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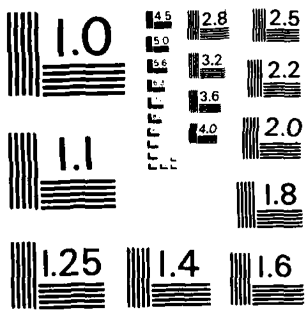

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ASYMMETRIC WIENER-POISSON CONTROL

BY

HOWARD WEINER

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Asymmetric Wiener-Poisson Control

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1. Introduction

Let $W(t)$, $t \geq 0$, $W(0) = 0$ be a standard Wiener process, independent of $N(t)$, $t \geq 0$, $N(0) = 0$, a Poisson process with constant unit jumps, and $EN(t) = \lambda t$, $\lambda > 0$. Let their sigma fields be $F(t) = \sigma(W(s), 0 \leq s \leq t)$ and $G(t) = \sigma(N(s), 0 \leq s \leq t)$. Let a stochastic process $X(t)$ be defined in terms of $u(t) \equiv u(t, X(t))$, a non-anticipating control, and $W(t)$, $N(t)$, for $0 \leq t \leq T$, $T > 0$ a constant, by the equation

$$dX(t) = u(t)dt + dW(t) + dN(t) \qquad (1)$$

$$x(0) = x, \text{ constant},$$

where $u(t)$ is measurable with respect to $\sigma(F(t) \cup G(t))$, (i.e. u is non-anticipative), and satisfies for constants $A, B > 0$,

$$|u-A| \leq B \qquad \text{all } 0 < t \leq T. \qquad (2)$$

The cost function for a given u satisfying (2), is, for $\alpha > 0$.

$$J(u) = \int_0^T e^{-\alpha y} E(X^2(y))dy. \qquad (3)$$

The object of this paper is to exhibit sufficient conditions so that a solution of a resultant Bellman equation yields an optimal admissible control $u_0(t)$, $0 \leq t \leq T$ which minimizes (3). The sufficient conditions are that the solutions to two homogeneous, constant coefficient partial differential-difference equations have solutions of certain growth, and that the Bellman function satisfy certain matching and boundary conditions.

The method follows Ref. 1. See also Ref. 3.

2. Finite Interval Control

Lemma 1 Let D, $\lambda > 0$, $\alpha > 0$ be constants.

The partial differential-difference equation in $V(t,x)$ given by

$$x^2 + D\frac{\partial}{\partial x}V(x,t) + \frac{1}{2}\frac{\partial^2}{\partial x^2}V(x,t) - \alpha V(x,t)$$

$$-\frac{\partial}{\partial t}V(x,t) + \lambda(V(x+1,t) - V(x,t)) = 0 \tag{4}$$

has a particular solution expressible as

$$J(D,x,t) \equiv \int_0^t e^{-\alpha y}E(Dy + N(y) + W(y) + x)^2 dy =$$

$$\frac{x^2}{\alpha}(1-e^{-\alpha t}) + \left(\frac{1-\alpha t e^{-\alpha t} - e^{-\alpha t}}{\alpha^2}\right)(\lambda + 1 + 2Dx + 2\lambda x)$$

$$+ \frac{(D+\lambda)^2}{\alpha^3}(2 - 2\alpha t e^{-\alpha t} - 2e^{-\alpha t} - \alpha^2 t^2 e^{-\alpha t}), \tag{5}$$

where $N(y)$, $W(y)$ are as in section 1.

The differential-difference equation in $V(x)$ given by

$$x^2 + DV'(x) + \frac{1}{2}V''(x) - \alpha V(x) + \lambda(V(x+1) - V(x)) = 0 \tag{6}$$

has a particular solution expressible as

$$J(D,x) \equiv \int_0^\infty e^{-\alpha y}E(Dy + N(y) + W(y) + x)^2 dy =$$

$$\frac{x^2}{\alpha} + \frac{\lambda+1+2Dx+2\lambda x}{\alpha^2} + \frac{2(D+\lambda)^2}{\alpha^3}. \tag{7}$$

Proof The proofs are direct computation upon expansion and evaluation of (5), (7) respectively.

Remark: The solutions represent the respective costs, if the constant control $u(t) \equiv D$ is employed.

Theorem 1. Let $X(t)$ be given, for $0 \leq t \leq T$, by

$$dX(t) = u\,dt + dN(t) + dW(t)$$

$x(0) = x$ with assumptions of section 1, with cost function, for $0 < T < \infty$ constant,

$$J(u) = \int_0^T e^{-\alpha y} E(X^2(y))dy.$$

The optimal control $u_0(t)$ which satisfies

$$|u_0(t) - A| \leq B \qquad 0 \leq t \leq T$$

is given by

$$u_o(t) = \begin{cases} A-B & \text{if } X_0(t) \geq b(T-t) \\ A+B & \text{if } X_0(t) < b(T-t) \end{cases} \tag{8}$$

where

$$dX_0(t) = u_0 dt + dN(t) + dW(t)$$

and it is assumed that $b(t)$ satisfies transcendental equations (13), (20)-(21), given below.

<u>Proof.</u> The Bellman equation for

$$V(t,x) \equiv \inf_{|u-A| \leq B} \int_0^t e^{-\alpha y} E(X^2(y))dy \tag{9}$$

with $X(0) = x$

is seen by heuristics or from Ref. 2 pp. 179-180 to be, where now $\frac{\partial}{\partial x} V \equiv V_x$, $\frac{\partial^2}{\partial x^2} V \equiv V_{xx}$, etc.,

$$x^2 + \inf_{|u-A| \leq B} (uV_x(x,t)) + \frac{1}{2} V_{xx}(x,t) - \alpha V(x,t)$$

$$- V_t(x,t) + \lambda(V(x+1, t) - V(x,t)) = 0. \tag{10}$$

Intuitive considerations suggest that a function $b(t)$ be sought such that

$V_1(x,t)$ satisfies

$$x^2 + (A-B)\ V_x(x,t) + \frac{1}{2}\ V_{xx}(x,t) - \alpha V(x,t) - V_t(x,t)$$

$$+ \lambda(V(x+1,t) - V(x,t)) = 0 \qquad (11)$$

$$\text{if } V_x(x,t) > 0 \text{ and } x > b(t),$$

and $V_2(x,t)$ satisfies

$$x^2 + (A+B)\ V_x(x,t) + \frac{1}{2}\ V_{xx}(x,t) - \alpha V(x,t) - V_t(x,t)$$

$$+ \lambda(V(x+1,t) - V(x,t)) = 0 \qquad (12)$$

$$\text{if } V_x(x,t) < 0, \text{ and } x < b(t).$$

The boundary conditions are, for $0 \le t \le T$,

$$V_1(x,0) = V_2(x,0) = 0 \quad \text{all} \quad x,$$

$$\frac{\partial}{\partial x}\ V_1(b(t),t) = \frac{\partial}{\partial x}\ V_2(b(t),t) = 0$$

$$V_1(b(t),t) = V_2(b(t),t) \qquad (13)$$

By Lemma 1, $J(A-B,x,t)$, $J(A+B,x,t)$ are particular solutions of (11), (12) respectively.

<u>Assumption 1</u>. There is a non-zero solution $H_1(x,t)$ to (omitting (x,t) arguments)

$$(A-B)H_x + \frac{1}{2}H_{xx} - \alpha H - H_t + \lambda(H(x+1,t)-H) = 0 \qquad (14)$$

such that

$$H_1(x,0) = 0 \qquad (15)$$

$$H_1(x,t,y) = O(e^{-\beta x})$$

$$H_{1,xx}(x,t) = O(e^{-\delta x}) \qquad (16)$$

for some $\beta > 0$, $\delta > 0$, each t, as $x \to \infty$.

Also, there is a non-zero solution $H_2(x,t,y)$ with

$$H_2(x,0) = 0$$

to

$$(A+B)H_x + \frac{1}{2}H_{xx} - \alpha H - H_t + \lambda(H(x+1,t)-H) = 0 \tag{17}$$

such that

$$H_2(x,t) = O(e^{\gamma x}) \tag{18}$$

$$H_{2,xx}(x,t) = O(e^{\lambda x}) \tag{19}$$

for some $\gamma > 0$, $\lambda > 0$, each t, as $x \to -\infty$.

Then one has

$$V_1(x,t) = J(A-B,x,t) + H_1(x,t) \tag{20}$$

$$V_2(x,t) = J(A+B,x,t) + H_2(x,t) \tag{21}$$

and b(t) is determined by (13), (20)-(21).

Lemma 2

$$V_{xx}(x,t) \equiv \begin{cases} V_{1,xx}(x,t) > 0, & x > b(t) \quad (22) \\ V_{2,xx}(x,t) > 0, & x < b(t) \quad (23) \end{cases}$$

Proof Let $W(x,t) = V_{xx}(x,t)$.

Then from (11), (12),

$$(A-B)\ W_x(x,t) + \frac{1}{2} W_{xx}(x,t) - (\alpha+\lambda)\ W(x,t)$$

$$- W_t(x,t) = -2 - \lambda W(x+1,t) \qquad (24)$$

$$x > b(t)$$

$$(A+B)\ W_x(x,t) + \frac{1}{2} W_{xx}(x,t) - (\alpha+\lambda)\ W(x,t)$$

$$- W_t(x,t) = -2 - \lambda W(x+1,t) \qquad (25)$$

$$x < b(t)$$

By construction of $V_1(x,t)$, $V_2(x,t)$ in (14)-(21), $W(x,t) = V_{xx}(x,t) > 0$ for each t, for all x sufficiently large.

Suppose there is an x_0 finite, possibly depending on t, such that $W(x_0,t) < 0$, and $W(x,t) \geq 0$, $x > x_0$.

Then the left sides of (24), (25) are negative for $x > x_0 - 1$. By ref. 4, Lemma 1, p. 34, $W(x,t)$ cannot have a negative minimum for $x > x_0 - 1$. But since if $W(x_0,t) < 0$, and $W(x,t) \geq 0, x > x_0$, $W(x,t) \geq 0$, $x \to -\infty$, then W would have a negative minimum for $x > x_0 - 1$, a contradiction to the existence of x_0, hence Lemma 2 is proved.

To complete the proof it is required to show that $u_0(t)$, $0 \le t \le T$ is optimal, given as a separate lemma.

Lemma 3. $u_o(t)$ of (8) is optimal.

Proof Define for an admissible u, where $|u-A| \le B$,

$$dX(t) = udt + dN(t) + dW(t)$$
$$X(0) = x$$

and let

$$H(X(t),t) \equiv \begin{cases} e^{-\alpha t}V_1(X(t),T-t) & \text{if } X(t) > b(T-t) \\ e^{-\alpha t}V_2(X(t),T-t) & \text{if } X(t) < b(T-t) \end{cases}$$

or
$$H(X(t),t) = e^{-\alpha t}V(X(t),T-t) \tag{26}$$

Using Ito's formula, (Ref. 2, p. 126)

$$H(X(T),T) = 0, \quad H(X(0),0) = V(x,T), \tag{27}$$

one obtains that, upon integrating from 0 to T,

$$\int_0^T e^{-\alpha y}(X^2(y))dy - V(x,T) = \int_0^T e^{-\alpha y}(-\alpha V(X(y),y) - V_t(X(y),y)$$
$$+ u(X(y),y)\ V_x(X(y),y) + X^2(y) + \frac{1}{2} V_{xx}(X(y),y))dy$$
$$+\int_0^T e^{-\alpha y}(V(X(y),y)dN(y)$$
$$+\int_0^T e^{-\alpha y}V_x(X(y),y)dW(y). \tag{28}$$

Upon taking expectations in (28), one obtains

$$\int_0^T e^{-\alpha y} E(X^2(y))dy - V(x,T) =$$

$$E\int_0^T e^{-\alpha y}(-\alpha V(X(y),y) - V_t(X(y),y) + \inf_{|u-A|\le B}(u(X(y),y)V_x(X(y),y))$$

$$+ X^2(y) + \frac{1}{2} V_{xx}(X(y),y) +$$

$$+ \lambda(V(X(y) + 1, y) - V(X(y),y))dy$$

$$+E\int_0^T [u(X(y),y)V_x(X(y),y) - \inf_{|u-A|\le B}(u(X(y),y)V_x(X(y),y))]dy \tag{29}$$

The first integral on the right side of (29) is zero by (10), and the second integral on the right is non-negative, with equality for $u = u_0$.

Hence

$$\int_0^T e^{-\alpha y} E(X^2(y))dy \ge V(x,T) \tag{30}$$

for any admissible u, and

$$\int_0^T e^{-\alpha y} E(X_0^2(y))dy = V(x,T), \tag{31}$$

for $u = u_0$, so that u_0 is optimal.

Remark: There is no claim that u_0 above is unique.

3. Infinite Interval Control

Theorem 2. Let $X(t)$ be given by

$$dX(t) = udt + dN(t) + dW(t)$$

for all $t > 0$, and $X(0) = x$, satisfying the assumptions of section 1 with cost function

$$J(u) = \int_0^\infty e^{-\alpha y} E(X^2(y))dy. \tag{32}$$

The optimal control $u_0(t)$ which satisfies

$$|u_0(t) - A| \leq B \text{ all } t > 0 \text{ is}$$

given by

$$u_1(t) = \begin{cases} A - B & \text{for } X_1(t) > b \\ A + B & \text{for } X_1(t) < b \end{cases} \tag{33}$$

where

$$dX_1(t) = u_1 dt + dN(t) + dW(t),$$

and it is assumed that b *is a constant which satisfies trancendental relations* (38)-(42).

<u>*Proof*</u> *The Bellman equation for*

$$V(x) \equiv \inf_{|u-A|\leq B} \int_0^\infty e^{-\alpha y} E(X^2(y))dy \tag{34}$$

with $X(0) = x$ is (Ref. 2, pp. 179-180)

$$x^2 + \inf_{|u-A|\leq B} (uV'(x)) + \frac{1}{2} V''(x) - \alpha V(x) + \lambda(V(x+1)-V(x)) = 0. \tag{35}$$

A solution of the following form is sought. $V_1(x)$ satisfies

$$x^2 + (A-B)V'(x) + \frac{1}{2} V''(x) - \alpha V(x) + \lambda(V(x+1)-V(x)) = 0. \tag{36}$$

for $V'(x) > 0$, $x > b$

and $V_2(x)$ satisfies

$$x^2 + (A+B)V'(x) + \frac{1}{2} V''(x) - \alpha V(x) + \lambda(V(x+1)-V(x)) = 0 \tag{37}$$

where

$$V'(x) < 0, \; x < b.$$

The matching conditions are

$$V_1'(b) = V_2'(b) = 0$$

$$V_1(b) = V_2(b). \tag{38}$$

By Lemma 1, J(A-B,x) is a particular solution to (36) and J(A+B,x) is a particular solution to (37). A solution to the homogeneous parts of (36),(37) is obtained as follows: let

$$f(r) = r^2 + 2(A-B)r - 2(\alpha + \lambda) + 2\lambda e^r \tag{39}$$

and

$$k(r) = r^2 + 2(A+B)r - 2(\alpha + \lambda) + 2\lambda e^r. \tag{40}$$

Since $f(0) = k(0) = -2\alpha < 0$, and $f(-\infty) = +\infty$, $k(\infty) = +\infty$, there exist $r_1 < 0$, $r_2 > 0$ such that $f(r_1) = k(r_2) = 0$.

Hence a solution to (36) is

$$V_1(x) = J(A-B,x) + Ce^{r_1 x} \tag{41}$$

for $x > b$

and $V_2(x) = J(A+B,x) + De^{r_2 x}$ (42)
for $x < b$,

and it is assumed that constants C,D,b are determined by conditions (38). It is required to show that (41), (42) solve the Bellman equation, that is, that (36),(37) hold.

<u>Lemma 4.</u> The function $V(x) = \begin{cases} V_1(x), & x > b \\ V_2(x), & x < b \end{cases}$ of (41), (42) satisfying (38) is a solution to the Bellman equation (36)-(37).

<u>Proof</u> Since $V_1'(b) = V_2'(b) = 0$, it suffices to show that $V_1'(x) > 0$ for $x > b$ and $V_2'(x) < 0$ for $x < b$. For this it suffices to show that $V''(x) > 0$ all $x \neq b$. Let $w(x) \equiv V''(x)$ and from (36),(37), one obtains that

$$(A-B)w'(x) + \frac{1}{2} w''(x) - (\alpha + \lambda)w(x) = -2 - \lambda w(x+1) \tag{43}$$

and for $x > b$

$$(A+B)w'(x) + \frac{1}{2} w''(x) - (\alpha + \lambda)w(x) = -2 - \lambda w(x+1) \tag{44}$$

for $x < b$.

By construction of the solution in (39)-(42),

$w(x) > 0$ for all x sufficiently large.

Suppose there is an x_0 such that $w(x_0) < 0$ and $w(x) > 0$, $x > x_0$. Then the right sides of (43),(44) are negative for $x > x_0-1$, and hence the left sides of (43),(44) are negative for $x > x_0-1$. By Ref. 4, p. 53, Theorem 19, $w(x)$ cannot have a negative minimum for $x > x_0-1$, a contradiction to the existence of x_0 such that $w(x_0) < 0$. This suffices to prove Lemma 4. To complete the proof of Theorem 2 it remains to show that $u_1(t)$ of (33) is optimal.

<u>Lemma 5</u> $u_1(t)$ is optimal.

<u>Proof</u> For a fixed u, $|u-A| \leq B$, let

$$dX(t) = u(t)dt + dW(t) + dN(t)$$

$$X(0) = x$$

and define

$$H(X(t),t) \equiv V(X(t))e^{-\alpha t} = \begin{cases} V_1(X(t))e^{-\alpha t}, & X(t) > b \\ V_2(X(t))e^{-\alpha t}, & X(t) < b. \end{cases} \tag{45}$$

Noting that $H(X(0),0) = V(x)$, an application of Ito's formula (Ref. 2, p. 126) yields, upon subsequent integration from 0 to t,

$$\int_0^t e^{-\alpha y}(X^2(y))dy + e^{-\alpha t}V(X(t))-V(x) =$$

$$\int_0^t e^{-\alpha y}(-\alpha V(X(y)) + u(X(y))V'(X(y)) + X^2(y) + \frac{1}{2} V''(X(y))dy$$

$$+ \int_0^t e^{-\alpha y}(V(X(y))dN(y) \qquad + \int_0^t e^{-\alpha y}V'(X(y))dW(y). \tag{46}$$

Upon taking expectations in (46), one obtains

$$\int_0^t e^{-\alpha y}E(X^2(y))dy - e^{-\alpha t}EV(X(t))-V(x) =$$

$$E\int_0^t e^{-\alpha y}(-\alpha V(X(y) + \inf_{|u-A|\leq B} u(X(y))V'(X(y)) + X^2(y)$$

$$+ \frac{1}{2} V''(X(y)) + \lambda(V(X(y)+1) - V(X(y)))dy$$

$$+E\int_0^t e^{-\alpha y}(u(X(y))V'(X(y)) - \inf_{|u-A|\leq B} u(X(y))V'(X(y))dy \tag{47}.$$

The first term on the right of (47) is zero by definition of V in (35)-(37). The second term on the right is non-negative.

By construction of $V(x)$ in (39)-(42), for t large,

$$EV(X(t)) \leq KE(x + N(t) + |W(t)| + (|A|+B)t)^2 \leq Mt^2, \tag{48}$$

for suitable constants K, M.

Hence, letting $t \to \infty$ in (47), in view of (48) one obtains that

$$\int_0^\infty e^{-\alpha y} E(X^2(y))dy \geq V(x) \tag{49}$$

with equality if $u = u_1$ and $X(t) = X_1(t)$, hence $u_1(t)$ is optimal. This completes Theorem 2.

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A one-dimensional Wiener plus independent Poisson control problem with asymmetric constant bounds on the control and integral discounted quadratic cost function is considered. The resultant Bellman equation is solved when two homogeneous partial differential-difference equations are solvable and when the Bellman function satisfies certain matching and boundary conditions. These sufficient conditions would allow the optimal control to be expressed in bang-bang form.

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