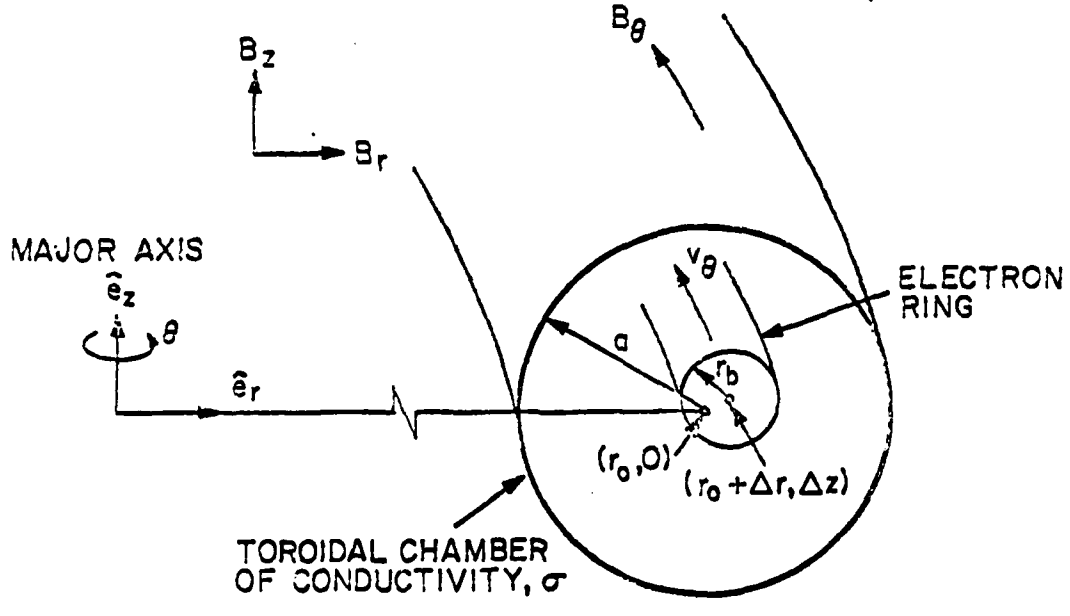
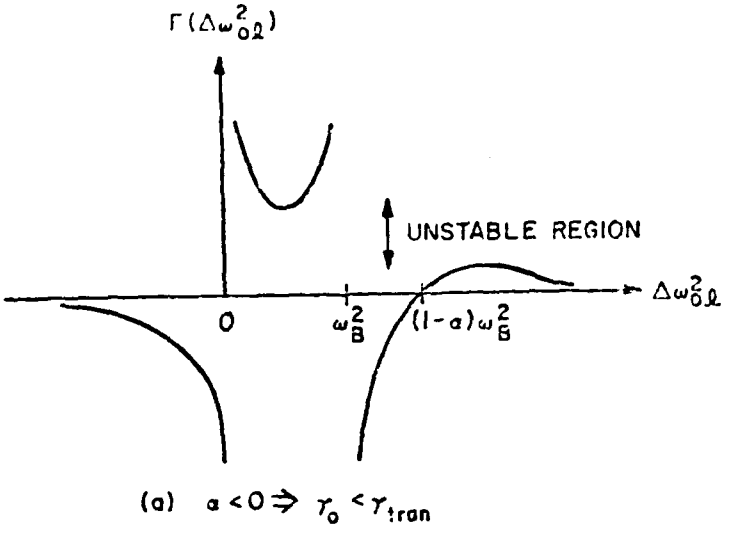


Fig. 2 — Location of the beam relative to the conducting toroidal chamber in a modified betatron accelerator. The conducting torus is centered at $(r_0, 0)$ and has a minor radius a. The beam has a circular cross-section of radius r_b and moves at a velocity v_θ. At time t and azimuthal angle θ, the center of the beam cross-section is located at $(r_0 + \Delta r, \Delta z)$.



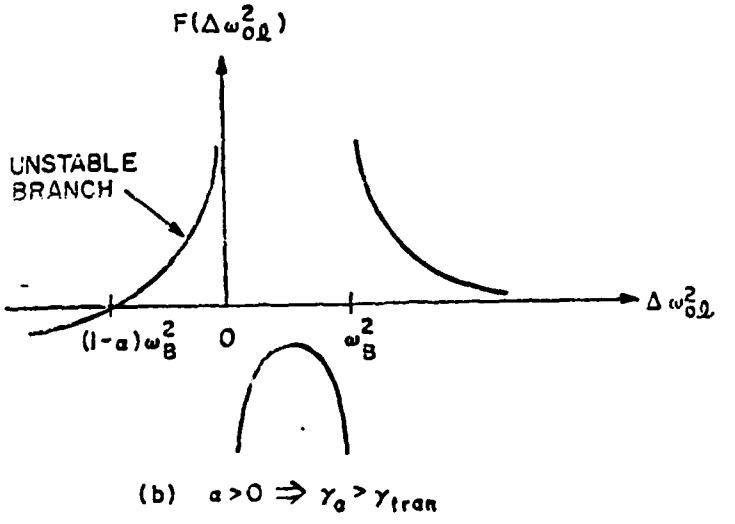


Fig. 3 — Determination of stability of the roots of the dispersion relation for a cold beam. (a) For $\alpha < 0$, i.e., $\gamma_0 < \gamma_{tran}$, the roots, $\Delta\omega^2_{0\ell}$, are unstable for intermediate values of $(\gamma_0/l\eta)^2$. (b) For $\alpha > 0$, i.e., $\gamma_0 > \gamma_{tran}$ both roots, $\Delta\omega^2_{0\ell}$, are real. One root is, however, negative resulting in instability.

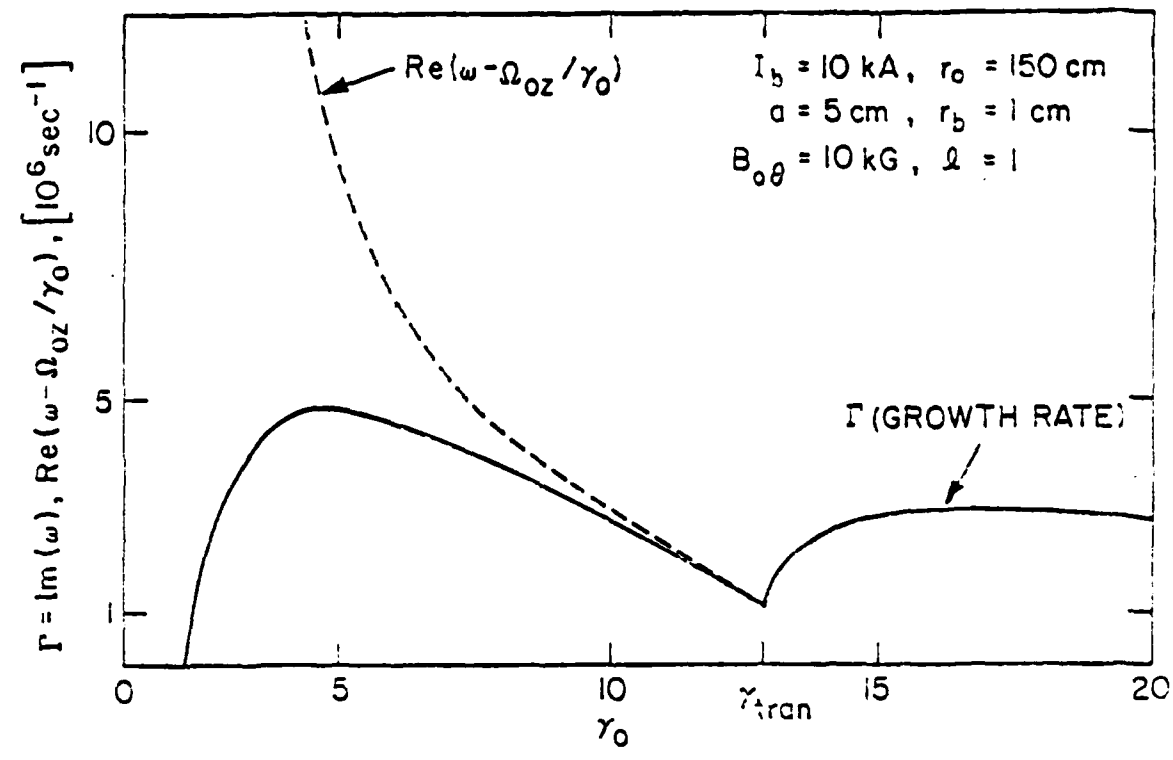


Fig. 4 — Growth rate Γ (solid curve) and real frequency shift Re $(\omega - \Omega_{0z}/\gamma_0)$ vs γ_0. For $\gamma_0 > \gamma_{tran}$ the real frequency shift is zero. The beam and betatron parameters are $I_b = 10$ kA, $r_0 = 150$ cm, $a = 5$ cm, $r_b = 1$ cm, $B_{0\theta} = 10$ kG, $n = 0.5$ and $\ell = 1$.

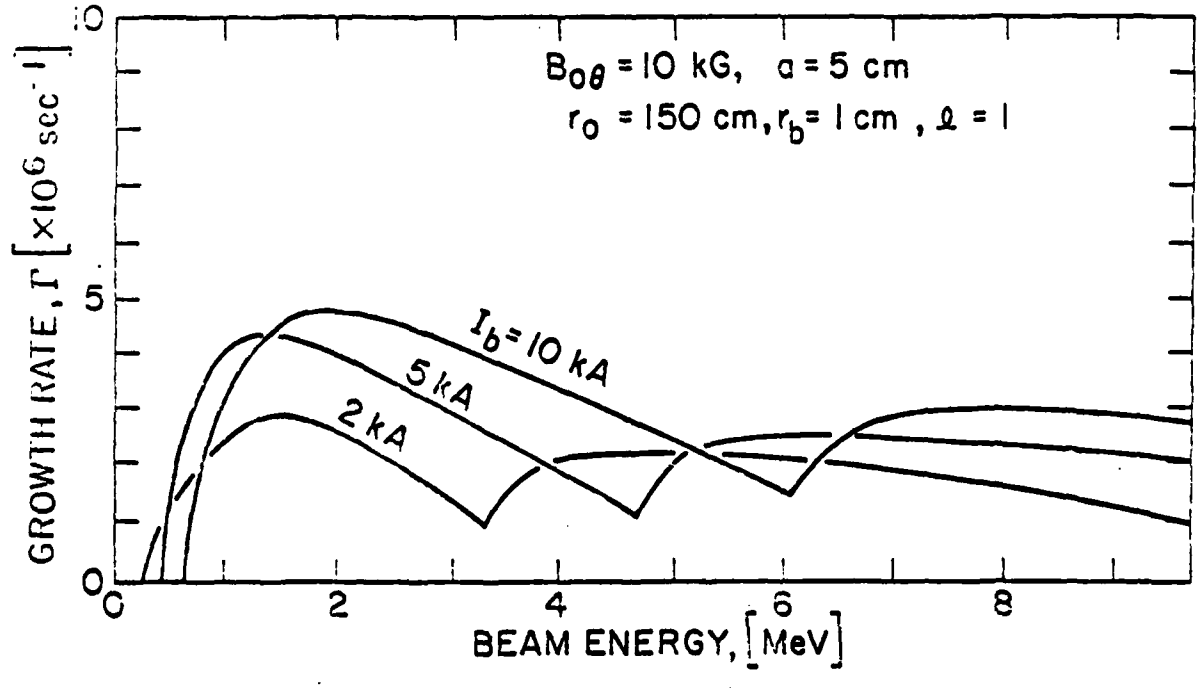


Fig. 5 — Growth rate Γ vs beam energy $E = (\gamma_0 - 1)m_0c^2$ for the parameters of Fig. 4 for various values of beam current $I_b = 2$ kA, 5 kA and 10 kA.

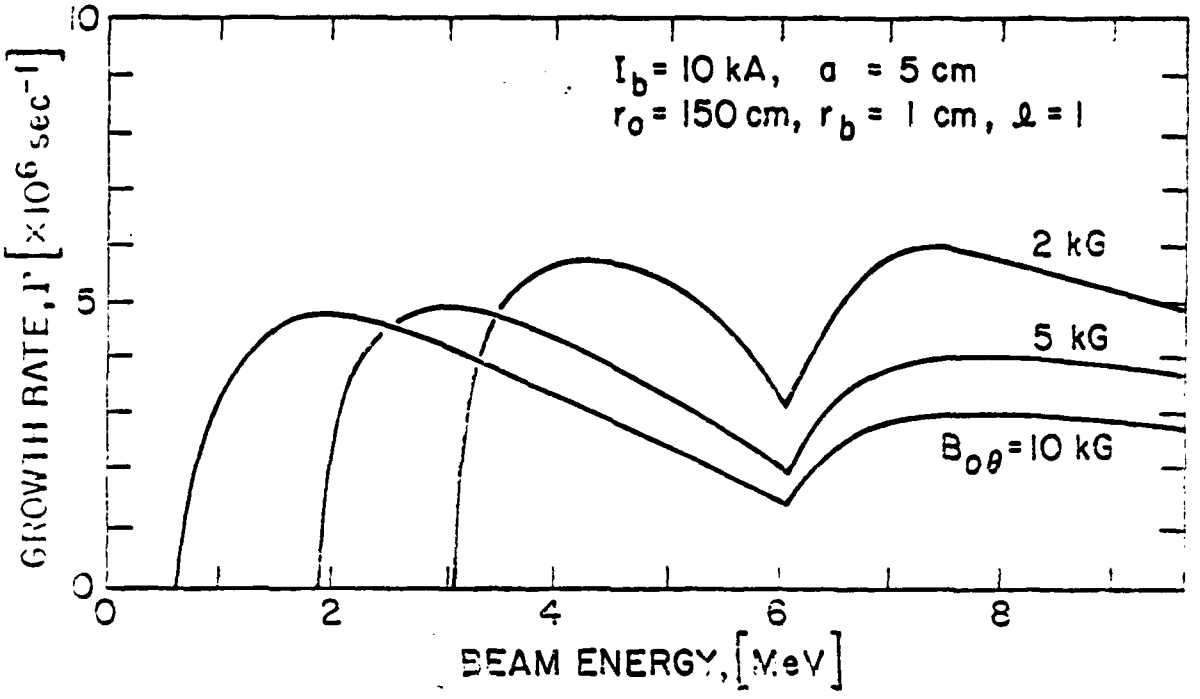


Fig. 6 — Growth rate Γ vs beam energy for the parameters of Fig. 4, for various values of the azimuthal magnetic field $B_{0\theta}$ = 2 kG, 5 kG and 10 kG.

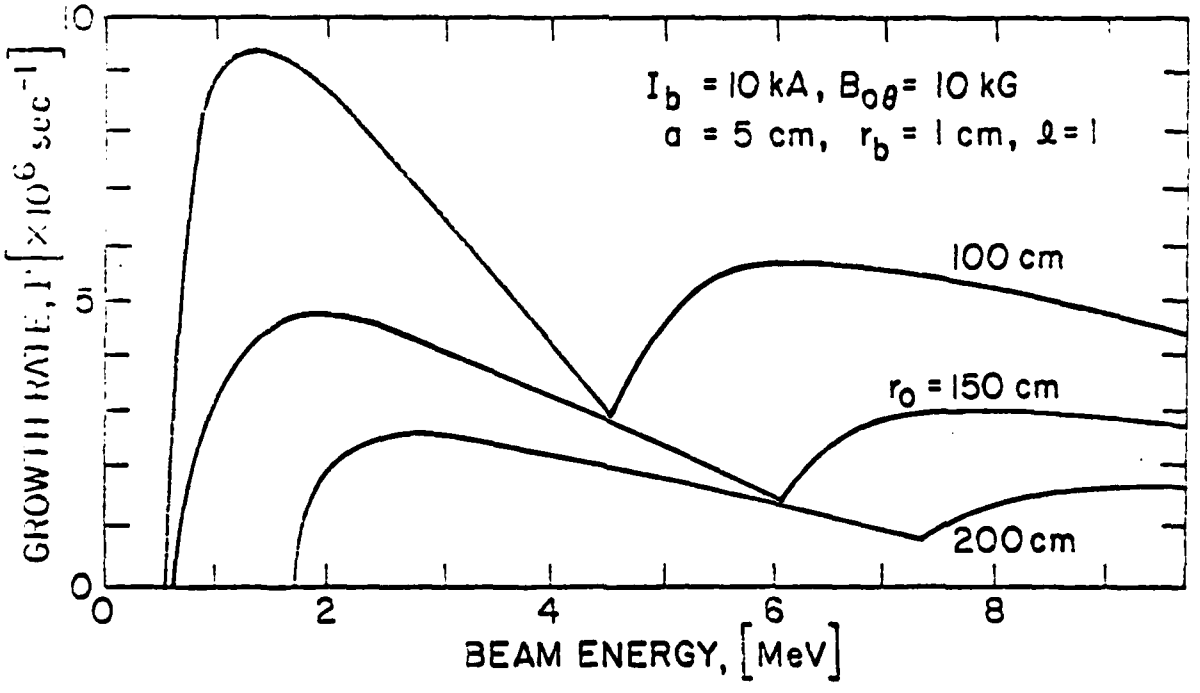


Fig. 7 — Growth rate Γ vs beam energy for the parameters of Fig. 4, for various values of the major radius r_0 = 100 cm, 200 cm and 150 cm.

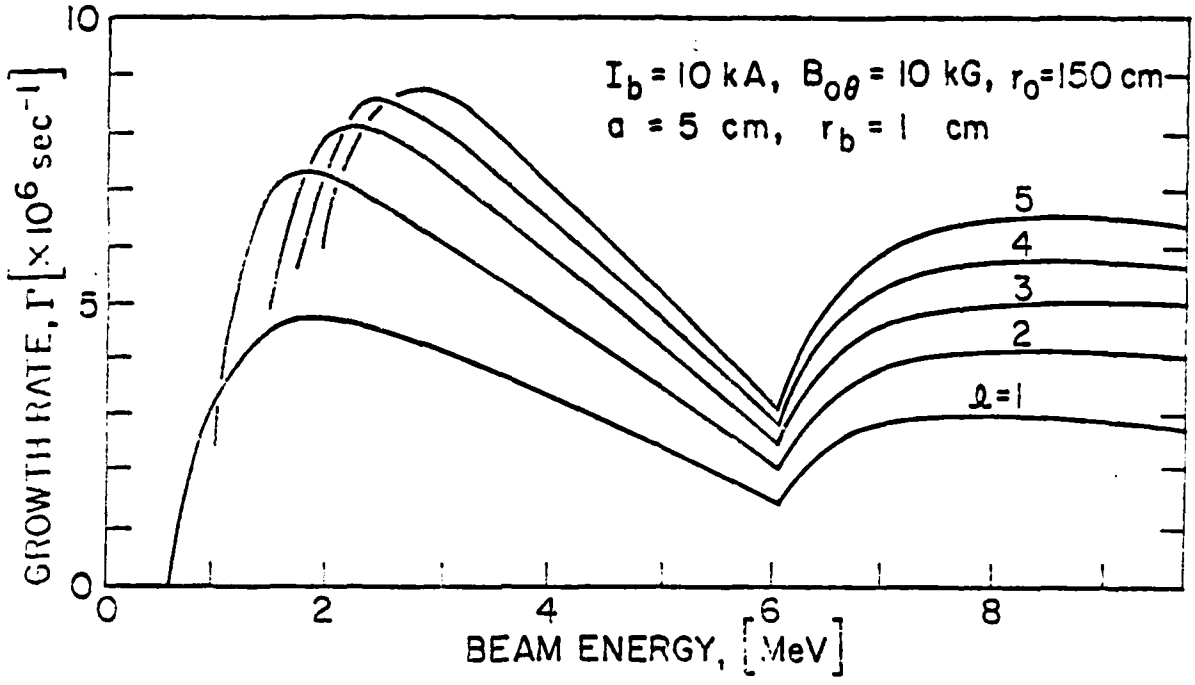


Fig. 8 — Growth rate Γ vs beam energy for the parameters of Fig. 4, for various values of the azimuthal harmonic number, l = 1, 2, 3, 4, and 5.

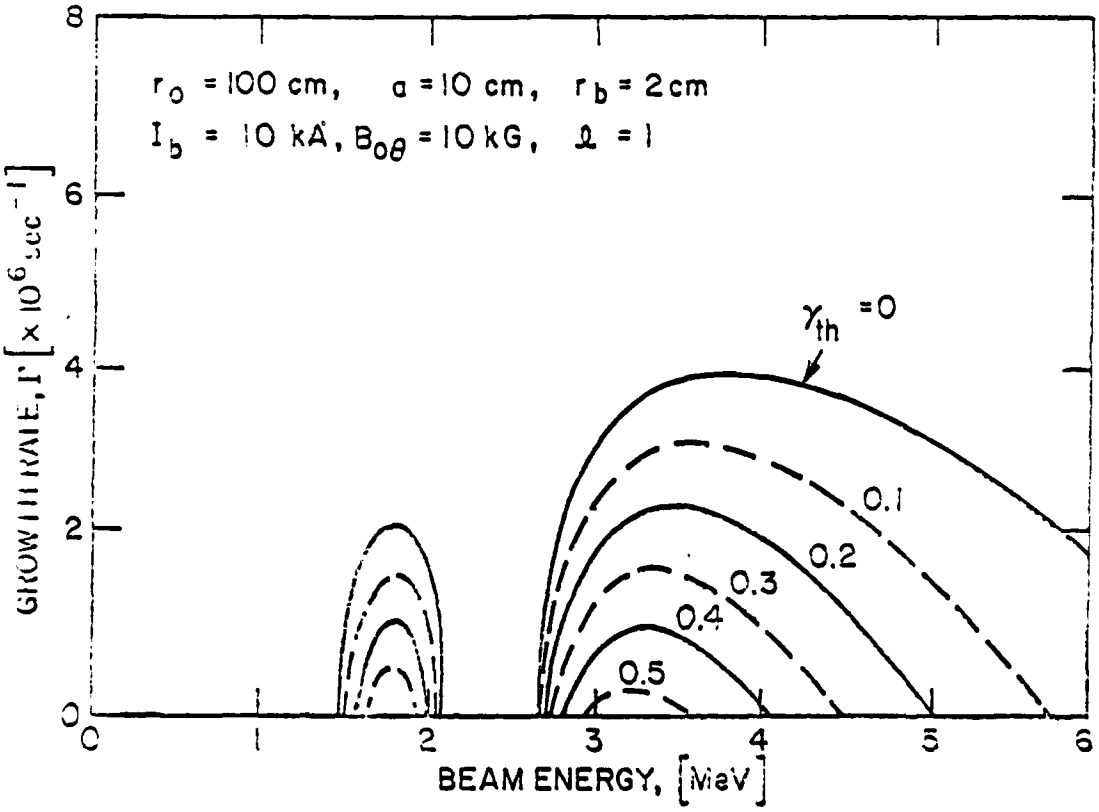


Fig. 9 — Stabilization of the instability with a beam thermal spread γ_{th} in a modified betatron with r_0 = 100 cm, a = 10 cm, r_b = 2 cm, I_b = 10 kA, $B_{0\theta}$ = 10 kG, n = 0.5 and l = 1. The distribution of the beam is taken to be Lorentzian in $\Delta P = m_0 c r_0 (\gamma - \gamma_0)$.

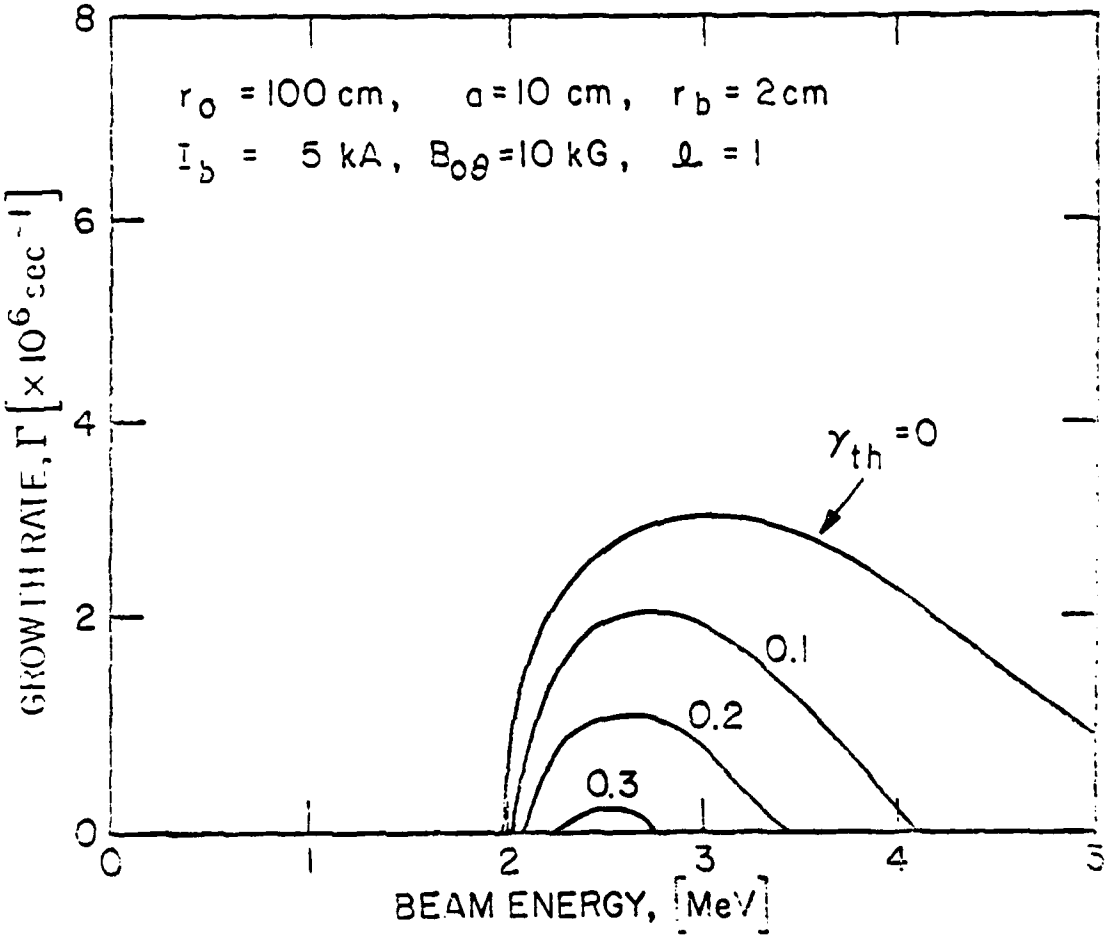


Fig. 10 — Stabilization of the instability with a thermal spread in a modified beatron with the parameters of Fig. 9 but with a current I_b = 5 kA.