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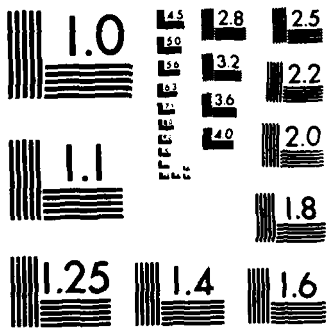

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LONGITUDINAL VIBRATIONS OF RODS OF FINITE LENGTH WITH RADIAL DEFORMATION

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US ARMY ARMAMENT RESEARCH AND DEVELOPMENT CENTER
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20. ABSTRACT (CONT'D)

the set of two partial differential equations is recapitulated together with appropriate boundary conditions. For vibration problems, two sets of eigenvalue problems are formulated to satisfy the simultaneous partial differential equations and the homogeneous boundary conditions. Suitable parameters are defined to describe the dispersion relations. These dual eigenvalue matrix equations are then solved numerically. For an infinite rod, a dispersion relation of frequency versus wave number which contains an imaginary branch has been obtained. The free vibration problem of a fixed-fixed Mindlin-Herrmann rod has been solved. The numerical values of six (6) lowest frequencies, the associated wave numbers and mode shapes are tabulated for three different slenderness ratios.

ACKNOWLEDGEMENTS

The authors wish to thank Mr. Kuen Ting for his help in obtaining the numerical results.

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LIST OF SYMBOLS

a	:	radius of the rod
w,u	:	displacements in z and r-directions, respectively
λ,μ	:	Lamé elastic constants
ρ	:	density
k,k_1	:	correction factors
γ	:	wave number
ω	:	frequency
L	:	wave length
c	:	phase velocity ($= \omega/\gamma$)
c_s	:	shear wave velocity ($= \sqrt{\mu/\rho}$)
ℓ	:	length of the rod
ω^*	:	dimensionless frequency
γ^*	:	dimensionless wave number
a^*	:	slenderness ratio
β	:	coefficients of the associated eigenfunctions for w
$\bar{\beta}$	:	coefficients of the associated eigenfunctions for u

INTRODUCTION

This report investigates the free vibration problem of a rod of finite length which models the axial as well as radial modes of motion. The formulation is based on a rod model of infinite length developed by Mindlin and Herrmann in 1951 (ref 1). The differential equations, boundary conditions, and the associated energy principle have all been developed in the original paper as well as dispersion relations (relations between wave number and velocity) for a rod of infinite length. Subsequently, Herrmann presented a solution formulation for free and forced vibration of rods of finite length (ref 2). No numerical results were given there, however. Miklowitz has obtained formed solutions to forced vibration problems of rods of semi-infinite and finite length by the use of Laplace transforms (ref 3). He has also obtained numerical results for the earlier formulations (ref 4). Nevertheless, no specific numerical data were provided for free vibrations of a finite rod. Since the free vibration information is fundamental to all linear and nonlinear wave propagation phenomena, our first step is to study this aspect for a Mindlin-Herrmann rod of finite length.

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[1]R. D. Mindlin and G. Herrmann, "A One-Dimensional Theory of Compressional Waves in an Elastic Rod," Proceedings of the First U.S. National Congress of Applied Mechanics, 1950, pp. 187-191.
[2]G. Herrmann, "Forced Motions of Elastic Rods," Journal of Applied Mechanics, September 1954, pp. 221-224.
[3]J. Miklowitz, "Travelling Compressional Waves in an Elastic Rod According to the More Exact One-Dimensional Theory," Proceedings of the Second U.S. National Congress of Applied Mechanics, June 1954, ASME, 1955, pp. 176-186.
[4]J. Miklowitz, "The Propagation of Compressional Waves in a Dispersive Elastic Rod," June 1957, pp. 231-239.

First, the governing partial differential equations and pertinent variables of the approximate theory of the rod are recapitulated. With the original equations, an eigenvalue problem based on a wave motion is formulated. The characteristic equation of this eigenvalue problem is shown to be identical to the one in Reference 1 in terms of phase velocity and wave number. However, the same equation can be written in terms of dimensionless frequency and wave number which is also allowed to assume imaginary values. Then, the dispersion relation of frequency versus wave number reveals another imaginary branch similar to the one corresponding to Timoshenko beam theory (ref 5). This new dispersion relation is shown here together with other results identical to the original paper by Mindlin and Herrmann. Next, solutions for finite rods are treated. As a first example, we take a rod with fixed-fixed ends. In order to satisfy these end conditions, another eigenvalue problem is established. This produces a characteristic equation for those particular frequencies so that free vibrations under the given boundary conditions are possible. The numerical results are presented; and among infinite number of discrete frequencies, several of the first lowest are recorded. Corresponding to each, the various modes (consisting of sine, cosine, hyperbolic sine, and cosine functions) and relative amplitudes are also presented in the last section.

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[1]R. D. Mindlin and G. Herrmann, "A One-Dimensional Theory of Compressional Waves in an Elastic Rod," Proceedings of the First U.S. National Congress of Applied Mechanics, 1950, pp. 187-191.

[5]S. P. Timoshenko, "On the Correction for Shear of the Differential Equation for Transverse Vibrations of Prismatic Bars," Philosophical Magazine, Series 6, Vol. 41, 1951, pp. 744-746.

GOVERNING EQUATIONS

It will be convenient here to recapitulate the full set of equations of the Mindlin-Herrmann rod model (ref 1). The displacement equations of motions are:

$$\left.\begin{aligned} a^2(\lambda+2\mu)w'' + 2a\lambda\mu' + 2aZ &= \rho a^2\ddot{w} \\ a^2k^2u'' - 8k_1{}^2(\lambda+\mu)u - 4ak_1{}^2\lambda w' + 4aR &= \rho a^2\ddot{u} \end{aligned}\right\} \qquad (1)$$

where $w = w(z,t)$ is the displacement component in z-direction; $u = u(z,t)$ so that $ru(z,t)/a$ is the displacement component in r-direction; z, r denote the axial and radial coordinates, respectively; t, the time. A prime (') denotes partial differential with respect to z and a dot (·), the same with respect to t. λ, μ are Lamé elastic constants, ρ is the density of the rod material, and a is the radius. R and Z denote the radial and axial components, respectively, of the traction on the cylindrical surface of the rod. k and k_1 are correction factors. They are introduced so that the dispersion relations from this approximate theory will better match the exact solution of an infinite rod. Their usage is discussed fully in the original paper by Mindlin and Herrmann (ref 1).

The stress-displacement relations are

$$\left.\begin{aligned} 2P_r = 2P_\theta &= k_1{}^2[2a(\lambda+\mu)u + a^2\lambda w'] \\ 2P_z &= 2a\lambda u + a^2(\lambda+2\mu)w' \\ 4Q &= k^2a^2\mu u' \end{aligned}\right\} \qquad (2)$$

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[1]R. D. Mindlin and G. Herrmann, "A One-Dimensional Theory of Compressional Waves in an Elastic Rod," Proceedings of the First U.S. National Congress of Applied Mechanics, 1950, pp. 187-191.

where P_r, P_θ, P_z, and Q are the averaged stresses which are related to the Cauchy stress components σ_{rr}, etc., by the following relations:

$$\left.\begin{aligned} P_r &= \int_0^a \sigma_{rr} r dr; \qquad P_\theta = \int_0^a \sigma_{\theta\theta} r dr \\ P_z &= \int_0^a \sigma_{zz} r dr; \qquad Q = \int_0^a \frac{\sigma_{rz}}{a} r^2 dr \end{aligned}\right\} \tag{3}$$

Equation (2) will be used to obtain solutions of finite rods if stress boundary conditions at the ends are given.

In the following discussion, we shall assume that the cylindrical surface of the rod is free of stress. Hence

$$R = Z = 0 \tag{4}$$

and Eq. (1) becomes:

$$\left.\begin{aligned} a^2(\lambda+2\mu)w'' + 2a\lambda u' &= \rho a^2 \ddot{w} \\ a^2k^2u'' - 8k_1{}^2(\lambda+\mu)u - 4ak_1{}^2\lambda w' &= \rho a^2 \ddot{u} \end{aligned}\right\} \tag{1'}$$

SOLUTION FORMULATIONS FOR RODS OF INFINITE LENGTH

To solve Eq. (1'), we divide through by ρa^2 and obtain:

$$\left.\begin{aligned} \left(\frac{\lambda+2\mu}{\rho}\right)w'' + \frac{2\lambda}{\rho a} u' &= \ddot{w} \\ \left(\frac{k^2\mu}{\rho}\right)u'' - \frac{8k_1{}^2(\lambda+\mu)}{\rho a^2} u - \frac{4k_1{}^2\lambda}{\rho a} w' &= \ddot{u} \end{aligned}\right\} \tag{5}$$

Let

$$\left.\begin{aligned} w(z,t) &= Ae^{i\gamma z}e^{-i\omega t} \\ u(z,t) &= Be^{i\gamma z}e^{-i\omega t} \end{aligned}\right\} \tag{6}$$

and substitute Eq. (6) into Eq. (5). One has

$$\left.\begin{aligned} &[-(\frac{\lambda+2\mu}{\rho})\gamma^2 + \omega^2]A + i\gamma(\frac{2\lambda}{\rho a})B = 0 \\ &-i\gamma(\frac{4k_1{}^2\lambda}{\rho a})A + [\frac{k^2\mu}{\rho}\gamma^2 - \frac{8k_1{}^2(\lambda+\mu)}{\rho a^2} + \omega^2]B = 0 \end{aligned}\right\} \quad (7)$$

Divide Eq. (7) through by γ^2:

$$\left.\begin{aligned} &[-\frac{(\lambda+2\mu)}{\rho} + \frac{\omega^2}{\gamma^2}]A + i(\frac{2\lambda}{\rho\gamma a})B = 0 \\ &-i(\frac{4k_1{}^2\lambda}{\rho\gamma a})A + [-\frac{k^2\mu}{\rho} - \frac{8k_1{}^2(\lambda+\mu)}{\rho\gamma^2 a^2} + \frac{\omega^2}{\gamma^2}]B = 0 \end{aligned}\right\} \quad (7')$$

From Eq. (7'), one obtains the following characteristic equation:

$$(\frac{\lambda+2\mu}{\rho} - \frac{\omega^2}{\gamma^2})(\frac{k^2\mu}{\rho} + \frac{8k_1{}^2(\lambda+\mu)}{\rho\gamma^2 a^2} - \frac{\omega^2}{\gamma^2}) + \frac{8k_1{}^2\lambda^2}{\rho^2\gamma^2 a^2} = 0 \quad (8)$$

and the amplitude ratio

$$\begin{aligned} \alpha = \frac{B}{A} &= -[-\frac{\lambda+2\mu}{\rho} + \frac{\omega^2}{\gamma^2}]/[i(\frac{2\lambda}{\rho\gamma a})] \\ &= +[i(\frac{4k_1{}^2\lambda}{\rho\gamma a})]/[-\frac{k^2\mu}{\rho} - \frac{8k_1{}^2(\lambda+\mu)}{\rho\gamma^2 a^2} + \frac{\omega^2}{\gamma^2}] \end{aligned} \quad (9)$$

where Eq. (8) is exactly the same as Eq. (22) in Mindlin-Herrmann's paper of 1951 (ref 1).

It would be convenient for the present analysis to introduce some dimensionless parameters. But first, let

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[1]R. D. Mindlin and G. Herrmann, "A One-Dimensional Theory of Compressional Waves in an Elastic Rod," Proceedings of the First U.S. National Congress of Applied Mechanics, 1950, pp. 187-191.

$$\ell = \text{length of the rod}$$

$$L = \text{wave length}$$

and hence the wave number can be defined as

$$\gamma = 2\pi/L \tag{10}$$

Now, define

$$\left.\begin{aligned} a^* &= a/\ell \quad , \quad \gamma^* = \gamma\ell = 2\pi\ell/L \\ (\omega^*)^2 &= \omega^2 a^2/(\mu/\rho) \quad , \quad \lambda^* = \lambda/\mu \end{aligned}\right\} \tag{11}$$

If, from Eq. (6), we define wave (phase) velocity as

$$c = \omega/\gamma \tag{12}$$

and

$$c_s = (\mu/\rho)^{1/2}$$

one has the relation

$$\frac{c^2}{c_s^2} = \frac{\omega^2}{\frac{\mu}{\rho}\gamma^2} = \frac{\omega^2 a^2}{\frac{\mu}{\rho}\gamma^2 a^2} = \frac{\omega^{*2}}{\gamma^{*2} a^{*2}} \tag{13}$$

With Eqs. (11) and (13), Eq. (8) becomes the following when divided through by $\mu^2/\rho^2 = c_s^2$:

$$\left(\lambda^* + 2 - \frac{\omega^{*2}}{\gamma^{*2}a^{*2}}\right)\left(k^2 + \frac{8k_1^2(\lambda^*+1)}{\gamma^{*2}a^{*2}} - \frac{\omega^{*2}}{\gamma^{*2}a^{*2}}\right) - \frac{8k_1^2\lambda^{*2}}{\gamma^{*2}a^{*2}} = 0 \tag{14}$$

Equation (9) can also be written as

$$\alpha = \frac{B}{A} = -i\left(\lambda^* + 2 - \frac{\omega^{*2}}{\gamma^{*2}a^{*2}}\right)\Big/\left(\frac{2\lambda^*}{\gamma^* a^*}\right) =$$

$$-i\left(\frac{4k_1^2\lambda^*}{\gamma^* a^*}\right)\Big/\left(k^2 + \frac{8k_1^2(\lambda^*+1)}{\gamma^{*2}a^{*2}} - \frac{\omega^{*2}}{\gamma^{*2}a^{*2}}\right) \tag{15}$$

Now, we shall solve $\gamma^* a^*$ for any given ω^* in Eq. (14). Let

$$\left.\begin{aligned} x &= \omega^{*2} \\ y &= \frac{1}{\gamma^{*2}a^{*2}} \end{aligned}\right\} \tag{16}$$

In terms of x and y, Eq. (14) becomes

$$(\lambda^*+2-xy)[k^2 + 8k_1{}^2(\lambda^*+1)y - xy] - 8k_1{}^2\lambda^{*2}y = 0$$

Or, upon expanding,

$$[x^2 - 8k_1{}^2(\lambda^*+1)x]y^2 - [(k^2+\lambda^*+2)x - 8k_1{}^2(3\lambda^*+2)]y + k^2(\lambda^*+2) = 0 \tag{17}$$

Or,

$$Ay^2 - By + C = 0 \tag{18}$$

with

$$\left.\begin{aligned} A &= x^2 - 8k_1{}^2(\lambda^*+1)x \\ B &= (k^2+\lambda^*+2)x - 8k_1{}^2(3\lambda^*+2) \\ C &= k^2(\lambda^*+2) \end{aligned}\right\} \tag{19}$$

Thus, one can write

$$y_{1,2} = \frac{B \pm \sqrt{B^2 - 4AC}}{2A} \tag{20}$$

Before we proceed to the actual computations, some observations on the sign of A, B, and $B^2 - 4AC$ should be helpful.

1. $A = x[x - 8k_1{}^2(\lambda^*+1)]$, and, since $x = \omega^{*2}$ is always positive or zero, for $\omega^* \neq 0$, A will be negative, zero, or positive depending on whether x is less, equal, or greater than $8k_1{}^2(\lambda^*+1)$:

$$A \lesseqgtr 0 \text{ if } x \lesseqgtr 8k_1{}^2(\lambda^*+1) \sim 16$$

2.

$$B = (k^2+\lambda^*+2)x - 8k_1{}^2(3\lambda^*+2)$$

$$= (k^2+\lambda^*+2)\left[x - \frac{8k_1{}^2(3\lambda^*+2)}{k^2+\lambda^*+2}\right]$$

Hence

$$B \lesseqgtr 0 \text{ if } x \lesseqgtr \frac{8k_1{}^2(3\lambda^*+2)}{k^2+\lambda^*+2} \sim 10$$

where the symbol ~ indicates "in the neighborhood when k, k_1 and λ^* are recognized to have an order of magnitude of unity."

3. Finally, let us consider $B^2 - 4AC$:

$$B^2 = (k^2+\lambda^*+2)^2x^2 + 64k_1{}^4(3\lambda^*+2)^2 - 16k_1{}^2(3\lambda^*+2)(k^2+\lambda^*+2)x$$

$$4AC = 4[x^2 - 8k_1{}^2(\lambda^*+1)x]k^2(\lambda^*+2) = 4k^2(\lambda^*+2)x^2 - 32k^2k_1{}^2(\lambda^*+1)(\lambda^*+2)x$$

$$= 4k^2(\lambda^*+2)x^2 - 32k_1{}^2(\lambda^{*2}+3\lambda^*+2)k^2$$

$$B^2 - 4AC = [k^2 - (\lambda^*+2)]^2x^2 + 64k_1{}^4(3\lambda^*+2)^2$$

$$- 16k_1{}^2x[3\lambda^*+2)(\lambda^*+2) - k^2(2\lambda^{*2}+3\lambda^*+2)]$$

a. For x being very large compared with unity, the first term dominates, hence, $B^2 - 4AC > 0$.

b. For x being very small compared with unity, the second term dominates, and hence, $B^2 - 4AC > 0$ again.

c. For x being in the neighborhood of unity, $B^2 - 4AC \sim 4 + 1600 - 128 = 1476 > 0$.

Hence, in most likelihood, $B^2 - 4AC > 0$ for all values of x. But we shall examine this carefully in actual computations.

NUMERICAL RESULTS FOR RODS OF INFINITE LENGTH

The solution formulations of the previous section are now carried out for specific numerical parameters. The results are shown in Figures 1, 2, and 3. In Figure 1, two dispersion curves (i.e., c/c vs. a/L) reported in Reference 1 are reproduced here for the purpose of numerical verifications. As in Reference 1, we have set $\nu = 0.29$. The curve 1a corresponds to $k_1 = 1$ and

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[1]R. D. Mindlin and G. Herrmann, "A One-Dimensional Theory of Compressional Waves in an Elastic Rod," Proceedings of the First U.S. National Congress of Applied Mechanics, 1950, pp. 187-191.

$k = 0.9258$ which is obtained from Eq. (25) in Reference 1; the curve 1'a corresponds to the same k, but k_1 is obtained by Eq. (28) of the same reference. In Figure 2, the coupling effect of the axial and radial modes through Poisson's ratio given in Reference 1 is also reproduced here. As shown in Figure 2, there exist two velocities for each wave number a/L in its full range. However, it will be realized that the full range of frequency (ω^*) is not covered in Figure 2. Since ω^* is related to the velocity c/c_s by Eq. (13), it is necessary to solve ω^* for given $a/L = 2\pi\gamma^* a^*$ in the characteristic equation (14). Hence, another dispersion relation is obtained and is shown in Figure 3 for ω^* vs. $2\pi\gamma^* a^*$ curves. Evidently, in this figure, full range of frequency is covered. It is observed that for ω^* smaller than a value near 4.3, one root of $\gamma^* a^*$ is imaginary, while the other is real. The eigenfunctions corresponding to the imaginary root are hyperbolic sine and cosine functions. They, together with the sine and cosine functions associated with the real root of $\gamma^* a^*$, will be needed for vibration solutions for a given set of end conditions of a finite rod.

SOLUTION FORMULATIONS OF RODS OF FINITE LENGTH

From solutions of rods of infinite length, one observes that there are four fundamental solutions in the form of sines and cosines or hyperbolic sine and cosine functions corresponding to any given frequency ω. Now, we wish to find a particular ω for a given specific length of the rod and for a given set of boundary conditions. It is also observed that the fundamental solutions of

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[1]R. D. Mindlin and G. Herrmann, "A One-Dimensional Theory of Compressional Waves in an Elastic Rod," Proceedings of the First U.S. National Congress of Applied Mechanics, 1950, pp. 187-191.

ω and u are matched as follows:

$$\left.\begin{array}{lll} w = \bar{\bar{\omega}}_1 \cos \gamma^* z^* e^{-i\omega t} & \text{with} & u = \bar{\bar{u}}_1 \sin \gamma^* z^* e^{-i\omega t} \\ w = \bar{\bar{\omega}}_2 \sin \gamma^* z^* e^{-i\omega t} & \text{with} & u = \bar{\bar{u}}_2 \cos \gamma^* z^* e^{-i\omega t} \\ w = \bar{\bar{\omega}}_3 \cosh \gamma^* z^* e^{-i\omega t} & \text{with} & u = \bar{\bar{u}}_3 \sinh \gamma^* z^* e^{-i\omega t} \\ w = \bar{\bar{\omega}}_4 \sinh \gamma^* z^* e^{-i\omega t} & \text{with} & u = \bar{\bar{u}}_4 \cosh \gamma^* z^* e^{-i\omega t} \end{array}\right\} \quad (21)$$

where

$$\bar{\bar{u}}_i = a_i \bar{\bar{\omega}}_i \quad , \quad i = 1,2,3,4 \quad (22)$$

and

$$\left.\begin{array}{l} \alpha_1 = -[-(\lambda^*+2) + \dfrac{\omega^{*2}}{\gamma^{*2} a^{*2}}]/(\dfrac{2\lambda^*}{\gamma^* a^*}) \\ \alpha_2 = -\alpha_1 \\ \alpha_3 = [(\lambda^*+2) + \dfrac{\omega^{*2}}{\gamma^{*2} a^{*2}}]/(\dfrac{2\lambda^*}{\gamma^* a^*}) \\ \alpha_4 = -\alpha_3 \end{array}\right\} \quad (23)$$

From the dispersion relations one observes that $(\gamma^* a^*)^2$ always has two roots for any given ω^*: either (1) both of these roots are real, or, (2) one is real and the other, imaginary.

For case (1), the general solution has the form

$$w(z,t) = (\bar{\bar{w}}_1 \cos \gamma_1^* z^* + \bar{\bar{w}}_2 \sin \gamma_1^* z^* + \bar{\bar{w}}_3 \cosh \gamma^* z^* + \bar{\bar{w}}_4 \sinh \gamma^* z^*) e^{-i\omega t} \quad (24)$$

$$u(z,t) = (\bar{\bar{u}}_1 \sin \gamma_1^* z^* + \bar{\bar{u}}_2 \cos \gamma_1^* z^* + \bar{\bar{u}}_3 \sinh \gamma^* z^* + \bar{\bar{u}}_4 \cosh \gamma^* z^*) e^{-i\omega t} \quad (25)$$

As our first example, the boundary conditions of a rod with both ends fixed will be considered:

$$\left.\begin{array}{l} w(0,t) = w(\ell,t) = 0 \\ u(0,t) = u(\ell,t) = 0 \end{array}\right\} \quad (26)$$

Using Eq. (26) for Eqs. (24) and (25), one has, respectively

$$\left.\begin{aligned} &\bar{\bar{w}}_1 + 0 + \bar{\bar{w}}_3 + 0 = 0 \\ &\bar{\bar{w}}_1 \cos \gamma_1^* + \bar{\bar{w}}_2 \sin \gamma_1^* + \bar{\bar{w}}_3 \cos \gamma_3^* + \bar{\bar{w}}_4 \sin \gamma_3^* = 0 \\ &0 + \alpha_2 \bar{\bar{w}}_2 + 0 + \alpha_4 \bar{\bar{w}}_4 = 0 \\ &\alpha_1 \bar{\bar{w}}_1 \sin \gamma_1^* + \alpha_2 \bar{\bar{w}}_2 \cos \gamma_1^* + \alpha_3 \bar{\bar{w}}_3 \sin \gamma_3^* + \alpha_4 \bar{\bar{w}}_4 \cos \gamma_3^* = 0 \end{aligned}\right\} \quad (27)$$

with

$$\left.\begin{aligned} \alpha_1 = -\alpha_2 = -[-(\lambda^*+2) + \frac{\omega^{*2}}{\gamma_1^{*2} a^{*2}}]/(\frac{2\lambda^*}{\gamma_1^* a^*}) \\ \alpha_3 = -\alpha_4 = -[-(\lambda^*+2) + \frac{\omega^{*2}}{\gamma_3^{*2} a^{*2}}]/(\frac{2\lambda^*}{\gamma_3^* a^*}) \end{aligned}\right\} \quad (28)$$

and

$$\left.\begin{aligned} &\bar{\bar{w}}_1 + 0 + \bar{\bar{w}}_3 + 0 = 0 \\ &\bar{\bar{w}}_1 \cos \gamma_1^* + \bar{\bar{w}}_2 \sin \gamma_1^* + \bar{\bar{w}}_3 \cosh \gamma_3^* + \bar{\bar{w}}_4 \sinh \gamma_3^* = 0 \\ &0 + \alpha_2 \bar{\bar{w}}_2 + 0 + \bar{\alpha}_4 \bar{\bar{w}}_4 = 0 \\ &\alpha_1 \bar{\bar{w}}_1 \sin \gamma_1^* + \alpha_2 \bar{\bar{w}}_2 \cos \gamma_1^* + \bar{\alpha}_3 \bar{\bar{w}}_3 \sinh \gamma_3^* + \bar{\alpha}_4 \bar{\bar{w}}_4 \cos \gamma_3^* = 0 \end{aligned}\right\} \quad (29)$$

with

$$\bar{\alpha}_3 = -\bar{\alpha}_4 = [(\lambda^*+2) + \frac{\omega^{*2}}{\gamma^{*2} a^{*2}}]/(\frac{2\lambda^*}{\gamma_3^* a^*}) \quad (30)$$

while α_1 and α_2 are the same as above.

For a nontrivial solution of Eq. (27), one has

$$\Delta_1 = \begin{bmatrix} 1 & 0 & 1 & 0 \\ \cos\gamma_1^* & \sin\gamma_1^* & \cos\gamma_3^* & \sin\gamma_3^* \\ 0 & \alpha_2 & 0 & \alpha_4 \\ \alpha_1 \sin\gamma_1^* & \alpha_2 \cos\gamma_1^* & \alpha_3 \sin\gamma_3^* & \alpha_4 \cos\gamma_3^* \end{bmatrix} = 0$$

$$\Delta_1 = \begin{bmatrix} \sin\gamma_1^* & \cos\gamma_3^* - \cos\gamma_1^* & \sin\gamma_3^* \\ \alpha_2 & 0 & \alpha_4 \\ \alpha_2 \cos\gamma_1^* & \alpha_3 \sin\gamma_3^* - \alpha_1 \sin\gamma_1^* & \alpha_4 \cos\gamma_3^* \end{bmatrix} \qquad (31)$$

$$\begin{aligned} \Delta_1 = {} & \alpha_2 \sin\gamma_3^*(\alpha_3 \sin\gamma_3^* - \alpha_1 \sin\gamma_1^*) \\ & + \alpha_2\alpha_4 \sin\gamma_1^*(\alpha_3 \sin\gamma_3^* - \alpha_4 \sin\gamma_1^*) \\ & - \alpha_4 \sin\gamma_1^*(\alpha_3 \sin\gamma_3^* - \alpha_1 \sin\gamma_1^*) \\ & - \alpha_2\alpha_4 \cos\gamma_3^*(\cos\gamma_3^* - \cos\gamma_1^*) \end{aligned}$$

Or,

$$\begin{aligned} \Delta_1 = {} & (\alpha_3 \sin\gamma_3^* - \alpha_1 \sin\gamma_1^*)(\alpha_2 \sin\gamma_3^* - \alpha_4 \sin\gamma_1^*) \\ & - \alpha_2\alpha_4(\cos\gamma_3^* - \cos\gamma_1^*)^2 = 0 \end{aligned} \qquad (32)$$

For a nontrivial solution of Eq. (29):

$$\Delta_2 = \begin{bmatrix} 1 & 0 & 1 & 0 \\ \cos\gamma_1^* & \sin\gamma_1^* & \cosh\gamma_3^* & \sinh\gamma_3^* \\ 0 & \alpha_2 & 0 & -\alpha_4 \\ \alpha_1 \sin\gamma_1^* & \alpha_2 \cos\gamma_1^* & \alpha_3 \sinh\gamma_2^* & \alpha_4 \cosh\gamma_3^* \end{bmatrix} = 0 \qquad (33)$$

Or,

$$\Delta_2 = (\bar{\alpha}_3 \sinh \gamma_3{}^* - \alpha_1 \sin \gamma_1{}^*)(\alpha_2 \sinh \gamma_3{}^* - \bar{\alpha}_4 \sin \gamma_1{}^*)$$
$$- \bar{\alpha}_2\alpha_4(\cosh \gamma_3{}^* - \cos \gamma_1{}^*)^2 = 0 \tag{34}$$

With Eqs. (33) and (34), one can write

$$\Delta = \Delta(\omega^*) = \begin{cases} \Delta_1(\omega^*) \text{ if } \gamma^{*2}a^{*2} \text{ has two real roots} \\ \Delta_2(\omega^*) \text{ if } \gamma^{*2}a^{*2} \text{ has one real and one imaginary root} \end{cases} \tag{35}$$

Hence, for a given ω^*, one must first find two values for $\gamma^{*2}a^{*2}$ and α_1, α_2, α_3, and α_4 (or $\bar{\alpha}_3$ and $\bar{\alpha}_4$). Then with given $a^* = a/\ell$, one obtains $\gamma_1{}^*$ and $\gamma_3{}^*$. Thus one can plot (ω^*) with respect to ω^* and find $\bar{\omega}^*$'s at which $\Delta(\bar{\omega}^*) = 0$.

Finally in this section, we will write down the formulas for computing the relative amplitudes of various mode shapes. Equation (27) for case (1) can be written as

$$\begin{bmatrix} 0 & 1 & 0 \\ \sin \gamma_1{}^* & \cos \gamma_3{}^* & \sin \gamma_3{}^* \\ \alpha_2 & 0 & \alpha_4 \end{bmatrix} \begin{bmatrix} \bar{\bar{w}}_2 \\ \bar{\bar{w}}_3 \\ \bar{\bar{w}}_4 \end{bmatrix} = - \begin{bmatrix} 1 \\ \cos \gamma_1{}^* \\ 0 \end{bmatrix} \bar{\bar{w}} \tag{36}$$

From Eq. (36), one has

$$\left.\begin{aligned} \bar{\bar{w}}_2 &= \frac{D_2}{D} \bar{\bar{w}}_1 = \beta_2\bar{\bar{w}}_1 \\ \bar{\bar{w}}_3 &= \frac{D_3}{D} \bar{\bar{w}}_1 = \beta_3\bar{\bar{w}}_1 \\ \bar{\bar{w}}_4 &= \frac{D_4}{D} \bar{\bar{w}}_1 = \beta_4\bar{\bar{w}}_1 \end{aligned}\right\} \tag{37}$$

where

$$D = \begin{bmatrix} 0 & 1 & 0 \\ \sin \gamma_1^* & \cos \gamma_3^* & \sin \gamma_3^* \\ \alpha_2 & 0 & \alpha_4 \end{bmatrix} = \alpha_2 \sin \gamma_3^* - \alpha_4 \sin \gamma_1^*$$

$$D_2 = \begin{bmatrix} -1 & 1 & 0 \\ -\cos \gamma_1^* & \cos \gamma_3^* & \sin \gamma_3^* \\ 0 & -1 & \alpha_4 \end{bmatrix} = \alpha_4(\cos \gamma_1^* - \cos \gamma_3^*)$$

$$D_3 = \begin{bmatrix} 0 & -1 & 0 \\ \sin \gamma_1^* & -\cos \gamma_1^* & \sin \gamma_3^* \\ \alpha_2 & 0 & 0 \end{bmatrix} = \alpha_4 \sin \gamma_1^* - \alpha_2 \sin \gamma_3^*$$

$$D_4 = \begin{bmatrix} 0 & 1 & -1 \\ \sin \gamma_1^* & \cos \gamma_3^* & -\cos \gamma_1^* \\ \alpha_2 & 0 & 0 \end{bmatrix} = \alpha_2(\cos \gamma_3^* - \cos \gamma_1^*) \tag{38}$$

For case (2), the above equation can still be used with $\cos \gamma_3^*$ and $\sin \gamma_3^*$ replaced by $\cosh \gamma_3^*$ and $\sinh \gamma_3^*$. Now, for $\bar{\bar{u}}_i$, $i = 1,2,3,4$, we observe Eqs. (22), (23), (28), and (30), and write

$$\bar{\bar{u}}_i = \alpha_i \bar{\bar{w}}_i = \bar{\beta}_i \bar{\bar{w}}_i \quad , \quad i = 1,2,3,4 \tag{39}$$

with

$$\bar{\beta}_i = \alpha_i \beta_i \quad , \quad i = 1,2,3,4 \tag{40}$$

where $\beta_1 = 1$ when $\bar{\bar{w}}_1$ is used as the basis of normalization,

$$\beta_i = \frac{D_i}{D_1} \quad , \quad i = 2,3,4$$

$$\beta_1 = 1 \tag{41}$$

In Eq. (40), $\bar{\alpha}_3$ and $\bar{\alpha}_4$ will be used in place of α_3 and α_4 if the mode shape involves hyperbolic cosine and sine as stated in the earlier discussion.

NUMERICAL RESULTS ON VIBRATIONS OF FINITE RODS WITH AXIAL AND RADIAL MOTIONS

The formulations obtained in the previous section are now used for specific numerical examples. In conjunction with a fixed-fixed rod (i.e., w and u are zero at both ends), several lowest vibration frequencies and the associated eigenfunctions have been obtained. The results are tabulated in Tables I through III for slenderness ratio $a^* = a/\ell = 0.05$, 0.10, and 0.20, respectively. In these tables, β_i, $i = 1,2,3,4$ are the coefficients for w(x,t) and β_i for u(x,t). They are normalized with respect to the one with largest absolute value. For any given ω^*, a pair of γ^*a^* is obtained. These are indicated by γ_1^*a and γ_3^*a, respectively, as described in the previous section. If one of them (γ_1^*a) is imaginary, i.e., $\bar{\gamma}_1^*a = i\bar{\gamma}_1^*a$, then γ_1^*a is given and it is indicated by parentheses. The coefficients of the associated eigenfunctions (hyperbolic sine and cosine functions) are also indicated by parentheses. For example, in Table I for $a^* = a/\ell = 0.5$, both the lowest frequencies have one imaginary root each. For all higher ω^*'s, both roots of γ^*a^* are real.

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TABLE I. VIBRATION SOLUTIONS TO A FINITE MINDLIN-HERRMANN ROD WITH FIXED-FIXED END CONDITIONS (ν = 0.29, Slenderness Ratio = 0.05)

Freq. Relative Magnitude of Modes	ω_1^* = 0.1575	ω_2^* = 3.6600	ω_3^* = 4.4813	ω_4^* = 4.6810	ω_5^* = 4.8480	ω_6^* = 5.1647
$\gamma_1^*a^*$	(82.29)	(39.81)	15.58	25.31	31.03	39.29
β_1	(-0.3380)	(1.0000)	-0.0018	0.3102	-0.0406	-0.1538
β_2	(0.2707)	(0.5580)	0.0128	0.0588	0.1139	0.2496
γ_3^*a	1.962	51.36	68.71	73.41	77.47	85.31
β_3	0.3380	-1.0000	-0.0018	-0.0310	0.0406	0.1538
β_4	0.2720	0.0000	1.0000	1.0000	1.0000	1.0000
$\gamma_1^*a^*$	(82.29)	(39.81)	15.58	25.31	31.03	39.29
$\bar{\beta}_1$	(-0.1005)	(0.0000)	0.0248	-0.0913	0.0604	0.1001
$\bar{\beta}_2$	(0.0805)	(0.0000)	0.1770	0.1731	0.1696	0.1625
$\gamma_3^*a^*$	1.962	51.36	68.71	73.41	77.47	85.31
$\bar{\beta}_3$	1.0000	-0.1903	0.0003	-0.0054	0.0069	0.0250
$\bar{\beta}_4$	-0.0805	0.0000	-0.1770	-0.1731	-0.1696	-0.1625

NOTE: The numerical values in parentheses are associated with imaginary wave numbers and their hyperbolic sine and hyperbolic cosine (mode shape) functions.

TABLE II. VIBRATION SOLUTIONS TO A FINITE MINDLIN-HERRMANN ROD WITH FIXED-FIXED END CONDITIONS (ν = 0.29, Slenderness Ratio = 0.10)

Freq. Relative Magnitude of Modes	ω_1^* = 3.6600	ω_2^* = 4.4391	ω_3^* = 4.6787	ω_4^* = 4.7515	ω_5^* = 4.8801	ω_6^* = 4.9863
$\gamma_1^* a^*$	(19.91)	6.247	12.63	13.96	16.00	17.47
β_1	(1.0000)	-0.0022	0.0980	0.0842	0.0809	0.0897
β_2	(0.2601)	0.0065	0.0584	0.0865	0.1259	0.1686
$\gamma_3^* a$	25.68	33.87	36.69	37.56	39.13	40.44
β_3	-1.0000	0.0022	-0.0980	-0.0842	-0.0809	-0.0897
β_4	0.0000	1.0000	1.0000	1.0000	1.0000	1.0000
$\gamma_1^* a^*$	(19.91)	6.247	12.63	13.96	16.00	17.47
$\bar{\beta}_1$	(0.0000)	0.0602	-0.2903	-0.1793	-0.1086	-0.0886
$\bar{\beta}_2$	(0.0000)	0.1778	0.1732	0.1717	0.1689	0.1666
$\gamma_3^* a^*$	25.68	33.87	36.69	37.56	39.13	40.44
$\bar{\beta}_3$	-0.1903	0.004	-0.0170	-0.0145	-0.0137	-0.0149
$\bar{\beta}_4$	-0.0000	-0.1778	-0.1732	-0.1717	-0.1689	-0.1666

NOTE: The numerical values in parentheses are associated with imaginary wave numbers and their hyperbolic sine and hyperbolic cosine (mode shape) functions.

TABLE III. VIBRATION SOLUTIONS TO A FINITE MINDLIN-HERRMANN ROD WITH FIXED-FIXED END CONDITIONS (ν = 0.29, Slenderness Ratio = 0.20)

Freq. Relative Magnitude of Modes	ω_1^* = 0.4156	ω_2^* = 3.6600	ω_3^* = 4.4396	ω_4^* = 4.6785	ω_5^* = 4.9000	ω_6^* = 5.1331
$\gamma_1^*a^*$	(20.48)	(9.952)	3.132	6.313	8.142	9.640
β_1	(-0.8876)	(-0.1279)	-0.0092	0.2121	0.2383	0.3925
β_2	(-0.1488)	(1.0000)	0.0066	0.0584	0.1336	0.2345
γ_3^*a	1.295	12.84	16.94	18.34	19.69	21.13
β_3	0.8876	0.1279	0.0092	-0.2121	-0.2383	-0.3925
β_4	-0.0393	-0.0000	1.0000	1.0000	1.0000	1.0000
$\gamma_1^*a^*$	(20.48)	(9.952)	3.132	6.313	8.142	9.640
$\bar{\beta}_1$	(-0.2645)	(0.0000)	0.2496	-0.6294	-0.3006	-0.2733
$\bar{\beta}_2$	(-0.4433)	(0.0000)	0.1778	0.1732	0.1685	0.1633
$\gamma_3^*a^*$	1.295	12.84	16.94	18.34	19.69	21.13
$\bar{\beta}_3$	1.0000	0.0243	0.0016	-0.0367	-0.0402	-0.0641
$\bar{\beta}_4$	0.4433	0.0000	-0.1778	-0.1732	-0.1685	-0.1633

NOTE: The numerical values in parentheses are associated with imaginary wave numbers and their hyperbolic sine and hyperbolic cosine (mode shape) functions.

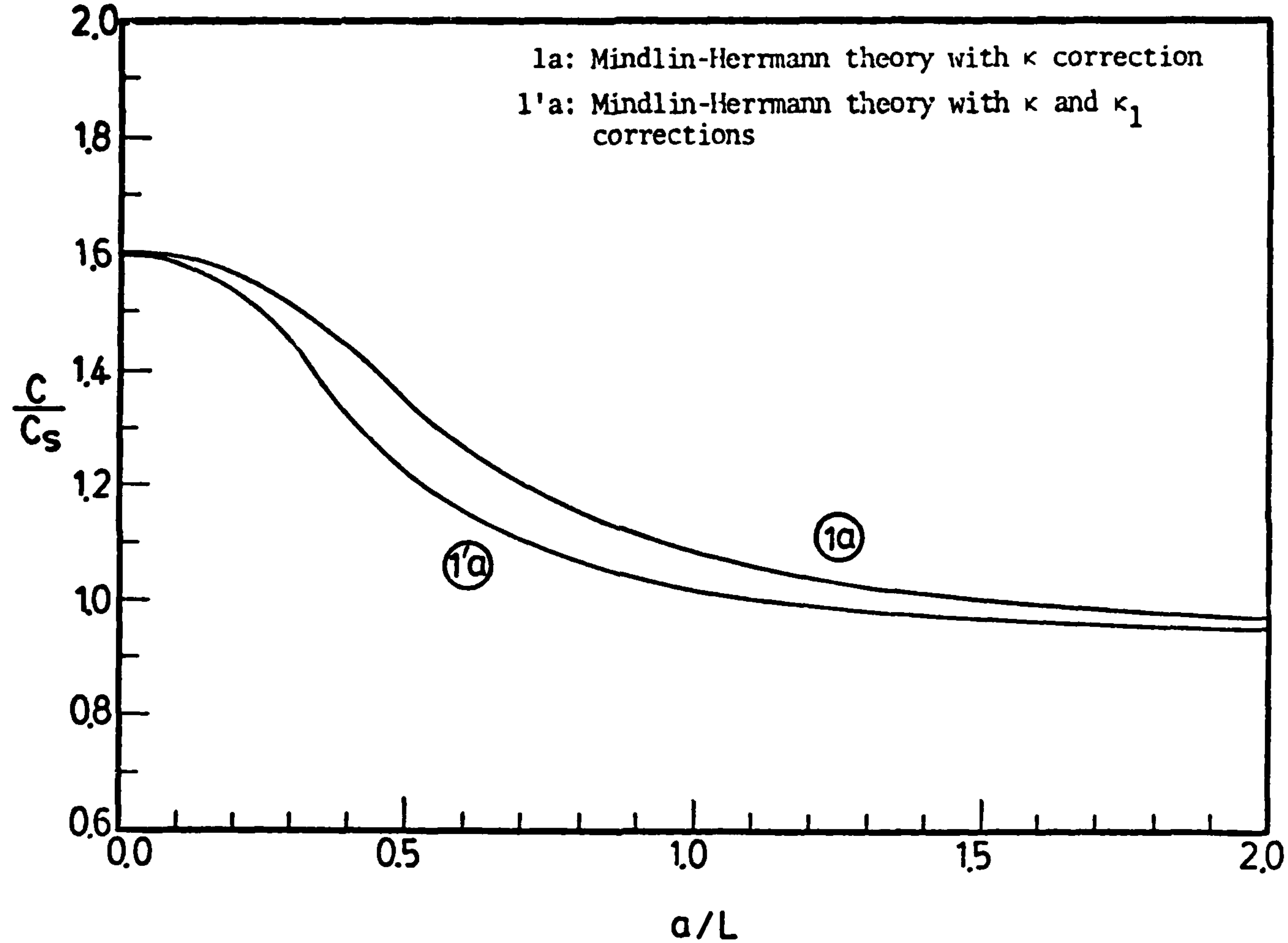

Figure 1. Longitudinal wave velocities of first mode of motion according to Mindlin-Herrmann Rod Theory (same as curves 1a and 1'a in Figure 1, Reference 1).

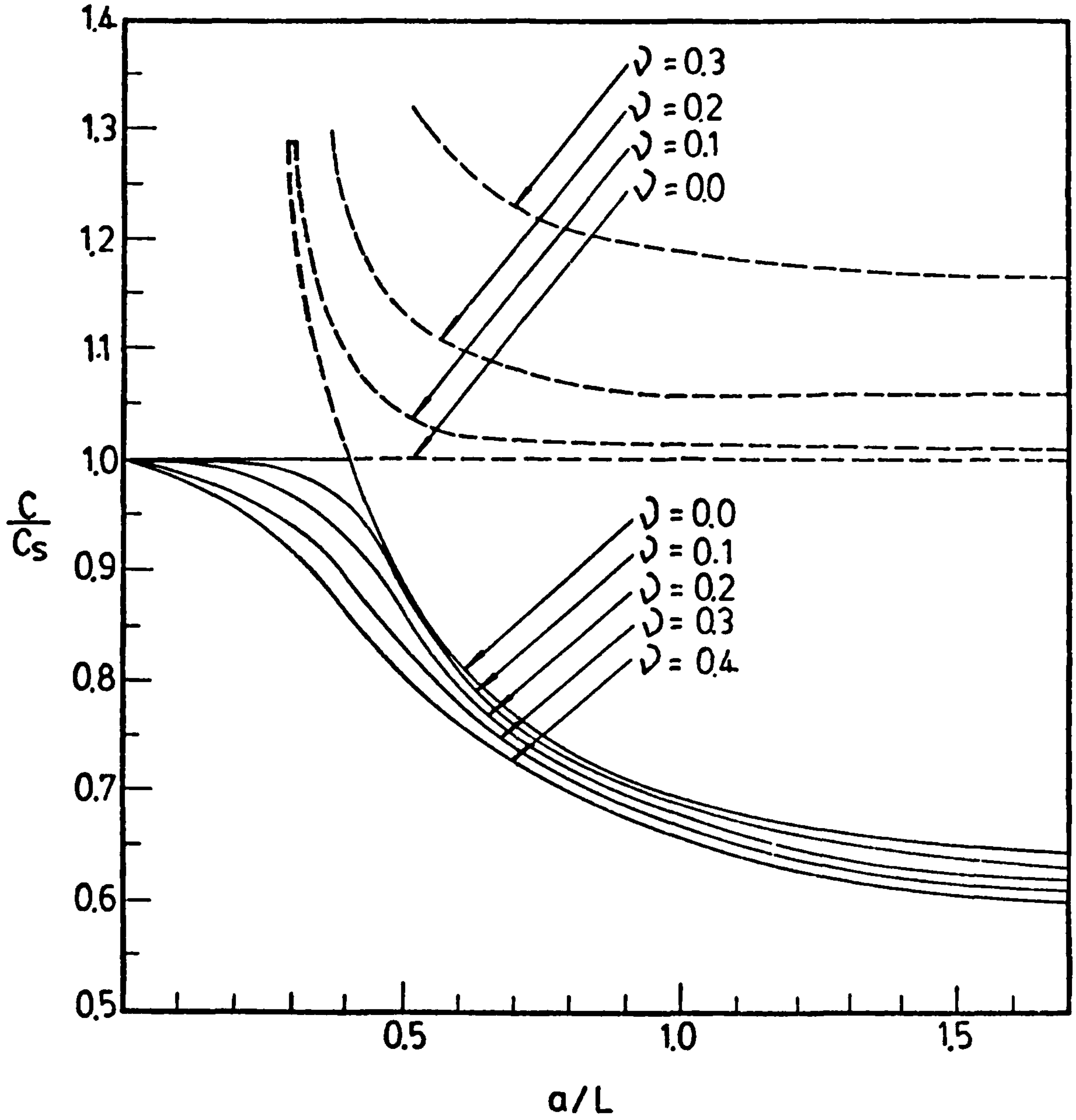

Figure 2. Effect of coupling through Poisson's Ratio, on wave velocities of axial and radial modes (same as Figure 2, Reference 1).

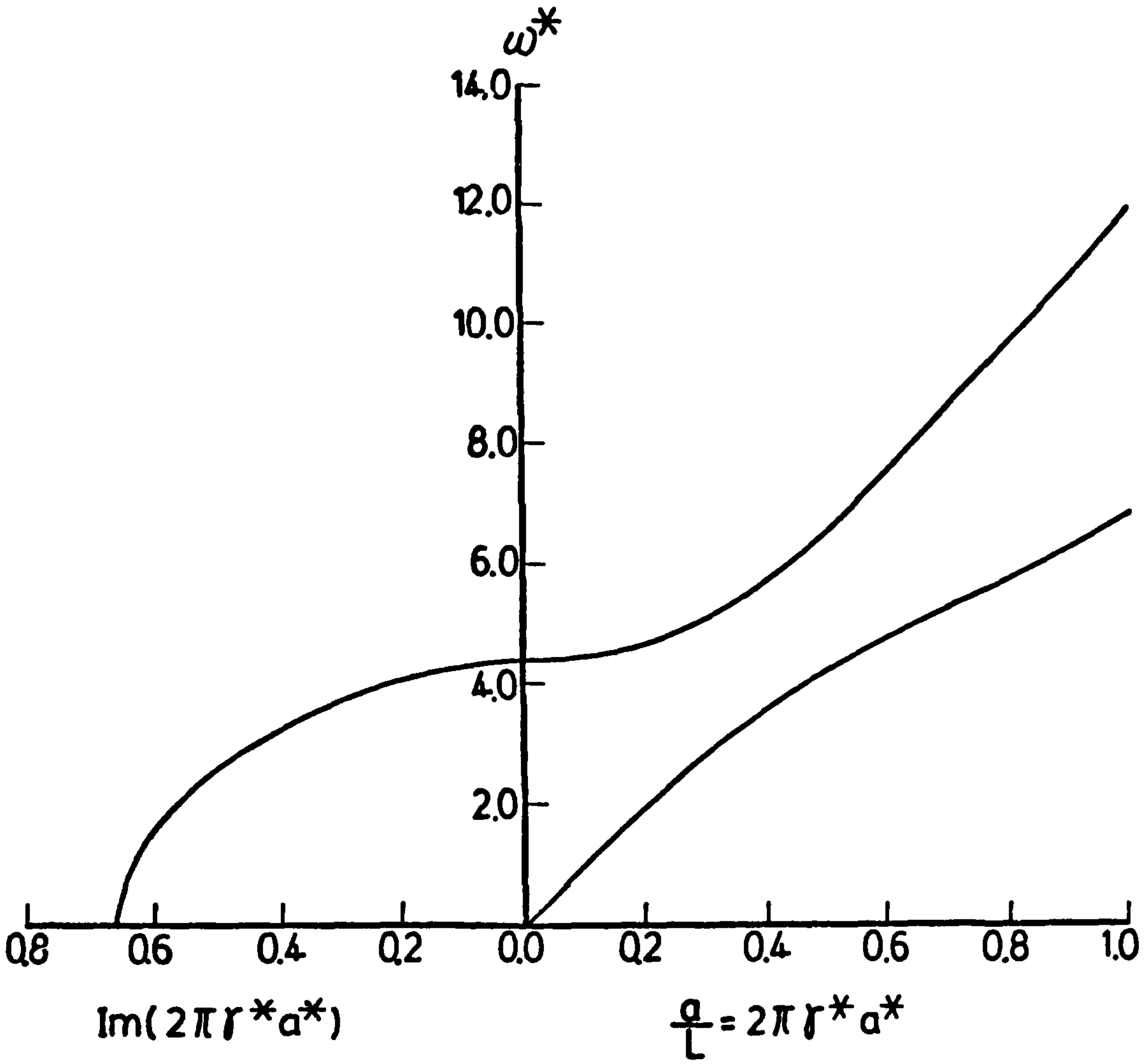

Figure 3. Frequency versus real and imaginary "wave number" in an infinite Mindlin-Herrmann Rod.

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