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DEFENCE SCIENCE AND TECHNOLOGY ORGANISATION

WEAPONS SYSTEMS RESEARCH LABORATORY

DEFENCE RESEARCH CENTRE SALISBURY
SOUTH AUSTRALIA

TECHNICAL MEMORANDUM

WSRL-0292-TM

ON THE IMPLEMENTATION OF NOISE IN THE DISCRETE SIMULATION OF CONTINUOUS SYSTEMS

K.H. LLOYD

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ON THE IMPLEMENTATION OF NOISE IN THE DISCRETE SIMULATION OF CONTINUOUS SYSTEMS

K.H. Lloyd

SUMMARY

This report discusses the implementation of system and process noise, including sensor errors and atmospheric turbulence, in the simulation of continuous models by digital computers. The concept of discrete white noise is first introduced in a manner in which it reduces to continuous white noise as the integration time interval reduces to zero. The derivation of first and second order Markov processes from white noise is then discussed. The discussion includes the consideration of a suitable difference algorithm to approximate differentiation. The treatment throughout has been aimed at providing the reader with the tools to implement a required power spectral density (or autocorrelation function) as noise in any discrete digital simulation model.

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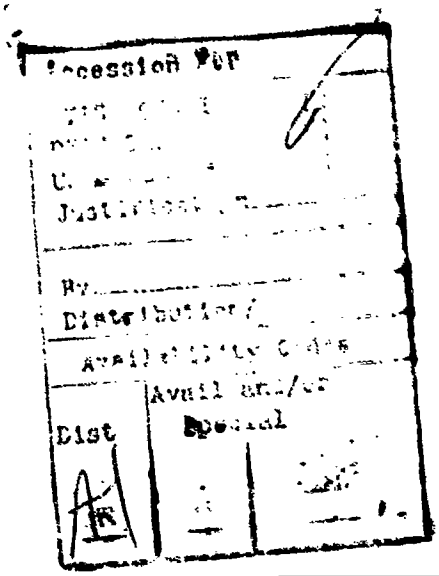

1. INTRODUCTION

In this Report we discuss the implementation of noise in the discrete simulation, on a computer, of a continuous process. We will develop the concepts necessary for our purposes, but only in sufficient depth to preserve continuity of the text. References are given to books in which a rigorous mathematical treatment can be found.

In the next section we briefly examine continuous noise, starting with white noise and then showing how noise of any desired spectral density can be obtained by using white noise as the input variable to a suitable (stochastic) differential equation.

Section 3 examines discrete noise, using the z transform formalism, in a manner suitable for implementation in digital computer simulation of a system. Particular care is taken to develop a formalism which reduces to the continuous case as the discrete interval tends to zero. The z transform, as normally defined, does not have this property.

The discrete equivalent of white noise is discussed, and then the spectrum of Markov processes generated by using white noise as the forcing function to a difference equation is examined. The relative merits of various algorithms for the discrete approximation of the continuous differential are considered in this treatment.

Approximating a continuous noise process by an equivalent discrete sequence introduces errors, which increase with the integration step size. Expressions are derived relating the maximum step size to the fractional error in the power spectral density and the parameters of the system.

Appendix I gives a worked example evaluating the parameters of a given noise spectrum, and implementation in a discrete simulation. Appendix II shows how atmospheric turbulence can be simulated by the methods described in the Report. Appendix III discusses the discrete simulation of the Wiener process.

2. NOISE IN CONTINUOUS SYSTEMS

2.1 Methods for describing stochastic processes

If v is some variable, eg acceleration roll rate or wind speed, and m is the measurement of that variable, then these are related by

$$m = v + n \tag{1}$$

where n is the measurement noise or turbulence, represented by a stochastic process. n is described by the probability density function $p(a,t)$, or equivalently, by the statistical moments. Gaussian random processes are completely described by the first and second moments, and, since most random processes are found to be nearly Gaussian, it is common not to examine any moment higher than the second. The first moment, or mean, is given by:

$$\mu = E[n(t)] = \int_{-\infty}^{\infty} a\, p(a,t)\,da = \lim_{T\to\infty} \frac{1}{T} \int_{-T/2}^{T/2} n(t)\,dt \tag{2}$$

where p is the probability density function for the process with amplitude α at time t. The last equality follows from the assumption that the process is ergodic. In this memorandum we assume the noise is stationary, so that p does not depend on time.

The second order moment can be given in terms of either the autocorrelation function or the covariance function:

$$C(\tau) = E[n(t+\tau)\, n(t)] = \int_{-\infty}^{\infty} \alpha\beta\, p(\alpha, t; \beta, t+\tau)\, d\alpha\, d\beta$$

$$= \lim_{T \to \infty} \frac{1}{T} \int_{-T/2}^{T/2} n(t+\tau)\, n(t)\, dt \tag{3}$$

$$R(\tau) = E[\{n(t+\tau) - \mu\}\{n(t) - \mu\}] \tag{4}$$

where $p(\alpha,t;\beta,t+\tau)$ is the joint probability distribution function. The autocorrelation and covariance functions are related by

$$R(\tau) = C(\tau) - \mu^2 \tag{5}$$

In this paper we will be solely concerned with zero mean processes for which $\mu = 0$. When dealing with a non-zero-mean process, it is assumed that the mean is first removed from the process.

The second moment can be equivalently described by the power spectral density(ref.1):

$$\Phi(\omega) = \int_{-\infty}^{\infty} C(\tau) e^{-j\omega\tau}\, d\tau \tag{6}$$

which is the Fourier (or bilateral Laplace transform with $s = j\omega$) transform of the autocorrelation function. Note that we are using a two-sided spectrum. This will be the case throughout. The inverse of equation (6) is

$$C(\tau) = \frac{1}{2\pi} \int_{-\infty}^{\infty} \Phi(\omega) e^{j\omega\tau}\, d\omega \tag{7}$$

We will now examine Parseval's theorem, in preparation for comparing its expressions in the discrete and continuous cases. Applying Parseval's theorem to N_T, the Fourier transform of n_T, where $n_T = n$ for $-T/2 < t < T/2$ and $= 0$ for $|t| > T/2$, gives(ref.1):

$$\lim_{T \to \infty} \frac{1}{T} \int_{-T/2}^{T/2} n_T^2(t)\, dt = \lim_{T \to \infty} \frac{1}{2\pi T} \int_{-\infty}^{\infty} N_T(\omega)\, N_T(-\omega)\, d\omega \tag{8}$$

This theorem can be regarded as a statement of equality of energy in the

hand side of equation (8) is the autocorrelation at zero lag, ie the variance. Also, it can be shown that

$$\Phi(\omega) = \lim_{T \to \infty} \frac{1}{T} N_T(\omega) N_T(-\omega) \tag{9}$$

so that equation (8) can be written as

$$C(0) = \frac{1}{2\pi} \int_{-\infty}^{\infty} \Phi(\omega)\, d\omega \tag{10}$$

Equation (10) also follows from equation (7) by setting $\tau = 0$; however, we have derived it using Parseval's theorem for comparison with the discrete analysis of the next section. Note that for continuous white noise both sides of the equation are infinite unless $\Phi(\omega) = 0$.

2.2 White noise and the generation of Markov processes

A stochastic process commonly used in the study of noise is Gaussian white noise which has autocorrelation and power spectral density defined by

$$C_u(\tau) = \Psi\, \delta_D(\tau), \Phi_u(\omega) = \Psi \tag{11}$$

where δ_D is the Dirac impulse function, and Ψ is the strength of the noise.

We shall refer to this process as continuous white noise, to distinguish it from discrete white noise introduced in the next section. The words 'continuous' and 'discrete' are used to describe the system to which the noise is applied.

If we have a measurement noise which is other than white, this random process can be generated by having it as the dependent variable in a differential equation whose forcing function is white noise, and augmenting this differential equation to the system equations(ref.2). For example, if x is given by

$$\dot{x} = -\nu x + a u \tag{12}$$

where u is white noise whose autocorrelation and spectral density is given by equation (11), then x is also a Gaussian process whose autocorrelation and spectrum are given by

$$C_x(\tau) = \frac{a^2}{2\nu} e^{-\nu|\tau|} \Psi, \Phi_x(\omega) = \frac{a^2}{\nu^2 + \omega^2} \Psi \tag{13}$$

There is an arbitrariness in the definition of a and u in equation (12). We shall remove this be defining the white noise to have unit power spectral density, viz $\Psi = 1$.

The process defined by equation (12) is a first order Markov process(ref.2). This noise is also known as exponentially correlated noise from the shape of its autocorrelation function.

A second order Markov process is generated by the pair of equations:

$$\dot{y} = \omega_0^2 x - 2\zeta \omega_0 y + cu \tag{14}$$

$$\dot{x} = y + au \tag{15}$$

Note that the same white noise source is input to both equations.

The autocorrelation function and power spectral density of x which result from this process are:

$$C_x(\tau) = \frac{a^2 \omega_0^2 + b^2}{4\zeta \omega_0^3 \cos\eta} \cos[\sqrt{1-\zeta^2} \cdot \omega_0 |\tau| - \eta] e^{-\zeta \omega_0 \tau} \Psi \tag{16}$$

$$\Phi_x(\omega) = \frac{a^2 \omega^2 + b^2}{\omega^4 + 2\omega_0^2 (2\zeta^2 - 1)\omega^2 + \omega_0^4} \cdot \Psi \tag{17}$$

where, once more, u is taken to have unit power spectral density, and $b = c + 2a\zeta\omega_0$.

3. DISCRETE SIMULATION OF NOISE

3.1 Methods for describing discrete random processes

In a digital computer program which simulates a continuous system, the states are evaluated only at discrete time intervals. For the case of uniform sampling intervals, δ, equation (1) becomes

$$m(k\delta) = v(k\delta) + n(k\delta) \tag{18}$$

whereby the values of the continuous functions are only sampled at set times. The formal expression of sampling may be written in different ways. One common method (see, for example, reference 3) is to introduce Dirac delta functions to represent the sampling process. However, this is unsatisfactory for computer simulations, which require numbers and not impulses. In addition, by expressing sampling in terms of numbers, as is done in equation (18), the analysis reduces to the continuous case as the sampling interval goes to zero. This is further discussed after equation (21).

Given the sampling process expressed by equation (18), the question which has to be answered is: what is the relation between the autocorrelation

function and power spectrum of this discrete system and of the continuous system to which it approximates? We commence by defining the autocorrelation function and spectral density for discrete time random processes:

$$C(\ell\Delta) = E[n(|k+\ell|\Delta)\, n(k\Delta)]$$

$$= \lim_{K\to\infty} \frac{1}{K+1} \sum_{k=-K/2}^{K/2} n(|k+\ell|\Delta)n(k\Delta) \tag{19}$$

$$\Phi(z) = \Delta \sum_{-\infty}^{\infty} C(\ell\Delta)z^{-\ell} \tag{20}$$

where Δ is the discrete time interval between samples. The power spectral density is identified as the two sided z transform(ref.4,5) of the autocorrelation function. Note that the spectral density of this sampled process is only defined for $-\pi/\Delta < \omega < \pi/\Delta$. The inverse of equation (20) is

$$C(\ell\Delta) = \frac{1}{2\pi j\Delta} \oint \Phi(z) z^{\ell-1}\, dz \tag{21}$$

which may be also written as

$$C(\ell\Delta) = \frac{1}{2\pi} \int_{-\pi/\Delta}^{\pi/\Delta} \Phi(e^{j\omega\Delta})\, e^{j\omega\ell\Delta}\, d\omega \tag{21(a)}$$

by substituting $z = e^{j\omega\Delta}$.

The factor Δ outside the summation in equation (20) and outside the integral in equation (21) has been included to give continuity with the Laplace transform as $\Delta \to 0$, $k\Delta \to t$. This factor is not generally included in the definition of the z transform, but is vital for the treatment of simulation in this text. We will therefore digress briefly on this matter by means of example. Consider the damped exponential defined by

$$x = \exp(-at) \qquad t \geq 0$$

$$= 0 \qquad t < 0$$

whose Laplace transform is

$$X_s = \mathcal{L}[x] = (a+s)^{-1}$$

Taking the z transform of x using the Δ multiplied definition (equation (20)) gives

$$X_z = \mathcal{Z}[x]$$

$$\triangleq \Delta \mathcal{L}[x^*]$$

$$= \Delta/(1 - \exp[-(a + s)\Delta]),\ z = e^{s\Delta}$$

where $x^*(t) = \Sigma x(t)\delta_D(t-n\Delta)$.

As $\Delta \to 0$, we have $X_z \to X_s$, which would not have been the case if the Δ factor had not been included. In general (eg in sampling systems), this continuity from the z transform to the Laplace transform is not required, so the factor Δ is omitted. However, we feel that the z transform including a Δ factor is the more natural definition. Confirmation of this is suggested by considering the expression for the z transform as a sum of the values of the Laplace transform over the primary and complementary strips. Using the usual definition of the z transform this relation is

$$X_z = \frac{1}{\Delta} \sum_{-\infty}^{\infty} X_s(s + 2\pi jn/\Delta)$$

If the Δ multiplied expression for the z transform is used, the above relation becomes

$$X_z = \sum_{-\infty}^{\infty} X_s(s + 2\pi jn/\Delta)$$

As $\Delta \to 0$, $X_s(s + jn.2\pi/\Delta) \to 0$ for all n except n = 0 and we are left with the simple relation

$$[X_z]_{\Delta \to 0} \to X_s$$

when we use the Δ multiplied definition of the z transform. We may heuristically view the Δ factor as moderating the impulses in the sampling process, so that, as $\Delta \to 0$, we regain a continuous function. The benefits of the Δ multiplied definition will become manifest in Section 3.2 where it enables us to define a discrete equivalent of continuous white noise. As we have said earlier, the Δ factor is usually omitted, since it is of no consequence when this limit is not of interest.

Further insight to this definition of the z transform can be gained by considering Parseval's theorem, which relates the energy in the signal to the energy in the spectrum. The discrete equivalent of Parseval's theorem is

$$\lim_{K \to \infty} \frac{\Delta}{K} \sum_{-K/2}^{K/2} n^2(k\Delta) = \lim_{K \to \infty} \frac{1}{2\pi K} \int_{-\pi/\Delta}^{\pi/\Delta} N_K(e^{j\omega\Delta}) N_K(e^{-j\omega\Delta}) d\omega \quad (22)$$

We identify the left hand side with correlation at zero shift (vide equation (19)) and it can be shown that

$$\Phi(z) = \lim_{K \to \infty} \frac{1}{K} N_K(z)\, N_K(z^{-1}),\ z = e^{j\omega\Delta} \qquad (23)$$

where N_K is the z transform of $n(k\Delta)$, k = -K/2 to K/2.

Substituting equation (23) into (22) yields equation (21(a)) with ℓ=0.

Contrary to the case for continuous white noise, Parseval's theorem does have application in the discrete equivalent of white noise, as we shall see.

3.2 Discrete white noise

Suppose for our noise in equation (18) we generate random numbers $u(k\Delta)$ with a Gaussian distribution and variance σ^2. This can be regarded as discrete 'white' noise with autocorrelation function and power spectral density given by

$$C_u(\ell\Delta) = \sigma^2 \delta_{Kr}(\ell),\ \Phi_u(z) = \sigma^2\Delta \qquad (24)$$

where δ_{Kr} is the Kronecker delta. We wish to show that $u(k\Delta)$ reduces to continuous white noise defined by equation (11) as $\Delta \to 0$(ref.1). Although the steps which follow are not mathematically rigorous, they are included as an heuristic derivation of the required relations.

If we define u(t) as

$$u(t) = \lim_{\Delta \to 0,\ \ell\Delta \to t} u(\ell\Delta) \qquad (25)$$

where the limit is assumed to exist in some generalized sense, then we wish to show that u(t) has the properties of "continuous" white noise as defined by equation (11). Now

$$\begin{aligned} C_u(\tau) &= E[u(t)\, u(t+\tau)] \\ &= \lim_{\Delta \to 0,\ k\Delta \to t,\ \ell\Delta \to \tau} E[u(k\Delta)\, u((k+\ell)\Delta)] \\ &= \lim_{\Delta \to 0,\ |\tau - \ell\Delta| < \Delta/2} \sigma^2\, \delta_{Kr}(\ell\Delta) \\ &= \lim_{\Delta \to 0} \sigma^2 \{h(\tau - \Delta/2) - h(\tau + \Delta/2)\} \\ &= \lim_{\Delta \to 0} \sigma^2\, \Delta\, \delta_D(\tau) \end{aligned}$$

where h(t) denotes the unit step function. Comparing this result with equation (11):

$$C_{\eta}(\tau) = \Psi\delta_{D}(\tau) \tag{11}$$

we see that $\eta(t)$ has the characteristics of continuous white noise, with

$$\Psi = \lim_{\Delta \to 0} \sigma^2\Delta \tag{28}$$

ie we have shown that if we implement discrete white noise with the autocorrelation function given by $\sigma^2\delta_{Kr}(\ell)$ then this is equivalent to a continuous white noise process with autocorrelation function $\Delta\sigma^2\delta_{D}(\tau)$.

Let us continue this equivalence of continuous and discrete processes into the frequency domain. Continuous white noise has the uniform power spectrum Ψ (equation (11)). The discrete white noise has the spectral density $\sigma^2\Delta$, which extends from $-\pi/\Delta < \omega < \pi/\Delta$. Using relation (28) it is seen that the amplitude of the power spectral density function for the discrete case becomes Ψ, which is identical to that for the continuous; these relations are shown in figure 1. Note that the discrete equivalent of Parseval's theorem is obeyed, so our use of the factor, Δ, has been consistent.

3.3 Discrete algorithms for the differential operator

As can be done for continuous processes, it is possible to generate a discrete random process with a desired power spectral density by a difference equation with a discrete white noise forcing function. Since we will be wishing to relate the continuous and discrete processes thus produced, in particular, to see how well the power spectral density of the latter approximates to that of the former, we will briefly digress and examine the derivation of difference equations from differential equations.

The simplest discrete approximation to the derivative is the forward difference defined by:

$$x[(n+1)\Delta] = x[n\Delta] + \dot{x}[n\Delta].\Delta \tag{29}$$

The difference equation thus obtained is unsatisfactory for an integration routine, as solutions obtained using it rapidly diverge. This is because the magnitude of the transfer function is greater than the $1/\omega$ value of continuous integration, especially at the higher frequencies. This is shown in figure 2 which plots the ratio of the magnitude of the transfer function for this case,

$$\text{viz.} \left| \frac{\Delta}{1 - z^{-1}} \right| = \frac{\Delta}{\sqrt{2(1 - \cos\theta)}}, \quad z = e^{j\omega\Delta} \tag{30}$$

as a function of angular frequency.

All this is well known, and forward differences are rarely used in discrete integration routines. Commonly used integration algorithms are fourth order Runge-Kutta and predictor-corrector, but both of these are difficult to implement with white noise as the input. Runge-Kutta is impossible to implement because it requires the evaluation of the function at

intersequence values of time, and we then have the choice of using a new value for the random variable or of using the value at one end of the time step. In either case we will be inputting a noise spectrum different from that of the discrete white noise discussed in Section 3.2. Predictor-corrector methods do not suffer from this problem; however, when they are being implemented, the values of the noise variable at the several time steps involved in each integration step must be consistently maintained. This involves careful programming, so it is preferable to use a simpler integration routine which is yet well behaved. One such is the modified Euler algorithm given by

$$x[n + 1)\Delta] = x[n\Delta] + \tfrac{1}{2}\Delta\{\dot{x}[(n + 1)\Delta] + \dot{x}[n\Delta]\} \qquad (31)$$

This is also known as second order Runge-Kutta algorithm. It gives for the discrete transfer function:

$$\frac{\Delta(1 + z^{-1})}{2(1 - z^{-1})} = \frac{\Delta}{2}\sqrt{\frac{1 + \cos\omega\Delta}{1 - \cos\omega\Delta}} \qquad (32)$$

This function, which is also plotted in figure 2, is seen to damp out high frequencies, which is why this routine is stable. Therefore, provided that the integration time step is sufficiently small that none of the natural modes of the system is severely damped by the transfer function of this integration routine, it is a suitable integration method. It also has the advantages of being simple to implement in a computer program (only requiring two iterations per step interval) and of being simple enough to examine analytically to see what effect it has on given noise inputs. The well known tendency of integration using this algorithm to 'drift' can be seen to arise because the curve in figure 2 starts to depart from unity at the lowest frequencies.

The accuracy of the modified Euler method can be increased by iteration for $x(n\Delta)$ and continuing until the difference between successive iterations is as small as desired. When this is done, the method can be regarded as a second-order prediction-corrector. A further virtue of this simple algorithm is that it is suitable for a set of stiff equations(ref.6), that is, a set of equations with a wide range of values for its eigenvalues. Such sets of equations occur in flight simulation where the time constant can vary from 100 s (for phugoid) to 0.01 s (for actuators). References 7 and 8 discuss the benefits of the modified Euler integration routine when the set of equations are stiff, it is frequently superior to far more sophisticated algorithms.

Although it is not the purpose of this memorandum to analyse integration procedures, we will end this Section with a mention of an alternative to formal integration routines. This procedure, discussed in reference 9, enables large time steps to be used in discrete simulation by using a difference equation which accurately maps the poles of the original differential equation. This technique is valuable for stiff systems because the time step interval can be made comparable with the time constant of the fastest mode (instead of much less, as is required for standard integration routines). However, when the coefficients of the difference equation are derived by requiring correct representation of the poles of the original differential equation, a power series representation of the input with time over each interval must be assumed (for example, quadratic). Since white noise can not be represented by a power series this procedure must introduce an approximation. It is generally advisable

to check if the consequent error be significant.

Another example of the analogue-digital technique for discrete simulation of continuous systems is given in reference 10. In the method given therein the coefficients of the difference equation are chosen so that the first and second moments of the discrete correlated noise are the same as for the correlated noise generated by the differential equation which is being modelled. The resulting expressions, which are rather lengthy, are essentially in terms of the transition matrix of the differential equation integrated over the discrete time interval.

We will now study the generation of discrete Markov processes, first by difference equations derived using the forward difference algorithm and then by using the modified Euler algorithm.

3.4 Generation of discrete Markov processes

3.4.1 Forward difference algorithm

By using the forward difference approximation to the derivative (equation (29)) the discrete equivalent of the first order Markov process (equation (12)) is

$$x[(n+1)\Delta] = (1-\nu\Delta)\, x[n\Delta] + a\Delta.u(n\Delta) \qquad (33)$$

where, as has been shown above, $u(n\Delta)$ is a sequence of uncorrelated random numbers with variance $\sigma^2 = \Psi/\Delta$.

The power spectral density of x is given by the product of the modulus of the frequency transfer function of the equation with the power spectral density of u(ref.11), so we have for the discrete first order Markov process a power spectral density:

$$\Phi_x(z) = \frac{a^2\,\Delta^2\,\Psi}{(z - 1 + \nu\Delta)(z^{-1} - 1 + \nu\Delta)} \qquad (34)$$

or

$$\Phi_x(\omega) = \frac{a^2\,\Delta^2\,\Psi}{1 + (1 - \nu\Delta)^2 - 2(1 - \nu\Delta)\,\cos\omega\Delta} \qquad (35)$$

Taking the limit $\Delta \to 0$ gives the power spectral density for the continuous first order Markov process, equation (13), as expected.

In figure 3, we have plotted the power spectral density for the continuous Markov process, which should be compared with that of the discrete process plotted in figure 4. It is seen that the power for the discrete process is greater than for the continuous process, but tends to it either as $\Delta \to 0$ or as $\omega \to 0$.

When simulating continuous processes by discrete sampling, it is important to know what errors are introduced as a function of the sampling interval. We will now derive expressions relating the maximum permissible discrete time integration interval for a desired accuracy, in terms of the parameters of the first order Markov process.

There are two constraints on the sampling interval, Δ. The first is that the power spectral density decrease to a negligible value at the Nyquist frequency ($=\pi/\Delta$). For a first order Markov process, the ratio of power spectral density at the Nyquist frequency to its peak value is

$$\varepsilon = \upsilon^2/[(\pi/\Delta)^2+\upsilon^2]$$

We can interpret this equation as stating that a fractional error, ε, results from a sampling interval Δ, due to aliasing. Rearranging gives

$$\Delta < \pi\sqrt{\varepsilon}/\upsilon \tag{36}$$

as the contraint on Δ, where we have used the condition $\varepsilon<<1$ in deriving equation (36). The other constraint is that the sampling interval be small enough that the discrete power spectral density approximates the continuous one. Let ε denote the ratio of the difference between the discrete and continuous power spectral densities to the maximum of the power spectral density. This ratio maximises at $\omega = \upsilon$, and gives the following constraint on Δ, for the error ratio to be less than ε:

$$\Delta < 2\varepsilon/\upsilon \tag{37}$$

Note that the two constraints both involve υ. This is hardly surprising as we have only one parameter, viz. υ. Since $\varepsilon<<1$, equation (37) imposes tighter constraint on Δ than does equation (36).

We now turn to second order Markov processes. Substituting equation (33) into equations (14) and (15), eliminating y, and taking the z transform of the resulting difference equation yields

$$[z^2 + 2(\zeta\omega_0\Delta - 1)z + (1 - 2\zeta\omega_0\Delta + \omega_0^2\Delta^2)]X = [a\Delta.z + (b\Delta^2 - a\Delta)]U \tag{38}$$

where $b = c + 2a\zeta\omega_0$ and X, U are the z transforms of x and u (the z transform of u being purely formal).

We can now write the power spectral density of x as

$$\Phi_x = \frac{\Delta^2[\{a^2 + (b\Delta - a)^2\} + 2a(b\Delta - a)\cos\omega\Delta]}{1 + \phi_1^2 + \phi_2^2 + 2\phi_1(1 + \phi_2)\cos\omega\Delta + 2\phi_2\cos 2\omega\Delta} \tag{39}$$

where

$$\phi_1 = 2(\zeta\omega_0\Delta - 1)$$

$$\phi_2 = 1 - 2\zeta\omega_0\Delta + \omega_0^2\Delta^2$$

As before, we compare the continuous second order Markov process, plotted in figure 6, with the forward difference discrete approximation, figures 7(a) and (b). It is seen the discrete power spectral density diverges from the continuous more rapidly with integration step interval than it did for the first order Markov case. This can be attributed to a Δ "resonance" in the power spectrum (for any given ω), regarding ϕ now as a function of Δ, and is illustrated in figure 7(c).

As before, we have constraints on Δ imposed by the requirements of the continuous power spectral density being small at the Nyquist frequency, and of the discrete power spectral density being close to the continuous. These are more difficult to apply to the second order Markov process. After some manipulation it can be shown that, for the aliasing and discrete approximation errors to be less than a fraction ϵ, the sampling interval must satisfy

$$\Delta < \pi\, b\sqrt{\epsilon}/a\omega_0^2 \tag{40}$$

and

$$\Delta < \frac{(b^2 + a^2\,\omega_0^2)\zeta\, e}{(b^2 - ab\zeta\omega_0 + a^2\,\omega_0^2)\omega_0} \tag{41}$$

3.4.2 Modified Euler algorithm

We will now derive the modified Euler approximations to the first and second order Markov generating equations and compare them with the results of the Section 3.4.1.

Substituting equation (31) into equation (12) and taking the z transform gives

$$[(1+\upsilon\nabla)\, z - (1-\upsilon\nabla)]\, X = a\nabla(1+z)U \tag{42}$$

where $\nabla = \Delta/2$.

We can immediately write down the power spectral density of x to be

$$\Phi_x(\omega) = \frac{2a^2\nabla^2\,(1 + \cos\omega\Delta)\,\Psi}{1 + \nu^2\,\Delta^2 - (1 - \nu^2\,\Delta^2)\,\cos\omega\Delta} \tag{43}$$

This function is plotted as a function of ω for $a = \upsilon = 1$ and $\Delta = 0,1,2$ in figure 5. Comparing with figure 4 we see there is a substantial (about four times in this case) improvement on accuracy for a given Δ. Note that the forward difference approximation amplifies the power spectral density at high frequencies, while the modified Euler approximation diminishes it. These properties are to be expected from their respective frequency response functions (see figure 2).

Limitations on the size of Δ exist from the same constraints as

discussed in Section 3.4.1 and lead to:

$$\Delta < \pi\sqrt{\varepsilon}/\upsilon \tag{44}$$

$$\Delta < 3\sqrt{\varepsilon}/\upsilon \tag{45}$$

Comparing equation (45) with (37) shows the modified Euler approximation allows larger time intervals for a given accuracy.

We now turn to our final case: the modified Euler approximation to the differential equations defining the second order Markov process. Substituting equation (31) into equations (14) and (15), and taking the z transform gives

$$[(1 + 2\zeta\omega_0\nabla + \omega_0^2\nabla^2)z^2 + 2(\nabla^2\omega_0^2 - 1)z + (1 - 2\zeta\omega_0\nabla)]X$$

$$= \nabla[(b\nabla + a)z^2 + 2b\nabla.z + (b\nabla - a)]U \tag{46}$$

where $b = c + 2a\zeta\omega_o$

which gives for the power spectral density

$$\Phi_x = \frac{2\nabla^2[a^2 + 3b^2\nabla^2] + 4b^2\nabla^2\cos\omega\Delta + (b^2\nabla^2 - a^2)\cos 2\omega\Delta]\Psi}{\phi_3^2 + \phi_4^2 + \phi_5^2 + 2\phi_4(\phi_3 + \phi_5)\cos\omega\Delta + 2\phi_3\phi_5\cos 2\omega\Delta} \tag{47}$$

where

$$\phi_3 = 1 + 2\zeta\omega_0\nabla + \omega_0^2\nabla^2$$

$$\phi_4 = 2(\omega^2\nabla^2 - 1)$$

$$\phi_5 = 1 - 2\zeta\omega_0\nabla + \omega_0^2\nabla^2$$

Equation (47) is plotted as a function of ω, for several values of Δ, in figure 8. Comparing with figure 7 shows that the modified Euler method is able to use much larger values of Δ before there is a significant difference between the continuous and discrete spectra. It is interesting that the modified Euler approximation produces not only a diminution of power at higher frequencies, but also a lowering of the frequency of maximum power. In place of equations (40) and (41) we have the following inequalities which must be satisfied for the discrete approximation to model the continuous second order Markov noise adequately:

$$\Delta < \pi b\sqrt{\epsilon}/a\omega_0^2 \tag{48}$$

$$\Delta < \frac{2\zeta}{\omega_0}\left[\frac{6(b^2 + \omega_0^2 a^2)\epsilon}{b^2(1 + 4\zeta^2) + \omega_0^2 a^2}\right]^{\frac{1}{2}}$$

$$\lesssim 2\zeta\sqrt{6\epsilon} \qquad (49)$$

We have summarised the results of Sections 2 and 3 in Tables 1 through 3. We have included the expressions for autocorrelation function in Table 1, because it is sometimes this, rather than the power spectral density of the random process, which is measured and to which the theoretical noise process is fitted to determine the parameters a, υ or a, c, ω_o, ζ.

3.5 Summary

A continuous white noise stochastic process, with power spectral density Ψ (or, equivalently with autocorrelation function $\Psi\delta_D$) can be modelled in a discrete simulation by a sequence of uncorrelated random numbers at the simulation times, whose variance is $\sigma^2 = \Psi/\Delta$ where Δ is the discrete time interval of the simulation. This discrete white noise also has a uniform power spectral density equal to Ψ, but it is band limited to the angular frequency range $-\pi/\Delta < \omega < \pi/\Delta$ (see figure 1).

Although white noise never occurs in nature, its benefit lies in the fact that noise of any given power spectral density can be generated analytically by the solution of differential equations whose forcing function is white noise. The first order Markov process has the power spectral density (see figure 3)

$$\Phi = \frac{2a^2}{\upsilon^2 + \omega^2} \cdot \Psi$$

and is generated by the first order differential equation

$$\dot{x} = -\upsilon x + au$$

where u is the white noise input, and is always chosen so that Ψ = 1. The second order Markov process (see figure 6)

$$\Phi_x = \frac{a^2\omega^2 + (c + 2\zeta\omega_0 a)^2}{\omega^4 + 2\omega_0^4(2\zeta^2 - 1)\omega^2 + \omega_0^4} \cdot \Psi$$

is generated by the coupled pair of equations

$$\dot{x} = y + au$$

$$\dot{y} = -\omega_0^2 x - 2\zeta\omega_0 y + cu$$

Given an observed power spectrum for noise we are then able to model the stochastic process which has this power spectrum by choosing the

coefficients of the equations (viz: a, υ; or a, c, ω_o, ζ) so the resultant power spectral density fits the observed power spectrum.

In discrete simulation of these processes, the differential equations are replaced by difference equations. As is discussed in the text, care must be taken in choosing a suitable algorithm with which to implement the differential operator. The best compromise between accuracy and ease of implementation appears to be the modified Euler method [...] which

$$x[(n+1)\Delta] = x[n\Delta] + \tfrac{1}{2}\Delta\{\dot{x}[(n+1)\Delta] + \dot{x}[n\Delta]\}$$

where the derivatives, $\dot{x}$, are replaced by the equations to be integrated. The numerical procedure which implements this relation is a two step iteration, because an estimate of $\dot{x}(n+1)$ must be made. When writing the code for the discrete simulation, care must be taken to be consistent in the use of the random numbers generated to simulate the white noise forcing function; viz: the number generated for u at the second step in the calculation of y(n+1) should also be used in the first step in the calculation of y(n+2).

As discussed in the text, integration routines such as fourth order Runge-Kutta, which require calculations at times intermediate to those at which the difference equation is evaluated, are unsuitable for the implementation of noise. This is because of the problem of what value to assign to the random variable at the intermediate steps. How this is done makes a large difference to the power spectral density of the process. Hence we recommend simpler integration routines, thereby obviating the problem.

A further matter which must be considered is that the power spectral density of a discrete Markov process calculated by difference equations differs from that given by the exact solution of the differential equations. This suggests that the Markov coefficients should be calculated by fitting the observed spectrum to the power spectral density of the discrete process. This is not recommended, because the fitment is no longer valid if the time step, Δ, of the simulation is changed. It is preferable to fit the continuous spectrum to the observations and note the error introduced by discrete simulation for various time step intervals.

4. ACKNOWLEDGEMENT

I am grateful to Dr D.P. Brown of Flight Research Group and Mr L.A. Nicholls of System Modelling Group for many valuable discussions.

TABLE 1. CONTINUOUS RANDOM PROCESSES

Process	White noise	1st order Markov	2nd order Markov
Power spectral density	Ψ	$\dfrac{a^2\ \Psi}{\omega^2 + \upsilon^2}$	$\dfrac{(a^2\ \omega^2 + b^2)\ \Psi}{\omega^4 + 2\omega_o^2 (2\zeta^2 - 1)\omega^2 + \omega_o^4}$
Integrated power spectral density	∞	$\dfrac{\pi\ a^2\ \Psi}{\upsilon}$	$\dfrac{\pi (a^2\ \omega_o^2 + b^2)\ \Psi}{2\ \zeta\ \omega_o^3}$
Autocorrelation function	$\Psi\ \delta_D \lvert \tau \rvert$	$\dfrac{a^2\ \Psi}{2\upsilon}\ e^{-\upsilon \lvert \tau \rvert}$	$\dfrac{(a^2\ \omega_o^2 + b^2)\ e^{-\zeta \omega_o \lvert \tau \rvert}}{4\ \zeta\ \omega_o^3 \cos\eta}$ $\cdot \cos[\sqrt{1-\zeta^2}\,\omega_o \lvert \tau \rvert - \eta]\ \Psi$
Generating equation *	$x = u$	$\dot{x} = -\upsilon x + au$	$\dot{y} = -\omega_c^2 x - 2\zeta \omega_o y + cu$ $\dot{x} = y + au$

* u is white noise with variance Ψ. In the text we take $\Psi = 1$.

$b = c + 2a\ \zeta \omega_o$

$$\tan \eta = \frac{\zeta}{\sqrt{1-\zeta^2}} \cdot \frac{b^2 - a^2\ \omega_o^2}{b^2 + a^2\ \omega_o^2}$$

TABLE 2. DISCRETE RANDOM PROCESSES - GENERATION BY FORWARD DIFFERENCE APPROXIMATION TO DIFFERENTIAL

Process	White noise	1st order Markov	2nd order Markov
Power spectral density	Ψ	$\dfrac{a^2\ \Delta^2\ \Psi}{1 + (1 - \upsilon\Delta)^2 - 2(1 - \upsilon\Delta)\ \cos\omega\Delta}$	$\dfrac{\Delta^2[\{a^2 + (b\Delta - a)^2\} + 2a(b\Delta - a)\cos\omega\Delta]}{1 + \phi_1^2 + \phi_2^2 + 2\phi_1(1 + \phi_2)\cos\omega\Delta + 2\phi_2\cos 2\omega\Delta}$
Integrated power spectral density	$\dfrac{2\pi\ \Psi}{\Delta}$	$\dfrac{2\pi\ a^2\ \Psi}{\upsilon(2 - \upsilon\Delta)}$	
Error criteria *	-	$\Delta < \pi\sqrt{\epsilon}/\upsilon$	$\Delta < \pi b\sqrt{\epsilon}/a\omega_o^2$
		$\Delta < 2\epsilon/\upsilon$	$\Delta < \dfrac{(b^2 + a^2\ \omega^2)\zeta\ \epsilon}{(b^2 - ab\omega_o\zeta + a^2\ \omega_o^2)\omega_o}$

* Δ must be less than this value in order that the discrete simulation model the continuous process to a fractional error less than ϵ.

$$\phi_1 = 2(\zeta\ \omega_o\ \Delta - 1)$$

$$\phi_2 = 1 + \omega_o^2\ \Delta^2 - 2\zeta\ \omega_o\ \Delta$$

TABLE 3. DISCRETE RANDOM PROCESSES - GENERATION BY MODIFIED EULER APPROXIMATION TO DIFFERENTIAL

Process	White noise	1st order Markov	2nd order Markov
Power spectral density	Ψ	$\dfrac{a^2 \nabla^2 (1 + \cos\omega\Delta)\ \Psi}{1 + v^2\nabla^2 - (1 - v^2\nabla^2)\cos\omega\Delta}$	$\dfrac{2\nabla^2\{(a^2 + 3b^2\nabla^2) + 4b^2\nabla^2 \cos\omega\Delta + (b^2\nabla^2 - a^2)\cos 2\omega\Delta\}}{\phi_3^2 + \phi_4^2 + \phi_5^2 + 2\phi_4(\phi_3 + \phi_5)\cos\omega\Delta + 2\phi_3\phi_5 \cos 2\omega\Delta}$
Integrated power spectral density	$\dfrac{2\pi\ \Psi}{\Delta}$	$\dfrac{\pi\ a^2\ \Psi}{v(1 + v\Delta)}$	
Error criteria	-	$\Delta < \pi\sqrt{\epsilon}/v$	$\Delta < \pi b\sqrt{\epsilon}/a\ \omega_o^2$
		$\Delta < 3\sqrt{\epsilon}/v$	$\Delta < \dfrac{2\zeta}{\omega_o}\left[\dfrac{6(b^2 + \omega_o^2 a^2)\epsilon}{b^2(1 + 4\zeta^2) + \omega_o^2 a^2}\right]^{\frac{1}{2}} \approx \dfrac{2\zeta}{\omega_o}\sqrt{6\epsilon}$

$$\phi_3 = 1 + \omega_o^2 \nabla^2 + 2\zeta\omega_o \nabla$$

$$\phi_4 = 2(\omega_o^2 \nabla^2 - 1)$$

$$\phi_5 = 1 + \omega_o^2 \nabla^2 - 2\zeta\ \omega_o \nabla$$

NOTATION

a	coefficient of white noise in differential equation for Markov process
b	$= c + 2 a \zeta \omega_c$
c	coefficient of white noise in differential equation for Markov process
C	correlation function
E	expected value of a variation
m	measured value of a state variable (=v+n)
n	continuous noise process
p	probability density function
P	power (= integrated spectral density from -∞ to ∞)
R	covariance function
s	Laplace transform variable (=jω)
t	time
T	time interval for integration (in the limit $\|T\| \to \infty$)
u	white noise stochastic process
v	true value of a state variable
x	Markov process derived from white noise (a) first order = exponentially correlated noise (b) second order = damped cosine correlated noise
y	intermediate variable in deriving second order Markov process
z	z transform variable (= exp(sΔ))
δ_D	Dirac delta function. $\int_a^b \delta_D(c-x)dx = 1$ if $a<c<b$; = 0 otherwise
δ_{Kr}	Kronecker delta. $\delta_{Kr}(m) = 1$ if $m = 0$; = 0 otherwise
Δ	time interval of discrete process
∇	= Δ/2
ε	fractional error in discrete approximation to continuous power spectral density
ζ	damping ratio in second order Markov process
μ	mean value
υ	correlation time in first order Markov process

σ	standard deviation of Gaussian random number sequence used for discrete simulation of white noise
τ	time lag, argument of autocorrelation function
Φ	power spectral density (two sided and function of ω)
ϕ	subsidiary expressions involving ζ, ω_o, Δ
ω	Fourier transform variable. Frequency in radian s^{-1}
ω_o	undamped natural frequency in second order Markov process
Ψ	spectral density of continuous white noise (taken = 1)

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APPENDIX I

AN EXAMPLE OF DISCRETE NOISE SIMULATION

Suppose we wish to model, in a discrete simulation computer program, the noise on the output of an angle measuring device whose measured noise power spectral density is given in figure 9. This approximates to a second order Markov process, and so we will assume that it can be generated from white noise, u, by the following pair of stochastic differential equations.

$$\dot{y} = \omega_0^2 x - 2\zeta\omega_0 y + cu \qquad \text{deg s}^{-2} \qquad (I.1)$$

$$\dot{x} = y + au \qquad \text{deg s}^{-1} \qquad (I.2)$$

As shown in the text, the power spectrum of x generated by these equations is

$$\Phi_x = \frac{(a^2\omega^2 + b^2)}{\omega^4 + 2\omega_0^2(2\zeta^2 - 1)\omega^2 + \omega_0^4} \cdot \Psi \qquad \text{deg}^2/(\text{rad s}^{-1})$$

where $b = c + 2a\zeta\omega_o$, and Ψ is the power spectral density of u. Since c and a appear as multipliers of u, defining Ψ will fix c and a. It is convenient to take Ψ as unity, and we will therefore do this:

viz: $$\Psi = 1 \text{ deg}^2/(\text{rad s}^{-1})$$

The parameters a, b, ω_o, ζ are then chosen to fit Φ_x to the observed power spectral density. This is partly trial and error, partly intelligent guesswork; for example the frequency and relative amplitude of the maximum suggests that $\omega_o \sim 2.5$ and $\zeta \sim 0.5$. We have plotted in figure 9 the calculated curve for the following values of the parameters, which give a satisfactory fit:

$$a = 1.6 \text{ rad s}^{-1}$$

$$c = 0.7 (\text{rad s}^{-1})^2$$

$$\omega_o = 2.6 (\text{rad s}^{-1})$$

$$\zeta = 0.6$$

$$b = c + 2a\zeta\omega_o = 5.7 (\text{rad s}^{-1})^2$$

To generate the discrete 2nd order Markov process corresponding to x, we replace the differentials in equations (I.1) and (I.2) by a finite difference algorithm. As explained in the text, a good compromise between convenience and accuracy is given by the modified Euler method. Using this method, and assuming that we require an accuracy of better than 20% in the discrete modelling of the continuous process, equations (48) and (49), give the following constraints on the integration step size:

$$\Delta < \pi b \sqrt{\epsilon} / a \omega_0^2 = 0.52 \text{ s}$$

$$\Delta < 2\zeta \sqrt{\epsilon} / \omega_0 = 0.50 \text{ s}$$

We have plotted in figure 9 the power spectrum of the discrete process corresponding to a 0.5 s time step, out to the Nyquist frequency ($\pi/\Delta = 6.3 \text{ sec}^{-1}$).

In figure 10 we have plotted 100 s of the discrete random process generated by the modified Euler algorithm applied to the differential equations, with the calculated values for the parameters, generated from discrete white noise given by Gaussian random number sequence of variance:

$$\sigma^2 = \Psi/\Delta = 2 \text{ deg}^2$$

The theoretical variance of the second order Markov process derived by these means is given by (see Table 1):

$$\sigma^2_{\text{theory}} = \frac{(a^2 \omega_0^2 + b^2)}{4\zeta \omega_0^3} \cdot \Psi$$

$$= 1.18 \text{ deg}^2$$

This value should be compared with the variance of the calculated random number sequence plotted in figure 10:

$$\sigma^2_{\text{calc}} = \frac{1}{N} \Sigma x^2 = 1.0 \text{ deg}^2$$

where the sum has been taken over 500 steps. It is wise to make this comparison as a check on the discrete simulation program. These values could also have been obtained by integrating under the corresponding curves in figure 9. The 20% accuracy between the continuous and discrete power is generally adequate for most applications. It is not often that the estimate of an error is required with greater accuracy.

In this example we started with a given power spectral density. If the measurements of the noise process had been of the autocorrelation function, we could determine the parameters a, c, ω_0, ζ by fitting the theoretical to observed autocorrelation function, and then proceed to model the noise in the same manner.

APPENDIX II

DISCRETE MODELLING OF ATMOSPHERIC TURBULENCE

The purpose of this Appendix is to demonstrate how the methods of this Report can be applied to the simulation of atmospheric turbulence. In simulating turbulence it is necessary to use an analytical expression for the power spectral density of the lateral turbulent atmospheric motions. We shall apply the Dryden formula, which is commonly used; it has the advantages of being analytically simple, whilst at the same time fitting the observed spectra very well.

In this Appendix we do not intend to give, nor is this the place for, an exegesis on atmospheric turbulence, its variability and the validity of the several analytical expressions for its power spectral density. These topics are covered, for example, in references 12 and 13.

The Dryden expressions for the power spectral density of the longitudinal and lateral components of atmospheric turbulence are:

$$\Phi_{long} = \frac{2\,\sigma^2 L}{1 + L^2\,\Omega^2} \quad (ms^{-1})^2/(rad\ m^{-1}) \qquad (II.1)$$

$$\Phi_{lat} = \sigma^2 L \cdot \frac{1 + 3L^2\,\Omega^2}{[1 + L^2\,\Omega^2]^2} \qquad (II.2)$$

where

Ω = spatial frequency (rad m^{-1})

σ^2 = mean square gust velocity $(ms^{-1})^2$

L = scale of turbulence (m/rad)

The longitudinal component refers to those random atmospheric motions along the line to which the spatial frequency is referred; the lateral component refers to motions (vertical or horizontal) perpendicular to this line.

If the above equations are compared with the Dryden formulae usually quoted, it will be noted that they contain an extra factor of $(2\pi)/2$. The factor 2π is because the power spectral density of atmospheric turbulence is commonly defined so that its integral gives the variance, σ^2. However, we prefer to retain the relation between power spectral density and variance given by equation (7), since this is in line with the standard definitions of the Laplace and Fourier transforms. The factor 2 arises because we are using a two-sided power spectral density of to be consistent with the main text, rather than the one-sided density which is commonly used. Finally, note that L and Ω are in radians and not cycles.

Numerous observations of turbulence have shown that σ typically lies in the range 0.5 to 3 ms^{-1}, with the lower values occurring more frequently(ref.14). L varies from 150 m/rad to 500 m/rad, except near the ground when L is proportional to altitude. Reference 12 recommends using 300 m/rad for L in the free atmosphere.

When considering the effect of turbulence on an aircraft, it is standard practice to assume that the aircraft moves through a spatial field given by equation (II.1) which is "frozen". This approximation, called the Taylor hypothesis, holds as long as the aircraft velocity is much greater than all the turbulent motions. The aircraft then sees these spatial structures as an equivalent frequency, ω, given by

$$\omega = V\Omega \text{ rad s}^{-1}$$

where V is the velocity of the aircraft. Substituting into equations (II.1) and (II.2) gives the power spectral density of turbulence, seen by an aircraft, at a frequency ω rad s^{-1} as

$$\Phi_{long} = \frac{2\sigma^2 L}{V} \cdot \frac{1}{1 + (L\omega/V)^2} \quad (ms^{-1})^2/(\text{rad } s^{-1}) \qquad \text{(II.3)}$$

$$\Phi_{lat} = \frac{\sigma^2 L}{V} \cdot \frac{1 + 3(L\omega/V)^2}{[1 + (L\omega/V)^2]^2} \qquad \text{(II.4)}$$

We now wish to show how these spectra can be generated by using the methods developed in this report. Comparing with equations (13) and (17) it is seen that the Dryden spectra, chosen because they give a good fit to observations while at the same time being similar in form to the theoretically exact von Karmen turbulent spectrum, can be modelled exactly by the first and second order Markov processes:

for longitudinal motions: $\dot{x} = -ux + a'u \quad ms^{-1}$

for lateral motions: $\dot{x} = y + au \quad ms^{-1}$

$\dot{y} = -\omega_0^2 x - 2\zeta\omega_0 y + cu \quad ms^{-2}$

where x is the desired output and u is white noise of power spectral density Ψ. The parameters in these equations are related to those of the Dryden spectra by

$$u = V/L$$

$$a'^2 = [2\sigma^2 V/L]/\Psi$$

$$\zeta = 1$$

$$\omega_o = V/L$$

$$a^2 = [3\sigma^2 V/L]/\Psi$$

$$b^2 = [\sigma^2 V^3/L^2]/\Psi$$

where

$$b = c + 2a\zeta\omega_o$$

and, as discussed in the text, we let the white noise input have unit power spectral density : $\Psi = 1\ (ms^{-1})^2/rad\ s^{-1}$.

The table below shows the value of these parameters for L = 300 m/rad and three values of RMS gust velocity, as observed by an aircraft travelling at 170 ms^{-1}:

σ	$m\ s^{-1}$	0.5	1.5	3.0
a'	$rad\ s^{-1}$	0.53	1.60	3.20
a	$rad\ s^{-1}$	0.65	1.96	3.92
b	$(rad\ s^{-1})^2$	0.21	0.65	1.29
c	$(rad\ s^{-1})^2$	-0.53	-1.59	-3.19

$\omega_o = 0.57\ rad\ s^{-1}$ for all σ.

Note that this approach to modelling atmospheric turbulence produces motions of all scales. There is no need to add long period sinusoids to the model in order to simulate the large scale motions, as we have seen done elsewhere.

For completeness we will note the autocorrelation functions corresponding to the Dryden longitudinal and lateral power spectral densities (equations (II.1), (II.2)).

$$C_{long} = \sigma^2 \exp[-|\zeta|/L]\ (m/s)^2$$

$$C_{lat} = \sigma^2 (1-|\zeta|/L)\exp(-|\zeta|/L)\ (m/s)^2$$

These equations give a better indication of the meaning of L than do the expressions for the power spectral density.

APPENDIX III

DISCRETE SIMULATION OF THE WIENER PROCESS

The Wiener process, also known as random walk and the process of uncorrelated increments, is formally defined as the integral of white noise (references 1 and 11):

$$\dot{x} = au \tag{III.1}$$

This process differs from those examined in the text by having an autocorrelation function whose expectation does not converge to a constant value for large time. On the contrary, the autocorrelation of the Weiner process increases linearly with time:

$$C(t,\tau) = a^2\Psi t \qquad t<\tau \tag{III.2}$$

where t is the elapsed time from the start of the process, and we will again take the white noise process to be of unit strength, $\Psi = 1$.

The discrete equivalent of the Wiener process, using the modified Euler integration algorithm, is

$$x[(n+1)\Delta] = x(n\Delta) + \tfrac{1}{2}\, a\Delta\{u[(n+1)\Delta] + u(n\Delta)\} \tag{III.3}$$

where, as discussed in the text, $u(n\Delta)$ is a sequence of random numbers whose variance is $\sigma^2 = \Psi/\Delta$. We can rewrite this equation as

$$x[(n+1)\Delta] = a\Delta\left\{\tfrac{1}{2}\, u(0) + \sum_{1}^{n} u(m\Delta) + \tfrac{1}{2}\, u[(n+1)\Delta]\right\}$$

The autocorrelation function can therefore be written as, letting $m<n$:

$$\begin{aligned} C_x(m,n) &= E[x(m\Delta)\, x(n\Delta)] \\ &= \Sigma\Sigma(a\Delta)^2\, E[u(m\Delta)\, u(n\Delta)] \\ &= (a\Delta)^2\, \Sigma\Sigma\, \sigma^2\, \delta_{Kr}(m-n) \\ &= (m-\tfrac{1}{2})(a\sigma\Delta)^2 \end{aligned}$$

where the $\frac{1}{2}$ arises from the end point weighting of u.

Substituting the relation $\sigma^2\Delta = \Psi$ and setting $t=m\Delta$ gives

$$C_x(m,n) = (m-\tfrac{1}{2})/m.a^2\Psi t \qquad m<n.$$

We see that discrete simulation gives the same autocorrelation as the

continuous process, except for the factor $(m - \frac{1}{2})/m$, ie the accuracy of the discrete simulation increases with the number of steps. The step size can therefore be chosen to give the desired accuracy at any specified time.

Because of the time dependent property of the autocorrelation function, the power spectral density is not defined.

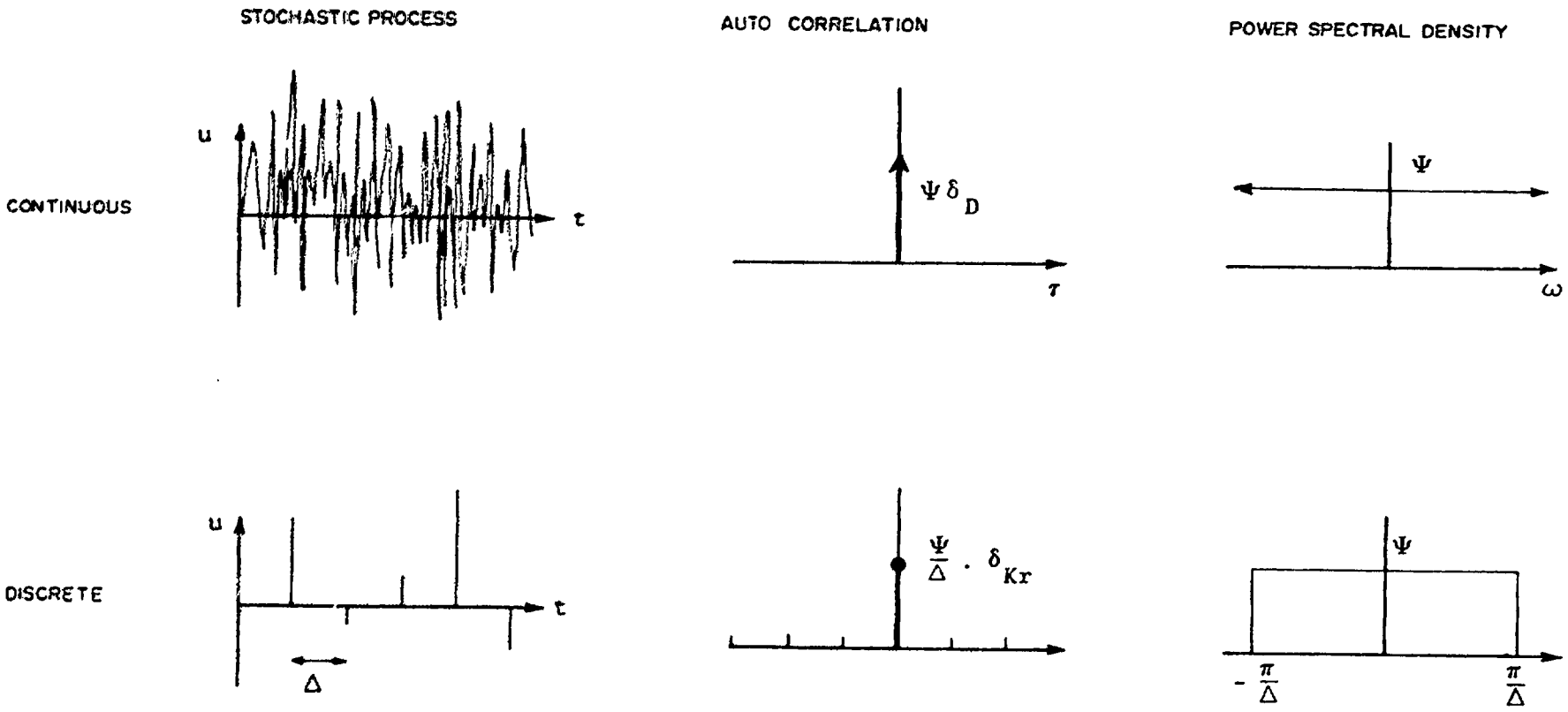
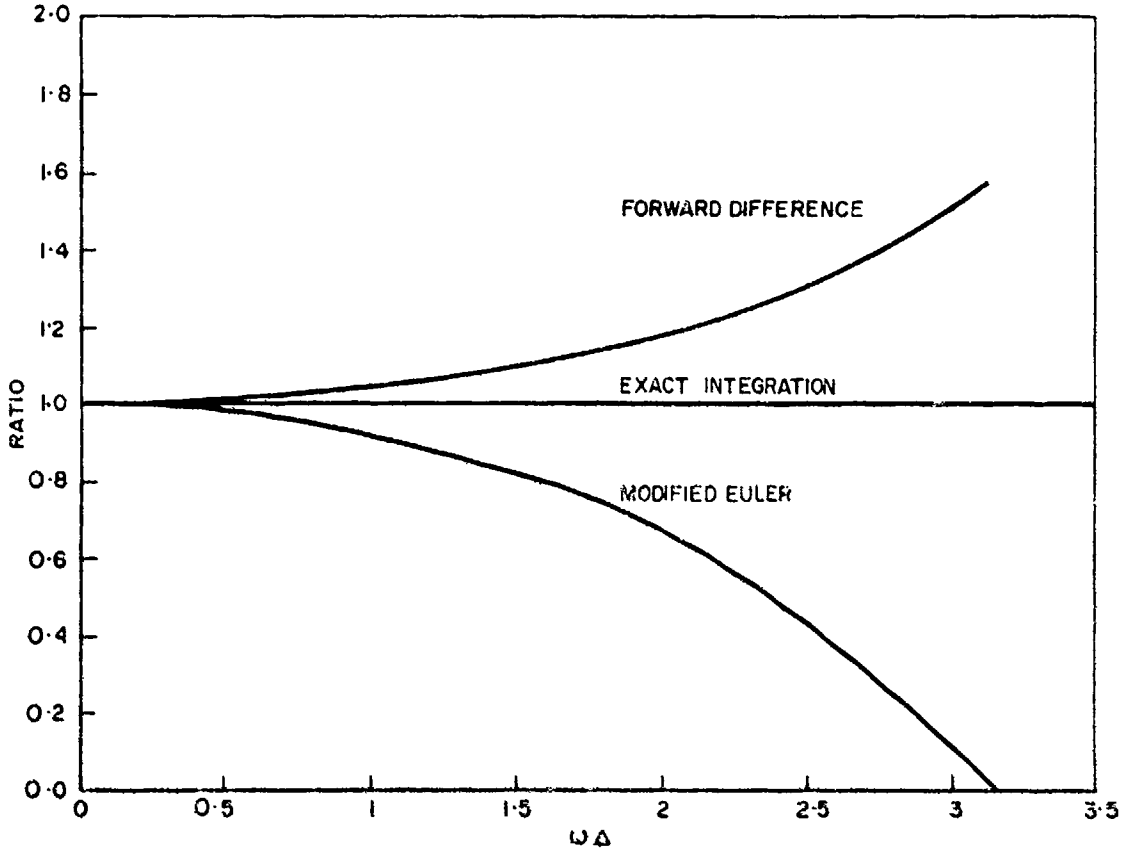
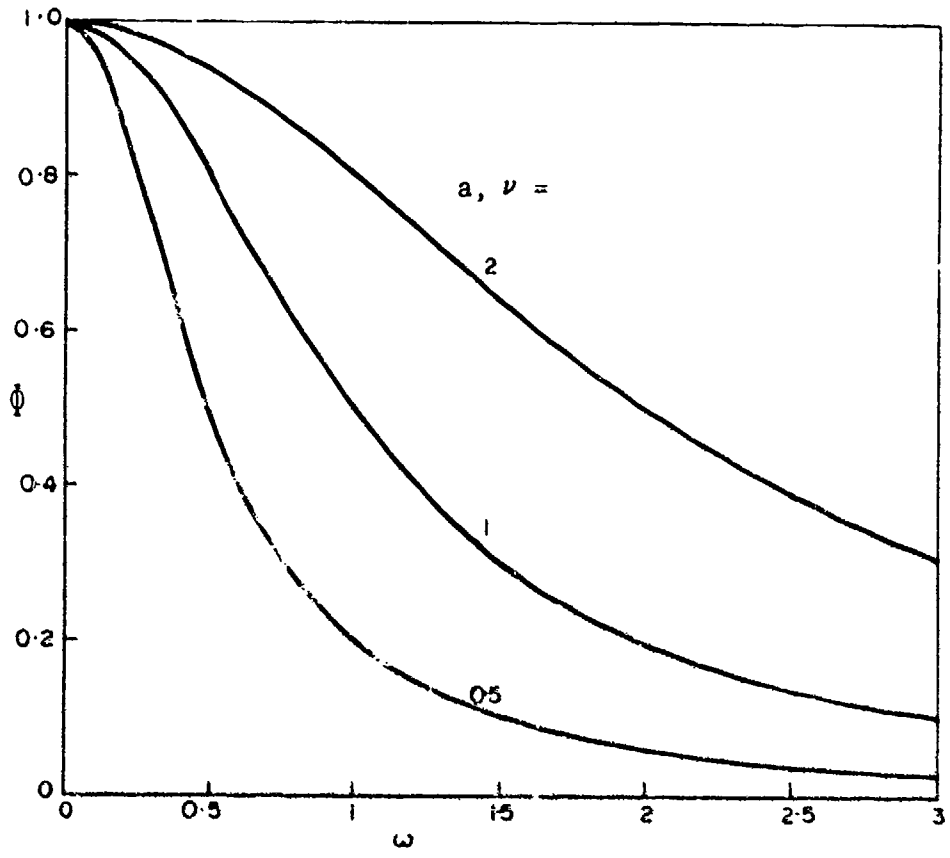


Figure 1. Comparison of continuous and discrete white noise



The curves give the ratio of the frequency response of the given discrete integration algorithm to the frequency response of exact integration ($= 1/\omega$)

Figure 2. Comparison of frequency response of two discrete integration algorithms



Note: We have defined power spectral density as being two sided, but in this and subsequent figures, only the positive half is plotted

Figure 3. Power spectral density of first order Markov process: Continuous

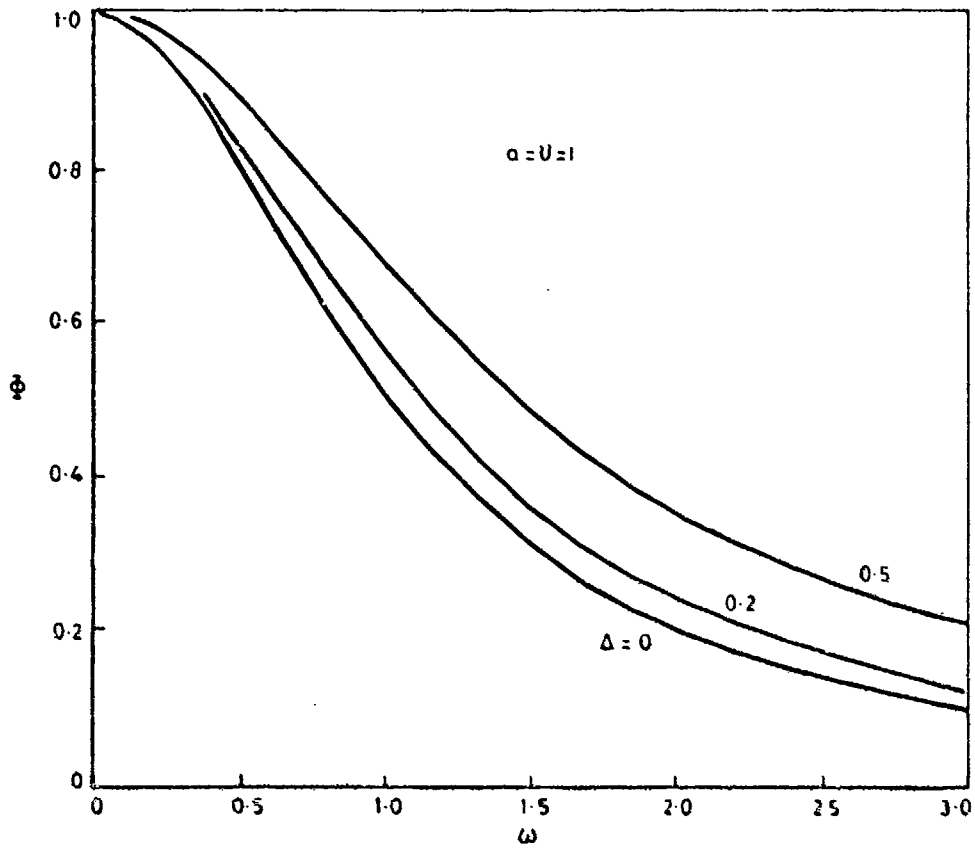


Figure 4. Power spectral density of first order Markov process: Discrete with forward difference algorithm

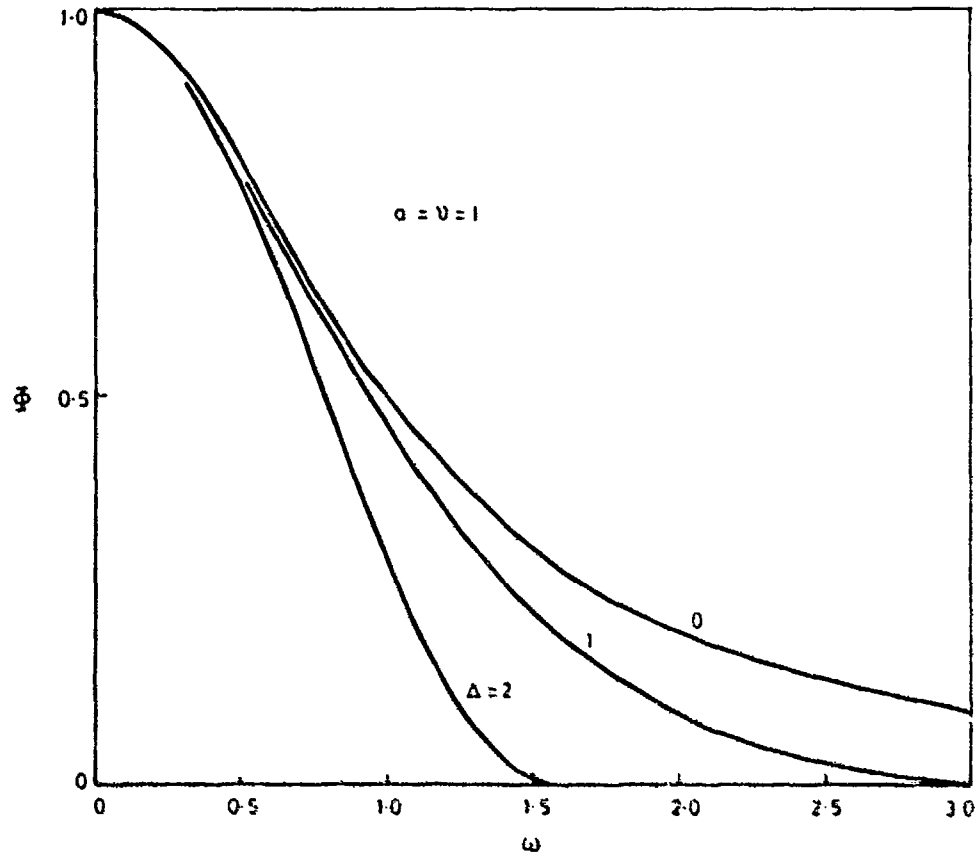


Figure 5. Power spectral density of first order Markov process: Discrete with modified Euler algorithm

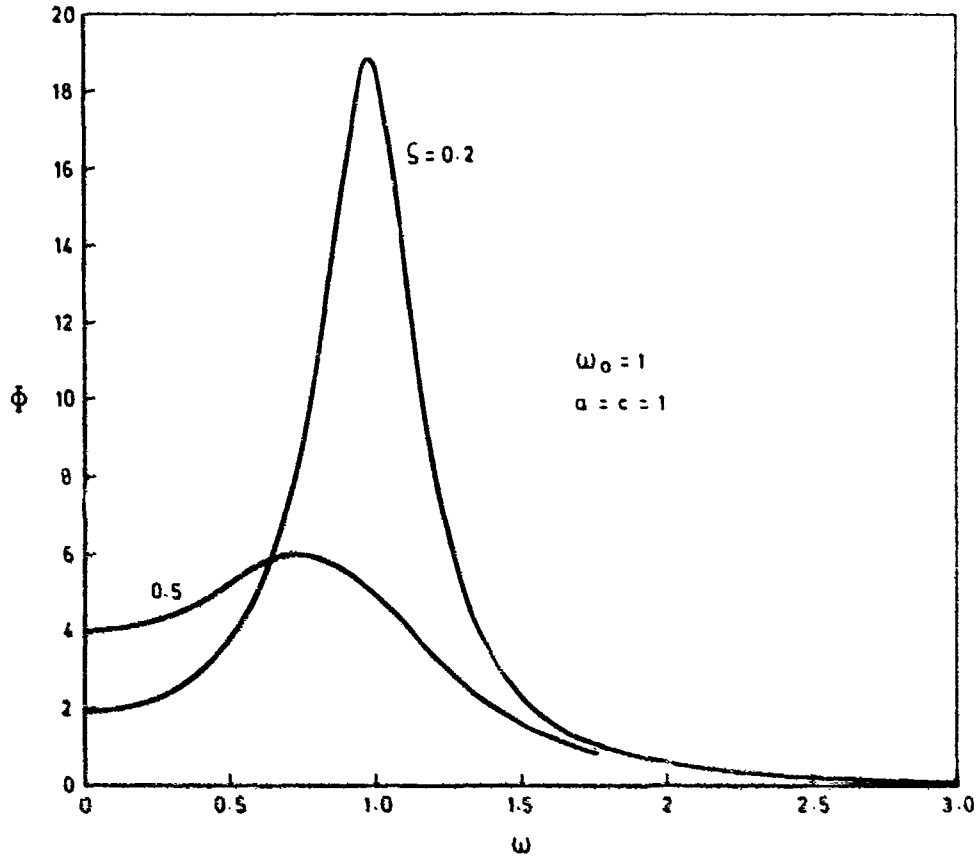


Figure 6. Power spectral density of second order Markov process: Continuous

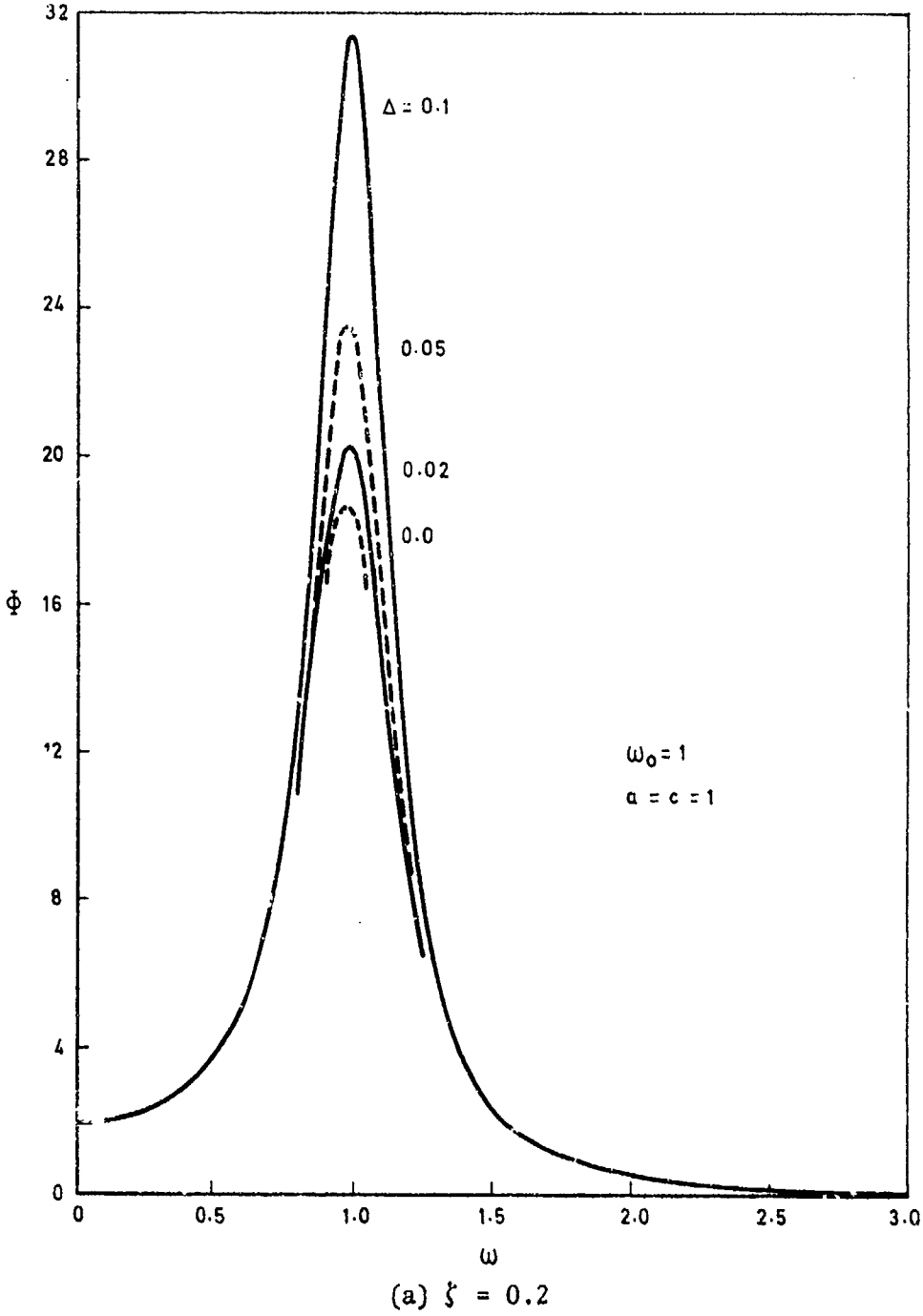
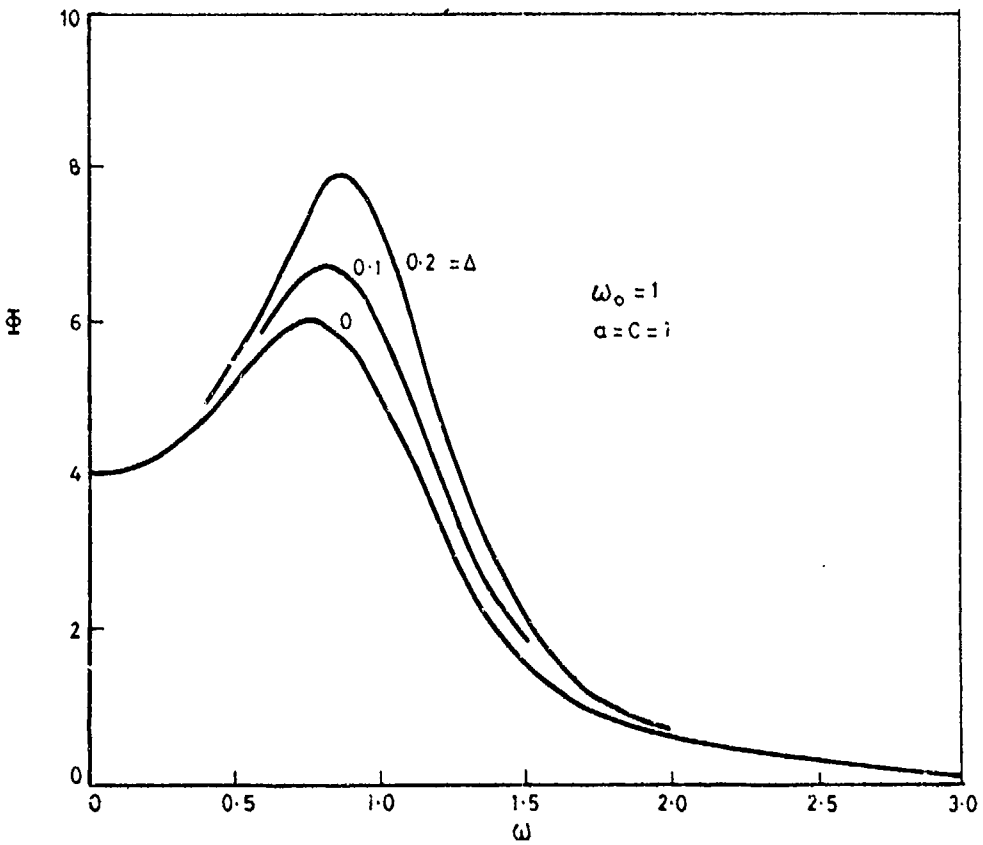
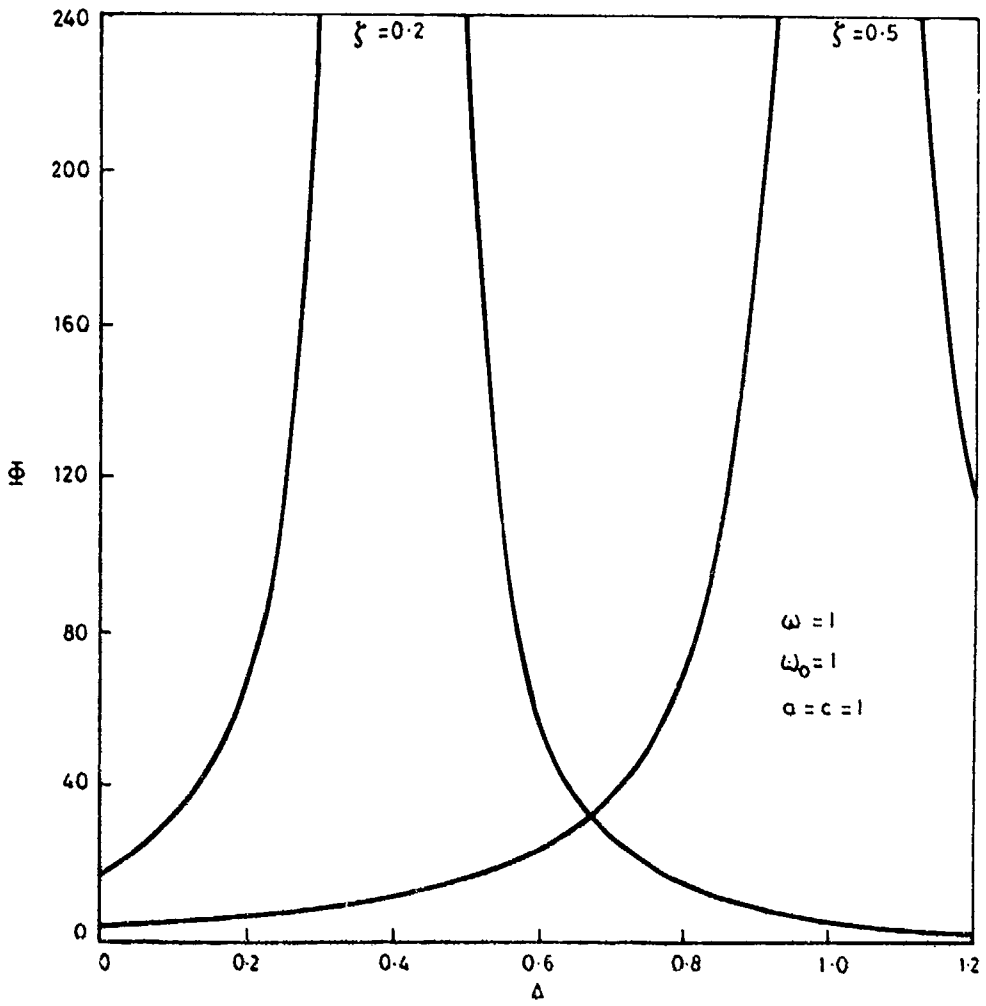


Figure 7. Power spectral density of second order Markov process: Discrete with forward difference algorithm



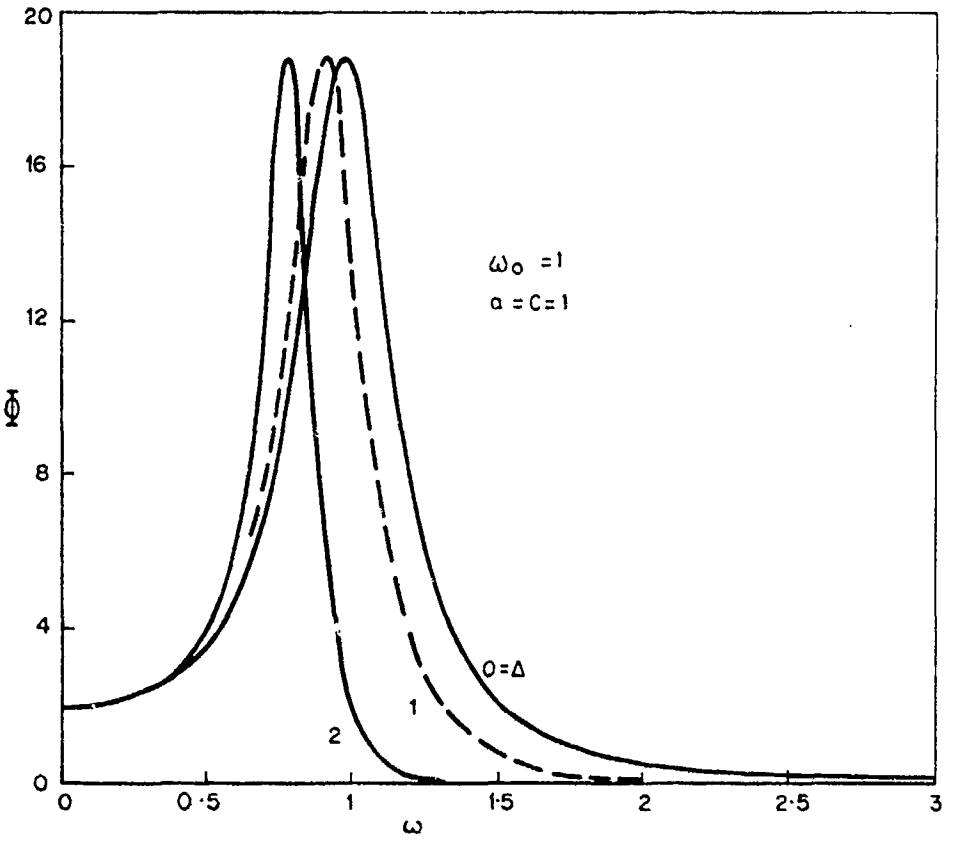
(b) $\zeta = 0.5$

Figure 7. Power spectral density of second order Markov process: Discrete with forward difference algorithm

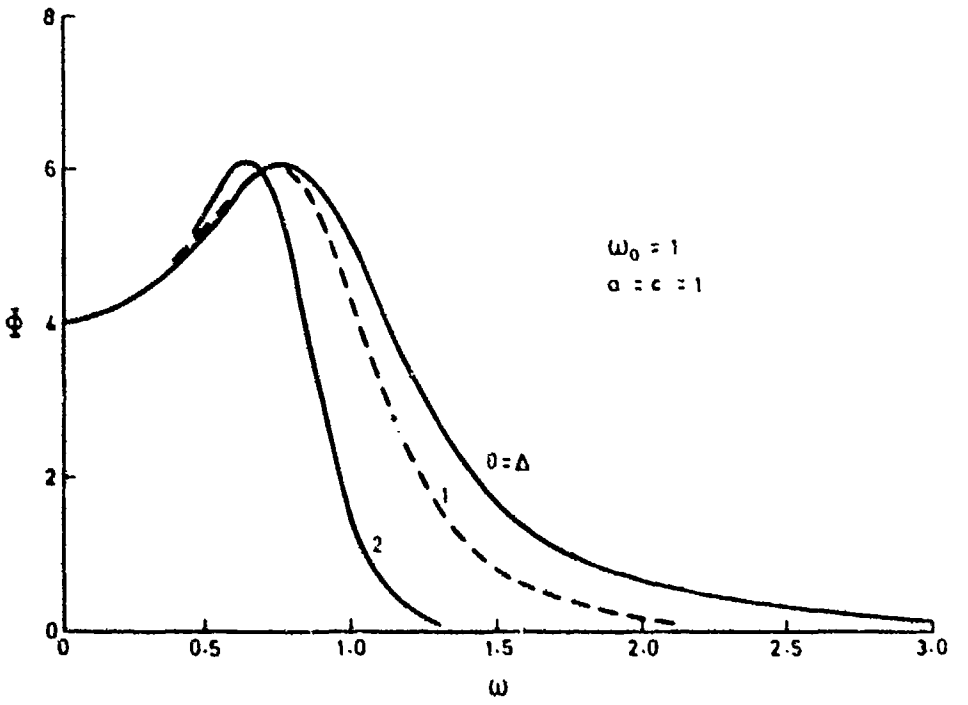


(c) Variation with step size

Figure 7. Power spectral density of second order Markov process: Discrete with forward difference algorithm



(a) $\zeta = 0.2$



(b) $\zeta = 0.5$

Figure 8. Power spectral density of second order Markov process: Discrete with modified Euler algorithm

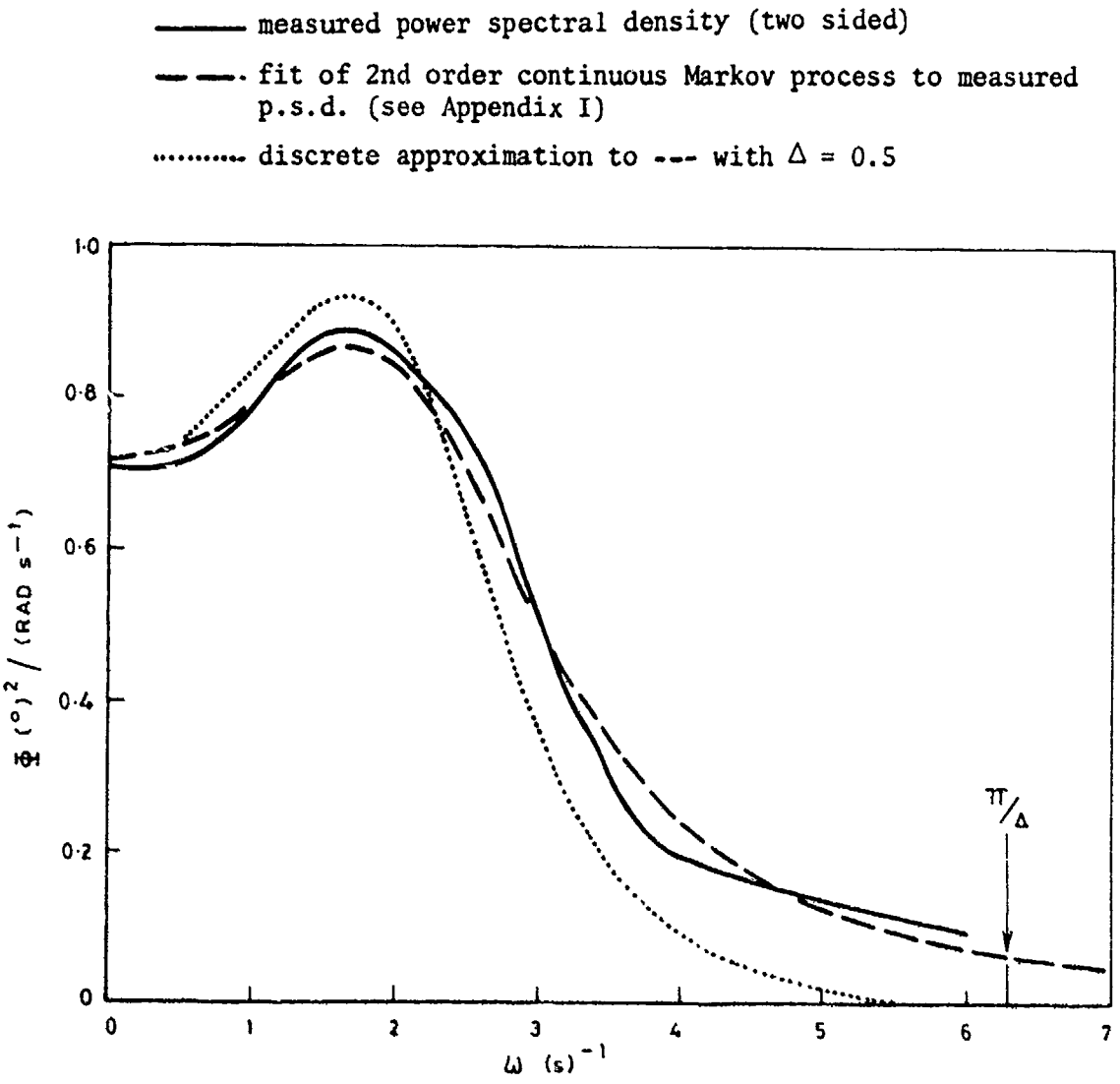
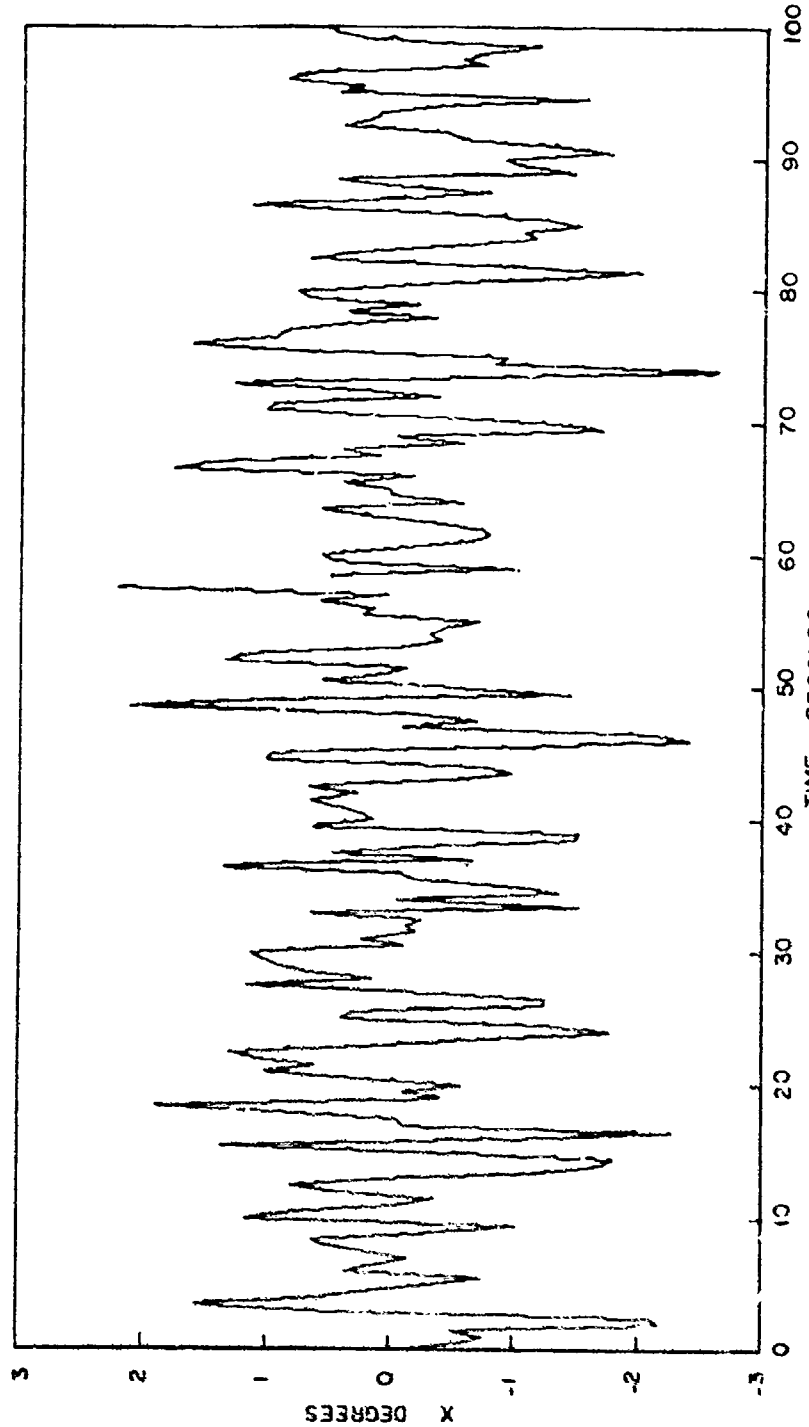


Figure 9. Measured power spectral density fitted by second order Markov process



This has the power spectral density plotted in figure 9.
$a = 1.6\ sec^{-1}$, $c = 0.7\ sec^{-2}$, $\omega_o = 2.6\ sec^{-1}$, $\zeta = 0.6$

Figure 10. Discrete second order Markov process

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16 SUMMARY OR ABSTRACT:
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This report discusses the implementation of system and process noise, including sensor errors and atmospheric turbulence, in the simulation of continuous models by digital computers. The concept of discrete white noise is first introduced in a manner in which it reduces to continuous white noise as the integration time interval reduces to zero. The derivation of first and second order Markov processes from white noise is then discussed. The discussion includes the consideration of a suitable difference algorithm to approximate differentiation. The treatment throughout has been aimed at providing the reader with the tools to implement a required power spectral density (or autocorrelation function) as noise in any discrete digital simulation model.