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Properties of gravity waves at the interface between two unbounded fluids of different densities in relative motion are examined. The conditions which limit the existence of steady waves are determined.

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FINITE-AMPLITUDE INTERFACIAL WAVES

P. G. Saffman

1. INTRODUCTION.

We consider gravity waves at the interface between two uniform, unbounded fluids of different densities in the presence of a current or relative horizontal velocity U. The fluids are supposed to be immiscible, incompressible and inviscid, and the motion is assumed to be irrotational. We are concerned with the properties and existence of finite amplitude two-dimensional, periodic waves of permanent form which propagate steadily without change of shape. By two-dimensional, we mean that the flow field depends only on the horizontal direction of propagation, which will be the x-axis, and the vertical y-direction. In the field of surface gravity waves, which is the limit of the present study when the density of the upper fluid is zero, it has been found recently that three-dimensional waves of permanent form exist and are observed experimentally (see e.g. [5]). It is expected that such waves will also exist and be important for interfacial waves, but they will not be considered in the present work.

For the purpose of calculating steady waves, there is no loss of generality in taking the speed of propagation c parallel to the current U, as an arbitrary constant transverse velocity may be linearly superposed on any

two-dimensional steady wave without affecting *its properties*
(the stability characteristics would, however, be
affected). The wave can be reduced to rest by choosing a
frame of reference moving with the wave. The problem is
then to calculate steady irrotational solutions of the Euler
equations which satisfy continuity of pressure across a
common streamline. It follows from dimensional analysis
that apart from scaling factors all flow variables will
depend upon three dimensionless parameters:

$$\frac{h}{L} \ , \quad \frac{\rho_2}{\rho_1} \ , \quad \frac{\rho_2 U^2}{\rho_1 g L} \ , \tag{1.1}$$

where h is the height of the wave defined as the vertical
distance between crest and trough, L is the wavelength
(the horizontal distance over which the flow field repeats
itself which in the present work will be the distance
between crests), ρ_2 and ρ_1 are the densities of the
upper and lower fluid respectively, and g is the
acceleration due to gravity. For example, the speed of the
waves is given by

$$c = (gL/2\pi)^{1/2} C\left(\frac{h}{L}, \ \frac{\rho_2}{\rho_1}, \ \frac{\rho_2 U^2}{\rho_1 g L}\right) \tag{1.2}$$

where C is a dimensionless function of its arguments. For
surface waves, where $\rho_2 = 0$ and there is dependence on
only one parameter, namely h/L, it is known that many
interesting and unexpected phenomena exist, especially when
the wave steepness becomes large. When there is dependence
on three parameters, it is to be expected that many more
phenomena are likely. However, in the absence of exact
solutions for large h/L, it is a highly non-trivial task
to search a three-dimensional parameter space. The results
to be presented below are limited to those phenomena which
seem currently to be of the most interest.

In contrast to the voluminous work on surface waves,
relatively little seems to have been done on interfacial
waves of permanent form, and that work seems to have been
confined to the case of zero current, i.e. U = 0. Tsuji

and Nagata [7] calculated Stokes type expansions to order
$(h/L)^5$, and Holyer [3] used the computer to compute the
coefficients in such an expansion to order $(h/L)^{37}$, and
then used Padé approximants to estimate the behavior for
large h/L. We are not aware of any work for waves with
current.

For the mathematical formulation, there is no loss of
generality in taking $g = 1$, $L = 2\pi$, and $\rho_1 = 1$. The
mathematical problem is to determine the x-periodic velocity
potentials and stream functions, for the lower and upper
fluid respectively, which satisfy Laplace's equation and are
harmonic conjugate pairs, so that at the unknown interface
$y = Y(x)$,

$$\psi_1(x,Y(x)) = 0, \quad \psi_2(x,Y(x)) = 0 , \qquad (1.3)$$

$$\frac{1}{2} (\nabla\phi_1)^2 + Y(x) + b = \frac{1}{2} \rho_2(\nabla\phi_2)^2 + \rho_B Y(x) . \qquad (1.4)$$

In general, $\rho_B = \rho_2$, but we allow for the possibility of
Boussinesq waves (in which the inertia of the two fluids is
the same and density differences only matter when multiplied
by g) by setting $\rho_2 = 1$ and $\rho_B = 0$. Surface tension is
neglected throughout. The quantity b is the Bernoulli
constant, which by suitable choice of the origin of pressure
may be set equal to zero in the lower fluid. Infinitely far
from the interface, we have

$$\phi_1 \sim -Cx, \quad \phi_2 \sim (U - c)x . \qquad (1.5)$$

The vertical origin is set by requiring that the mean
elevation of the interface is zero and the horizontal origin
can be fixed by placing the crest at $x = 0$. This problem
now appears to be free of arbitrary constants and the wave
is determined by the crest to trough height h. It is
expected that isolated families of solutions exist in
connected regions in (h, ρ_2, U) space, although this does
not yet appear to have been proved.

One question of considerable interest is the domain of
parameter space in which solutions exist. Suppose that we
consider a fixed value of ρ_2 and vary h and U. It is
found that as U increases with h kept constant the

system of equations describing steady solutions fails to
have a solution, even though the 'limiting' wave profile is
smooth and exhibits no singular properties. For $U > 0$
there are, when solutions exist, at least two physically
distinct waves corresponding to the two wave speeds for
propagation with and against the current. As U increases,
the wave propagating against the current is 'entrained' by
the current and at a certain value of U, which depends
on h and ρ_2, the two waves become identical and for
larger U there are no real solutions of the equations.
Mathematically, this is like the disappearance of roots of a
quadratic). We shall term this factor which limits
existence a 'dynamical limit'.

The second factor is what we term a 'geometrical
limit'. The mathematical formulation remains well-behaved
but the solutions cease to make physical sense as the wave
profiles cross themselves. This occurs for fixed U and
increasing h. Examples of this phenomenon are found in
pure capillary and capillary-gravity waves [2,1] for which
the wave profile crosses itself at a critical value of h.
If $U \neq 0$, this limit is going to be different for the two
solutions of waves moving with and against the current. In
the case of surface waves, this limit corresponds to a 120°
cusp. It is easy to see that except for two special cases
(see §4), this cannot happen for interfacial waves. Holyer
[3] identified the geometrical limit for $U = 0$ with the
existence of a vertical tangent. We shall present evidence
that waves can exist with a vertical tangent and significant
overhang, and the evidence indicates that the geometrical
limit is associated with the wave crossing itself when it is
sufficiently high for $U > 0$.

2. WEAKLY NONLINEAR WAVES.

The properties of weakly nonlinear steady waves may be
obtained by using the Stokes expansion in which all
variables are expanded as power series in h/L. However,
the algebra can be simplified somewhat by using Whitham's
variational approach. Proceeding in the usual manner, one
finds after some algebra that the average Lagrangian is

$$L = \frac{1}{4} (\rho_B - 1)(\frac{1}{4} h^2 + a_2^2)$$

$$+ \frac{h^2}{16k} [\omega^2 + \rho_2(Uk-\omega)^2] - (\frac{a_2^2}{2k} + \frac{kh^4}{256})[\omega^2 + \rho_2(Uk-\omega)^2]$$

$$- \frac{h^2 a_2}{16} [\omega^2 - \rho_2(Uk-\omega)^2] + o(h^6) \qquad (2.1)$$

for the wave with interface shape

$$Y(x) = \frac{h}{2} \cos(kx - \omega t) + a_2 \cos 2(kx - \omega t) \qquad (2.2)$$

The value of a_2 is found from $\partial L/\partial a_2 = 0$ to be

$$a_2 = \frac{h^2}{8(1 - \rho_B)}[c^2 - \rho_2(U - c)^2] + o(h^4) ,$$

where $C = \omega/k$ is the phase speed. The dispersion relation for the weakly nonlinear wave then follows from $\partial L/\partial h = 0$:

$$c^2 + \rho_2(U - c)^2$$

$$= (1 - \rho_B)[1 + \frac{h^2}{8} (\frac{2c^2}{1 - \rho_B} - 1)^2 + \frac{h^2}{8}] + o(h^4) . \qquad (2.4)$$

For $U = 0$, the values of C agree with those in [7]

The values of the energy, momentum and action densities and fluxes follow from the expression (2.1) for L in the usual way. In particular, the total energy density E is given by

$$E = kCL_\omega - L . \qquad (2.5)$$

It is to be noted that for $U > 0$, the energy is measured relative to the energy of the uniform state with a flat interface. Negative energies may therefore exist and mean that the energy of the state with waves is less than that of the undisturbed flow.

It follows from the dispersion relation (2.4) that for linear waves $(h \to 0)$ and given values of ρ_2 and U, there are two solutions corresponding to the two roots of the quadratic equation for C in terms of ρ_2 and U. We denote these two solutions by C_+ and C_-, where

$C_+ > C_-$. For the linear case, steady solutions cease to exist when U exceeds a critical value U_{c0} given by

$$U_{c0} = [(1 + \rho_2)(1 - \rho_B)/\rho_2]^{1/2} \qquad (2.6)$$

for which the two wave speeds are equal with the value $C_+ = C_- = \rho_2 U_{c0}/(1 + \rho_2)$.

The values of C_+ and C_- are

$$\rho_2 U \pm [\rho_2^2 U^2 - (1 + \rho_2)(\rho_2 U^2 - 1 + \rho_B)]^{1/2} . \qquad (2.7)$$

For $U = 0$, the values are equal and opposite. As U increases, the speed of the wave propagating with the current originally increases but eventually decreases. The speed of the wave propagating against the current increases monotonically (in the algebraic sense), becomes zero when $U = [(1 - \rho_B)/\rho_2]^{1/2}$ and then increases to equal C_+ when U is given by (2.6). According to the linear approximation, the energy density E equals $\frac{1}{8} h^2 [c^2(1+r) - rCU]$, and it is interesting that the energy becomes negative when the direction of the C_- waves changes.

For finite amplitude waves, the two solutions corresponding to C_+ and C_- waves continue into two families of solutions marked by wave speeds $C_+(h, \rho_2, U)$ and $C_-(h, \rho_2, U)$. For any given value of h and ρ_2, there will again be a critical current U_c beyond which steady solutions no longer exist. For the weakly nonlinear approximation, this value is given by

$$U_c = U_{c0} \left[1 + \frac{h^2}{4} \frac{(1 + \rho_2^2)}{(1 + \rho_2)^2} \right]^{1/2} . \qquad (2.9)$$

It is noteworthy that increasing h increases U_c.

3. NUMERICAL METHODS.

For values of h that are not small, it is necessary to employ numerical methods. Three different techniques were employed. The first was to compute in physical space, i.e. the interface, potentials and stream function were expanded as Fourier series in x with coefficients which

are exponential in y. The series were truncated to N
modes and the boundary conditions were then satisfied at
N + 1 equally horizontally spaced points on the
interface. This procedure gives 3N + 4 equations for
3N + 4 unknowns. These equations were solved by Newton's
method, using continuation in either U or h to give the
first guesses. Note that this formulation is essentially
equivalent to calculating numerically the coefficients of
the Stokes expansion as done in [3].

The second method used the potential and stream
function as the independent variables and expands the
physical coordinates as series in these. The boundary
conditions are now satisfied at equally spaced values of the
velocity potential and the resulting system of 3N + 3
equations in 3N + 3 variables, the expansions being
truncated to N modes, was also solved by Newton's method
with continuation in U and h employed to give a first
guess.

The third method used a vortex sheet representation in
which the unknowns are the shape of the interface and the
dipole strength of the equivalent double layer. This gives
a nonlinear integrodifferential equation, which was solved
by discretization and collocation, the resulting system of
nonlinear equations again being solved by Newton's method
with continuation.

For details, see [4,5]. All methods worked extremely
well for small values of h/L, which generally meant
h < 0.6, with some dependence on ρ_2 and U. (With our
scaling, the surface wave of greatest height has h = 0.89).
The first method was the first to fail as h increased. It
is of course clear that this approach of working in physical
space must fail when the wave becomes very steep, but the
failure, marked by the apparent failure of the Fourier
series to converge, seemed to be due to other causes. What
actually happened was that the singularities of the analytic
continuation of the lower velocity potential, say, into the
upper half plane moved down below the crest. In this case,
the expansion of the velocity potential would have to
diverge near the crest, even though the solution was

perfectly well behaved and physically meaningful. This
difficulty would not affect the other two methods which were
used for values of h up to 1.2 for various values of ρ_2
and U. For large values of h, 100 modes were used in the
second method and this seemed adequate except for the
largest h. The vortex sheet method with 65 intervals was
employed for this case. This method offers in principle the
advantage of being able to concentrate points near regions
of high curvature, although this was not done.

The accuracy of the calculations was checked by
comparing the results of the somewhat different methods with
each other in regions of apparent validity and by performing
the usual tests of internal consistency by investigating the
dependence on number of retained modes. The calculations
were carried out on a PRIME 750 and the CRAY-1 at NCAR.

4. A SPECIAL CLASS OF SOLUTIONS.

It is interesting to note that a special class of
solutions exist which are simple transformations of the
well-known surface permanent wave solutions, which have been
extensively studied both numerically and theoretically by
many authors. For each value of ρ_2, these solutions
describe the shape of the interface for the C_+ case when

$$C_+ = U = (1 - \rho_B)^{1/2} C_s(h) \tag{4.1}$$

where $C_s(h)$ is the wave speed of the surface wave of
permanent frm for the given wave height h. Since $C_+ = U$,
the upper fluid is stagnant in the wave-fixed coordinates.
The dynamic boundary condition for the motion in the lower
fluid then becomes that for surface waves with a reduced
gravity $g(1 - \rho_B)^{1/2}$. The velocities and wave speed are
therefore those of the surface wave multiplied by the factor
$(1 - \rho_B)^{1/2}$.

For the C_- branch, special solutions exist with

$$C_- = 0, \quad U = [(1 - \rho_B)/\rho_2]^{1/2} C_s(h) . \tag{4.2}$$

In this case, the lower fluid is stagnant and the dynamic
boundary condition on the motion of the upper fluid is that
with a reduced upside down gravity. The wave profiles are

inverted surface waves, with negative gravity multiplied by the factor $[(1 - \rho_B)/\rho_2]^{1/2}$.

These special solutions have geometrical limits when $h = 0.892$, where the waves have a corner at the crest for the C_+ wave, and a corner at the trough for the C_- wave, with an interior angle of $120°$. However, these special geometrical limits are only for the case when one of the fluids is moving with the wave. In general, it is expected (see below) that the geometrical limit is associated with the wave surface crossing itself.

5. RESULTS.

The equations have been solved numerically for various values of h, $r = \rho_2/\rho_1$, and U. The existence of the dynamical limit was confirmed, and it was found that the weakly nonlinear approximation (2.9) is a good approximation for values of h up to 0.6. A typical set of results is shown in figure 1. These results are for $r = 0.5$ and $h = 0.6$, and show the wave speeds C, total energies E, and kinetic energies T for both the $+$ and $-$ waves as functions of the current velocity U. The existence of the dynamical limit where $C_+ = C_-$ is clearly demonstrated.

One feature of remarkable interest is the existence of a region of negative energy. This implies that there will be a range of parameters in which the energy of the state with finite amplitude waves is less than that with the same current and a flat surface. Spontaneous generation of such flows is then a definite possibility. The computed results and linear analysis suggest that negative energies appear when C_- is zero and continue for values of U up to that for which the dynamical limit is reached. This aspect of the solutions needs to be explored further in detail.

The geometrical limit or the shape of the wave of greatest height has been addressed for the case of zero current. Solutions were obtained for three values of r (1.0, 0.9, 0.1), using the vortex sheet method as this seemed to provide the best resolution when the waves are large. Values of the wave speed C for $+$ waves are shown in figure 2. For the larger values of r, it was possible to calculate solutions with vertical slope and the shapes
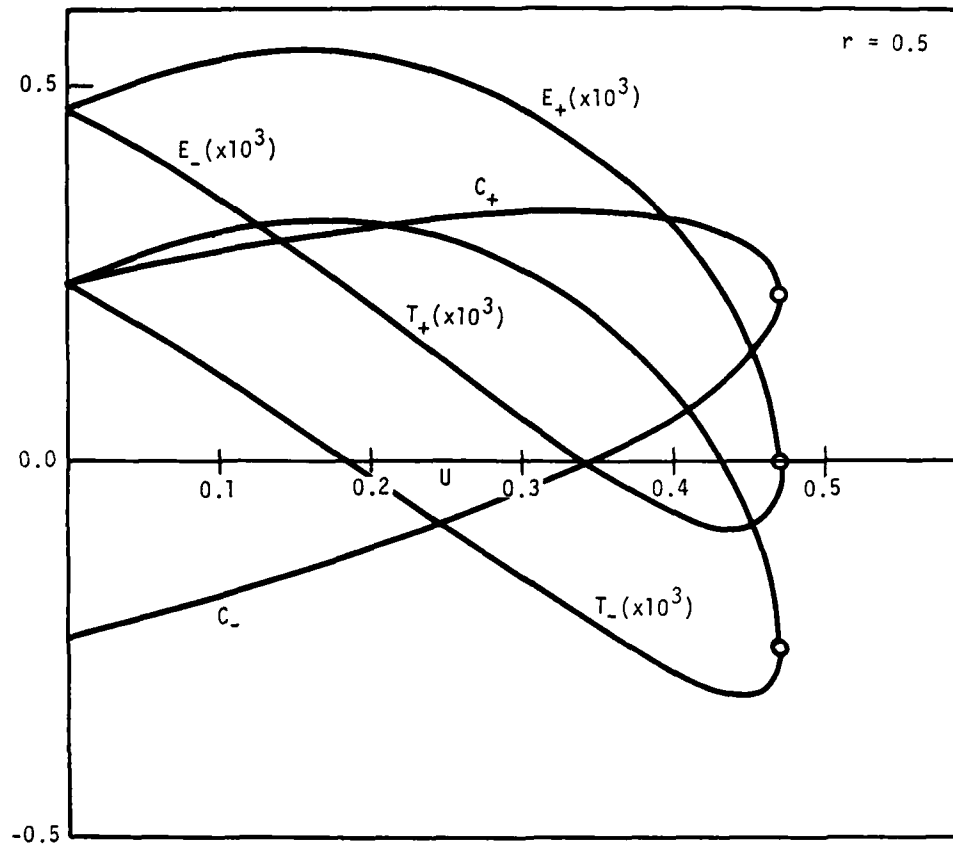
Figure 1. Properties of interfacial waves as functions of current velocity U for a fixed wave height h = 0.6 and density ratio r = 0.5. Wave speed C, total energy E, and kinetic energy T are shown for both branches.
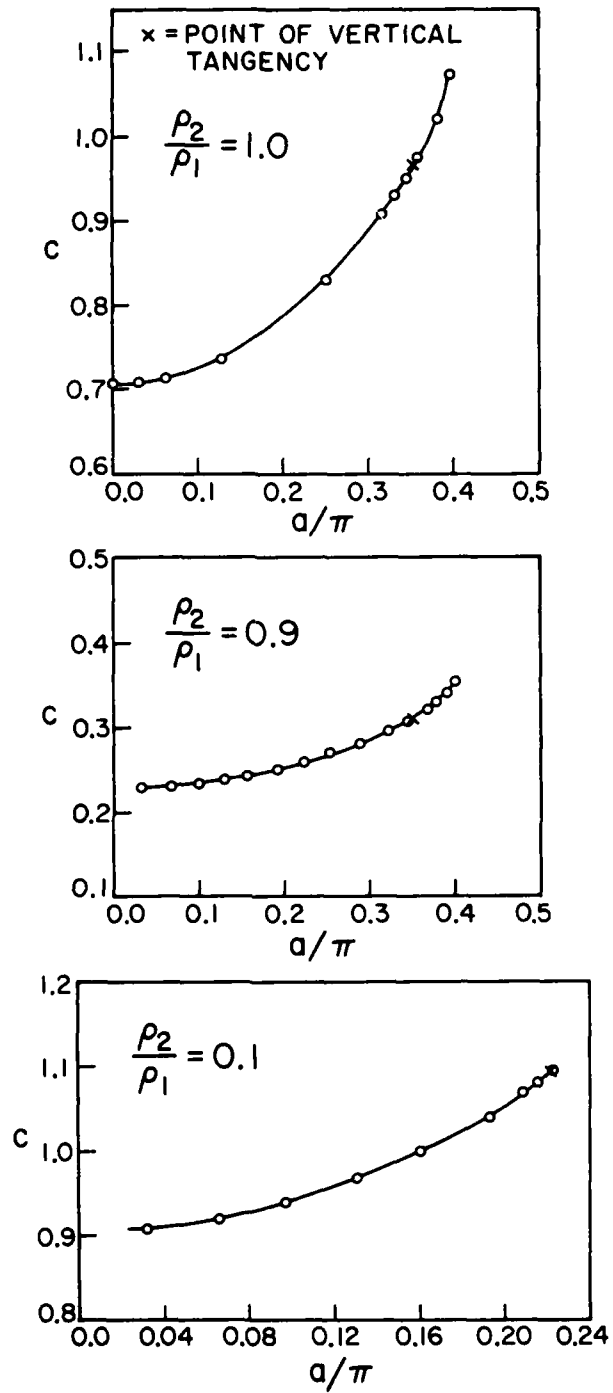
Figure 2.   Phase speed C vs wave steepness $a/\pi = h/L$
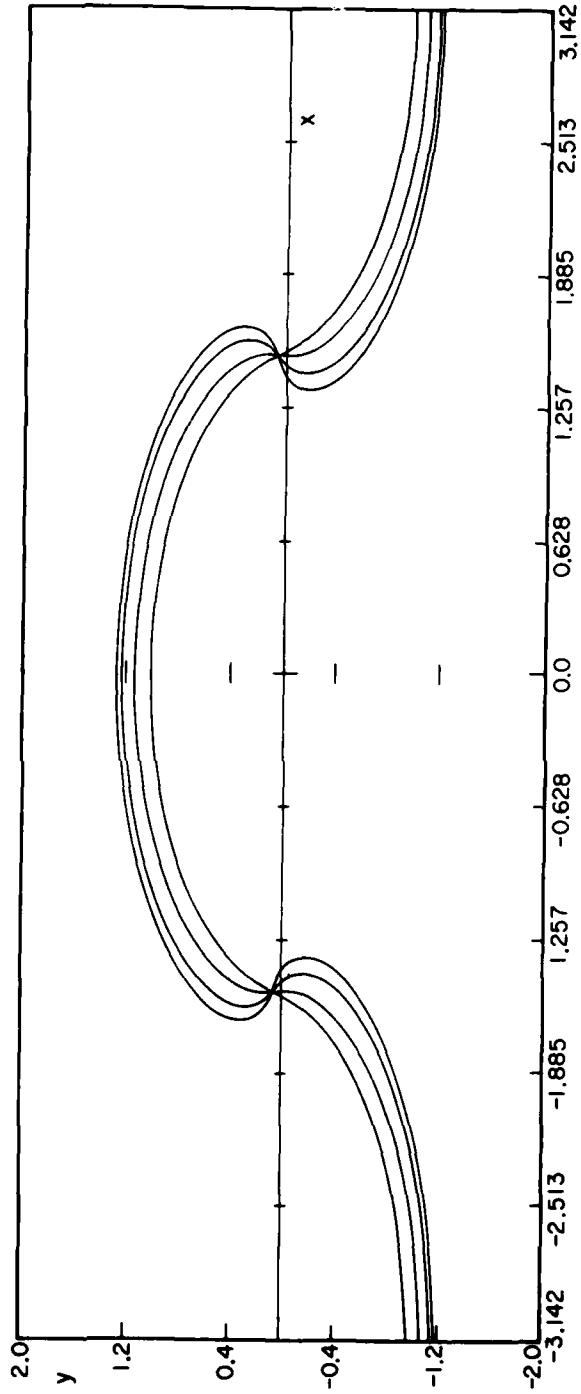for $r = 0.1, 0.9, 1.0$.   x   denotes point of
vertical tangency.

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Figure 3. Profiles of steady interfacial waves for r = 0.9.
The x-axis is the mean level.
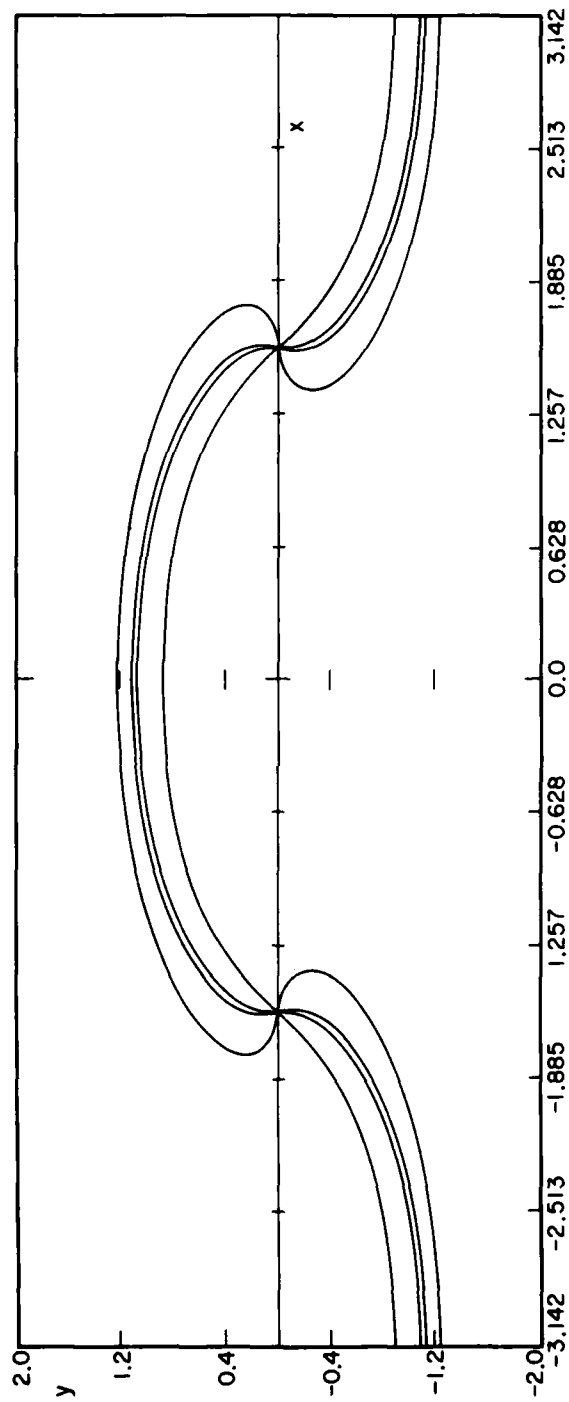
Figure 4. Profiles of steady interfacial waves for r = 1.0. The x-axis is the mean level. These waves are symmetrical about y = 0.

are shown in figures 3 and 4 for $r = 0.9$ and $r = 1.0$,
respectively. These demonstrate clearly the existence of
waves of permanent form with a substantial overhang region
in which heavy fluid lies on top of light fluid. It is
interesting to note that the fluid particles on the
interface between the points of vertical tangency in the
overhang region are moving faster than the wave. For the
smallest value of the density ratio, we did not have
sufficient resolution to distinguish the wave shape near the
geometrical limit. This difficulty is to be expected, since
the smaller the density ratio, the closer the geometrical
limit will be to the 120° cusp and the smaller the size of
the overhang region.

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