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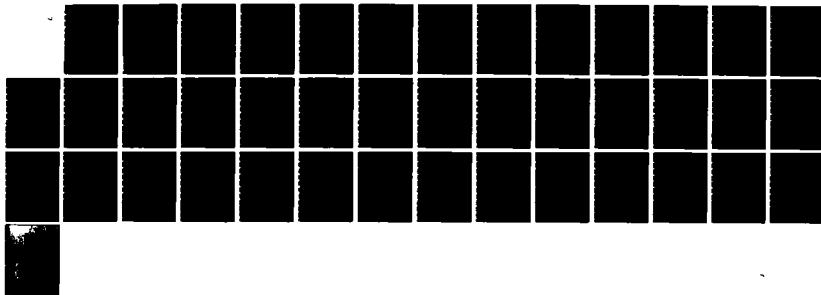
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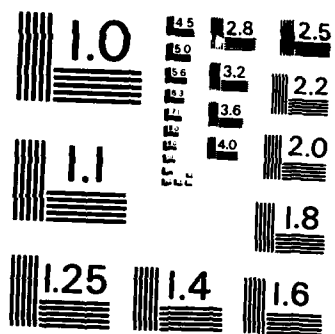
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THE GENERATION OF A MASS POINT  
MODEL FROM SURFACE GRAVITY DATA

HANS SÜNDEL

DEPARTMENT OF GEODETIC SCIENCE AND SURVEYING  
THE OHIO STATE UNIVERSITY  
COLUMBUS, OHIO 43210

DECEMBER 1983

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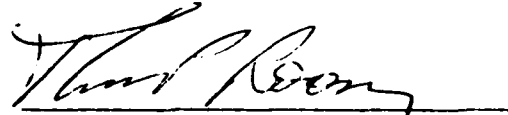
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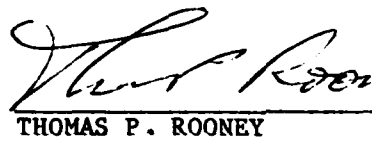
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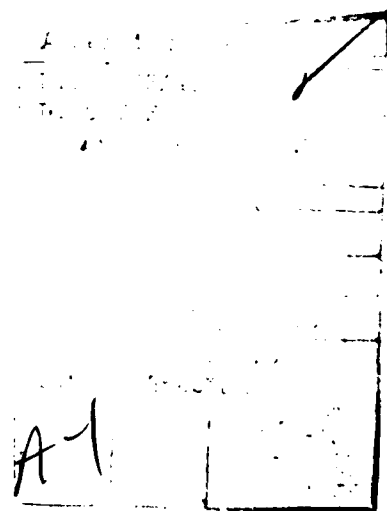
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20. ABSTRACT (Continue on reverse side if necessary and identify by block number) The generation of a point mass model from surface gravity data requires the employment of sophisticated techniques mainly because a large number of unknowns has to be determined from a large number of data. The method proposed relies on the transformation of the problem into the frequency domain. The calculation of the coefficients of the transformation matrix requires the evaluation of an integral of an isotropic kernel with respect to a limited area on the unit sphere, if mean values are used as		

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## FOREWORD

This report was prepared by Dr. Hans Sünkel, Institute of Mathematical Geodesy, Technical University at Graz, Austria, under Air Force Contract No. F19628-82-K-0017, The Ohio State University Research Foundation, Project No. 714255, Project Supervisor, Urho A. Uotila, Professor, Department of Geodetic Science and Surveying. The contract covering this research is administered by the Air Force Geophysics Laboratory (AFGL), Hanscom Air Force Base, Massachusetts, with Christopher Jekeli, Contract Monitor.

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## I. INTRODUCTION

Among the various methods for the representation of the earth's anomalous gravity field, the point mass model representation is particularly attractive both because of its conceptual simplicity and probably because of its closest "neighborhood" to geological reality among all proposed methods. The generation of a point mass model, however, is not only non-unique, but also not quite simple. The difficulty lies in a) the non-uniqueness of the problem, b) the computationally demanding relation between mean values and point values, c) the use of mean values of various kinds like  $5^{\circ} \times 5^{\circ}$ ,  $1^{\circ} \times 1^{\circ}$ , and so on.

In a former report (Sünkel, 1982) we demonstrated the reasons for a multi-level mass point model employing known statistical properties of the earth's gravity field. In the present contribution the conceptual mathematical problems have been solved and a conceivable algorithm for the actual generation of a mass point model presented. Frequency domain methods have been frequently used throughout this paper.

The evaluation of the kernel which relates point masses and mean gravity disturbances, requires integration on the sphere over a limited area - a time-consuming process. In order to speed up calculation, a fast approximation has to be designed and the approximation error to be estimated. Using Peano's theorem on the sphere and the method of Sard, "best" approximations have been derived for various kernel approximation functions and the corresponding approximation errors have been estimated.

The relation between the depth of the point mass level and a set of mean gravity data at zero level has been derived from the principle of minimum deviation of 2 operators: the smoothing operator (which transforms point data into mean values) and the operator which turns point masses at

a certain level into its gravitation.

The algorithm for the mass point model generation is entirely based on fast Fourier transform methods:

the data and the operator are transformed into the frequency domain, the solution of the linear system performed, and the result re-transformed into the space domain yielding the required set of point masses. Due to the use of mean values of various kind, a recursive procedure had to be designed.

## 2. POINT MASSES AND MEAN GRAVITY DISTURBANCES

The gravitational potential  $T$  generated by a set of point masses  $\{m_j\}$  is given by

$$T(P) = \sum_{j=1}^J \frac{Gm_j}{l(P, Q_j)} ; \quad (2-1)$$

$G$  denotes the gravitational constant,  $l(P, Q_j)$  the spatial distance between the calculation point  $P$  and the location  $Q_j$  of the point mass  $m_j$ . The negative radial component of the corresponding gravitational vector  $\nabla T$  will be denoted gravity disturbance and can easily be derived from (2-1),

$$\delta g(P) = - \frac{\partial T}{\partial r} = \sum_{j=1}^J \frac{r_P - r_j \cos \psi_j}{l^3(P, Q_j)} Gm_j , \quad (2-2)$$

where  $r$  stands for radius. Given  $\{m_j\}$ , the calculation of  $\delta g$  is straightforward.

Let us now investigate the relation between point masses and mean gravity disturbances, assuming that all the point masses are located on a sphere with radius  $\alpha < 1$  with the gravity disturbance determined for a point on the unit sphere  $\alpha = 1$  concentric to the former sphere. Then  $r_P = 1$  and  $r_j = \alpha$  and the position of  $P$  on the unit sphere is given by the unit vector  $\xi$ , the position of  $Q_j$  by  $\alpha \eta$  with the unit vector  $\eta$ .

The kernel in equation (2-2) is homogeneous and isotropic and can be represented in terms of a series of Legendre polynomials  $P_n(\cos \psi)$  (Heiskanen & Moritz, 1967, p.35),

$$\frac{(1 - \alpha \cos \psi)}{l^3} = \sum_{n=0}^{\infty} (n+1) \alpha^n P_n(\cos \psi) . \quad (2-3)$$

Representing  $P_n$  in terms of orthonormal surface spherical harmonics  $\{\phi_{nm}\}$ ,  $n = 0, \dots$ ;  $m = -n, \dots, n$ ,

$$\int_{\sigma} \phi_{nm}(\xi) \phi_{kl}(\xi) d\sigma(\xi) = \delta_{nk} \delta_{ml} \quad +) \quad (2-4)$$

with the decomposition formula

$$P_n(\xi, \eta) = \frac{4\pi}{2n+1} \sum_{m=-n}^n \phi_{nm}(\xi) \phi_{nm}(\eta) \quad , \quad (2-5)$$

the dependence of the kernel (2-3) on the position of  $P$  and  $Q$  on the unit sphere, represented by the unit vectors  $\xi$  and  $\eta$ , is given by

$$\frac{(1-\alpha \cos \psi_{PQ})}{r_{PQ}^3} = 4\pi \sum_{n=0}^{\infty} \frac{n+1}{2n+1} \alpha^n \sum_{m=-n}^n \phi_{nm}(\xi) \phi_{nm}(\eta) \quad . \quad (2-6)$$

If we want to determine the kernel between a point mass and a mean gravity disturbance, we have to calculate the mean of (2-6) with respect to  $P$ ,

$$M_P \left\{ \frac{(1-\alpha \cos \psi_{PQ})}{r_{PQ}^3} \right\} = 4\pi \sum_{n=0}^{\infty} \frac{n+1}{2n+1} \alpha^n \sum_{m=-n}^n \phi_{nm}(\eta) M_P \{ \phi_{nm}(\xi) \}, \quad (2-7)$$

where  $M$  stands for the mean value operator

$$M \{ (\cdot) \} = \omega^{-1} \int_{\omega} (\cdot) d\sigma \quad (2-8)$$

taken over a limited part  $\omega \subset \sigma$  of the unit sphere  $\sigma$ . The mean gravity disturbance  $\tilde{\delta g}$ ,

---

+ ) Note the difference between our definition and the definition of fully normalized spherical harmonics of (Heiskanen & Moritz, p.31 )

$$\tilde{\delta g}_P = M_P \{\delta g\}$$

can then be expressed by

$$\tilde{\delta g}_P = 4\pi \sum_{j=1}^J Gm_j \sum_{n=0}^{\infty} \frac{n+1}{2n+1} \alpha^n \sum_{m=-n}^n \phi_{nm}(r_j) \tilde{\phi}_{nm}(\xi). \quad (2-9)$$

It is evident that the rate of convergence of (2-9) depends on  $\alpha$  and therefore on the depth  $(1-\alpha)$  of the point mass sphere. Shallow point masses will cause a slow convergence, deep point masses a rapid convergence. (If  $\alpha = 0$ , all the point masses are concentrated at the center of the sphere and all terms  $n > 0$  are annihilated by the powers of  $\alpha$ .) Therefore, for very deep masses the summation can be terminated at a modest  $n = N$ ; however, in our study even for the most favourable depth ( $D = 550\text{km}$ ) the summation has to be carried out at least up to the degree  $n = 150$  in order to obtain 6 significant digits. Consequently, the use of equation (2-9) for the calculation of mean gravity disturbances is prohibitive despite the existence of a very smart algorithm for the calculation of integrals of associated Legendre functions (Gerstl, 1980).

As an alternative, we suggest a local approximation of the kernel by a low degree polynomial which, restricted to the sphere, is a linear combination of low degree spherical harmonics. Such an approximation allows us to estimate the error of the kernel's mean value for the area in consideration by taking advantage of Peano's theorem and moreover, to estimate the best possible approximation in the sense of Sard.

### 3. LOCAL KERNEL APPROXIMATIONS

In a paper by Sünkel and Rummel (1981) one possible local kernel approximation has been proposed; an integration error estimate has been given for the very special case that the calculation point coincides with the pole. In this particular case, the one-dimensional Peano theorem can be applied. Considerable effort in overcoming this deficiency has been made by W. Freeden culminating in a particularly beautiful elaborate investigation "On Spherical Spline Interpolation and Approximation" (1981).

In the sequel, the results of this paper are used from an application point of view. In particular, we are interested in both the estimate of the error which we commit, if for the integration a local low degree polynomial approximation of the kernel is employed, and in the best approximation by such a polynomial. Since the author of this paper favors the inductive approach, the simplest case will be studied first. For the sake of simplicity we further assume in the following derivations that the calculation point is kept fixed; then the kernel to be studied is a function of the integration point only.

#### 3.1. Approximation of degree zero ( $J = 0$ )

The simplest approximation of a mean value of a function  $f$  is obviously given by a single function value which is usually taken at the center of the area in consideration. The mean value of a function (for a limited area  $\omega \subset \sigma$  of the unit sphere) is a linear functional  $L$  applied to the function  $f$  and can be represented in terms of

$$Lf = \int_{\sigma} g(n)f(n)d\sigma(n) \quad (3-1)$$

with the characteristic  $g$

$$g(n) = \begin{cases} \omega^{-1} & \text{for } n \in \omega \\ 0 & \text{else .} \end{cases} \quad (3-2)$$

This mean value is approximated by a single function value at a preselected point  $\xi_1$ , which is again a linear functional  $\hat{L}$ ,

$$\hat{L}f = a_1 f(\xi_1) ; \quad (3-3)$$

the remainder  $Rf$  (another linear functional) is consequently given by

$$\begin{aligned} Rf &= Lf - \hat{L}f = (L - \hat{L})f \\ &= \int_{\sigma} g(n)f(n)d\sigma(n) - a_1 f(\xi_1) . \end{aligned} \quad (3-4)$$

For the following we assume that  $f \in C^2(\sigma)$ , the space of all twice continuously differentiable functions on  $\sigma$ .

### Peano's Theorem

Let us assume that  $R$  annihilates all zero degree polynomials,  $Rh = 0 \quad \forall h \in \mathbb{P}_0$ . Then, according to Peano's theorem,  $Rf$  can be represented in terms of

$$Rf = \int_{\sigma} K_0(n) \Delta_n^* f(n) d\sigma(n) \quad (3-5)$$

with the spherical Peano kernel

$$K_o(n) = - \frac{1}{4\pi} R_\xi G_o^{(1)}(\xi, n) \quad (3-6)$$

and Green's function (Freedon, p.566)

$$G_o^{(1)}(\xi, n) = \sum_{n=1}^{\infty} \frac{2n+1}{n(n+1)} P_n(\xi \cdot n) \quad (3-7)$$

In this context  $\Delta_\xi^*$  denotes the Beltrami-operator, which is simply the restriction of the Laplace-operator  $\Delta$  to the unit sphere  $\sigma$ ,

$$\Delta_\xi^* = \frac{\partial}{\partial t} (1-t^2) \frac{\partial}{\partial t} + \frac{1}{1-t^2} \left( \frac{\partial}{\partial l} \right)^2, \quad t: = \cos\theta \quad (3-8)$$

applied at the point  $\xi \in \sigma$  having spherical polar coordinates  $\theta$  and  $l$ .

Using Schwarz's inequality, an upper bound for the approximation error can be given by

$$(Rf)^2 \leq \int_{\sigma} [K_o(n)]^2 d\sigma(n) \cdot \int_{\sigma} [\Delta_n^* f(n)]^2 d\sigma(n) \quad (3-9)$$

The first integral on the right hand side can be shown (Freedon, p.558) to be equal to

$$A: = \int_{\sigma} [K_o(n)]^2 d\sigma(n) = \frac{1}{(4\pi)^2} R_\xi R_\zeta G_o^{(2)}(\xi, \zeta) \quad (3-10)$$

with the iterated Green's function

$$G_o^{(2)}(\xi, \zeta) = 4\pi \sum_{n=1}^{\infty} \frac{2n+1}{[n(n+1)]^2} P_n(\xi \cdot \zeta) \quad (3-11)$$



According to the decomposition theorem, the Legendre polynomials  $P_n(\xi, \zeta)$  can be expressed in terms of the normalized surface spherical harmonics  $\phi_{nm}$ ,  $m = -n, \dots, n$ ,

$$G_o^{(2)}(\xi, \zeta) = (4\pi)^2 \sum_{n=1}^{\infty} \frac{1}{[n(n+1)]^2} \sum_{m=-n}^n \phi_{nm}(\xi) \phi_{nm}(\zeta). \quad (3-11)'$$

The application of (3-10) yields

$$A \leq \sum_{n=1}^{\infty} \frac{1}{[n(n+1)]^2} \sum_{m=-n}^n [\tilde{\phi}_{nm}^2 - 2\tilde{\phi}_{nm} a_1 \phi_{nm}(\xi_1) + a_1^2 \phi_{nm}^2(\xi_1)]. \quad (3-10)'$$

The second integral on the right hand side is equally simple to evaluate: since our original kernel  $f$  to be investigated is homogeneous and isotropic and since the integration in (3-9) has to be carried out over the whole unit sphere  $\sigma$ , the position of the calculation point is immaterial; therefore, we choose the pole  $\theta = 0$ . In this case the Beltrami-operator reduces to the Legendre-operator

$$\frac{d}{dt} (1-t^2) \frac{d}{dt}, \quad (3-12)$$

and with our expression (2-3) for the kernel and with the use of the Legendre differential equation

$$\frac{d}{dt} (1-t^2) \frac{d}{dt} P_n(t) = -n(n+1)P_n(t) \quad (3-13)$$

the operation  $\Delta^* f$  yields

$$\Delta^* f = - \sum_{n=1}^{\infty} n(n+1)^2 \alpha^n P_n(t). \quad (3-14)$$

Taking into account the orthogonality of the Legendre polynomials

$$\int_{-1}^1 P_n(t)P_m(t)dt = \frac{2}{2n+1} \delta_{nm} \quad (3-15)$$

( $\delta_{nm}$ ... Kronecker symbol), we obtain

$$B := \int_{\sigma} [\Delta_n^* f(n)]^2 d\sigma(n) = 4\pi \sum_{n=1}^{\infty} \frac{n^2(n+1)^4}{2n+1} \alpha^{2n} \quad (3-16)$$

and the estimation for the upper bound of the approximation error

$$|Rf| \leq \sqrt{AB} \quad (3-17)$$

Let us investigate this estimate in more detail: for the calculation of A we have a) to calculate the mean value of surface spherical harmonics for the area in consideration and b) to evaluate the surface spherical harmonics at a (single) point. In this respect there is practically no difference between (3-10)' and (2-7); however, due to the strong damping factor  $[n(n+1)]^{-2}$ , the series (3-10)' converges strongly and, as a consequence, only relatively few terms are necessary compared to (2-7) which, even under optimal circumstances ( $D = 550\text{km}$ ), has to be evaluated up to a very high degree (about 3 times higher if the result is to be accurate to 6 significant digits); the situation is getting even worse with smaller D. The calculation of the factor B is very simple and has to be carried out only once for each D. It is evident that the convergence of (3-16) is faster for a larger D. The behavior of (3-16) is quite similar to the one shown in (Sünkel, 1981, Fig.2.1, pp. 9,10) with the maximum shifted to the right due to the higher power in n.

The next step will be to find the best value for the coefficient  $a_1$  such that the remainder (3-17) is minimized under the constraint  $Rh = 0 \quad \forall \quad h \in \mathbb{P}_0$ .

Since  $B$  is independent of  $a_1$ , it is a mere scale factor and can be put equal to 1. Therefore, the optimization problem can be formulated as follows:

$$\begin{aligned} \frac{\partial}{\partial a_1} \left\{ \frac{A}{Z} - \sqrt{4\pi\lambda} \left[ \int_{\sigma} g(n)\phi_{o,o}(n)d\sigma(n) - a_1\phi_{o,o}(\xi_1) \right] \right\} &= 0 \\ \frac{\partial}{\partial \lambda} \left\{ \begin{array}{c} - \text{''} - \end{array} \right\} &= 0. \end{aligned} \quad (3-18)$$

With  $\phi_{o,o} = (4\pi)^{-\frac{1}{2}}$  and (3-2) the first expression simplifies to

$$\frac{\partial}{\partial a_1} \left[ \frac{A}{Z} - \lambda(1-a_1) \right] = 0.$$

With the constraint mentioned before (second line of (3-18)), this leads to the following linear system with the two unknowns,  $a_1$ , and the Lagrange multiplier  $\lambda$ :

$$\begin{bmatrix} \sum_{n=1}^{\infty} \frac{1}{[n(n+1)]^2} \sum_{m=-n}^n \phi_{nm}^2(\xi_1), & 1 \\ 1, & 0 \end{bmatrix} \begin{bmatrix} a_1 \\ \lambda \end{bmatrix} = \begin{bmatrix} \sum_{n=1}^{\infty} \frac{1}{[n(n+1)]^2} \sum_{m=-n}^n \tilde{\phi}_{nm}\phi_{nm}(\xi_1) \\ 1 \end{bmatrix} \quad (3-19)$$

Since

$$\sum_{m=-n}^n \phi_{nm}^2(\xi) = \frac{2n+1}{4\pi} \quad \forall \xi \in \sigma$$

and

$$\sum_{n=1}^{\infty} \frac{2n+1}{[n(n+1)]^2} = \sum_{n=1}^{\infty} \left[ \frac{1}{n^2} - \frac{1}{(n+1)^2} \right] = 1, \quad (3-20)$$

the above linear system reduces to

$$\begin{bmatrix} \frac{1}{4\pi} & , & 1 \\ 1 & , & 0 \end{bmatrix} \begin{bmatrix} a_1 \\ \lambda \end{bmatrix} = \begin{bmatrix} \sum_{n=1}^{\infty} \frac{1}{[n(n+1)]^2} \sum_{m=-n}^n \tilde{\phi}_{nm} \phi_{nm}(\xi_1) \\ 1 \end{bmatrix} \quad (3-21)$$

with the solution

$$\begin{aligned} a_1 &= 1 \\ \lambda &= -\frac{1}{4\pi} + \sum_{n=1}^{\infty} \frac{1}{[n(n+1)]^2} \sum_{m=-n}^n \tilde{\phi}_{nm} \phi_{nm}(\xi_1) \end{aligned} \quad (3-22)$$

Therefore, the best approximation of the kernel's integral by a single kernel function value is obtained by

$$\hat{L}f = f(\xi_1)$$

and the a posteriori error estimate of this optimal solution is given by

$$(Rf)^2 \leq 4\pi \sum_{n=1}^{\infty} \frac{1}{[n(n+1)]^2} \sum_{m=-n}^n [\tilde{\phi}_{nm} - \phi_{nm}(\xi_1)]^2 \cdot \sum_{k=1}^{\infty} \frac{k^2(k+1)^4}{2k+1} \alpha^{2k} \quad (3-23)$$

A special case is obviously given if the area of averaging  $\omega$  reduces to zero and if  $\xi_1$  is (as usual) located at the center of the area. In this case  $\tilde{\phi}_{nm} = \phi_{nm}(\xi_1)$  and the error becomes evidently zero as one should expect. This concludes the error estimation of a zero degree local kernel approximation.

How do our formulas change if  $Lf$  is approximated by a linear combination of  $I > 1$  function values,

$$\hat{L}f = \sum_{i=1}^I a_i f(\xi_i) \quad (3-24)$$

such that  $Rh = (L - \hat{L})h = 0 \quad \forall h \in P_0$  ? As a matter of fact the definition of the remainder will now be

$$Rf = \int_{\sigma} g(n)f(n)d\sigma(n) - \sum_{i=1}^I a_i f(\xi_i) \quad (3-25)$$

and as a consequence

$$A \leq \sum_{n=1}^{\infty} \frac{1}{[n(n+1)]^2} \sum_{m=-n}^n \{ \tilde{\phi}_{nm}^2 - 2 \sum_{i=1}^I a_i \tilde{\phi}_{nm} \phi_{nm}(\xi_i) + \sum_{i=1}^I \sum_{j=1}^I a_i a_j \phi_{nm}(\xi_i) \phi_{nm}(\xi_j) \}. \quad (3-26)$$

Now  $A$  depends on  $I$  parameters  $\{a_i\}$  and therefore, the optimization problem requires the solution of a linear system of dimension  $I + 1$  :

$$\frac{\partial}{\partial a_1} \left\{ \frac{A}{2} - \sqrt{4\pi\lambda} \left[ \int_{\sigma} g(n) \phi_{\infty\infty}(n) d\sigma(n) - \sum_{j=1}^I a_j \phi_{\infty\infty}(\xi_j) \right] \right\} = 0 \quad (3-27)$$

$$\frac{\partial}{\partial \lambda} \left\{ \begin{array}{c} - \\ - \\ - \end{array} \right\} = 0 ,$$

which we will write in the form

$$Ux = y$$

or

$$\begin{pmatrix} U_1 & U_2 \\ U_1^T & 0 \end{pmatrix} \begin{pmatrix} x_1 \\ x_2 \end{pmatrix} = \begin{pmatrix} y_1 \\ y_2 \end{pmatrix} \quad (3-28)$$

with the  $(I \times I)$  symmetric positive definite Gram matrix  $U_1$ ,

$$U_1 = \left\{ \frac{1}{4\pi} \sum_{n=1}^{\infty} v_n^{(0)} P_n(\xi_i \cdot \xi_j) \right\}, \quad i, j, = 1, \dots, I; \quad v_n^{(0)} = \frac{2n+1}{[n(n+1)]^2}, \quad (3-29a)$$

the  $I$ -dimensional vectors  $U_2, x_1, y_1$

$$U_2^T = \{1, 1, \dots, 1\}. \quad (3-29b)$$

$$x_1^T = \{a_1, a_2, \dots, a_I\}, \quad (3-29c)$$

$$y_1^T = \left\{ \sum_{n=1}^{\infty} v_n^{(0)} \frac{1}{2n+1} \sum_{m=-n}^n \tilde{\phi}_{nm} \phi_{nm}(\xi_i) \right\}, \quad i = 1, \dots, I, \quad (3-29d)$$

and the scalars  $x_2$  and  $y_2$ ,

$$x_2 = \lambda, \quad (3-29e)$$

$$y_2 = 1. \quad (3-29f)$$

The solution of the system (3-28) can be easily found by Cholesky's method.

### 3.2. Approximation of degree one ( $J = 1$ )

The next simple approximation is given by a first degree fit which requires a minimum of 4 function values to be linearly combined.<sup>1)</sup> However, let us consider an arbitrary number  $I \geq (J + 1)^2$  of function values; then  $Lf$  will be approximated by

$$\hat{L}f = \sum_{i=1}^I a_i f(\xi_i).$$

1) ... = the dimension of the space including zero and first degree polynomials

Then Peano's theorem states that the remainder  $Rf$  can be represented by

$$Rf = \int_{\sigma} K_1(n) [(\Delta_n^*)(\Delta_n^*)_1 f(n)] d\sigma(n) \quad (3-30)$$

with  $(\Delta_n^*)_1 = \Delta_n^* + 2$  (3-31)

and the spherical Peano kernel

$$K_1(n) = \frac{1}{(4\pi)^2} R_{\xi} G_1^{(1)}(\xi, n) \quad (3-32)$$

and Green's function

$$G_1^{(1)}(\xi, n) = 4\pi \sum_{n=2}^{\infty} \frac{2n+1}{[n(n+1)][n(n+1)-2]} P_n(\xi \cdot n) . \quad (3-33)$$

With Schwarz's inequality we obtain the estimate for  $|Rf|_{\max.}$  ,

$$(Rf)^2 \leq \int_{\sigma} [K_1(n)]^2 d\sigma(n) \cdot \int_{\sigma} [(\Delta_n^*)(\Delta_n^*)_1 f(n)]^2 d\sigma(n) . \quad (3-34)$$

As in section 3.1. the first integral will be denoted by  $A$  and is equal to

$$A = \int_{\sigma} [K_1(n)]^2 d\sigma(n) = \frac{1}{(4\pi)^4} R_{\xi} R_{\zeta} G_1^{(2)}(\xi, \zeta) \quad (3-35)$$

with the iterated Green's function

$$G_1^{(2)}(\xi, n) = (4\pi)^3 \sum_{n=2}^{\infty} \frac{2n+1}{[n(n+1)]^2 [n(n+1)-2]^2} P_n(\xi \cdot n) . \quad (3-36)$$

The application of (3-35) yields

$$A \leq \sum_{n=2}^{\infty} \{[n(n+1)][n(n+1)-2]\}^{-2} \sum_{m=-n}^n \left\{ \tilde{\phi}_{nm}^2 - 2\tilde{\phi}_{nm} \sum_{i=1}^I a_i \phi_{nm}(\xi_i) \right. \\ \left. + \sum_{i=1}^I \sum_{j=1}^I a_i a_j \phi_{nm}(\xi_i) \phi_{nm}(\xi_j) \right\} \quad (3-37)$$

The factor  $\{[n(n+1)][n(n+1)-2]\}^{-2}$  goes very rapidly to zero and as a consequence only very few terms of the series have to be considered. Let us now turn to the evaluation of the second integral in (3-34) which we again denote by B,

$$B = \int_{\sigma} [(\Delta_n^*)(\Delta_n^* + 2)f(n)]^2 d\sigma(n) \quad (3-38)$$

With the representation (2-3) for our kernel we obtain

$$(\Delta^*)(\Delta^* + 2)f = \sum_{n=2}^{\infty} n(n+1)^2 [n(n+1) - 2] \alpha^n P_n(t)$$

and considering the orthogonality relation (3-15) we obtain

$$B = 4\pi \sum_{n=2}^{\infty} \frac{n^2(n+1)^4 [n(n+1)-2]^2}{2^{n+1}} \alpha^{2n} . \quad (3-39)$$

The relation between the two components A and B deserves to be discussed: from (3-10)' and (3-37) we conclude that the number of spherical harmonics and its local integral to be computed decreases rapidly with increasing degree of approximation; at the same time the number of terms needed for the calculation of B increases with comparable speed (cp. (3-16) with (3-39)) and with  $(Rf)^2 \leq A \cdot B$  we have once more a beautiful example of the balance of difficulties: blessing-burden = constant.

The best estimation of the coefficients  $\{a_i\}$ ,  $i=1, \dots, I$  is also getting more laborious for two reasons: first, because



of the relation  $I \geq (J + 1)^2$ , and second, because of the  $(J + 1)^2$  constraints, the linear system (3-28) has, for the first order approximation case ( $J = 1$ ) studied in this section, at least the dimension  $2(J + 1)^2 = 8$ . The  $(J + 1)^2$  constraints are due to the required annihilation  $Rh = 0 \quad \forall h \in P_1$ , where  $P_1$  is the linear space of all zero and first degree polynomials in three variables, restricted to  $\sigma$ . Therefore, the optimization problem is formulated as follows:

$$\begin{aligned} \frac{\partial}{\partial a_i} \left\{ \frac{A}{2} - \sqrt{4\pi} \sum_{n=0}^I \sum_{m=-n}^n \lambda_{nm} \left[ \int_{\sigma} g(n) \phi_{nm}(n) d\sigma(n) - \sum_{j=1}^I a_j \phi_{nm}(\xi_j) \right] \right\} &= 0 \\ \frac{\partial}{\partial \lambda_{nm}} \left\{ \begin{array}{c} - \text{''} - \end{array} \right\} &= 0 \end{aligned} \quad (3-40)$$

which leads to the linear system (3-28) with

$$\begin{aligned} U_1 &= \left\{ \frac{1}{\sqrt{4\pi}} \sum_{n=2}^{\infty} v_n^{(1)} P_n(\xi_i \cdot \xi_j) \right\}, \quad i, j = 1, \dots, I; \quad v_n^{(1)} := \frac{2n+1}{[n(n+1)]^2 [n(n+1)-2]^2} \\ U_2^T &= \sqrt{4\pi} \left\{ \frac{1}{\sqrt{4\pi}}, \phi_{1,-1}(\xi_i), \phi_{1,0}(\xi_i), \phi_{1,1}(\xi_i) \right\}, \quad i = 1, \dots, I, \end{aligned} \quad (3-41)$$

$$x_1^T = \{a_i\}, \quad i = 1, \dots, I,$$

$$y_1^T = \left\{ \sum_{n=2}^{\infty} v_n^{(1)} \frac{1}{2n+1} \sum_{m=-n}^n \tilde{\phi}_{nm} \phi_{nm}(\xi_i) \right\}, \quad i = 1, \dots, I.$$

Note that  $x_2$  and  $y_2$  are no longer scalars; they are now vectors of length 4, (because of the 4 constraints in the linear model we need 4 Lagrange multipliers),

$$x_2^T = \{ \lambda_{0,0}, \lambda_{1,-1}, \lambda_{1,0}, \lambda_{1,1} \},$$

$$y_2^T = \sqrt{4\pi} \{ \tilde{\phi}_{0,0}, \tilde{\phi}_{1,-1}, \tilde{\phi}_{1,0}, \tilde{\phi}_{1,1} \} .$$

As in section 3.1. the solution can be obtained by applying Cholesky's algorithm.

### 3.3. Approximation of arbitrary degree J

The general case of approximation degree J requires a linear combination of  $I \geq (J + 1)^2$  function values and the error estimation is based on the requirement that the function  $f \in C^{2(J+1)}(\sigma)$ . According to Peano's theorem the remainder Rf can be represented by

$$Rf = \int_{\sigma} K_J(\eta) \left[ (\Delta_{\eta}^*)_{J} f(\eta) \right] d\sigma(\eta) . \quad (3-42)$$

with

$$(\Delta_{\eta}^*)_{J} = \prod_{j=0}^J (\Delta_{\eta}^* + \kappa_j) = (\Delta_{\eta}^* + \kappa_0) \cdots (\Delta_{\eta}^* + \kappa_J), \quad (3-43)$$

and the eigenvalues  $\kappa_j = j(j + 1)$  of the Beltrami differential operator (Freeden, 1981, p. 553). The spherical Peano kernel  $K_J(\eta)$  is given by

$$K_J(\eta) = \frac{1}{(-4\pi)^{J+1}} R_{\xi} G_J^{(1)}(\xi, \eta) \quad (3-44)$$

and Green's function by

$$G_J^{(1)}(\xi, \eta) = (4\pi)^J \sum_{n=J+1}^{\infty} (2n+1) \left[ \prod_{j=0}^J (\kappa_n - \kappa_j) \right]^{-1} P_n(\xi \cdot \eta) \quad (3-45)$$

(Freedon, 1981, p. 556). Evidently, for  $J = 0$  we obtain (3-7), for  $J = 1$  equation (3-33). Schwarz's inequality provides an estimate for  $|Rf|_{\max}$ ,

$$(Rf)^2 \leq \int_{\sigma} [K_J(n)]^2 d\sigma(n) \cdot \int_{\sigma} [(\Delta_n^*)_J f(n)]^2 d\sigma(n). \quad (3-46)$$

The first integral is equal to

$$A = \int_{\sigma} [K_J(n)]^2 d\sigma(n) = (4\pi)^{-2(J+1)} R_{\xi} R_{\zeta} G_J^{(2)}(\xi, \zeta) \quad (3-47)$$

with the iterated Green's function

$$G_J^{(2)}(\xi, \zeta) = (4\pi)^{2(J+1)} \sum_{n=J+1}^{\infty} (2n+1) \left[ \prod_{j=0}^J (\kappa_n - \kappa_j) \right]^{-2} P_n(\xi \cdot \zeta); \quad (3-48)$$

note that  $G_J^{(2)}$  is simply the result of a convolution of  $G_J^{(1)}$  with itself,

$$G_J^{(2)} = G_J^{(1)} * G_J^{(1)}. \quad (3-49)$$

With (3-47) and (3-48)  $A$  can be shown to be equal to

$$A \leq \sum_{n=J+1}^{\infty} \left[ \prod_{j=0}^J (\kappa_n - \kappa_j) \right]^{-2} \sum_{m=-n}^n \left\{ \tilde{\phi}_{nm}^2 - 2\tilde{\phi}_{nm} \sum_{i=1}^I a_i \phi_{nm}(\xi_i) + \sum_{i=1}^I \sum_{j=1}^I a_i a_j \phi_{nm}(\xi_i) \phi_{nm}(\xi_j) \right\}. \quad (3-50)$$

The second integral

$$B = \int_{\sigma} [(\Delta_n^*)_J f(n)]^2 d\sigma(n) \quad (3-51)$$

reduces to a very simple representation if  $f$  is a homogeneous and isotropic function. This is true for the kernel function discussed here (equation (2-3)). With

$$(\Delta^*)_J f(t) = \sum_{n=J+1}^{\infty} \left[ \prod_{j=0}^J (\kappa_n - \kappa_j) \right] (n+1) \alpha^n P_n(t) \quad (3-52)$$

and the orthogonality relation (3-15) we obtain

$$P = 4\pi \sum_{n=J+1}^{\infty} \left\{ \left[ \prod_{j=0}^J (\kappa_n - \kappa_j) \right] (n+1) \alpha^n \right\}^2 (2n+1)^{-1}. \quad (3-53)$$

The best estimates of the coefficients  $\{a_i\}$  are obtained from the solution of the optimization problem

$$\frac{\partial}{\partial a_i} \left\{ \frac{A}{2} - \sqrt{4\pi} \sum_{n=0}^J \sum_{m=-n}^n \lambda_{nm} \left[ \int_{\sigma} g(\eta) \phi_{nm}(\eta) d\sigma(\eta) - \sum_{j=1}^I a_j \phi_{nm}(\xi_j) \right] \right\} = 0 \quad (3-54)$$

$$\frac{\partial}{\partial \lambda_{nm}} \left\{ \begin{array}{c} - \\ - \\ - \end{array} \right\} = 0,$$

which leads to the linear system (3-28) with

$$U_1 = \left\{ \frac{1}{4\pi} \sum_{n=J+1}^{\infty} \left[ \prod_{j=0}^J (\kappa_n - \kappa_j) \right]^{-2} (2n+1) P_n(\xi_i \cdot \xi_j) \right\}, \quad i, j = 1, \dots, I,$$

$$U_2^T = \sqrt{4\pi} \left\{ \phi_{nm}(\xi_i) \right\}, \quad n = 0, \dots, J, \quad m = -n, \dots, n, \quad i = 1, \dots, I,$$

$$x_1^T = \{a_i\}, \quad i = 1, \dots, I, \quad (3-55)$$

$$y_1^T = \left\{ \sum_{n=J+1}^{\infty} \left[ \prod_{j=0}^J (\kappa_n - \kappa_j) \right]^2 \sum_{m=-n}^n \tilde{\phi}_{nm} \phi_{nm}(\xi_i) \right\}, \quad i = 1, \dots, I,$$

$$x_2^T = \{\lambda_{nm}\} , \quad n = 0, \dots, J, \quad m = -n, \dots, n ,$$

$$y_2^T = \sqrt{4\pi}\{\tilde{\phi}_{nm}\} , \quad n = 0, \dots, J, \quad m = -n, \dots, n .$$

The advantage of a fast converging  $A$  is balanced by the disadvantages of a slowly converging  $B$  and the solution of a large linear system  $Ux = y$ . Since the convergence of  $B$  depends strongly on the value of  $\alpha$  and therefore on the depth  $D$ , we conclude that a first (probably even a second) degree approximation is advantageous for a small  $\alpha$  (large  $D$ ). If  $\alpha$  is very close to 1 (small  $D$ ), we are obviously in troubles because due to the poor convergence of  $B$  we have to keep the degree of approximation low (probably at zero); as a consequence, the convergence of  $A$  will be very poor and the estimation of the approximation error a laborious expensive task.

#### 4. GRAVITY DISTURBANCE MEAN VALUES VERSUS DEPTH OF POINT MASSES

In this chapter we shall investigate at which depth the point masses are most likely to be located if they are to be derived from mean gravity disturbances of various size like  $5^\circ \times 5^\circ$ ,  $1^\circ \times 1^\circ$ , etc. In other words we look for (gravity disturbances) averaging operators with a spectrum (eigenvalues) as close as possible to the spectrum of the operator transforming the point masses into gravity disturbances.

For this purpose we will assume a continuous (rather than discrete) anomalous mass distribution on a geocentric sphere with radius  $\alpha < 1$ . The gravity disturbances are assumed to refer to the geocentric sphere  $\alpha = 1$ .

According to equation (2-3), the isotropic integral kernel  $K$  of the operator which transforms mass anomalies into gravity disturbances, can be represented in terms of

$$K(\xi, \eta) = \frac{1}{2\pi} \sum_{n=0}^{\infty} (n+1)\alpha^n P_n(\xi \cdot \eta) \quad (4-1)$$

We employ the Funk-Hecke theorem (Müller, 1966), which states that the eigenvalues  $k_n$  of an isotropic integral kernel  $K$  on the unit sphere are given by

$$k_n = 2\pi \int_{-1}^1 K(t) P_n(t) dt \quad (4-2)$$

with  $P_n$  denoting the Legendre polynomial of degree  $n$  and  $t = \cos\psi$ . Due to the orthogonality of the Legendre polynomials

$$\int_{-1}^1 P_n(t) P_m(t) dt = \frac{2}{2n+1} \delta_{nm} \quad (4-3)$$

the eigenvalues of the kernel (4-1) are obtained as

$$k_n = \frac{n+1}{n+\frac{1}{2}} \alpha^n, \quad (4-4)$$

which behaves obviously like  $\alpha^n$  for high degree  $n$ .

The question is, can we find a kind of moving average operator which behaves similarly?

Let us investigate the moving average operator acting on a circular cap of radius  $\psi_0$ ,

$$\tilde{f}(\xi) = \int_{\sigma} B(\xi \cdot \eta) f(\eta) d\sigma(\eta) \quad (4-5)$$

with the isotropic integral kernel

$$B(\xi \cdot \eta) = \frac{1}{2\pi(1-\cos\psi_0)} \begin{cases} 1 & \text{for } \xi \cdot \eta \geq \cos\psi_0 \\ 0 & \text{else} \end{cases} \quad (4-6)$$

According to the above referenced Funk-Hecke formula, the eigenvalues  $\beta_n$  of  $B$  are given by

$$\beta_n = \frac{1}{1-\cos\psi_0} \int_{t_0}^1 P_n(t) dt. \quad (4-7)$$

The integral yields  $\frac{1}{2n+1} [P_{n-1}(t_0) - P_{n+1}(t_0)]$ , which can be used to design a recursion formula for  $\beta_n$  (Sjöberg, 1980),

$$\begin{aligned} \beta_0(t_0) &= 1 \\ \beta_1(t_0) &= \frac{1}{2} (1+t_0) \end{aligned} \quad (4-8)$$

$$\beta_n(t_0) = \frac{1}{n+1} [(2n-1)t_0 \beta_{n-1}(t_0) - (n-2)\beta_{n-2}(t_0)], \quad n \geq 2.$$

These eigenvalues approach zero by oscillating around zero, dependent on  $t_0$ . For the extreme case  $\psi_0 = 0$  (no smoothing), all eigenvalues are equal to 1, for the extreme  $\psi_0 = \pi$  (total smoothing), all eigenvalues are zero apart from the zero degree eigenvalue which is equal to 1.

Now, the behavior of these eigenvalues does not quite agree with that of  $k_n$  given by (4-4); an approximation procedure could be designed which fits (4-4) to (4-8) in the best possible way yielding a best possible value for  $\alpha$  and, therefore, for the depth of the mass anomaly layer. For the purpose of obtaining a closed expression for the covariance function of mean values such an approximation is actually used; this is basically equivalent to replacing the mean values at zero level by point values at a certain altitude dependent on  $\alpha$ . This technique provides us with information on how to select the depth of a mass layer corresponding to gravity disturbance mean values of rectangular blocks of a certain size:

$$\begin{aligned} \psi_0 &\Rightarrow \alpha \\ \text{block size} &\Rightarrow \text{depth} . \end{aligned}$$

Denoting the harmonic coefficients of the mass-disturbance implied gravity disturbance  $\delta g$  by  $g_{nm}$  and the harmonic coefficients of the distribution of mass disturbance  $2\pi\delta\mu$  by  $\mu_{nm}$ , we conclude from (4-4) that they are related to each other by

$$g_{nm} = \frac{n+1}{n+2} \alpha^n \mu_{nm} . \quad (4-9)$$

Since  $\alpha < 1$ , the behavior of the mass-disturbance implied gravity disturbance at zero level will be smoother than that of the mass disturbances with the degree of smoothness depen-



ding on the depth of the mass-layer. Vice versa, deriving mass disturbances from unsmoothed gravity disturbances is an unsmoothing process and comparable to differentiation. A stable transformation from one quantity to another one can be expected if the frequency behaviour of the output function is smoother than or at least as smooth as that of the input function. Therefore it is a good advice to use mean gravity disturbances  $\delta\tilde{g}$  with harmonic coefficients  $\tilde{g}_{nm}$  related to  $g_{nm}$  by

$$\tilde{g}_{nm} = \beta_n g_{nm} \quad (4-10)$$

to determine  $\delta_\mu$  such that the behaviour of  $k_n$  is comparable to that of (4-8). In other words, the unsmoothing  $\delta g \rightarrow \delta\mu$  has to be balanced by a sufficiently smooth  $\delta\tilde{g}$ .

As far as the determination of a "best" value for the factor  $\psi_0$  is concerned, one could follow the ideas of Schwarz (1976) and fit the eigenvalues  $\{k_n\}$  best (in the sense of least-squares, e.g.) to the eigenvalues  $\{\beta_n\}$  with  $n \leq N(\alpha)$ . If  $\alpha$  belongs to a large  $D$ ,  $N(\alpha)$  will be low (on the order of a few hundred for  $D \sim 500$  km, if  $\alpha$  belongs to a small  $D$ ,  $N(\alpha)$  will be high (on the order of a few thousand for  $D \sim 10$  km); the degree of truncation  $N(\alpha)$  depends on the depth of the point mass layer  $D$ . (See chapter 2 of Sünkel (1981).)

## 5. AN ALGORITHM FOR THE DETERMINATION OF POINT MASSES

In (Sünkel, 1982) it was argued that a mass distribution confined to a single layer is inadequate for the representation of the disturbing potential. Mass distributions on a few layers at various depths are preferable. In chapter 4 we have discussed how to determine the most appropriate depths provided a certain mean value information on gravity disturbances at zero level is available.

Let us now describe the principle of the procedure which could be used to derive point masses from known gravity disturbances. For the sake of simplicity we will assume that the various mean values can be approximated sufficiently well by a corresponding moving average as described in the previous chapter. Then the actual gravity disturbance harmonic coefficients (unsmoothed) can be represented in terms of

$$g_{nm} = \left[ \sum_{l=0}^L (\beta_n^{(l)} - \beta_n^{(l+1)}) \right] g_{nm} \quad (5-1)$$

with  $\beta_n^{(0)} = 1 \quad \forall n \in N_o$  and  $\beta_n^{(L+1)} = \delta_{n,0}$ . Let us illustrate equation (5-1) by an example: consider  $\beta_n^{(1)}$  as the eigenvalues of the moving average operator corresponding to a mean value of  $10' \times 10'$ ,  $\beta_n^{(2)}$  corresponding to  $1^\circ \times 1^\circ$ , and  $\beta_n^{(3)}$  corresponding to  $5^\circ \times 5^\circ$ . Then  $(\beta_n^{(L)} - \beta_n^{(L+1)})g_{nm} = \beta_n^{(3)}g_{nm}$  would be the coefficients of the  $5^\circ \times 5^\circ$  mean values,  $(\beta_n^{(2)} - \beta_n^{(3)})g_{nm}$  the coefficients of the  $1^\circ \times 1^\circ$  mean values referred to the  $5^\circ \times 5^\circ$  mean values,  $(\beta_n^{(1)} - \beta_n^{(2)})g_{nm}$  the coefficients of the  $10' \times 10'$  mean values referred to the  $1^\circ \times 1^\circ$  mean values, and  $(\beta_n^{(0)} - \beta_n^{(1)})g_{nm} = (1 - \beta_n^{(1)})g_{nm}$  the coefficients of the actual gravity disturbance field referred to the  $10' \times 10'$  mean values.

According to (4-9) the coefficients  $g_{nm}$  are related to the coefficients of the mass disturbances by

$$g_{nm} = \frac{n+1}{n+\frac{1}{2}} \sum_{l=0}^L \alpha_l^n \mu_{nm}^{(l)}. \quad (5-2)$$

The coefficients  $\mu_{nm}^{(l)}$  refer to the mass disturbances at the layer  $l$ ; the layer  $l = 0$  is assumed to be the most shallow and  $l = L$  the deepest layer. Comparing (5-2) to (5-1),

$$\sum_{l=0}^L (\beta_n^{(l)} - \beta_n^{(l+1)}) g_{nm} = \sum_{l=0}^L \frac{n+1}{n+\frac{1}{2}} \alpha_l^n \mu_{nm}^{(l)}, \quad (5-3)$$

then a "separation by layer and mean value" presents itself as a practicable method to determine mass disturbances from mean gravity disturbances,

$$(\beta_n^{(l)} - \beta_n^{(l+1)}) g_{nm} = \frac{n+1}{n+\frac{1}{2}} \alpha_l^n \mu_{nm}^{(l)}. \quad (5-4)$$

In other words, we intend to determine the mass disturbances located at the layer  $l$  from gravity disturbance mean values corresponding to the smoothing  $l$  and referred to the  $(l+1)$  smoothed values. For our previous example this means that the point masses at depth  $D = D_3$  ( $l = L = 3$ ) should be determined from  $5^\circ \times 5^\circ$  means, the point masses at depth  $D = D_2$  ( $l = 2$ ) from residual  $1^\circ \times 1^\circ$  means (referred to  $5^\circ \times 5^\circ$  means), the point masses at depth  $D = D_1$  ( $l = 1$ ) from residual  $10' \times 10'$  means (referred to  $1^\circ \times 1^\circ$  means), and the point masses at zero depth  $D = 0$  from the residual gravity disturbances (referred to the  $10' \times 10'$  mean values).

5.1. Point masses of  $5^\circ \times 5^\circ$  and  $1^\circ \times 1^\circ$ , etc. global mean gravity disturbances

It is natural to put mass points at the center of each mean value block. Therefore, to each  $5^\circ \times 5^\circ$  mean value there corresponds one mass point at depth  $D = D_L$ ; given the vector of  $5^\circ \times 5^\circ$  mean gravity disturbances  $g$ , we have to find the vector of point masses  $\mu$  by solving the linear system of equations

$$g = C\mu \quad (5-5)$$

Let us now investigate the structure of the matrix  $C$  in detail. We assume that the sphere is subdivided into  $I$  (equally spaced) parallels and  $J$  equally spaced meridians and that to each grid element  $[\phi_i, \phi_{i+1}; \lambda_j, \lambda_{j+1}]$  there corresponds one mean gravity disturbance to its center of point mass. Then  $g$  and  $\mu$  consist of  $I$  subvectors (corresponding to  $I$  parallels) consisting of  $J$  mean values and point masses, respectively, (corresponding to  $J$  meridians); in the same way the matrix  $C$  consists of  $I^2$  submatrices of dimension  $J^2$  each, corresponding to the subdivision of the data vector  $g$  and the solution vector  $\mu$ ,

$$g = \begin{bmatrix} g^1 \\ g^2 \\ \vdots \\ g^I \end{bmatrix}, \quad \mu = \begin{bmatrix} \mu^1 \\ \mu^2 \\ \vdots \\ \mu^I \end{bmatrix}, \quad C = \begin{bmatrix} C^{11} & C^{12} & \dots & C^{1I} \\ C^{21} & C^{22} & & C^{2I} \\ \vdots & & & \\ C^{I1} & C^{I2} & & C^{II} \end{bmatrix} \quad (5-6)$$

$$g^{Ti} = (g^{ij}) \quad , \quad j = 0, \dots, J-1 \quad ,$$

$$\mu^{Ti} = (\mu^{ij}) \quad , \quad j = 0, \dots, J-1 \quad ,$$

$$C_{jj'}^{ii'} = (c_{jj'}^{ii'}) \quad , \quad j, j' = 0, \dots, J-1 \quad .$$

$C^{ii'}$  is the matrix which relates  $g^i$  to  $\mu^{i'}$ ,  $c_{jj'}^{ii'}$  is the coefficient which relates  $g^{ij}$  to  $\mu^{i'j'}$ . From the regular distribution of the data and the unknowns with respect to longitude it is evident that  $c_{jj'}^{ii'}$  depends only on the longitude difference  $\lambda_{j'} - \lambda_j$  for each pair of parallels  $(\phi_i, \phi_{i'})$ . Matrices of this kind are known as Toeplitz circulant or briefly circular matrices, because row number  $j$  equals row number  $j-1$  rotated by 1 element. Circular matrices become diagonal matrices under a discrete Fourier transformation. This property can be used very advantageously to design a very efficient algorithm for the solution of the very large linear system (5-5), using frequency domain methods. In the sequel, we shall briefly outline this method following closely the fundamental work by Colombo (1979).

The first row (and all the others) of every submatrix is an equispaced sample of an even function; therefore, it has only a non-vanishing cosine-spectrum with Fourier-coefficients  $\bar{c}_k^{ii'}$  (an overbar denotes a quantity in the spectral domain,  $k$  stands for a discrete frequency),

$$\bar{c}_k^{ii'} = \frac{1}{H} \sum_{j'=0}^{J-1} c_{oj'}^{ii'} \cos \omega_k j' \quad , \quad (5-7a)$$

and vice versa, the elements of this particular row are obtained by an inverse Fourier transform,

$$c_{oj'}^{ii'} = \sum_{k=0}^K \bar{c}_k^{ii'} \cos \omega k j' \quad (5-7b)$$

with  $\omega = 2\pi/J$  ,  
 $K = \text{int}(J/2)$  , (5-7c)

$$H = \begin{cases} J & \text{if } k = 0 \text{ or } k = K \text{ (J even)} \\ J/2 & \text{if } 0 < k < K \text{ (=K if J odd)} \end{cases} .$$

The row number  $j$  equals row number  $j-1$  , rotated by one element, and therefore  $c_{jj'}^{ii'}$  is synthesized by

$$c_{jj'}^{ii'} = \sum_{k=0}^K \bar{c}_k^{ii'} \cos \omega k (j' - j) \quad (5-8)$$

Using the orthogonality relations between equispaced sampled trigonometric functions

$$\sum_{j=0}^{J-1} \cos \omega k j \cos \omega k' j = \begin{cases} J & \text{if } k = k' = 0 \text{ or } k = k' = K \text{ (J even)} \\ J/2 & \text{if } 0 < k = k' < K \text{ (=K if J odd)} \\ 0 & \text{if } k \neq k' \end{cases} ,$$

$$\sum_{j=0}^{J-1} \sin \omega k j \sin \omega k' j = \begin{cases} J/2 & \text{if } 0 < k = k' < K \text{ (=K if J odd)} \\ 0 & \text{else} \end{cases}$$

$$\sum_{j=0}^{J-1} \cos \omega k j \sin \omega k' j = 0 \quad \forall k, k' , \quad (5-9)$$

we obtain

$$\bar{c}_k^{ii'} H \begin{Bmatrix} \cos \omega k j \\ \sin \omega k j \end{Bmatrix} = \sum_{j'=0}^{J-1} c_{jj'}^{ii'} \begin{Bmatrix} \cos \omega k j' \\ \sin \omega k j' \end{Bmatrix} , \quad 0 \leq k \leq K \quad (5-10)$$

Consequently

$$\begin{aligned} c_k^T &:= \{\cos \omega k j\} \\ &\quad , j = 0, \dots, J-1 \\ s_k^T &:= \{\sin \omega k j\} \end{aligned} \tag{5-11}$$

are eigenvectors and  $\bar{c}_k^{ii'H}$  eigenvalues of the submatrix  $C^{ii'}$ .

Let us now transform the vectors  $g$  and  $\mu$  into the frequency domain  $(\bar{g}, \bar{\mu})$ ,

$$\bar{g}_k = \begin{bmatrix} \bar{g}_k^1 \\ \bar{g}_k^2 \\ \vdots \\ \bar{g}_k^I \end{bmatrix} , \quad \bar{\mu}_k = \begin{bmatrix} \bar{\mu}_k^1 \\ \bar{\mu}_k^2 \\ \vdots \\ \bar{\mu}_k^I \end{bmatrix} \tag{5-12}$$

$$\begin{aligned} \text{with } \bar{g}_k^i &= \gamma_k^i \begin{Bmatrix} C_k \\ S_k \end{Bmatrix} \\ \text{and } \bar{\mu}_k^i &= \delta_k^i \begin{Bmatrix} C_k \\ S_k \end{Bmatrix} ; \end{aligned} \tag{5-13}$$

observing (5-5) and (5-10), we find immediately the relation between  $\gamma_k^i$  and  $\delta_k^i$ ,

$$\gamma_k^i = \sum_{i'=1}^I \bar{c}_k^{ii'H} \delta_k^{i'} \tag{5-14}$$

Denoting the I.I matrix of eigenvalues  $\bar{c}_k^{ii'H}$  by  $E_k$ , equation (5-14) is given by

$$\begin{aligned} \gamma_k &= E_k \delta_k \\ \text{and } \delta_k &= E_k^{-1} \gamma_k , \quad k = 0, \dots, K . \end{aligned} \tag{5-15}$$

Here we have already the algorithm at hand which permits the fast transformation from the known mean gravity disturbance vector  $g$  to the unknown point masses  $\mu$ :

- 1) Calculate the first row of each submatrix  $C^{ii'}$  and transform it into the frequency domain by fast Fourier transformation;
- 2) Establish the matrices  $E_k$  for  $k = 0, \dots, K$  and calculate the inverses;
- 3) transform the data vector  $g$  into the frequency domain ( $\gamma_k$ );
- 4) perform the matrix-vector product  $\delta_k = E_k^{-1} \gamma_k$  for  $k = 0, \dots, K$ ;
- 5) Perform the inverse Fourier transformation of  $\delta_k$  (synthesis) and obtain  $\mu$ , the vector of point masses,

$$\mu^i = \sum_{k=0}^K \mu_k^i \quad (5-16)$$

The advantages of using this frequency domain method are obvious:

(a) Instead of solving the linear system (5-5) with dimension  $I \cdot I$ , we have to solve the  $K+1$  linear systems (5-15) of dimension  $I \cdot I$  each. Since the solution time is approximately proportional to the 3rd power of the dimension of the linear system, the computation time reduces by a factor of about  $J^2$  (~5200 for the  $5^\circ \times 5^\circ$  case, ~130 000 for the  $1^\circ \times 1^\circ$  case).

(b) Since all submatrices  $C^{ii'}$  are Toeplitz circulant, we need only calculate the first row, and moreover, since the first row represents an equispaced sample of an even function, only half of the elements have to be calculated. This reduces the (anyway expensive) calculation effort for the setup of  $C$



by another factor  $2J$  compared to the straight-forward algorithm.

(c) The storage requirements are drastically reduced due to redundancy in the  $C$ -matrix .

(d) The calculation of all required elements requires the evaluation of an integral. In chapter 3 we have proposed optimal approximation algorithms for its calculation, which can be easily implemented.

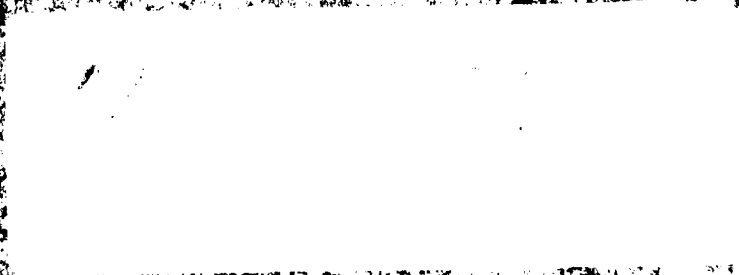
Note that in the spherical case  $C$  is not symmetric due to the convergence of meridians ( $c_{jj}^{ii'} \neq c_{jj}^{ii}$  because the coefficient which relates a mean value in row  $i$  to a point mass in row  $i' \neq i$  is different from the coefficient relating a point mass in row  $i$  to a mean value in row  $i'$  ; for a cylinder,  $C$  would be symmetric.) However, if the arrangement of mean values and point masses is symmetric with respect to the equator, the number of different elements of  $C$  is once more reduced by a factor of 2 .

REFERENCES

- COLOMBO, O.L. (1979): Optimal estimation from data regularly sampled on a sphere with applications in geodesy. Report No. 291, Department of Geodetic Science, The Ohio State University, Columbus, Ohio.
- FREEDEN, W. (1981): On spherical spline interpolation and approximation. Math. Meth. in the Appl. Sci. 3, pp. 551-575.
- GERSTL, M. (1980): On the Recursive Computation of the Integrals of the Associated Legendre Functions. Manuscripta Geodaetica Vol.5, pp. 181-199.
- HEISKANEN, W.A. and H. MORITZ (1967): Physical Geodesy. Freeman, San Francisco.
- MÖLLER, C. (1966): Spherical Harmonics, Lecture Notes in Mathematics, No.17, Springer-Verlag, Berlin.
- SJÖBERG, L. (1980): A recurrence relation for the  $\beta_n$  - function. Bull. Géod. 54, pp. 69-72.
- SÖNKEL, H. and R. RUMMEL (1981): On the Numerical Evaluation of Spherical Integral Formulas based on Mean Values. Boll. di Geodesia e Scienze Affini, No.2, 1981.
- SÖNKEL, H. (1982): Point mass models and the anomalous gravitational field. Report No. 327, Department of Geodetic Science and Surveying, The Ohio State University, Columbus, Ohio.

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