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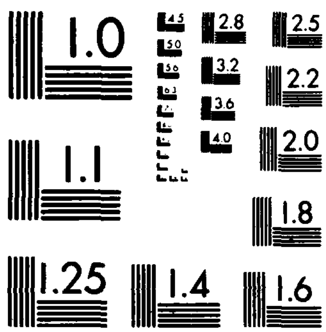

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APPLICATIONS OF COMBINATORICS TO THE BUSY PERIOD IN SEVERAL QUEUEING MODELS

BY

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APPLICATIONS OF COMBINATORICS TO THE BUSY PERIOD IN SEVERAL QUEUEING MODELS

S.G. Mohanty and J.L. Jain

Summary

Combinatorial methods are used to derive the distribution of the busy period in a Markovian tandem queue and queues with finite capacity in five models each involving batches.

1. Introduction

Takács in his book (see Takács(1967)) and many of his papers amply demonstrated his pioneering work on the application of combinatorial methods for deriving the distribution of the busy period. He (see Takács (1962)) gave a new direction of approaching the problem through the so-called urn problem (which is a generalization of the ballot problem). The technique basically requires the following two steps:

(i) Write down a finite set of relations in the form of inequalities and equalities on a sequence of random variables which a busy period when suitably conditioned must satisfy. The conditions are such that each sequence has the same weight (in terms of either probability mass or probability density).

(ii) Count the number of possibilities or find the measure of the set satisfying the inequalities.

The random variables are in terms of the number of arrivals or departures or the arrival or departure instants. The Markovian assumption of either interarrival times or service times plays the central role in

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the combinatorial technique in the sense that it assures the requirement of uniformity as suggested in (i) and thus leads to the enumeration problem in (ii).

The purpose of this paper is to further explore the scope of applications of the same technique. With this objective in mind, in Section 2 we present three enumerative propositions which find applications in the remaining part of the paper.

Although queueing network models abound in applications (such as communication networks, computer time sharing and multiprogramming systems) the study of such systems has not progressed at a pace commensurate with its importance (see Jackson (1957), Jackson (1963), Disney (1980)). In Section 3, we consider the simplest form of a queueing network, viz., a Markovian queue in series (also called a tandem queue) and obtain the joint distribution of the length of a busy period and the number of arrivals.

Queues with finite capacity is rather common in practice. However, the literature on busy period with finite capacity is not large (e.g. Cohen (1971), Enns (1969), Neuts (1964), Mohanty (1972), Takács (1976)). The direct combinatorial approach worked well on determining the distribution of a busy period with infinite capacity. Never-the-less, it has limited success for queues with finite capacity (Mohanty (1972)). Not surprisingly, most often the finite capacity case is treated through the Laplace-Steiltjes transform (Cohen (1971), Enns (1969), Takács (1976)) rather than the exact distribution with the help of the combinatorial method because part (ii) evaluation in general is not simple. However, the method can be applied to more situations than in the past and in Section 4 and Section 5 we obtain the disbribution of a busy period with finite capacity (i.e., fixed maximum queue length or maximum work load) in four different models, each involving batches.

2. Some Auxiliary Combinatorial Results

In this section we present three combinatorial results each of which is applied in the subsequent sections.

Proposition 1. (Kreweras (1965), p. 35)

Let $N(k; \underline{a}, \underline{b})$ denote the number of k-dimensional lattice paths from the point $\underline{a} = (a_1,\ldots,a_k)$ to the point $\underline{b} = (b_1,\ldots,b_k)$ such that every lattice point $(x_1,\ldots,x_k)$ on the path satisfies the condition $x_1 \geq \cdots \geq x_k$. Then

(1) $N(k; \underline{a}, \underline{b}) = [\underline{b} - \underline{a}]! \| c_{ij} \|_{k\times k}$

where $[\underline{n}] = \sum_{i=1}^{k} n_i$, and $\| c_{ij} \|_{k\times k}$ is the $k\times k$ determinant with (i,j)th element

$$c_{ij} = \frac{1}{(b_i - a_j - i + j)!} .$$

Kreweras' original result is in terms of Young chains which when converted into lattice paths has the above interpretation.

Before stating the next proposition we need some definitions and notations. Consider $(k + 1)$-dimensional lattice paths from the origin to the point $(n_o, n_1,\ldots,n_k)$. By the $\underline{r}(=(r_1,\ldots,r_k))$th level we mean the set of points $\{(x_o, n_1 - r_1,\ldots,n_k - r_k): 0 \leq x_o \leq n_o\}$.

Let

$$x(\underline{r}) = n_o - \min\{x_o\} \quad \text{if the path reaches the } \underline{r}\text{th level and } (x_o, n_1 - r_1, \ldots, n_k - r_k) \text{ is a point on the path;}$$
$$= 0 \quad \text{otherwise.}$$

Clearly $0 \leq x(\underline{r}) \leq n_o$ and $x(\underline{n}) = n_o$. Denote by $a(\underline{r})$ and $b(\underline{r})$ the upper and lower restrictions at the $\underline{r}$th level, by which we mean the path at the $\underline{r}$th level can pass only through points in the set

$$\{(x_o, n_1 - r_1, \ldots, n_k - r_k): 0 \leq b(\underline{r}) \leq n_o - x_o \leq a(\underline{r}) \leq n_o\}.$$

The sets

$$A(\underline{n}) = \{a(\underline{r}): \underline{0} \leq \underline{r} \leq \underline{n}\}$$

and

$$B(\underline{n}) = \{b(\underline{r}): \underline{0} \leq \underline{r} \leq \underline{n}\}$$

are respectively called the upper and lower restrictions on the path. The order relation $\underline{x} \leq \underline{y}$ means $x_i \leq y_i$ for each i. Note that $a(\underline{r})$ and $b(\underline{r})$ are non-negative integers, and non-decreasing in each coordinate.

Let $\underline{x}\,\alpha\,\underline{y}$ mean $x_i < y_i$ for at least one i. For example, the lexicographic ordering $(\underline{u}_1, \ldots, \underline{u}_d)$ of the set $\{\underline{r}: \underline{0} \leq \underline{r} \leq \underline{n}\}$ such that $d = \prod_{i=1}^{k} (n_i + 1)$, $\underline{u}_1 = \underline{0}$ and $\underline{u}_d = \underline{n}$ is an α ordering in the sense that $\underline{u}_1 \alpha \ldots \alpha \underline{u}_n$. Remember that the sequence $((0,0), (0,1), (1,0), (1,1), (2,0), (2,1))$ is a lexicographic ordering of vectors $\{\underline{r}: (0,0) \leq (r_1, r_2) \leq (2,1)\}$.

Proposition 2. (Handa and Mohanty (1979))

Denote by $g_k(A(\underline{n})|B(\underline{n}))$ the number of paths with upper restriction $A(\underline{n})$ and lower restriction $B(\underline{n})$. Let

$$\binom{a}{\underline{n}} = \binom{a}{n_1,\ldots n_k} = \frac{a(a-1)\ldots(a-\sum_{i=1}^{k} n_i + 1)}{\underline{n}!}$$

$$\binom{a}{z}_+ = \binom{\max(a,o)}{z}$$

and

$$\underline{n}! = \prod_{i=1}^{k} n_i!.$$

Then $g_k(A(\underline{n})|B(\underline{n}))$ satisfies the recurrence relation

$$(2) \quad \sum_{\underline{0} \le \underline{r} \le \underline{n}} (-1)^{[\underline{n} - \underline{r}]} \binom{a(\underline{r}) - b(\underline{n}) + 1}{\underline{n} - \underline{r}}_+ g_k(A(\underline{r})|B(\underline{r})) = \delta^{\underline{n}}_{\underline{o}}$$

where

$$\delta^{\underline{n}}_{\underline{o}} = 1 \qquad \text{when } \underline{n} = \underline{0},$$

$$= 0 \qquad \text{otherwise.}$$

An explicit solution of (2) is the following:

$$(3) \quad g_k(A(\underline{n})|B(\underline{n})) = (-1)^{d-[\underline{n}]-1} \left\| \binom{a(\underline{u}_i) - b(\underline{u}_{j+1}) + 1}{\underline{u}_{j+1} - \underline{u}_i}_+ \right\|_{(d-1) \times (d-1)}$$

where $d = \prod_{i=1}^{k} (n_i + 1)$ and $\{\underline{u}_1,\ldots,\underline{u}_d\} = \{\underline{r}\colon \underline{0} \le \underline{r} \le \underline{n}\}$ such that

$\underline{0} = \underline{u}_1 \alpha \ldots \alpha \underline{u}_d = \underline{n}$.

The explicit expression (3) may be obtained by first using Cramer's rule to the system (2) of linear equations and then simplifying it For practical purpose, the lexicographic ordering of vectors is good enough. It is easily seen that if we take $(n_o, n_1,\ldots,n_k)$ as the origin and reverse the steps in the path, the upper and lower restrictions respectively become

$$A'(\underline{n}) = \{n_o - b(\underline{n} - \underline{r}): \underline{0} \leq \underline{r} \leq \underline{n}\}$$

and

$$B'(\underline{n}) = \{n_o - a(\underline{n} - \underline{r}): \underline{0} \leq \underline{r} \leq \underline{n}\}.$$

Then we get an alternative expression for the same number of paths as $g_k(A'(\underline{n})|B'(\underline{n}))$. In our applications, the alternative expression is used.

Next we formulate a continuous analogue of Proposition 2 which gives a generalization of Steck's result (1971). Though Steck has stated the result in terms of order statistics, its connection with paths is explained in Mohanty (1979) chapters 2 and 4; Mohanty (1980). In (k+1)-dimension with axes $x_o, x_1, \ldots x_k$, consider paths (not necessarily lattice paths) from the origin to $(n_o, n_1, \ldots, n_k)$ where n_o is a non-negative real number and $n_1, \ldots, n_k$ are non-negative integers. In this case, a path is like a lattice path except that the the number of units moved at any time on x_o-axis is a non-negative real number. As before, we may define the level $\underline{r}$, the upper restriction $A(\underline{n})$ and the lower restriction $B(\underline{n})$. Here, we may remember that $a(\underline{r})$ and $b(\underline{r})$ are non-negative real numbers. In the next assertion, we adopt some of the earlier notations.

Proposition 3.

Let $g_k^*(A(\underline{n})|B(\underline{n}))$ be the measure of the set of paths with upper restriction $A(\underline{n})$ and lower restriction $B(\underline{n})$. Then $g_k^*(A(\underline{n})|B(\underline{n}))$ satisfies the recurrence relation

(4) $$\sum_{\underline{0} \le \underline{r} \le \underline{n}} (-1)^{[\underline{n} - \underline{r}]} \frac{(a(\underline{r}) - b(\underline{n}))_+^{[\underline{n} - \underline{r}]}}{(\underline{n} - \underline{r})!} g_k^*(A(\underline{r})|B(\underline{r})) = \delta_{\underline{o}}^{\underline{n}}$$

where $(x)_+ = \max(0,x)$. An explicit solution of (4) is given by

(5) $g_k^*(A(\underline{n})|B(\underline{n}))$

$$= (-1)^{d-[\underline{n}]-1} \left\| \frac{(a(\underline{u}_i) - b(\underline{u}_{j+1}))_+^{[\underline{u}_{j+1} - \underline{u}_i]}}{(\underline{u}_{j+1} - \underline{u}_i)!} \right\|_{(d - 1) \times (d - 1)}$$

The proof is inductive, follows the similar line as in Proposition 2 and therefore is omitted. When $k = 1$, we get

(6) $$\sum_{r=o}^{n} (-1)^{n-r} \frac{(a(r) - b(n))_+^{n-r}}{(n - r)!} g_1^*(A(r)|B(r)) = \delta_o^n$$

and

(7) $$g_1^* = \left\| \frac{(a(u_i) - b(u_{j+1}))_+^{[u_{j+1} - u_i]}}{(u_{j+1} - u_i)!} \right\|_{n \times n}$$

(g_1^* written for $g_1^*(A(\underline{n})|B(\underline{n}))$).

In fact $u_r = r - 1$, $r = 1,\dots,n + 1$ and g_1^* represents the integral

(8) $$\int_{u_1}^{v_1} \int_{y_2}^{v_2} \dots \int_{y_n}^{v_n} dx_n \dots dx_1$$

where $y_i = \max(u_i, x_{i-1})$ (see Mohanty (1971)) such that $a(r - 1) = v_r$ and $b(r) = u_r$ $r = 1,\dots,n$. Thus the determinant solution (7) of g_1^* checks with the earlier one in Steck (1971) (see Mohanty (1979), page 56, Mohanty (1980)). Relation (6) which is needed in the inductive proof follows easily from (7).

The remarks following Proposition 2 are all valid for Proposition 3. For computational purpose, recurrence relations (2) and (4) are often more useful than the explicit expressions (3) and (5).

3. A Tandem Queue

The queueing model called Model I is described by the following properties:

Model I.

(a) There are r service counters numbered 1 to r and each counter has only one server. A customer arrives at the first service counter and moves from the ith counter to the $(i + 1)$st counter $i = 1,\ldots,r - 1$ for service. The service is completed just after the customer leaves the rth counter.

(b) Customers are arriving in accordance to a Poisson process with parameter λ.

(c) Service times at the ith counter are i.i.d. exponential random variables with parameter μ_i, $i = 1,\ldots,r$.

(d) Service times at various counters are mutually independent and are independent of arrival times.

The busy period ends when for the first time the first counter is empty while the other counters are busy and are never empty before.

We are interested in ascertaining

$G_1 = G_1(j_1,\ldots,j_r;\ k_1,\ldots,k_{r-1};\ n;\ t)$ which represents the probability that (i) there are initially $j_i + 1$ customers at the ith counter, $i = 1,\ldots,r$; (ii) n customers have arrived during the busy period; (iii) at the end of the busy period there are $k_{i-1} + 1$ customers including one being served, left at the ith counter, $i = 2,\ldots,r$;

(iv) the length of the busy period is $\leq t$.

Let the point of time when either there is an arrival or a completion of a service be called an instant. Denote by α_{oj} and $\alpha_{ij}(i = 1,\dots,r)$ the number of customers arrived at instant j and the number of units served by the ith server at instant j respectively. Observe that $\alpha_{ij} = 0$ or 1, $i = 0, 1,\dots,r$ such that $\sum_{i=o}^{r} \alpha_{ij} = 1$. Let $N+1$ be the total number of instants upto the end of the busy period. Then the event in question with conditions (i), (ii) and (iii) is equivalent to the following relations:

$$(9)\quad \begin{cases} j_i + \sum_{\ell=1}^{m} \alpha_{i-1,\ell} \geq \sum_{\ell=1}^{m} \alpha_{i\ell}, \; i = 1,\dots,r, \; m = 1,\dots,N, \\ \sum_{\ell=1}^{N+1} \alpha_{o\ell} = n, \; \sum_{\ell=1}^{N+1} \alpha_{1\ell} = n + j_1 + 1, \; \alpha_{1,N+1} = 1, \\ \sum_{\ell=1}^{N+1} \alpha_{i\ell} = n + \sum_{t=1}^{i} (j_t - k_t) + k_i + 1, \; i = 2,\dots,r. \end{cases}$$

Putting

$$j_{i+1}+\dots+j_r + \sum_{\ell=1}^{m} \alpha_{i,\ell} = x_{i,m}, \; i = 0,1,\dots, r-1, \; m = 1,\dots,N,$$

$$\sum_{\ell=1}^{m} \alpha_{r,\ell} = x_{r,m}$$

and

$$\underline{x}_o = (x_{oo}, x_{1o},\dots,x_{ro}) = \left(\sum_{t=1}^{r} j_t, \sum_{t=2}^{r} j_t,\dots,j_r,0\right),$$

we note that relations (9) become

$$x_{om} \geq x_{1m} \geq \dots \geq x_{rm}, \; m = 0, 1,\dots,N,$$

where

$$\underline{x}_N = (x_{oN}, x_{1N}, \dots, x_{rN})$$

with

$$x_{oN} = \sum_{t=1}^{r} j_t + n, \quad x_{1N} = \sum_{t=1}^{r} j_t + n, \quad \text{and} \quad x_{iN} = x_{oN} - \sum_{t=1}^{i-1} k_t + 1,$$

$i = 2, \dots, r.$

The number of sequences $((x_{om}, x_{1m}, \dots, x_{rm}), m = 0,1,\dots,N)$ satisfying the condition $x_{om} \geq x_{1m} \geq \dots \geq x_{rm}$ $m = 0,1,\dots,N$ is the number $N(r + 1; \underline{x}_o, \underline{x}_N)$ of paths as in Proposition 1.

Taking the superposition of $r + 1$ Poisson processes into consideration, one observes that there are

$n^* = (r + 1)n + j_1 + \sum_{i=2}^{r} \sum_{t=1}^{i} (j_t - k_t) + \sum_{i=2}^{r} k_i + r - 1$ occurrences in the combined Poisson process with parameter $\lambda + \sum_{j=1}^{r} \mu_j$ during the busy period of length y and one occurrence at the end of the period (i.e., during the interval $(y, y + dy)$, $dy \to 0$). The probability of this event is

$$\text{(10)} \quad e^{-y(\lambda + \sum_{j=1}^{r} \mu_j)} \frac{(y(\lambda + \sum_{j=1}^{r} \mu_j))^{n^*}}{n^*!} \left((\lambda + \sum_{j=1}^{r} \mu_j)dy + o dy\right).$$

Given the above event, the probability for any sequence of arrivals and completion of services at all counters during the busy period (i.e., any sequence $((x_{om}, x_{1m}, \dots, x_{rm}), m = 0,1,\dots,N$ subject to the condition $x_{om} \geq x_{1m} \geq \dots \geq x_{rm}$, $m = 0,1,\dots,N)$ is

$$\text{(11)}\quad A = p_o^n\, p_1^{\,n+j_1+1} \prod_{i=2}^{r} p_i^{\sum_{t=1}^{i}(j_t - k_t) + k_i + n + 1}$$

where

$$p_o = \frac{\lambda}{\lambda + \sum_{j=1}^{r} \mu_j}, \quad p_i = \frac{\mu_i}{\lambda + \sum_{j=1}^{r} \mu_j}, \quad i = 1,\ldots,r.$$

Thus combining these facts, we obtain

$$\text{(12)}\quad G_1 = N(r+1;\, \underline{x}_o,\, \underline{x}_N)\, A \int_o^t e^{-y(\lambda + \Sigma\mu_j)} \frac{(y(\lambda + \Sigma\mu_j))^{n^*}}{n^*!} (\lambda + \Sigma\mu_j)\, dy$$

where Σ refers to $\sum_{1}^{r}$.

4. Markovian queues

We deal with two heterogeneous Markovian models involving batches. With the aid of Proposition 2, the joint distribution of several characteristics one of which is the maximum queue length during the busy period is obtained for each model.

Model II.

The model is characterized by the following:

(a) Customers arrive at a counter from r sources in batches of size u_i from the ith source ($i = 1,\dots,r$) in accordance to the Poisson process with parameter λ_i.

(b) Customers are served in batches of size u_o and service times at the counter are i.i.d. exponential random variables with parameter μ.

(c) Service times are independent of arrival times from different sources. Arrival times from different sources are also independent.

Without loss of generality, we will first treat the case $r = 2$. Let $G_2 = G_2(\ell + u_o; n_1, n_2; k, t)$ be the probability that a busy period initiated by $\ell + u_o$ customers, consists of n_i batches arriving from the ith source ($i = 1, 2$) has a maximum queue length $\leq k$ and has length $\leq t$.

Let $n_o + 1$ be the number of batches served during the busy period. Then

$$(13)\quad n_o = \left[\frac{u_1 n_1 + u_2 n_2 + \ell}{u_o}\right],$$

$[z]$ being the largest integer less than or equal to z. Assume that $k \geq u_o$.

Let us define the (i,j)th arrival instant as that instant which is either the ith arrival instant from the first source and j arrival instants from the second source precede it or the jth arrival instant from the second source and i arrival instants from the first source precede it. Denote by X_{ij} the number of departures after the (i,j)th arrival instant and before the next arrival instant. Also use X_{n_1,n_2} to represent the number of departures between the (n_1, n_2)th arrival instant and the end of the busy period. Then for a busy period the following relations must be satisfied:

$$(14)\quad \ell - k + iu_1 + ju_2 \le u_o \sum_{\beta=0}^{j} \sum_{\alpha=0}^{i} X_{\alpha,\beta} \le iu_1 + ju_2 + \ell$$

for $i = 0, 1, \ldots, n_1$, $j = 0, 1, \ldots, n_2$ but $(i,j) \neq (n_1, n_2)$

and $\sum_{\beta=0}^{n_2} \sum_{\alpha=0}^{n_1} X_{\alpha,\beta} = n_o + 1.$

If we represent a departure by a unit on x_o-axis, an arrival from source i on x_i-axis $(i = 1, 2)$, then the sequences of arrivals and departures satisfying (14) correspond to the set of lattice paths from $(0, 0, 0)$ to $(n_o + 1, n_1, n_2)$ which do not cross the planes

$$(15)\quad u_o x_o = u_1 x_1 + u_2 x_2 + \ell$$

and

$$(16)\quad u_o x_o = u_1 x_1 + u_2 x_2 + \ell - k$$

except at the end when the plane (15) is crossed by a unit step on x_o-axis. The number of such paths as given in Proposition 2 is

$g_2(A(\underline{n})|B(\underline{n}))$, where

$$a(i,j) = \min(n_o, [\frac{iu_1 + ju_2 + \ell}{u_o}])$$

and

$$b(i,j) = \max(0, [\frac{\ell - k + iu_1 + ju_2}{u_o}]),$$

$i = 0, 1, \ldots, n_1$ and $j = 0, 1, \ldots, n_2$.

Using an argument similar to the last part of the previous section, we obtain

$$(17) \quad G_2 = g_2(A(\underline{n})|B(\underline{n}))\, p_o^{n_o+1}\, p_1^{n_1}\, p_2^{n_2} \int_o^t B\, dy$$

for $k \geq u_o$ and n_o given by (13)

where

$$p_o = \frac{u}{\mu + \lambda_1 + \lambda_2},\ p_1 = \frac{\lambda_1}{\mu + \lambda_1 + \lambda_2},\ p_2 = \frac{\lambda_2}{\mu + \lambda_1 + \lambda_2}$$

and

$$B = e^{-(\mu+\lambda_1+\lambda_2)y} \frac{((\mu+\lambda_1+\lambda_2)y)^{n_o+n_1+n_2}}{(n_o + n_1 + n_2)!} (\mu + \lambda_1 + \lambda_2).$$

When $k < u_o$, the busy period terminates with one service and the expression is obvious.

For the general case, we may write

$G_2^{(r)} = G_2(\ell + u_o; n_1, \ldots, n_r; k, t)$ to represent the probability that a busy period initiated by $\ell + u_o$ customers consists of n_i batches arriving from the ith source, has a maximum queue length $\leq k$ and has length $\leq t$. The value of n_o becomes

$$n_o = \left[\frac{\Sigma u_i n_i + \ell}{u_o}\right] .$$

The expression for the generalized G_2 is

$$(17) \quad G_2(r) = g_r(A(\underline{n}) | B(\underline{n})) p_o^{n_o+1} \left(\prod_{i=1}^{r} p_i^{n_i}\right) \int_o^t B_r\, dy$$

where

$$a(\underline{s}) = \min\left(n_o, \left[\frac{\Sigma s_i u_i + \ell}{u_o}\right]\right)$$

$$b(\underline{s}) = \max\left(o, \left[\frac{\Sigma s_i u_i + \ell - k}{u_o}\right]\right)$$

for $\underline{0} \leq \underline{s} \leq \underline{n}$,

$$p_o = \frac{\mu}{\mu + \Sigma\lambda_i} , \; p_j = \frac{\lambda_j}{\mu + \Sigma\lambda_i} , \; i = 1,\ldots,r,$$

and

$$B_r = e^{-(\mu+\Sigma\lambda_i)y} \frac{((\mu + \Sigma\lambda_i) y)^{n_o+\Sigma n_i}}{(n_o + \Sigma n_i)!} (\mu + \Sigma\lambda_i).$$

In the above discussion, Σ represents $\sum_1^r$.

Model III.

We consider another Markovian model which is described below:

(a) Customers arrive at a queueing system in batches of size u_o in accordance to a Poisson process with parameter λ.

(b) The service system consists of r counters. Customers are served in batches of size u_i at the ith counter $(i = 1,\ldots,r)$. The service times at the ith counter are i.i.d. exponential random variables with parameter $\mu_i (i = 1,\ldots,r)$.

(c) Service times at various counters are mutually independent and are independent of arrival times.

The busy period ends when for the first time any one counter becomes empty, and without loss of generality let us assume it to be the last counter.

Because of our observation in Model II, we will only deal with the case $r = 2$. Denote by $G_3 = G_3(\ell + u_1 + u_2; n_o, n_1; k, t)$ the probability that a busy period initially with $\ell + u_1 + u_2$ customers consists of n_o arrivals of batches of size u_o and n_1 batches served at the first counter, has a maximum queue length $\leq k$ and has length $\leq t$.

Letting $n_2 + 1$ to represent the number of services at the second counter during the busy period, it can be verified that

$$(18)\quad n_2 = [\frac{n_o u_o + \ell - n_1 u_1}{u_2}] .$$

For n_o, n_1 and n_2 to be meaningful, we assume $k \geq \max(u_1, u_2)$.

As in Model II, we define the (i,j)th departure instant as that instant which is either the ith departure instant at the first counter and j departure instants at the second counter precede it or the jth departure instant at the second counter and i departure instants precede it. Without ambiguity, we may again denote by X_{ij} the number of arrivals after the (i,j)th departure instant and before the next departure instant. The busy period will consist of n_o arrivals, n_1 batches served at the first counter and $n_2 + 1$ batches served at the second counter when

$$(19)\quad u_1 i + u_2 j - \ell \leq u_o \sum_{\beta=o}^{j} \sum_{\alpha=o}^{i} X_{\alpha\beta} \leq u_1 i + u_2 j - \ell + k \quad \text{for}$$

$i = 0, 1, \ldots, n_1$, $j = 0, 1, \ldots, n_2$,

such that

$$\sum_{\beta=o}^{n_2} \sum_{\alpha=o}^{n_1} X_{\alpha\beta} = n_o.$$

Representing an arrival by a unit on x_o-axis and a departure from the ith counter by a unit on x_i-axis, $i = 1, 2$, the number of sequences of arrivals and departures satisfying (19) can be seen to be equal to $g_2(A^*(\underline{n})|B^*(\underline{n}))$ where

$$a^*(i,j) = \min(n_o, [\frac{u_1 i + u_2 j - \ell + k}{u_o}]),$$

and

$$b^*(i,j) = \max(0, [\frac{u_1 i + u_2 j - \ell}{u_o}]),$$

$i = 0, 1, \ldots, n_1$ and $j = 0, 1, \ldots, n_2 + 1$.

Again, following the rest of the argument routinely we derive

(20) $G_3 = g_2(A^*(\underline{n})|B^*(\underline{n}))p_o^{n_o} p_1^{n_1} p_2^{n_2+1} \int_o^t B^* \, dy$ for $k \geq \max(u_1, u_2)$

and n_2 given by (18)

where

$$p_o = \frac{\lambda}{\lambda + \mu_1 + \mu_2}, \; p_1 = \frac{\mu_1}{\lambda + \mu_1 + \mu_2}, \; p_2 = \frac{\mu_2}{\lambda + \mu_1 + \mu_2}$$

and

$$B^* = e^{-(\lambda+\mu_1+\mu_2)y} \frac{((\lambda + \mu_1 + \mu_2)y)^{n_o+n_1+n_2}}{(n_o + n_1 + n_2)!} (\lambda + \mu_1 + \mu_2).$$

Note that when $k < \max(u_1, u_2)$, several possibilities arise. If $k < \min(u_1, u_2)$, one gets $n_1 = n_2 = 0$. On the other hand,

$n_2 = 0$ if $u_2 > u_1$ and $u_1 \leq k < u_2$ and $n_1 = 0$ if $u_1 > u_2$ and $u_2 \leq k < u_1$. Discussion for the first counter to be empty is similar.

The derivation for the general case is similar and is omitted.

It may be noted that the models in Mohanty (1972) are special cases of the models discussed in this section.

5. Non-Markovian Queues

Two models in which either the input or the service-time distribution is non-Markovian are considered. Setting inequalities similar to the last section in terms of either arrival instant or departure instant, an expression of the joint distribution involving the maximum workload in the first model and the maximum queue length in the second is derived with the help of Proposition 3. The advantage of this approach is mostly achieved when the non-Markovian distribution is deterministic.

Model IV.

The model is the same as Model II except that (b) is changed to the following:

(b') Customers are served individually and the service times $\{\psi_{i\alpha}\}$ of customers from the ith source are i.i.d. random variables with distribution function $H^{(i)}(t)$ $i = 1,\dots,r$. Service times for each source are independent.

Consider only the case $r = 2$. Let $G_4 = G_4(y;\, n_1,\, n_2;\, z,t)$ be the probability that a busy period with initial work load y consists of n_i batches arriving from the ith source $(i = 1, 2)$, has a maximum work load $\leq z$ and has length $\leq t$.

Denote by $\tau_{i,j}$ the (i,j)th arrival instant (see Model II for the definition) and by T_{ij} the instant of the next immediate arrival instant. A busy period with the given parameters in G_4 must satisfy the following inequalities:

$$(21)\quad \begin{cases} 0 \le T_{oo} \le y \\ \tau_{10} \le T_{10} \le y + \sum_{\alpha=1}^{u_1} \psi_{1\alpha} \\ \tau_{01} \le T_{01} \le y + \sum_{\alpha=1}^{u_2} \psi_{2\alpha} \\ \tau_{ij} \le T_{ij} \le y + \sum_{\alpha=1}^{iu_1} \psi_{1\alpha} + \sum_{\alpha=1}^{ju_2} \psi_{2\alpha} \\ i = 1,\ldots,n_1,\ j = 1,\ldots,n_2. \end{cases}$$

But for all (i,j) τ_{ij} is related to z as follows:

$$y - \tau_{ij} + \sum_{\alpha=1}^{iu_1} \psi_{1\alpha} + \sum_{\alpha=1}^{ju_2} \psi_{2\alpha} \le z.$$

Therefore (21) becomes

$$(22)\quad \begin{cases} 0 \le T_{oo} \le y \\ y - z + \sum_{\alpha=1}^{iu_1} \psi_{1\alpha} + \sum_{\alpha=1}^{ju_2} \psi_{2\alpha} \le T_{ij} \le y + \sum_{\alpha=1}^{iu_1} \psi_{1\alpha} + \sum_{\alpha=1}^{ju_2} \psi_{2\alpha} \\ i = 0,\ldots,n_1,\ j = 0,\ldots,n_2,\ (i,j) \ne (0,0). \end{cases}$$

(Notice the similarity between (14) and (22).)

Under the condition that there are n_i arrivals from the ith source ($i = 1, 2$) during a busy period of length t, T_{ij}'s form two independent sets of order statistics from the uniform distribution over $(0,t)$, the joint p.d.f. of which is

$$(23)\quad \frac{n_1!\, n_2!}{t^{n_1+n_2}}$$

Represent the arrival time on x_o-axis followed by a unit on x_i-axis if the arrival is from the ith source. Then the sequences of arrival instants of a busy period are represented by a set of paths in Proposition 3 each of which has the weight (23). Given the length of the busy period to be x, the upper and lower restrictions $A_1(\underline{n})$ and $B_1(\underline{n})$ of the set are given by

$$a_1(i,j) = \min(x,\ y + \sum_{\alpha=1}^{iu_1} \psi_{1\alpha} + \sum_{\alpha=1}^{iu_2} \psi_{2\alpha})$$

and

$$b_1(i,j) = \max(0,\ y - z + \sum_{\alpha=1}^{iu_1} \psi_{1\alpha} + \sum_{\alpha=1}^{iu_2} \psi_{2\alpha})$$

$i = 0,1,\ldots,n_1$ and $j = 0,1,\ldots,n_2$.

Therefore, given there are n_i batches of arrival from the ith source $(i = 1, 2)$ and

$$(24) \qquad y + \sum_{\alpha=1}^{u_1} \psi_{1\alpha} + \sum_{\alpha=1}^{u_2} \psi_{2\alpha} = x,$$

the probability of the busy period is

$$(25) \qquad C = \frac{n_1!\, n_2!}{x^{n_1+n_2}} \quad E_x\, g_2^*\, (A_1(\underline{n})\,|\,B_1(\underline{n}))$$

where E_x stands for the conditional expectation taken over $\{\psi_{1\alpha}\}$ and $\{\psi_{2\alpha}\}$ subject to the condition (24).

Now it is a matter of routine argument as before to establish that

$$(26) \qquad G_4 = \int_y^t C\, e^{-(\lambda_1+\lambda_2)x}\, x^{n_1+n_2}\, \frac{\lambda_1^{n_1}\lambda_2^{n_2}}{n_1!\,n_2!}\, d(H^{(1)}_{u_1n_1}(x-y) * H^{(2)}_{u_2n_2}(x-y))$$

where $H^{(i)}_n$ is the nth convolution of $H^{(i)}$ with itself and * represents the convolution operator.

For any general service time distribution $H^{(i)}$ expression (26) seems to be unwieldy. However, if the service times are deterministic, namely, the service time for the customer from the ith source is $a_i (i = 1, 2)$, then the probability that a busy period having the same specifications as in G_4 except that it has length $t = y + a_1u_1n_1 + a_2u_2n_2$ is

$$(27) \qquad g_2^*(A_1(\underline{n})\,|\,B_1(\underline{n}))\; e^{-(\lambda_1+\lambda_2)t}\, \lambda_1^{n_1}\, \lambda_2^{n_2}$$

where the specialized values of restrictions are given by

$a_1(i,j) = \min(t, y + a_1u_1i + a_2u_2j)$

and

$b_1(i,j) = \max(0, y - z + a_1u_1i + a_2u_2j)$.

<u>Model V.</u>

It is characterized by Model III except that (a) is modified as follows:

(a') Customers arrive one at a time at a queueing system. Let ϕ_n denote the arrival instant of the nth customer. The interarrival times $\{\phi_n - \phi_{n-1}\}$ $n = 1, 2, \ldots$ are i.i.d. random variables with distribution function $M(t)$.

Assume that the busy period ends when the first counter is empty. For $r = 2$, let $G_5 = G_5(\ell + u_1 + u_2; n_1 + 1, n_2; k, t)$ be the probability that a busy period initially with $\ell_1 + u_1 + u_2$ customers consists of serving $n_1 + 1$ batches at the first counter, n_2 batches at the second, has a maximum queue length $\leq k$ and has length $\leq t$.

Without any confusion, we may denote by τ_{ij} the (i,j)th departure instant (see Model III for definition) and by T_{ij} the instant of the next immediate departure after the (i,j)th departure instant. A busy period with the constraints of G_5 satisfies the following inequalities:

$$(28)\qquad \begin{cases} 0 \leq T_{oo} \leq \phi_{k+1-\ell} \\ \tau_{ij} \leq T_{ij} \leq \phi_{iu_1 + ju_2 + k+1-\ell} \\ \text{where} \quad \phi_{iu+ju-\ell} \leq \tau_{ij} \end{cases}$$

for $i = 0, 1, \ldots, n_1$, $j = 0, 1, \ldots, n_2$, $(i,j) \notin \{(0,0), (n_1, n_2)\}$.

If we fix the length of the busy period to be equal to x, we have

$$\tau_{n_1+1,\, n_2} = T_{n_1,\, n_2} = x.$$

Under the condition that there are $n_1 + 1$ departures from the first counter, n_2 from the second during the busy period of length x, the joint p.d.f. of T_{ij}'s is

$$\frac{n_1!\, n_2!}{x^{n_1+n_2}}\,.$$

Represent the departure time on x_o-axis followed by a unit on x_i-axis if the departure is from the ith counter. The sequences of departure instants become a set of paths in Proposition 3, the upper and lower restrictions $A_2(\underline{n})$ and $B_2(\underline{n})$ being

$$a_2(i,j) = \min(x, \phi_{iu_1+ju_2+k+1-\ell})$$

and

$$b_2(i,j) = \max(0, \phi_{iu_1+ju_2-\ell})$$

$i = 0, 1, \ldots, n_1$ and $j = 0, 1, \ldots, n_2$.

Therefore, given there are $n_1 + 1$ batches of departure from the first counter and n_2 batches from the second and the busy period is of length x, the conditional probability of the busy period is

$$D = \frac{n_1!\, n_2!}{x^{n_1+n_2}} \quad E_x(g_2^*[(A_2(\underline{n})|B_2(\underline{n})) \tag{29}$$

the expectation being over $\{\phi_\alpha\}$ subject to the condition that the busy period is of length x.

The busy period is of length x is equivalent to $\phi_N + \sigma = x$, where $N = n_1 u_1 + (n_2 - 1)u - \ell$ and σ is the time between the Nth arrival and the last departure. Thus

$$G_5 = \int_o^t D\, e^{-(\mu_1+\mu_2)x}\, x^{n_1+n_2} \frac{\mu_1^{n_1+1}\,\mu_2^{n_2}}{n_1!\, n_2!}\, d(\int_o^x (1 - M(x - u))d\, M_N(u)), \tag{30}$$

M_N being the Nth convolution of M with itself.

When the interarrival times are deterministic, i.e.

$\phi_n - \phi_{n-1} = 1$ (say), the probability that a busy period having the same specifications as in G_5 except that it has length t is

$$(31) \qquad g_2^* \, (A_2(\underline{n}) | B_2(\underline{n})) \, e^{-(\mu_1+\mu_2)t} \, \mu_1^{n_1+1} \, \mu_2^{n_2}$$

where the specialized values of the restrictions are given by

$a_2(i,j) = \min(t, iu_1 + ju_2 + k + 1 - \ell)$

and

$b_2(i,j) = \max(0, iu_1 + ju_2 - \ell)$.

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Combinatorial methods are used to derive the distribution of the busy period in a Markovian tandem queue and queues with finite capacity in five models each involving batches.

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