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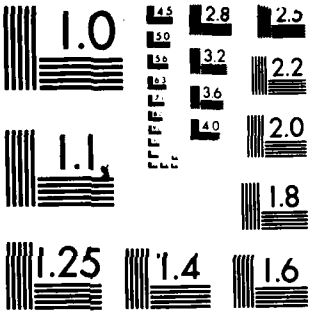

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THE FACTORIZATION METHOD FOR TWO POINT BOUNDARY VALUE PROBLEMS FOR ODE'S AND ITS RELATION TO THE FINITE DIFFERENCE METHOD

by

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The paper shows that the elimination method for solving the system of linear equations stemming from finite differences or finite elements is a special numerical method for solving certain initial value problems for an ODE. Various conclusions are drawn from this observation.

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THE FACTORIZATION METHOD FOR TWO POINTS BOUNDARY VALUE PROBLEMS FOR ODE'S AND ITS RELATION TO THE FINITE DIFFERENCE METHOD

I. Babuška[1*] and V. Majer[2]

1. INTRODUCTION

Finite difference and finite element methods for solving two point boundary value problems for systems of ordinary differential equations consist of

a) a discretization procedure which transforms the original problem into a family of finite dimensional systems of algebraic equations parametrized by the mesh size h, and

b) a solution procedure for the systems of algebraic equations.

For linear boundary value problems the algebraic equations are linear and step b) reduces to the selection of a matrix reduction scheme. In this paper we consider only direct (elimination) methods of matrix reduction.

By these two steps, taken together, the original problem is transformed into a sequential numerical process (§5) which depends on the mesh parameter h. A complete analysis of the numerical procedure must consider this underlying numerical process, not merely the discretization step a). In this paper we carry out such a complete analysis for a model singular perturbation

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problem of turning point type (§2) studied by H. O. Kreiss et al. in [1]. We show (§4) that the numerical process converges, as $h \to 0$, to the solution of initial value problems for certain differential equations. These limiting equations are the closure [2] of the process. Thus it is possible to interpret the numerical process as a special method (of low order) for solving these initial value problems.

This fact suggests that one should study directly a class of transformations of the original boundary value problem into systems of initial value problems, called factorizations, which class includes the above-mentioned closure. We address this question in §3, where we single out those transformations of this class which are stable in a precisely defined sense, and which therefore can be solved by proper numerical methods. One such method, of course, would lead to the identical numerical process as that stemming from the finite element or finite difference discretizations. Although we restrict ourselves here only to a linear model problem, the results described here hold for general systems of boundary value problems and can be applied also to nonlinear problems [3].

2. MODEL PROBLEM

2.1. Statement of the problem

Consider the scalar second-order boundary value problem

$$\text{(2.1a)} \qquad \varepsilon \frac{d^2}{ds^2} u(s) = \frac{d}{ds}(a(s)u(s)) + b(s)u(s) + g(s), \qquad s \in [\sigma,\sigma^*]$$

$$\text{(2.1b)} \qquad u(\sigma) = \gamma, \qquad u(\sigma^*) = \gamma^*.$$

The functions a, b, and g are smooth on $[\sigma,\sigma^*]$, and $0 < \varepsilon \ll 1$ is a constant parameter. The solutions of (2.1) can exhibit boundary or interior layers in which the solution is rapidly changing. A discussion of the range of possible behaviors may be found in [1].

We seek the values of an approximate solution of (2.1) on a partition

$$\pi: \sigma \equiv \sigma_0 < \sigma_1 < \cdots < \sigma_m \equiv \sigma^*$$

of <u>target points</u> of $[\sigma,\sigma^*]$. We would also like to determine the location of the interior or boundary layers and perhaps to resolve them.

2.2. Reformulation as a First Order System

It is possible to recast (2.1) as a first order system in a number of ways. We consider one of them here. Let $v' = bu + g$. Then

$$\text{(2.2a)} \qquad \begin{cases} u'(s) = \dfrac{a(s)}{\varepsilon} u(s) + \dfrac{1}{\varepsilon} v(s), \\ v'(s) = b(s)u(s) + g(s), \end{cases}$$

$$\text{(2.2b)} \qquad u(\sigma) = \gamma, \quad v(\sigma^*) = \gamma^*,$$

in which we have used $\frac{d}{ds} = '$. Written in matrix form (2.2) becomes

$$(2.3a) \qquad \begin{bmatrix} u'(s) \\ v'(v) \end{bmatrix} = \begin{bmatrix} a(s)/\varepsilon & 1/\varepsilon \\ b(s) & 0 \end{bmatrix} \begin{bmatrix} u(s) \\ v(s) \end{bmatrix} + \begin{bmatrix} 0 \\ g(s) \end{bmatrix}$$

subject to

$$(2.3b) \qquad [1 \quad 0] \begin{bmatrix} u(\sigma) \\ v(\sigma) \end{bmatrix} = \gamma, \qquad [1 \quad 0] \begin{bmatrix} u(\sigma^*) \\ v(\sigma^*) \end{bmatrix} = \gamma^*.$$

Equation (2.3) is but a special case of

$$(2.4a) \qquad \begin{bmatrix} u'(s) \\ v'(s) \end{bmatrix} = \begin{bmatrix} a(s) & c(s) \\ b(s) & d(s) \end{bmatrix} \begin{bmatrix} u(s) \\ v(s) \end{bmatrix} + \begin{bmatrix} f(s) \\ g(s) \end{bmatrix}$$

subject to

$$(2.4b) \qquad [\alpha \quad \beta] \begin{bmatrix} u(\sigma) \\ v(\sigma) \end{bmatrix} = \gamma_1, \qquad [\alpha^* \quad \beta^*] \begin{bmatrix} u(\sigma^*) \\ v(\sigma^*) \end{bmatrix} = \gamma^*.$$

The form (2.4) encompasses all the various reductions of (2.1) to first order form. In the sequel, it is assumed that the solution of (2.4) exists.

3. INDIRECT SOLUTION OF THE MODEL PROBLEM

The indirect method of solution of (2.4) is based upon the propagation across the interval $[\sigma,\sigma^*]$ of the boundary conditions (2.4b) in a manner consistent with the original equation. For this we require certain auxiliary initial value problems.

3.1. Auxiliary Initial Value Problems

Consider the initial value problems

$$(3.1)\qquad \begin{cases} \phi'(s) = -a(s)\phi(s) - b(s)\psi(s) + z(s)\phi(s), \\ \phi(\sigma) = \alpha, \end{cases}$$

$$(3.2)\qquad \begin{cases} \psi'(s) = -c(s)\phi(s) - d(s)\psi(s) + z(s)\psi(s), \\ \psi(\sigma) = \beta, \end{cases}$$

$$(3.3)\qquad \begin{cases} \chi'(s) = f(s)\phi(z) + g(s)\psi(s) + z(s)\chi(s), \\ \chi(\sigma) = \gamma, \end{cases}$$

$$(3.4)\qquad \begin{cases} \phi^{*\prime}(s) = -a(s)\phi^*(s) - b(s)\psi^*(s) + z^*(s)\phi^*(s), \\ \phi^*(\sigma^*) = \alpha^*, \end{cases}$$

$$(3.5)\qquad \begin{cases} \psi^{*\prime}(s) = -c(s)\phi^*(s) - b(s)\psi^*(s) + z^*(s)\psi^*(s), \\ \psi^*(\sigma) = \beta^*, \end{cases}$$

$$\chi^*(s) = f(s)\phi^*(s) + g(s)\psi^*(s) + z^*(s)\chi^*(s),$$

(3.6)

$$\chi^*(\sigma^*) = \gamma^*,$$

in which the **conditioning functions** $s \to z(s)$, $z^*(s)$ are continuous but, at the moment, otherwise arbitrary. Problems (3.1)-(3.3) are solved forward, while (3.4)-(3.6) are solved backward.

Theorem 3.1. Suppose that (3.1)-(3.3) have been solved on (σ,ξ) and (3.4)-(3.6) have been solved on $[\xi^*,\sigma^*]$. Then for $s \in [\sigma,\xi] \cap [\xi^*,\sigma^*]$, and $[u(s)v(s)]^\top$ satisfying (2.4),

$$(3.7) \qquad \begin{bmatrix} \phi(s) & \psi(s) \\ \phi^*(s) & \psi^*(s) \end{bmatrix} \begin{bmatrix} u(s) \\ v(s) \end{bmatrix} = \begin{bmatrix} \chi(s) \\ \chi^*(s) \end{bmatrix} . \qquad \blacksquare$$

Corollary 3.2. If $s \to [u(s) \quad v(s)]^\top$ is the unique solution of (2.4) and the hypotheses of Theorem 3.1 are satisfied, then

$$(3.8) \qquad \phi(s)\psi^*(s) - \phi^*(s)\psi(s) \neq 0$$

for $s \in [\sigma,\xi] \cap [\xi^*,\sigma^*]$. $\blacksquare$

It is possible to relax the hypotheses of Theorem 3.1. It is necessary only that z and z^* be piecewise continuous.

Theorem 3.3. Suppose that (3.1)-(3.6) are satisfied with $s \to z(s)$, $z^*(s)$ piecewise continuous and having points of discontinuity ξ_i, $i = 1,\ldots,n$ and ξ_i^*, $i = 1,\ldots,n^*$, respectively. If there exist non-zero constants k_i, $i = 1,\ldots,n$ and k_i^*, $i = 1,\ldots,n^*$ such that

$$\psi(\xi_i^+) = k_i\psi(\xi_i^-), \tag{3.9a}$$

$$\phi(\xi_i^+) = k_i\psi(\xi_i^-), \tag{3.9b}$$

$$\chi(\xi_i^+) = k_i\chi(\xi_i^-), \tag{3.9c}$$

for $i = 1,\ldots,n$, and such that

$$\phi^*(\xi_i^{*-}) = k_i^*\phi^*(\xi_i^{*+}), \tag{3.10a}$$

$$\psi^*(\xi_i^{*-}) = k_i^*\psi^*(\xi_i^{*+}) \tag{3.10b}$$

$$\chi^*(\xi_i^{*-}) = k_i^*\chi^*(\xi^{*+}) \tag{3.10c}$$

for $i = 1,\ldots,n^*$, then formula (3.7) continues to hold, as do the conclusions of Corollary 3.2. ∎

The collections of functions $\{\phi,\psi,\chi,z\}$, $\{\phi^*,\psi^*,\chi^*,z^*\}$, **breakpoints** $\{\xi_i\}$, $\{\xi_i^*\}$, and **multipliers** $\{k_i\}$, $\{k_i^*\}$ satisfying the conditions of Theorem 3.3 comprise a **factorizations** of the two point boundary value problem (2.4).

Theorem 3.3 is central to the viability of a computer implementation of a factorization procedure based on the solution of (3.1)-(3.6) followed by the solution of (3.7) at selected target points at which the values of u and v are desired. The conditioning functions z, z^*, the breakpoints ξ_i, ξ_i^* and the multipliers k_i, k_i^* are chosen adaptively by the computer during the computation in order to stabilize the factorization.

3.2. Stable Factorizations

Definition 3.4. A factorization for (2.4) is bounded if there exist constants $0 < \lambda \leq \Lambda < \infty$ such that

$$\lambda^2 \leq \phi^2(s) + \psi^2(s) \leq \Lambda^2 \tag{3.11}$$

for all $s \in [\sigma,\sigma^*]$ with a similar inequality for ϕ^*.

Bounded factorizations are stable in the sense of

Theorem 3.5. Let $\tilde{\phi}$, $\tilde{\psi}$ $\tilde{\chi}$ and $\tilde{\phi}^*$, $\tilde{\psi}^*$, $\tilde{\chi}^*$ be the solutions of the initial value problems (3.1)-(3.6) under the relaxed hypotheses of Theorem 3.3 and such that the right hand sides have been perturbed by functions of magnitute τ (in the L_∞ norm). Then $\tilde{u}$, $\tilde{v}$ solving (3.7) with $\tilde{\phi}$ replacing ϕ, etc., are solutions of a perturbed problem (2.4), for which the perturbations of the input data are of magnitude $c(\lambda,\Lambda)\tau$ where $c(\lambda,\Lambda)$ is independent of the problem. ∎

The significance of Theorem 3.5 lies in the fact that $c(\lambda,\Lambda)$ is independent of the data of the problem (e.g. of ε in (2.1a)), and in the fact that the numerical solution of the initial value problems (3.1)-(3.6) can be interpreted as the exact solution of perturbed equations. The magnitudes of the perturbations of the equations are under the direct control of the solution tolerance, for example, of an adaptive solver. The adaptive selection of the functions z, z^* the breakpoints ξ_i, ξ_i^* and the multipliers k_i, k_i^* by the computer itself is essential to ensure that Λ/λ and Λ are of reasonable magnitude.

Example 3.6. Normalized Factorization

Set

$$z(s) = \frac{a(s)\phi^2(s)+(b(s)+c(s))\phi(s)\psi(s)+d(s)\psi^2(s)}{\phi^2(s)+\psi^2(s)} . \tag{3.12}$$

It is not hard to see that ϕ, ψ satisfying (3.1) and (3. with this choice of z have the property that

$$\phi^2(s) + \psi^2(s) = 1 \tag{3.13}$$

if $\alpha^2 + \beta^2 = 1$ (which normalization we may assume without loss of generality). In this case z is continuous and the sets of breakpoints and multipliers are empty.

Example 3.7. Riccati Factorization

It is possible to normalize the boundary conditions (2.4b) so that either $\alpha = 1$ and $|\beta| < 1$ or $\beta = 1$ and $|\alpha| < 1$. If $\alpha = 1$ set

$$z(s) = a(s)\phi(s) + b(s)\psi(s). \tag{3.14}$$

In this case it follows that ϕ satisfying (3.1) is constant,

$$\phi(s) \equiv 1, \tag{3.15}$$

and equations (3.2) and (3.3) take the forms

$$\psi'(s) = -c(s) + (a(s)-d(s))\psi(s) + b(s)\psi^2(s) \tag{3.16}$$

$$\chi'(s) = f(s) + g(s)\psi(s) + a(s)\chi(s) + b(s)\psi(s)\chi(s). \tag{3.17}$$

If $\beta = 1$ set

(3.18) $$z(s) = d(s)\psi(s) + c(s)\phi(s).$$

In this case it follows that ψ satisfying (3.2) is constant,

(3.19) $$\psi(s) \equiv 1,$$

and equations (3.1) and (3.3) take the forms

(3.20) $$\phi'(s) = b(s) + (d(s)-a(s))\phi(s) + c(s)\phi^2(s)$$

(3.21) $$\chi'(s) = g(s) + f(s)\phi(s) + d(s)\chi(s) + c(s)\phi(s)\chi(s).$$

The constraints (3.15) or (3.19) are made explicitly in (3.1), (3.2), and (3.3). This reduces by one the number of initial value problems to be solved in each direction and the lower bound λ equals 1. Unfortunately the solutions of the resulting (Riccati) equations may not exist on the entire interval $[\sigma,\sigma^*]$. But if they do not, this fact is signalled by $\phi^2(s) + \psi^2(s) \to \infty$ (i.e., $1 + \psi^2(s) > \Lambda^2$ or $\phi^2(s) + 1 > \Lambda^2$) in which case a breakpoint can be defined and the solutions renormalized. That is, the factorization algorithm can switch adaptively between the ϕ-problem and the ψ-problem.

4. DIRECT SOLUTION OF THE MODEL PROBLEM

The direct method of solution of (2.1) is based upon the discretization of the problem by a finite difference (or finite element) scheme and the subsequent solution of the resulting matrix equations for the values of the discrete solution. The dicretization may be carried out in the second-order form (2.1) or in the transformed first-order forms (2.3) or (2.4). In either case, a complete analysis of the algorithm should consider the underlying numerical processes which result from the solution of the discrete matrix equations. Let us carry out such an analysis for a particular discretization based on (2.4).

4.1. Block-Diagonal Matrix Equations

A difference scheme for poblems of the type (2.4) with $a(s) \gg d(s)$ has been proposed in [1], where such problems are said to be **essentially diagonally dominant.** In the spirit of the particular form (2.3) let us restrict ourselves to the case $d(s) \equiv 0$.

Suppose that a mesh

$$\pi^{(h)}: \sigma \equiv \sigma_0^{(h)} < \alpha_1^{(h)} < \cdots < \sigma_{m^{(h)}}^{(h)} \equiv \sigma^*$$

with $\pi \subset \pi^{(h)}$ has been given. The parameter h characterizes the fineness of the partition; e.g., $h = \max_i(\sigma_{i+1}^{(h)} - \sigma_i^{(h)})$. Then the difference scheme proposed in [1] can be written

$$\text{(4.1a)} \qquad \frac{u_{i+1}^{(h)} - u_i^{(h)}}{h_i} = \kappa_i(a_i u_i^{(h)} + c_i v_i^{(h)}) + (1-\kappa_i)(a_{i+1} u_{i+1}^{(h)} + c_{i+1} v_{i+1}^{(h)})$$

$$\frac{v_{i+1}^{(h)} - v_i^{(h)}}{h_i} = \frac{1}{2}(b_{i+1} u_{i+1}^{(h)} + b_i u_i^{(h)}) + \frac{1}{2}(g_{i+1} + g_i)$$

where

$$\kappa_i = \begin{cases} 0 & \text{if } |a_i h_i| > 1, \quad a_i < 0 \\ \frac{1}{2} & \text{if } |a_i h_i| \leq 1 \\ 1 & \text{if } |a_i h_i| > 1, \quad a_i > 0. \end{cases} \tag{4.2}$$

Written as a system of linear equations for the vector $U^{(h)}$ $= [\cdots u_i^{(h)}, v_i^{(h)}, u_{i+1}^{(h)}, v_{i+1}^{(h)} \cdots]^T$ of nodal variables, (4.1) has the form

$$A^{(h)}U^{(h)} = F^{(h)} \tag{4.3}$$

where the structures of $A^{(h)}$ and $F^{(h)}$ are given in Table 4.1, based on the quantities

$$\begin{aligned} p_i &= -\frac{h_i}{2} b_i, \\ q_i &= -1, \\ r_i &= -\frac{h_i}{2} b_{i+1}, \\ s_i &= 1, \\ f_i &= \frac{h_i}{2} (g_i + g_{i+1}), \\ t_i &= -(1+\kappa_i h_i a_i), \\ w_i &= -\kappa_i h_i c_i, \\ x_i &= 1 - (1-\kappa_i) h_i a_{i+1}, \\ y_i &= -(1-\kappa_i) h_i c_{i+1}, \end{aligned} \tag{4.4}$$

for $i = 0, 1, \ldots, m^{(h)}-1$.

TABLE 4.1

Matrix $A^{(h)}$ and vector $F^{(h)}$.

α	β	0	0	0	0	0	. . .		γ
p_0	q_0	r_0	s_0	0	0	0	. . .		f_0
t_0	w_0	x_0	y_0	0	0	0	. . .		0
		p_1	q_1	r_1	s_1	0	. . .		f_1
		t_1	w_1	x_1	y_1	0	. . .		0
				.					.
				.					.
				.					.
				p_i	q_i	r_i	s_i	0 . . .	f_i
				t_i	w_i	x_i	y_i	0 . . .	0
						.			.
						.			.
						.			.
						p_{m-1}	q_{m-1}	r_{m-1} s_{m-1}	f_{m-1}
						t_{m-1}	w_{m-1}	x_{m-1} y_{m-1}	0
								α^* β^*	γ^*

$A^{(h)}$ $F^{(h)}$

Perform forward Gauss elimination with pivoting on $A^{(h)}$ in order to bring it into the upper triangular form shown in Table 4.2. A similar backward elimination beginning with the last row of $A^{(h)}$ brings it into the lower triangular form shown in Table 4.3. Observe that the nodal values of the solution vector $U^{(h)}$ then satisfy the local 2×2 matrix equations

$$\text{(4.5)} \qquad \begin{bmatrix} \phi_i & \psi_i \\ \phi^*_{i+1} & \psi^*_{i+1} \end{bmatrix} \begin{bmatrix} u^{(h)} \\ v^{(h)}_i \end{bmatrix} = \begin{bmatrix} \chi_i \\ \chi^*_{i+1} \end{bmatrix}$$

for $i = 0,1,\ldots,m^{(h)}$. Clearly (4.5) is a discrete analog of (3.7). However now the entries ϕ_i, ψ_i, χ_i and ϕ^*_i, ψ^*_i, χ^*_i satisfy explicit one-step recursion formulas which are determined by the nature of the elimination algorithm.

As an example, it is possible to perform the elimination in such a way that one or the other of the following two recursions hold:

ψ recursion:

$$\text{(4.6a)} \qquad \phi_{i+1} = 1,$$

$$\text{(4.6b)} \qquad \psi_{i+1} = \frac{\psi_i(1+\kappa_i h_i a_i)+\kappa_i h_i c_i-(1-\frac{h_n}{2} b_i\psi_i)(1-\kappa_i)h_i c_i}{-\frac{h_i}{2} b_{i+1}\left(\psi_i(1+\kappa_i h_i a_i)-\kappa_i h_i c_i\right)+(1-\frac{h_i}{2} b_i\psi_i)\left(1-(1-\kappa_i)h_i a_{i+1}\right)}$$

$$\text{(4.6c)} \qquad \chi_{i+1} = \frac{\frac{h_i}{2}(g_0+g_1+b_i\chi_i)(\psi_i(1+\kappa_i h_i a_i)-\kappa_i h_i c_i)+\chi_i(1+\kappa_i h_i a_i)}{-\frac{h_i}{2} b_{i+1}(\psi_i\left(1+\kappa_i h_i a_i\right)-\kappa_i h_i c_i)+(1-\frac{h_i}{2} b_i\psi_i)(1-(1-\kappa_i)h_i a_{i+1})};$$

TABLE 4.2

Forward Elimination

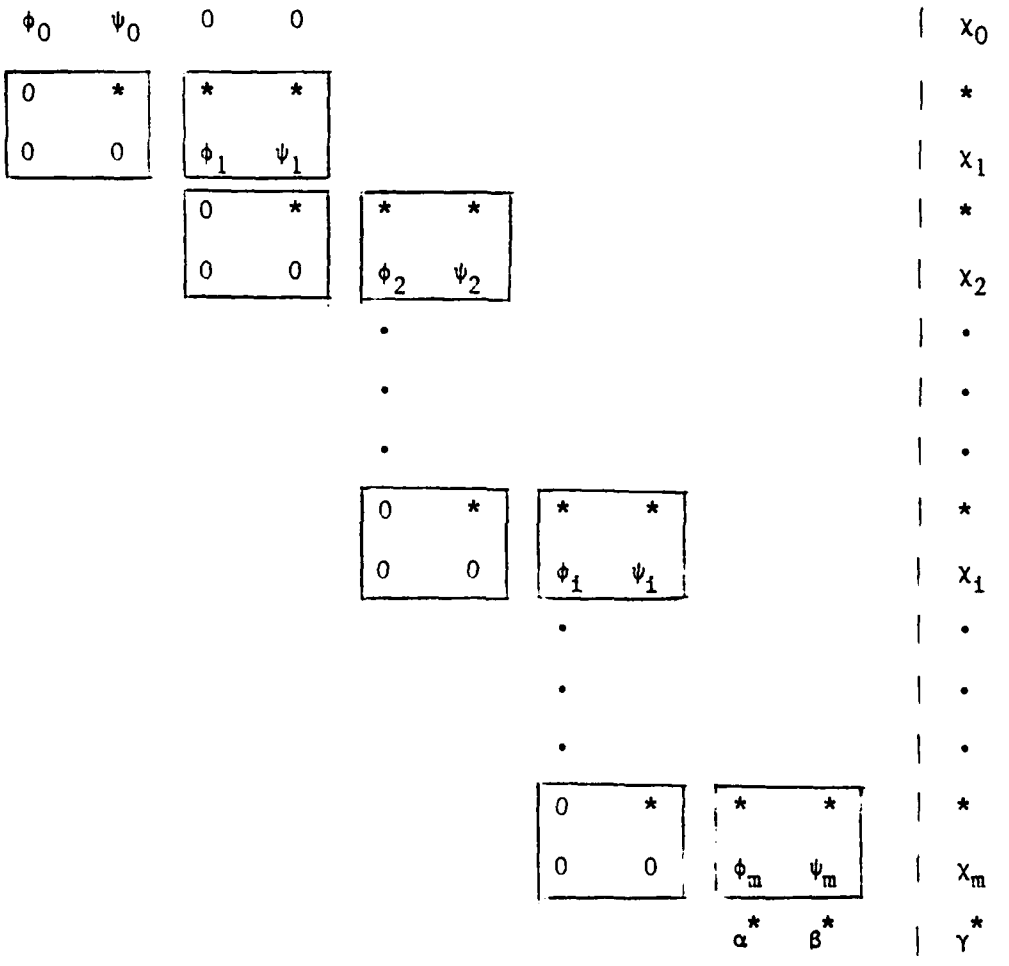

TABLE 4.3

Backward Elimination

α_1	β_1	0	0						γ_1
ϕ_1^*	ψ_1^*	0	0						x_1^*
*	*	*	0						*
		ϕ_2^*	ψ_2^*	0	0				x_2^*
		*	*	*	0				*
				•					•
				•					•
				•					•
				ϕ_i^*	ψ_i^*	0	0		x_i^*
				*	*	*	0		*
						•			•
						•			•
						•			•
						ϕ_m^*	ψ_m^*	0 0	x_m^*
						*	*	* 0	*
								ϕ_{m+1}^* ψ_{m+1}^*	x_{m+1}^*

ϕ recursion:

$$(4.7a)\quad \phi_{i+1} = \frac{-\frac{h_i}{2} b_{i+1}\left(1+\kappa_i h_i a_i-\kappa_i h_i c_i \phi_i\right)+\left(\phi_i-\frac{h_i}{2} b_i\right)\left(1-(1-\kappa_i) h_i a_{i+1}\right)}{\left(1+\kappa_i h_i a_i-\kappa_i h_i c_i \phi_i\right)-\left(\phi_i-\frac{h_i}{2} b_i\right)(1-\kappa_i) h_i c_{i+1}},$$

$$(4.7b)\quad \psi_{i+1} = 1,$$

$$(4.7c)\quad \chi_{i+1} = \frac{\left(-\frac{h_i}{2}\left(g_i+g_{i+1}\right)+\chi_i\right)+\chi_i\left(1+\kappa_i h_i a_i-\kappa_i h_i c_i \phi_i\right)+\left(\phi_i-\frac{h_i}{2} b_i\right)\kappa_i h_i c_i \chi_i}{\left(1+\kappa_i h_i a_i-\kappa_i h_i c_i \phi_i\right)-\left(\phi_i-\frac{h_i}{2} b_i\right)(1-\kappa_i) h_i c_{i+1}}.$$

It is readily seen (by letting $h \to 0$) that the explicit recursion formulas (4.6) and (4.7) can be regarded as special one-step methods for the initial value problems (3.15)-(3-17) and (3.19)-(3-21), respectively. That is, the closure of (4.6) is (3.15)-(3-17) and the closure of (4.7) is (3.19)-(3-21). Of course, it is also possible to devise an elimination scheme for (4.3) which yields the normalized factorization (z given by (3.12)) as its closure. Analogous recursions can be given for the backward elimination.

5. STABLE NUMERICAL PROCESSES

The explicit one-step recursion formulas (4.6) and (4.7) are examples of a general one-step numerical process (cf. [2]) of the form

$$y_{i+1} = \Phi_i(y_i,h_i), \qquad i = 0,1,2,... \tag{5.1}$$

where $y_i,\Phi_i \in R^N$. We have already noted that (4.6) and (4.7) are consistent with (i.e., the closures of) particular realizations of the initial value problems (3.1)-(3.3). They are also stable.

Definition 5.1 (Stability). The process (5.1) is stable if it is bounded,

$$\|y_i\| \leq K, \tag{5.2a}$$

and if there exist a neighborhood $B_\delta(y) = \{z: \|z-y\|_0 < \delta\}$ of y_0, and an $L \leq 1$ such that

$$\|z_{i+1}-y_{i+1}\| \leq L\|z_i-u_i\| \tag{5.2b}$$

for z_{i+1} given by

$$z_{i+1} = \Phi_i(z_i,h_i)$$

with $z_0 \in B_\delta(y_0)$.

Definition 5.1 is an adaptation of BN-statibility (cf. [4]). In contrast to the concepts of A-stability and B-stability, the definition is made without reference to a particular test equation. If $L < 1$ the process is strongly stable.

Consider the model problem (2.1) with

$$\frac{a(s)h}{\epsilon} < -1 \tag{5.3a}$$

and

(5.3b) $$b(s) > 0.$$

Under these conditions it is not hard to see that recursion (4.6) will hold with $\kappa_i = 0$ in (4.2). In fact, (4.6b) reduces to

(5.4) $$\psi_{i+1} = \frac{(\frac{h_i^2 b_i}{2} + \varepsilon)\psi_i - h_i}{(\frac{h_i^2 b_i a_{i+1}}{2} - \frac{\varepsilon h_i (b_i + b_{i+1})}{2})\psi_i + (\varepsilon - h_i a_{i+1})},$$

which can be shown to be an order 1 method for (3.16).

Theorem 5.2. For $\varepsilon < \varepsilon_0$, with ε_0 such that conditions (5.3) hold, the recursion (5.4) remains in the semi-infinite strip

(5.5) $$\psi < \frac{2 - (\frac{\psi \varepsilon_0}{\bar{a}h})^{1/2}}{h\bar{b}}, \quad \bar{a} \geq \max|a(s)|, \quad \bar{b} \geq \max|b(s)|$$

and is strongly stable, independently of ε. ∎

Results analogous to Theorem 5.2 hold for the χ recursion (4.6c), and also for the recursions (4.7) in the event that $|\frac{ha(s)}{\varepsilon}| > 1$. For the case $|\frac{ha(s)}{\varepsilon}| < 1$ it is not possible to say a priori whether recursion (4.6) or (4.7) will be used. It is likely that the matrix reduction algorithm will switch between them, depending upon the behavior of $a(s)$, $b(s)$, and on the pivoting strategy in a particular case. It is not our intent to give an exhaustive analysis here.

As a final example to illustrate the ideas discussed above, note, that another first order accurate, stable, explicit recursion for the initial value

problem (3.2) can be obtained by applying the implicit Euler method. Thus

$$\psi_{i+1} = \psi_i + h_i \left\{ -\frac{1}{\varepsilon} + \left(\frac{a_{i+1}}{\varepsilon}\right)\psi_{i+1} + b_{i+1}\psi_{i+1}^2 \right\}, \tag{5.6}$$

whence

$$\psi_{i+1} = \frac{\varepsilon - h_i a_{i+1}}{2\varepsilon h_i b_{i+1}} \left\{ 1 - \frac{\psi \varepsilon h_i b_{i+1} (\varepsilon \psi_i - h_i)}{(\varepsilon - h_i a_{i+1})^2}^{1/2} \right\}. \tag{5.7}$$

6. CONCLUSION

We have shown on a model problem

1) that there exist factorizations of the original two point boundary value problem into systems of initial value problems with guaranteed stability properties;

2) that there can be many methods--some specially designed--which solve these initial value problems. Some of these methods will produce the identical numerical processes as the finite difference (or finite element) methods applied to the original boundary value value problems.

In view of these two points, we suggest that it may be advantageous to avoid the finite element or finite difference approach altogether, and rather to study the class of stable factorizations directly, with the goal of selecting both an optimal factorization for the given boundary value problem, and also an optimal method (possibly a special one) for solving the initial value problems of the factorization. In this way one might employ the modern ideas and adaptive approaches widely used in today's software for the numerical treatment of initial value problems for ordinary differential equations.

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