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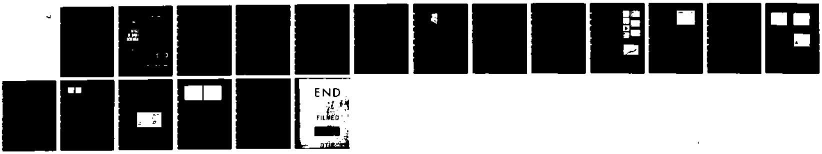

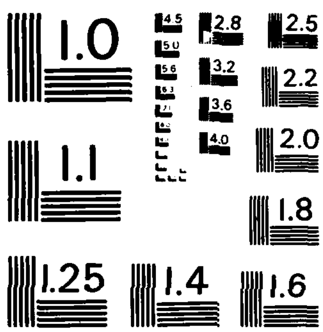

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OFFICE OF NAVAL RESEARCH

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EUROPE APPROACHES CHAOS WITH ELECTRICAL CIRCUITS

DAVID MOSHER

24 April 1984

UNITED STATES OF AMERICA

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In this report, the underlying concepts for the period-doubling route to chaos are presented, and the scope of recent research in a variety of physical systems is briefly noted. Then European research investigating chaotic behavior in nonlinear, driven electrical circuits is discussed in detail. These circuits--really nonlinear analog computers which solve the differential equations describing idealized physical systems--represent a bridge between the simple and highly abstract deterministic models and the experiments where complicating and competing effects can obscure the universal behavior.

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EUROPE APPROACHES CHAOS WITH ELECTRICAL CIRCUITS

In some nonlinear dynamical systems, behavior changes from simple and predictable to chaotic as some external control parameter is varied. An example of this behavior is hydrodynamic flow around an obstruction in a fluid stream (Kadanoff, 1983). When the fluid speed is low, the flow is laminar, and a downstream probe measures a constant velocity. As the speed is increased, eddies with a regular structure are formed behind the obstruction and are convected downstream to produce a periodic variation in velocity. At still higher speeds, the flow becomes fully turbulent and the downstream velocity varies erratically and unpredictably.

During the last few years, analogous transitions from simple to chaotic behavior have been observed in a variety of hydrodynamic, biological, electronic, chemical, and optical systems. Although these systems are physically very different, extremely simple deterministic mathematical models can reproduce the approach to chaos for all of them. Even more surprising, the models show that the observed "period-doubling" route to chaos possesses universal features that depend only on a few fundamental qualities that the systems share. This recent understanding has provided important and fundamental insights about turbulent and noisy behavior in many physical systems.

The application of this research to noise is of particular interest to the Office of Naval Research, Arlington, VA. The Nonlinear Dynamics Program within the Physics Division includes a basic research program to study the evolution of chaotic nonlinear systems using analytic theories, computer simulations, and experiments with simple electrical devices (Shlesinger, 1984). The specific objective of the program is to understand chaos-induced noise in sensors, transmission lines, computer memories, and communications systems--though, as already indicated, the research may also characterize turbulent behavior in other physical systems of interest to the Navy. Parallel US research efforts are supported by the Air Force Office of Scientific Research, the Department of Energy, the National Science Foundation, and the Army Research Office.

In this report, the underlying concepts for the period-doubling route to chaos are presented, and the scope of recent research in a variety of physical systems is briefly noted. Then, European research investigating chaotic behavior in nonlinear, driven electrical circuits is discussed in detail. These circuits--really nonlinear analog computers which solve the differential equations describing idealized physical systems--represent a bridge between the simple and highly abstract deterministic models and the experiments where complicating and competing effects can obscure the universal behavior.

The Basics

The period-doubling transition to chaos arises from simple cyclic behavior at small values of a control parameter R. As R is increased, a period-doubling bifurcation occurs such that the system state repeats only after two periods of the basic cycle. Additional bifurcations appear with further increases in R, causing increasingly complex behavior which repeats only after four periods, then eight, etc. At a certain value of R, an infinite number of doublings has been reached and the system becomes fully aperiodic.

A disarmingly simple mathematical model with this behavior was investigated by Mitchell Feigenbaum (1978) of the Los Alamos National Laboratory. Feigenbaum studied one-dimensional maps for which a variable x evolved in steps according to the quadratic transformation

$$x_{j+1} = Rx_j(1-x_j) \qquad (1)$$

and determined the character of the long-term (large j) behavior as R was varied.

Equation (1) is trimmed to the bare essentials required to study chaotic behavior. The nonlinear differential equations that model the continuous variations of a physical system have been replaced by a deterministic one-dimensional map that describes the state at time j+1 completely in terms of the state at time j. The linear term acts like a driving term, while the quadratic term is dissipative and provides the required nonlinearity. A vivid example of the behavior described by the equation comes from the study of population dynamics, where the insect population in year j+1 is determined by the number of eggs laid in year j. For this example, R is a measure of environmental suitability, such as the availability of food, and the quadratic term represents the reduction of natural growth caused by overcrowding. Without the quadratic term, equation (1) shows exponential growth for R greater than 1 and exponential decay for R less than 1 from any initial population x_0. With the quadratic term, the behavior becomes amazingly complex.

The long-term evolution of equation (1) can be studied by restricting consideration to $R \leqslant 4$, so that the population x_j is limited to the interval 0 to 1. Examples for R in this interval are shown in Figure 1 for which an HP-85 computer was used to iterate equation (1) and display the results. For R less than 1, the population decays to zero because of the inhospitable environment (Figure 1a). For R between 1 and 3, the population approaches a constant value X independent of x_0 for which the provisions of the environment are just adequate to support the long-term population (Figure 1b). This equilibrium value is determined by setting $x_{j+1} = x_j = X$ and solving to obtain $X = 1 - 1/R$.

As R is increased above 3, the simple asymptotic behavior is replaced by a 2-year cycle in which the population alternates between low and high values (Figure 1c). The Fourier power spectrum of this time variation contains two lines at the 1-year fundamental and the 2-year subharmonic. A low population in one year reproduces rapidly to leave a large number of eggs. This results in overcrowding the next year, few eggs, and a population that is again small the year after.

As R is increased above 3.4, the 2-cycle bifurcates to a 4-cycle (Figure 1d) so that the population assumes four values before repeating. Increasing R still further causes additional bifurcations until about 3.57, when a cycle of infinite period appears. At this point, the evolution is fully aperiodic but limited to a particular range of x values (Figure 1e), and the power spectrum contains an infinite number of subharmonic lines. As R is increased further, the bands of accessible x values broaden until R = 4, where the population variation becomes fully chaotic and covers the range 0 to 1 (Figure 1f). The power spectrum has evolved to broad-band continuum noise filling the frequency range from zero up to the fundamental.

Figure 1f demonstrates an important characteristic of chaotic behavior. For large j, the value of x_j depends sensitively on the initial value x_0. Two trajectories with arbitrarily close initial values rapidly diverge and become uncorrelated. Each trajectory seems to evolve randomly though equation (1) is fully deterministic--any x_j is defined precisely by x_0 and the iteration procedure. However, tiny initial and computational errors grow exponentially so that, for even the most accurate computers, the calculated evolution is a complete fiction after several tens of iterations. The long-term behavior is known in the classical sense but is not calculable in practice. This pseudorandom behavior has wide-ranging implications. For example, long-term weather forecasting may never be accurate because of the extreme sensitivity to initial conditions about which there is imprecise knowledge.

The full range of behavior displayed by equation (1) is summarized in the bifurcation diagram shown in Figure 2. A scatter plot of accessible x_j values is constructed by varying the control parameter in small steps, iterating the equation for each value of R a large number of times, and then

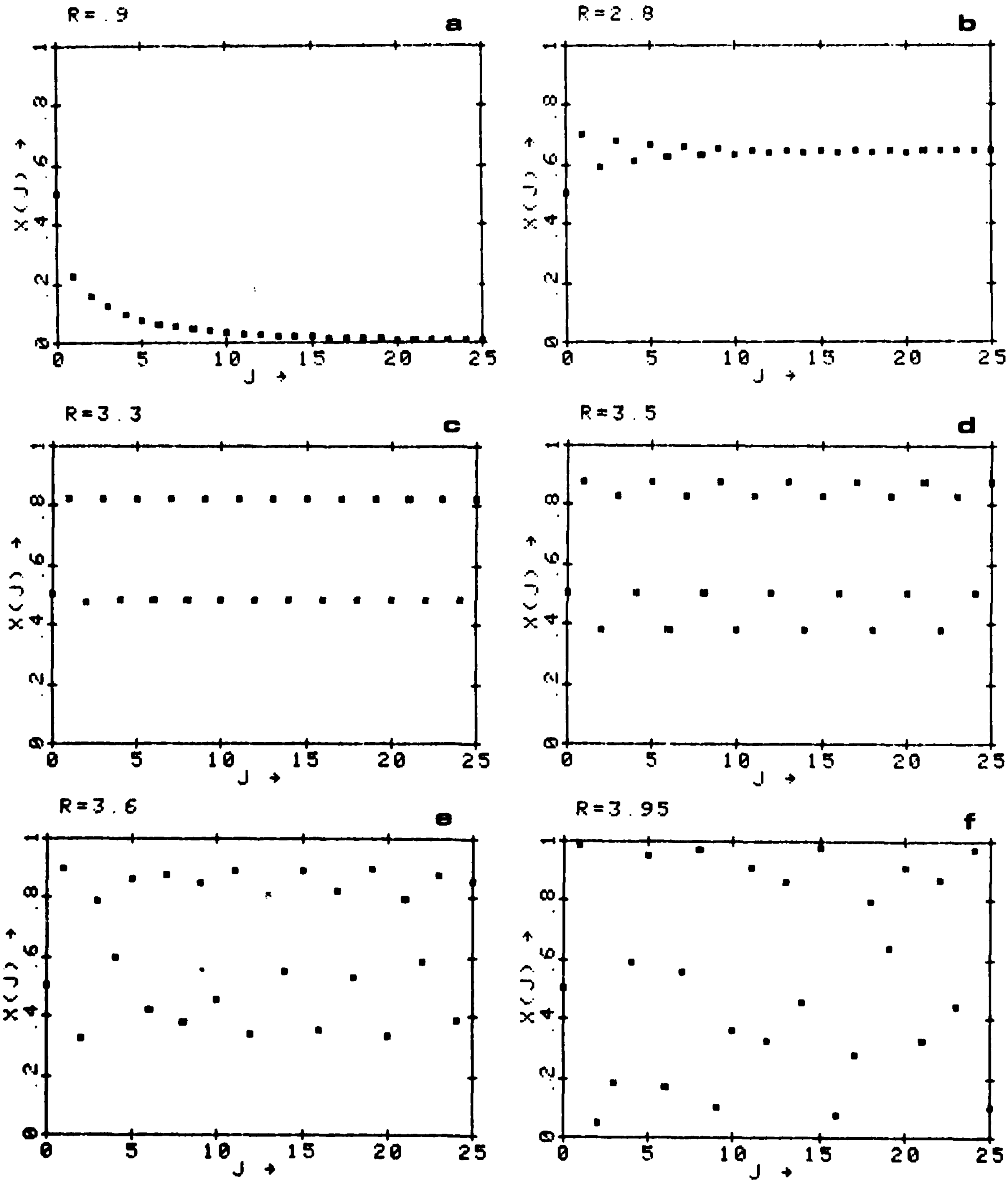

Figure 1. Population evolution for various values of the control parameter.

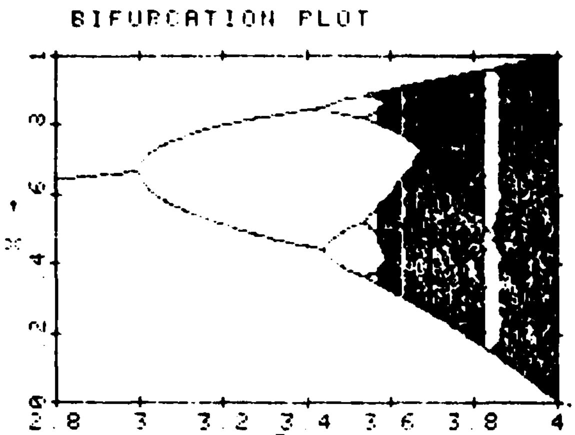

Figure 2. A bifurcation diagram for the quadratic map.

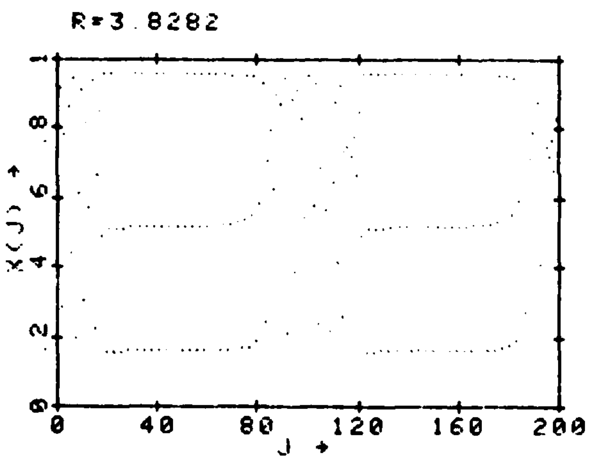

Figure 3. An intermittant 3-cycle.

plotting the next few hundred sequential values of x_j. The chaotic regime (dark, because of a continuum of x values) shows clear windows above 3.6 and 3.8. Other, narrower windows can be seen on higher resolution displays. These regions show a new behavior--a slowly varying 3-cycle interrupted by intermittent chaotic bursts (Figure 3). This intermittency has also been observed in experiments which display period doubling to chaos.

The rich dynamical behavior just described is not unique to equation (1). Feigenbaum showed that period-doubling to chaos occurs with any map of the form $x_{j+1} = Rp(x_j)$, provided that $p(x)$ is a smooth function with a single maximum between 0 and 1 and $p(0) = p(1)$. Furthermore, he demonstrated that the behavior of all such maps near the points of infinite bifurcation are the same. Defining R_n as the control parameter value at which the n-period cycle first appears, Feigenbaum showed that the quantity

$$\delta_n = (R_{n+1} - R_n)/(R_{n+2} - R_{n+1}) \qquad (2)$$

rapidly converges to a constant value $\delta = 4.669...$ as n increases above a few. A second property is that the subharmonic spectral lines of the n+1 cycle will have intensities about 8.2 dB below those of the n cycle. These two universal numbers should appear in all real systems exhibiting a period-doubling approach to turbulence and so provide key experimental tests.

Experimental Observations

In 1979, A. Libchaber and J. Maurer of the École Normale Superieure in Paris performed Rayleigh-Benard convection experiments using a liquid helium cell heated from below. As the temperature difference between the top and bottom of the fluid was increased, the induced convection showed subharmonic bifurcation until the system reached turbulence. The frequency spectrum showed lines up to those of the n = 4 or 16-period bifurcation. Using the intensity of the n = 2 bifurcation lines as a base, Feigenbaum (1979) demonstrated that the n = 3 and n = 4 intensities were each reduced by about the 8.2 dB predicted by the theory.

During the past few years, researchers on both sides of the Atlantic Ocean have demonstrated period-doubling to chaos in a variety of physical and chemical systems. Rayleigh-Benard experiments using room-temperature silicon oil in place of the exotic liquid helium have demonstrated bifurcation to turbulence at the Woods Hole Oceanographic Institution in Massachusetts (Whitehead, 1983) and at the

Centre d'Études Nucléaires de Scalay in France (Bergé and Dubois, 1983). The French research has shown intermittent behavior in the form of alternating sequences of spatial chaos interrupted by transient ordered states with well-defined convective structures. Researchers in Germany (Mullin et al., 1983) and the US (Brändstater et al., 1983) have reported chaotic behavior in the Couette-Taylor flow of fluid around a rotating cylinder. The relationship between broad-band noise and intermittent behavior was studied at the Centre National de la Récherche Scientifique in France in the electrohydrodynamic instability of an insulating liquid with an embedded ion space charge (Malraison and Atten, 1982). In that work, the intensity of a destabilizing electric field played the role of control parameter.

Rueben Simoyi and coworkers at the University of Texas at Austin (Simoyi et al., 1982) have observed forced Belousov-Zhabotinskii chemical reactions and have constructed one-dimensional maps that correspond to the periodic and chaotic states. These nonlinear-reacting chemical mixtures form coherent spatial structures with different component concentrations, and when stirred, the concentrations oscillate in time in a manner mimicking biological oscillators. Researchers at the Prague Institute of Chemical Technology in Czechoslovakia (Dolník et al., 1984) have disturbed the natural rhythm of the oscillation by periodically adding bromine ions and have observed alternating sequences of order and aperiodicity as functions of ion concentration and addition period. By studying how the added ions reset the chemical clock and introduce aperiodic behavior, the authors have gained insight into biological processes and disfunctions such as cardiac disrhythmia.

Chaotic behavior has been reported in several laser-driven, optically bistable systems. Bistability in the optical transmission of a cavity filled with a nonlinear medium is currently of great interest. The phenonemon governs the behavior of optical and opto-electronic devices that may be the building blocks for a new generation of large-scale computers (Smith et al., 1984). Japanese researchers calculated that transmitted light from a bistable ring cavity should undergo a transition from a stationary state to periodic and aperiodic states when the intensity of transmitted light is increased (Ikeda et al., 1980). F.T. Arrechi and coworkers (Arecchi et al., 1982) from the Istituto Nazionale di Ottica in Florence, Italy, were the first to report experimental evidence of bifurcation and chaos in such a quantum optical-molecular system. In that work, they were able to correlate the appearance of low-frequency noise in the power spectrum with jumps between the two system states of a Q-switched CO_2 laser and have proposed a chaotic mechanism as the source of the ubiquitous 1/f noise that limits the sensitivity of electrical circuits. An electronic analog of this system is described below. Soon after, R.G. Harrison and coworkers (Heriot-Watt University, Edinburgh, UK) pumped an ammonia cell resonator with a pulsed CO_2 laser and obtained results in excellent agreement with the Ikeda theory (Harrison et al., 1983). Agreement with the theory had been obtained earlier in experiments at the University of Arizona using a hybrid optical-electronic bistable device (Gibbs et al., 1981).

Chaotic behavior has also been reported in German experiments involving a parametrically forced mechanical pendulum (Koch et al., 1983), and in measurements of the conductance of cryogenically cooled germanium crystals at Harvard University (Teitsworth et al., 1983). Recently, chaos has been proposed as a solution to the puzzle of why some polyatomic molecules are so easily dissociated by infrared lasers (Ackerhalt et al., 1983).

The Japanese, possibly because of long-term applications to advanced computers, have taken the lead in the study of chaotic behavior in neurons. Hayashi and coworkers at Kyushu University have observed period doubling, chaos and intermittency by driving the pacemaker neuron of the mollusk *Onchidium verruculatum* with a sinusoidal current (Hayashi et al., 1983). Other researchers studied random nets of large numbers of neurons numerically and determined the chaotic behavior of the firing wave instability by treating each neuron as a threshold element (Aoki et al., 1983).

Electrical Circuits

Electrical circuits with nonlinear elements have distinct advantages over other systems for experimental study of chaotic behavior. The experiment is readily assembled with inexpensive components and easily diagnosed with standard, low-frequency oscilloscopes and frequency analyzers. Nonlinear dynamics can be studied over wide ranges of system parameters by simply changing circuit component values. The circuits are accurate representations of differential equations, and results are easy to interpret because extraneous effects, such as nonideal and multidimensional behavior, are absent. Thus, they lie between the one-dimensional logistical maps and experimental systems in complexity--they are rich enough to model interesting physics but simple enough to do so unambiguously. The electrical circuits are not physical systems but nonlinear analog computers which solve the differential equations that model the idealized behavior of other experiments. In that regard, they perform better than the large-scale digital computers which replaced their linear-circuit ancestors as equation solvers 30 years ago.

Most of the electrical experiments described below employ a series combination of resistance R, inductance L, and a nonlinear electronic element driven by a low-frequency oscillating voltage V whose amplitude is a control parameter. In terms of charge q (defined by $\dot{q} = I$, the circuit current), the governing equation is

$$L\ddot{q} + R\dot{q} + v(q) = V(t) \ , \qquad (3)$$

where v is the voltage across the nonlinear element. Equation (3) is equivalent to the dynamical equation

$$\ddot{x} + \eta\dot{x} + f(x) = F(t) \ , \qquad (4)$$

where η is the dissipation factor. Equation (4) describes the motion of a forced pendulum with a nonlinear restoring force and friction. With appropriately chosen f(x), it also describes the dynamics of other physical systems.

Chaotic effects are pronounced when the driving force F has a period close to that of a natural resonance in the unforced, free-running system. Under such conditions, the equation exhibits period-multiplying bifurcations leading to chaos. The connection with one-dimensional maps described above can be made by identifying x_j with sequential values of x at a particular phase of the driving oscillation. The dynamic evolution of the differential equation is then equivalent to the iteration scheme

$$\begin{aligned} x_{j+1} &= g(x_j, \dot{x}_j) \\ \dot{x}_{j+1} &= h(x_j, \dot{x}_j) \ , \end{aligned} \qquad (5)$$

where the nonlinear functions g and h must usually be determined from the solution of equation (4) (Ott, 1982). In electrical experiments, these functions can be determined electronically so that the observed chaotic behavior can be assessed in terms of Feigenbaum's p(x) requirements for universality.

Researchers in the US were first to demonstrate the experimental advantages of electrical circuits to study chaos. Paul Linsay (1981) of the Massachusetts Institute of Technology described a driven nonlinear electrical oscillator exhibiting period doubling and chaotic behavior. In his circuit, a varactor diode with a voltage-dependent capacitance was the nonlinear element. Linsay performed spectral analyses and demonstrated agreement with theory by measuring the 8.2 dB intensity change between subharmonic Fourier components. Six months later, James Testa and coworkers (1982) at the Lawrence Berkeley Laboratory used a similar circuit to directly record the bifurcation diagram with an x-y oscilloscope, performed a detailed comparison with the logistical model, and demonstrated excellent agreement with Feigenbaum's universal constant δ.

<u>A Reverse Bifurcation</u>

In Europe, a modified varactor circuit has been used by J. Cascais and coworkers of CAUL-CFMC (Av. Prof. Gama Pinto 2, 1699 Lisbon Codex, Portugal) to demonstrate a new effect: reverse bifurcation. Like the US researchers, they used a varactor circuit driven by an harmonic oscillator. The modification was a dc bias of b volts added to the sinusoid. The dc amplitude represents a second control parameter whose presence profoundly changes the chaotic behavior.

The group collected data using R = 5 ohms, L = 1 mH, and a Philip's silicon varactor diode with a voltage-dependent capacity of $50(0.6 + V)^{-1/2}$ pf. The ac amplitude A was kept constant at 10.5 V throughout the experiment, and the system behavior was measured as b was varied. Phase-space plots for which the instantaneous current was plotted against varactor voltage showed bifurcation, chaos, and period-five windows as b was increased. Further increases caused a reverse of the process marked by successive period halving. Figure 4 shows this sequence of transitions for a driving frequency ω of 360 kHz. Shown are period two (A), period four (B), chaos (C), a period-five window (D), reverse period eight (E), reverse period four (F), and reverse period two (G). Bifurcation diagrams for several values of driving frequency were obtained by plotting the peak voltage across the varactor against b, and an example for 496 kHz is shown in Figure 5. These diagrams show a strong dependence on driving frequency--as ω increases, the chaotic bands become narrower and finally disappear above about 550 kHz.

Although the quadratic map of equation (1) does not show reverse bifurcation, some one-dimensional maps in which the control parameter is an additive constant (rather than a multiplicative one) do. Cascais and coworkers calculated the bifurcation diagram in which the quadratic dependence is replaced by

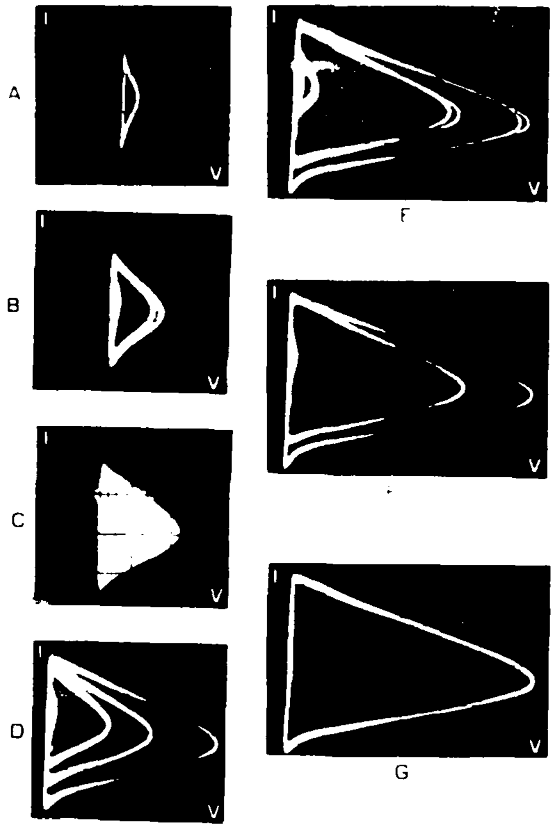

Figure 4. Phase space plots for ω = 360 kHz and various values of b.

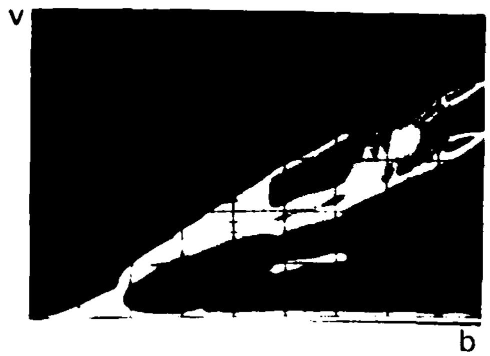

Figure 5. A bifurcation diagram for ω = 496 kHz.

$$p(x) = \exp(-a^2x^2/2\pi) + C . \qquad (6)$$

With a = 7, the map bifurcates to chaos and then reverse bifurcates to a stable two cycle as C is varied from -1 to 0. Though this behavior has yet to be observed in a physical system, reverse bifurcation in electrical circuits and one-dimensional maps demonstrates that in certain situations a chaotic system can decrease its dynamic complexity with increasing stress.

Researchers from the University of Göttingen in the Federal Republic of Germany have also examined bifurcation in varactor circuits and shown behavior fundamentally different from the logistic equation (Klinker et al., 1984). They summarized results in a phase plot of driving voltage amplitude versus frequency showing areas of period doubling, chaos, period five, and doubling from period three. A most interesting result is a hysteresis effect--the boundary between areas of different behavior depends on whether the transition is approached with increasing of decreasing amplitude. Bifurcation diagrams showing hysteresis agreed with numerical solution of equation (3), in which v was assumed to have the form exp(q)-1 for the forward-conducting diode.

An Application to Microwave Electronics

Researchers at the Clarendon Laboratory at Oxford University became interested in chaotic electrical systems when David Jefferies (1982) observed period doubling in the vibrational resonances of the piezoelectric Rochelle salt (sodium potassium tartrate tetrahydrate). In such ferro-electric crystals near the phase transition, the dielectric and elastic constants are strong functions of the electric field. Since the sound velocity in Rochelle salt is a function of the strain, the restoring force--and therefore the resonant frequency for acoustic oscillations--are nonlinear functions of displacement. As the amplitude of a sinusoidal driving field was increased, the voltage across the crystal first distorted and then broke discontinuously into a half-period waveform. Further increases led to chaos and regions of stable waveforms with integral multiple periods.

Following these initial piezoelectric experiments, Jefferies and his associate A. Usher (1983) became interested in electrical analogs. They

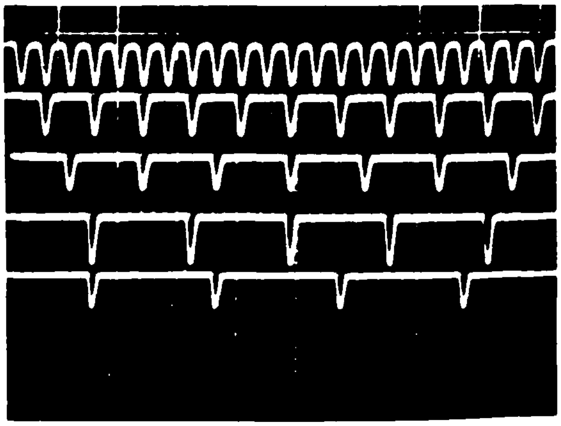

Figure 6. Frequency division by a diode circuit.

realized that a critical property of the varactor circuit for bifurcation was that the charge-storage time in the device had to be a significant fraction of the driving period and that ordinary silicon diodes also possessed that property. They then investigated a sinusoidally driven diode and series inductor to determine the circuit's ability to frequency divide (i.e., period multiply) the driver. They found that a 1N4001 diode with a charge storage time of about 3 μs could divide frequencies ranging from 10 kHz to 20 MHz for suitable drive amplitudes. Figure 6 shows successive division of a 1.2 MHz drive by 2,3,4, and 5 when the amplitude was increased from about 0.8 V to 11 V. Jefferies and Usher also used a four-stage cascade based on the diode circuit to frequency divide a 4 MHz signal by 16.

The three order-of-magnitude range of frequency that the diode could divide suggested that a Schottky barrier diode with a 100-ps charge-storage time might be used to divide 1 to 100 GHz signals since standard scaling techniques are difficult to apply at these frequencies. A Hewlett-Packard HSCH1001 diode was used to divide a 1.5-GHz input signal by two in a short, 230-ohm transmission line. The output waveform showed behavior similar to that of the low-frequency dividers, and the half-frequency near 750 MHz was accurate to 10 digits.

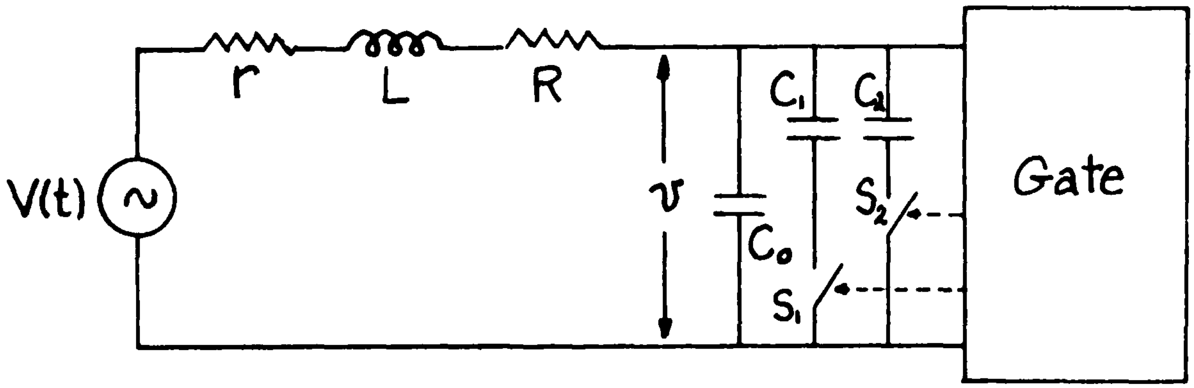

Figure 7. Gate-driven circuit for nonlinear studies.

Though the low-frequency results indicate that the circuit should be able to divide much higher frequencies, the waveform generator and detectors used in the experiment had an upper limit of about 2 GHz so that the full operating range could not be tested.

<u>A Comprehensive Study</u>

During a recent visit to the Clarendon, I spoke with F.N.H. Robinson, who developed the high-frequency version of the divider experiment. He is about to publish a major experimental study on chaotic behavior using more complex nonlinear elements than those described above. Robinson designed his electrical circuit to model equation (4) for a variety of nonlinear forces. He obtained power spectra, and phase and bifurcation diagrams for this range of f(x) forms and used the results to confirm approximate analytic solutions. The analytic solutions were then used to determine the force thresholds required to sustain motions of different periods, and the resulting scaling laws were also confirmed by the data.

Rather than a single semiconductor element, Robinson uses the circuit shown in Figure 7. The nonlinearity is provided by a gate circuit that senses the voltage v across the capacitor array and operates the switches S_1 and S_2 at predetermined threshold values. By changing the thresholds, five different forms for the force law were studied. All of the studied forms for f(x) consisted of three straight-line segments with the central one passing through the x-f (i.e., q-v) origin. The two outer line segments could have symmetric, greater or lesser slopes than the central one (in which cases the restoring forces are called symmetric hard or soft), or the negative-x outer segment could have the same slope as the central segment (asymmetric hard or soft), or the three segments could all have different slopes (totally asymmetric). The piece-wise nature of f(x) makes the differential equation analytically tractable with certain approximations, and the circuit has a wide dynamic range since the capacitance values can be varied over a range of 10^8. Highly nonlinear behavior can be driven, and stable motions with periods as high as 49 times that of the drive have been observed. The main disadvantage of this electrical system is the discontinuous first derivative of f which complicates comparison with physical systems.

The voltage across the small resistance r is proportional to the current, so that q is derived from it by electronic integration. The variables q and $\dot{q}$ are fed to x and y oscilloscope inputs to record the phase space motion. An auxiliary circuit is used to blank or brighten the oscilloscope trace at predetermined points in the drive cycle or at voltage threshold points. This circuit is used to construct bifurcation diagrams when the amplitude of the driving voltage is slowly changed and recorded as the horizontal oscilloscope input.

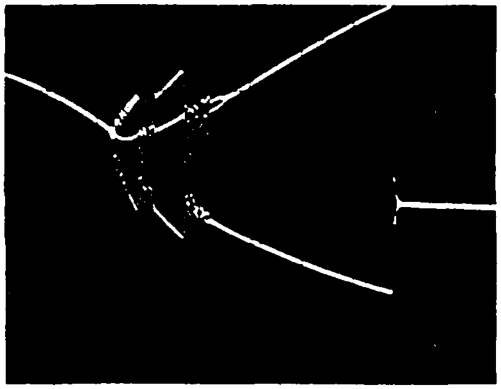

Figure 8. Bifurcation diagram for an asymmetric hardener.

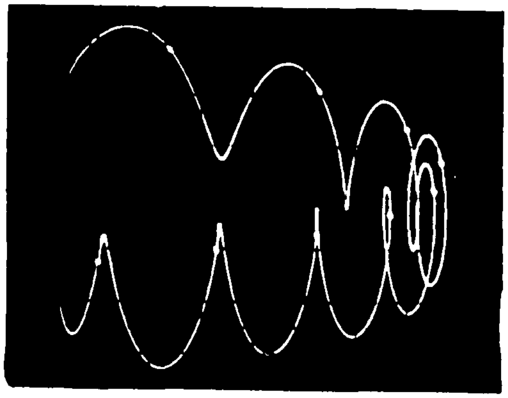

Figure 9. A stable 9 cycle for a softener restoring force.

Asymmetric hardening is achieved if a single switch is opened when v exceeds the threshold v_1 and closed otherwise. This force law corresponds to several important dynamical systems, such as a ship moored to a wharf or an oil rig tethered to the ocean bed. Figure 8 shows a bifurcation diagram for such a case at a 684-Hz driving frequency, C_o = 100 pF and C_1 = 47 nF. The single period breaks into a chaotic regime out of which period two emerges. After further chaos, a brief period four collapses to period two, and then via more chaos back to the fundamental. This rather simple behavior with reverse bifurcation can become much more complex with other parameter values--in one case a stable period-18 oscillation was observed.

An asymmetric softener is realized when a single switch is closed above a positive threshold and opened otherwise. It is made symmetric if closure also occurs for $v \leq -v_1$. Unlike hardeners, where asymmetry leads to a "bounce" against the nearer threshold and orbits very different from the symmetric case, asymmetric and symmetric softeners have similar orbits. With large changes in slope between segments ($C_0 + C_1 >> C_0$), softeners exhibit complex motions of high period. A phase diagram is shown in Figure 9 for a softener with C_0 = 1 nF, C_1 = 4.7 μF and a drive frequency of 4520 Hz. The trace is brightened

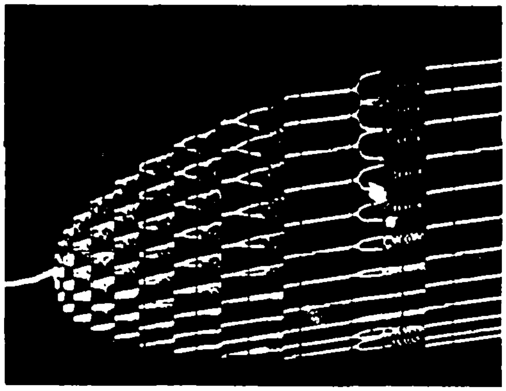

Figure 10. A softener bifurcation diagram.

once every drive period to show nine bright spots for period nine. Figure 10 is the bifurcation diagram for this type of motion showing the bifurcation sequence k, 2k, chaos, k+1, 2(k+1), chaos, k+2, etc.

Armed with the data provided by a large number of electrical experiments, Robinson could test the validity of approximate analytic solutions used to derive simple force requirements for sustained periodic behavior. Provided that the dissipation and driving force are not too large, the solution of equation (4) can be approximated by

$$x(t) = \sum_{n=1}^{\infty} x_n \cos(2\pi nt/k + \phi_n) \ , \qquad (7)$$

where k is an integer and the driving force has the form $F\sin(2\pi t+\psi)$. Integration of equation (4) then yields the energy balance equation

$$2\pi\eta \sum n^2 x_n^2 = kFx_k \cos(\psi-\phi_k) \ . \qquad (8)$$

To proceed farther, solutions for the specific force laws of the experiment must be employed. Here, Robinson's solution procedure for the symmetric softener is summarized. He starts by assuming a large slope ratio between the central and outer line segments of f(x) and vigorous motion, i.e., the excursions of oscillation are much greater than $\pm x_o$, the values at which the slope changes. Then, the motion can be approximated by

$$x(t) = A[a \cdot \sin(t/k) - \cos(t)] \ . \qquad (9)$$

This form fits the observed behavior quite well. In fact, a phase plot of equation (9) for the conditions of Figure 9 is nearly indistinguishable from that figure. Substituting the approximate form into the energy balance equation leads to an expression for the minimum force required to sustain an orbit of period k:

$$F = 4\eta(\alpha^2 x_o)k^3/\pi \ , \qquad (10)$$

where α^2 is the slope in the central region.

Robinson compared this expression to the experimental parameters at the onset of the k = 3,5,7, and 9 period motion. He demonstrated that F was indeed proportional to k^3 and that the constant of proportionality was accurate to a few percent. The analytic solutions also predicted the amplitude of the slow component to be $F/\eta k$, and the data confirmed the k^2 dependence. Similar comparisons between data and the analytic model were made for the asymmetric softener and hardener. Thus, Robinson has demonstrated that simple analytic forms can reproduce the observed periodic motion faithfully and that energy-balance considerations provide accurate predictions of control parameter thresholds for the appearance of the various periodicities. Unfortunately, the analysis cannot be used to gain understanding of the chaotic regime.

One important discovery about systems with softener-restoring forces with consequences for a number of physical systems is a shift in the power spectrum to lower frequencies. Robinson provides the example of lattice vibrations in crystals near the melting point. A striking feature of melting is the suddenness with which it occurs. A possible explanation is that at melting, lattice modes of high frequency reach an amplitude at which period multiplication begins. This shifts the power spectrum of thermal excitation to lower frequencies, causing a new group of modes to become unstable. By a series of such cascades, thermal energy in the form of latent heat might be extracted from the surroundings and be concentrated in low-frequency modes that suddenly disrupt the lattice at the melting point.

An Explanation of Noise

As indicated above, F.T. Arecchi's nonlinear optics group at the Istituto Nazionale di Ottica in Florence is well known for work on optical bistability. In one set of experiments (Arecchi et al., 1982), they correlated the appearance of 1/f noise with chaotic jumps between the two stable states of a CO_2 laser cavity system. As part of an ongoing theoretical effort to describe this phenomenon, Arecchi and F. Lisi have built an electronic analog using a field-effect transistor to simulate the cubic force law $f(x) = 4x^3-x$. This force has two potential valleys centered at $x = \pm 1/2$ that represent the two stable operating points of the laser cavity. The circuit is harmonically driven as in other experiments, and the driving voltage amplitude A is the control parameter. The circuit is characterized by a driving frequency varied in the vicinity of 560 Hz, a natural resonance frequency of about 459 Hz, and a fixed damping rate of $\eta = 0.154$.

For values of A below about 631 mV, motion is confined to one of the

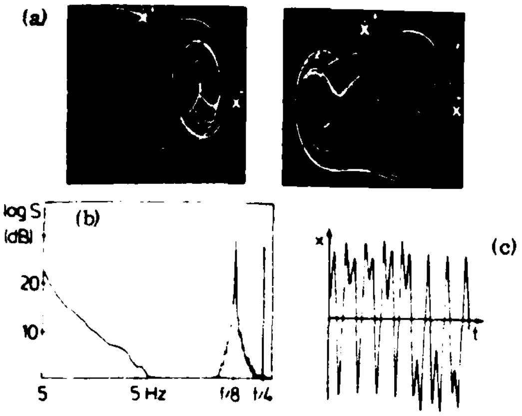

Figure 11. Phase space plots, noise spectrum, and time variation for the optically bistable analog.

valleys, and Arecchi and Lisi observe a standard sequence of subharmonic bifurcations leading to chaos. Doubling up to period 16 has been observed with the universal behavior predicted by Feigenbaum. As A is increased above 1.8 V, the energy of motion exceeds the potential barrier between the two valleys, and the system hops from one to the other after periods of chaotic activity in each. The two alternating phase plots are completely symmetric and show subharmonic line spectra (Figures 11a and 11c). However, as shown in Figure 11b, the hopping introduces a new feature in the power spectrum, an accurate 1/f noise continuum in the low-frequency 0.5- to 5-Hz band.

Arrechi and Lisi suggest that the noise may be heuristically explained by the long-period jumps between different states within each of which the short-period motion is chaotic. In addition to explaining the noise spectrum of the simulated laser experiments, they suggest that the well-studied flicker noise in electrical resistors may arise from similar random jumping of electrons between surface traps when driven by a current.

Chaos in Josephson Junctions

The noise structure of Josephson junctions also provides experimental evidence for chaotic behavior--although, unlike the nonlinear optical systems, direct observation of bifurcation has yet to be made. Still, researchers have develo,.-d electrical analogs of Josephson junctions with the objectives of studying the detailed dynamic behavior of the differential equations and learning how to control noise in experiments (Yeh and Kao, 1982). As in optical bistability research, a primary application of Josephson devices is advanced computers, because the digital elements can be made small, fast, and low-powered.

H. Seifert of the University of Tübingen, Federal Republic of Germany, has recently used a Josephson junction analog to study the effects of chaotic behavior on the alternating-current-induced step structure of the time-averaged current-voltage (I-V) characteristic. The circuit (described in Seifert, 1983) models the normalized current conservation equation for such devices

$$B\ddot{\phi} + \dot{\phi} + \sin\phi = i_0 + i_1\sin\xi t, \qquad (11)$$

where the left-hand side represents the sum of currents within the junction, and the right-hand side is the applied driving current. Equation (11) is similar to those considered above with a sinusoidal f(x) and a dc drive component. The dynamic variable ϕ is the phase difference between the junction electrodes and is related to the normalized voltage across the electrodes by $v = \dot{\phi}$.

Seifert chose the parameters $B = 0.5$, $i_1 = 0.8$, $\xi = 1$, and (similar to the Portuguese research) treated i_0 as the control parameter. The chosen parameter regime corresponds to resistively shunted tunnel junctions in which hysteresis effects are small and the damping is insufficient to suppress chaotic behavior. The resulting I-V characteristic, shown in Figure 12, is quite complex but models the behavior of superconducting microbridges quite well. Seifert studied the detailed behavior occurring on either side of the $v = 1/3$ voltage dip on the $v = 1/2$ plateau. Within these straight-line regions, the pair current is locked to subharmonics of the driving current (the time-averaged voltage is constant since it is proportional to $\dot{\phi}$).

Figure 13 shows a bifurcation diagram for this part of the I-V

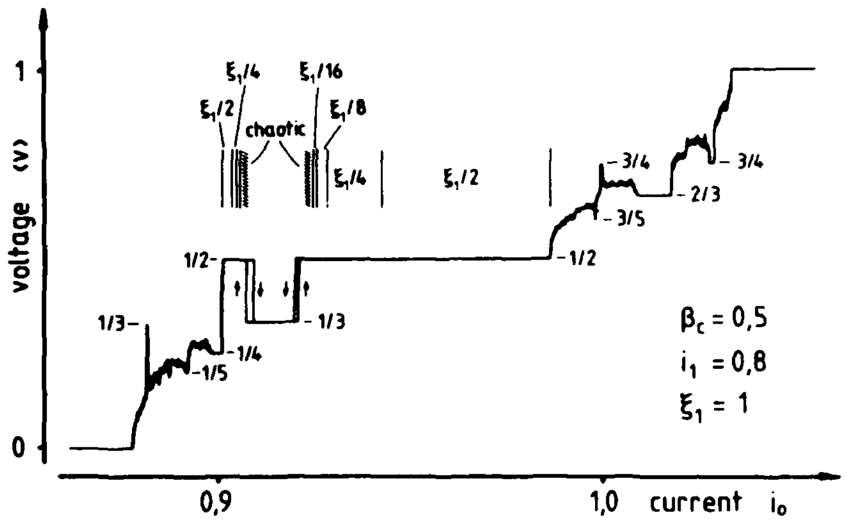

Figure 12. Josephson junction analog I-V characteristic.

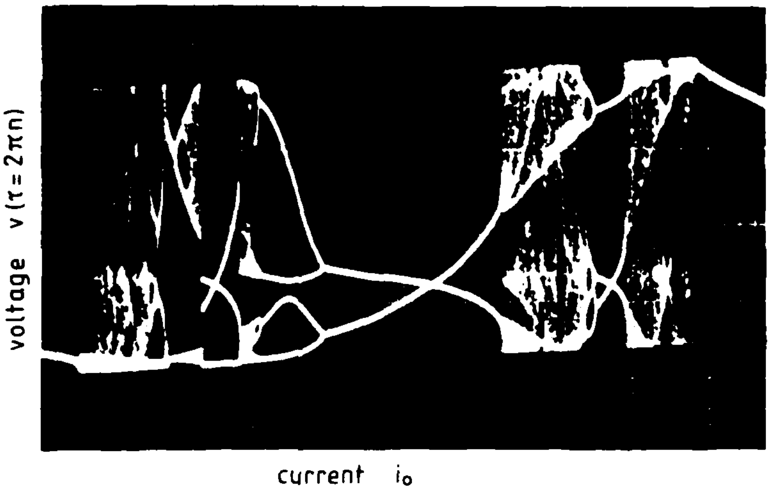

Figure 13. Bifurcation diagram near the v = 1/3 step.

characteristic obtained by plotting the voltage value at a particular phase of each driving period. The locked states are represented by discrete lines, and their number at a given i_0 determines the periodicity. Note the quite different behavior on the two sides of the voltage step.

Amplitude spectra in the multiple-period regime were examined for Feigenbaum universality, and considerable deviations were found. Seifert believes there are two reasons for the discrepancies. First, noise in the analog system might initiate chaotic behavior after only a few bifurcations so that the universal, high-periodicity behavior could not be observed. Second, the requirements on p(x) for Feigenbaum's one-dimensional maps might not be satisfied

Figure 14. Return maps constructed from the phase space plots.

in the Josephson analog. To test this conjecture, Seifert constructed the return maps $x_{j+1} = f(x_j)$ with $x = \sin\phi$ by electronically processing phase space plots for the two sides of the voltage step. The plots of x_{j+1} versus x_j are shown in Figure 14. Seifert argued that the central dip in Figure 14b violates Feigenbaum's single-maximum requirement so that deviations from the theory are to be expected.

However, the main results of the work do not depend on such details. Seifert has demonstrated that the ac-induced steps in the time-averaged I-V characteristic can be explained by period doubling to chaos and that the chaotic behavior can lead to the observed negative resistance regions in such Josephson junction devices.

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