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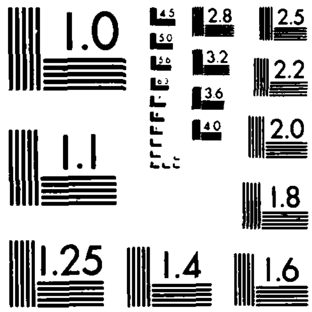

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Research Report CCS 459

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by

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This research was partly supported by USAF Contract F49642-82-C0129 and USARI Contract MDA903-81-C-0365 with the Center for Cybernetic Studies, The University of Texas at Austin. Reproduction in whole or in part is permitted for any purpose of the United States Government.

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Abstract

The construction and analysis of Pareto-optimal frontier production functions by a new Data Envelopment Analysis method is developed in the context of new theoretical characterizations of the inherent structure and capabilities of such empirical production functions. Contrasts and connections with other developments, including solutions of some remaining problems, are made re aspects such as informatics, economies of scale, isotonicity and non-concavity, discretionary and non-discretionary inputs, piecewise linearity, partial derivatives and Cobb-Douglas properties of the functions. Non-Archimedean constructs are not required.

Key Words

Pareto Optimality
Efficiency Analysis
Frontier Production Functions
Data Envelopment Analysis

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Original: "An Empirical DEA Production Function" by A. Charnes, W.W. Cooper & L. Seiford, April 1981.

INTRODUCTION

Classically, the economic theory of production is heavily based on the conceptual use of the Pareto-efficiency (or Pareto-optimal) frontier of production possibility sets to define "the" production function. The work of R. Shephard [18], [19] under severe restrictions on the mathematical structure of production possibility sets and cost relations, developed an elegant "transform" theory between production aspects and cost aspects [10]. This was applied to various classes of explicitly given parametric functional forms and problems of statistical estimation of parameters from data were considered in classical statistical contexts especially by successors such as R. Frisch, S. Afriat, D. Aigner, F. Forsund [1, 2, 16]. These efforts were almost exclusively for single output functions.

M.J. Farrell in [14], seeking to disentangle prices or costs from "technical" aspects of production, as well as to provide a more meaningful technical setting to statistical and empirical aspects of production, defined (for the <u>single</u> output case) a measure of "technical efficiency" of observed production units relative to the total units observed assuming that the production process of inputs to output conversion was linear and of constant returns to scale.

Building on the unit-by-unit evaluations of Farrell and the engineering ratio idea of efficiency measure for a single input and output, efficiency analysis in its managerial aspects and its constructible extensions to multi-input, multi-output situations was initiated by Charnes, Cooper and Rhodes in [8], [9]. Subsequent extensions and elaborations by the former pair with other students and colleagues were made in [7], [11], [12] . . . with more attention to classical economic aspects and to the production function side of the mathematical duality structure and Data Envelopment Analysis first discovered in the CCR work. The CCR ratio measures and the variants of Farrell, Shephard, Färe, Banker, et al. require, however, non-Archimedean constructs for rigorous theory and usage. Their solution methods also do not easily provide important needed properties of their associated empirical production functions.

Thus, in this paper we introduce as basic the idea of Pareto optimality with respect to an empirically defined production possibility set. We characterize the mathematical structures permitted under our minimal assumptions and contrast these with others' work. Properties such as isotonicity, non-concavity, economies of scale, piece-wise linearity, Cobb-Douglas forms, discretionary and non-discretionary inputs are treated through a new Data Envelopment Analysis *method and informatics* which permits a constructive development of an empirical production function and its partial derivatives without loss of efficiency analysis or use of non-Archimedean field extensions.

EMPIRICAL FUNCTION SETTING AND GENERATION

By an "empirical" function we shall mean a vector function whose values are known at a finite number of points and whose values at other points in its domain are given by linear (usually convex) combinations of values at known points. The points in the domain are "inputs," the component values of the vector function "outputs." We shall assume that inputs are so chosen that convex combinations of input values for each input are meaningful input values. We assume this for output values as well.

In efficiency analysis, observations are generated by a finite number of "DMU"s, or "productive," or "response" units, all of which have the same inputs and outputs. A relative efficiency rating is to be obtained for each unit. Typically, observations over time will be made of each unit and the results of efficiency analyses will be employed to assist in managing each of the units. We assume n units, s outputs and m inputs. The values are to be non-negative (sometimes positive) numbers.

A HYPOGRAPH EMPIRICAL PRODUCTION POSSIBILITY SET

Given the (empirical) points (X_j, Y_j), j=1,...,n with (mx1) "input" vectors $X_j \geqslant 0$ and (sx1) "output" vectors $Y_j \geqslant 0$, we define the "empirical production set" P_E to be the convex hull of these points i.e.

$$(2.1) \qquad P_E \triangleq \{(x,y) : x = \sum_{j=1}^{n} X_j \mu_j \ , \ y = \sum_{j=1}^{n} Y_j \mu_j \ , \ \forall \mu_j \geqslant 0 \ , \sum_j \mu_j = 1\}$$

We extend it to our "empirical production possibility set" Q_E by adding to P_E all points with inputs in P_E and outputs not greater than some output in P_E i.e.

$$(2.2) \qquad Q_E \triangleq \{(x,y) : x = \bar{x} \ , \ y \leqslant \bar{y} \text{ for some } (\bar{x},\bar{y}) \in P_E\}$$

Note that Q_E is contained in (e.g. is smaller than) <u>every</u> production possibility set heretofore employed, i.e. those studied by Farrell [14], Shephard [19], Banker, Charnes and Cooper [3], Färe, <u>et</u> <u>al</u>. [13], etc. The Farrell, Shephard, Färe sets are (truncated) cones; the BCC set (when not also a cone) adds to Q_E the set

$$\{(x,y) : x \geqslant \bar{x} \ , \ y = \bar{y} \text{ for some } (\bar{x},\bar{y}) \in Q_E\}.$$

These relations may be visualized in the schematic plot of Figure 1:

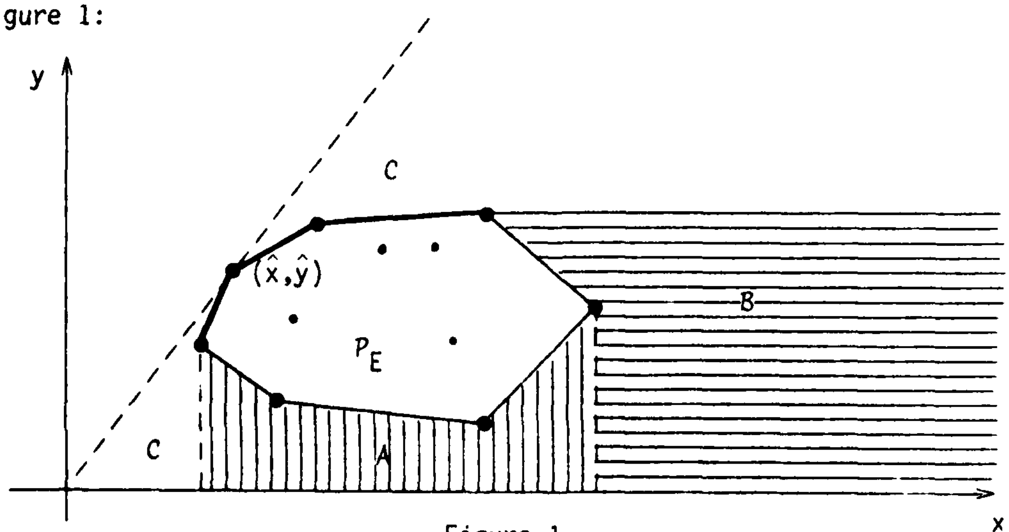

Figure 1

where $\mathcal{Q}_E = P_E \cup A$, the BBC set is $\mathcal{Q}_E \cup \mathcal{B}$, and the Farrell, Shephard, Färe set is $\mathcal{Q}_E \cup \mathcal{B} \cup C$.

Let $\mathcal{P}_E^r$, $\mathcal{Q}_E^r$ denote the sets corresponding to $\mathcal{P}_E$ and $\mathcal{Q}_E$ when only the output y_r is the ordinate. Evidently a frontier function $f_r(x)$ is determined by

(2.3) $$f_r(x) = \max y_r \text{ for } (x,y_r) \in \mathcal{Q}_E^r$$

Then,

<u>Theorem 0</u>: $\mathcal{Q}_E^r$ is the hypograph of $f_r(x)$ over $\{x : (x,y) \in \mathcal{Q}_E\}$

<u>Proof</u>: The hypograph H_r of $f_r(x)$ is the set

$$H_r \triangleq \{(x,y_r) : y_r \leqslant f_r(x) , (x,y) \in \mathcal{Q}_E\} \quad \blacksquare$$

Let $\mathcal{D}_E$ denote $\{x : (x,y) \in \mathcal{Q}_E\}$. It is the domain (the input set) of our empirical frontier functions.

<u>Theorem 1</u>: $f_r(x)$ is a <u>concave</u>, piecewise linear function on $\mathcal{D}_E$.

<u>Proof</u>: A necessary and sufficient condition for $f_r(x)$ to be concave is that its hypograph is a convex set (cf. Rockefellar [17], or Fenchel [15]). The piecewise linearity also follows from the construction of $\mathcal{Q}_E$ by all convex combinations of the empirical points (X_j,Y_j), j=1,...,n. $\blacksquare$

We observe explicitly further that <u>no</u> <u>use</u> whatever has been made of <u>non-negativity</u> of input and output values in the sets, functions or proof of Theorems 0 and 1. Therefore, they hold without this restriction--a fact we shall employ elsewhere.

Also, <u>no</u> assumptions have been made about the properties of <u>any</u> underlying function, or function hypograph, from which the (X_j,Y_j) of our empirical construct may be considered samples. Theorem 1 shows, therefore, that any <u>empirical</u> (maximum) frontier function is the "concave cap" function of its graph.

THE EMPIRICAL PARETO-OPTIMAL PRODUCTION FUNCTION

A Pareto-optimum for a finite set of functions $g_1(x),\ldots,g_K(x)$ is a point x^* such that there is no other point x in the domain of these functions such that

$$(3.1) \qquad g_k(x) \leqslant g_k(x^*) \ , \ k=1,\ldots,K$$

with at least one strict inequality. Charnes and Cooper in [5], Chapter IX, showed that x^* is Pareto-optimal iff x^* is an optimal solution to the mathematical (goal) program

$$(3.2) \qquad \min \sum_{k=1}^{K} g_k(x) \quad \text{subject to } g_k(x) \leqslant g_k(x^*) \ , \ k=1,\ldots,K$$

This was employed by Ben-Israel, Ben-Tal and Charnes in [4] to develop the currently strongest necessary and sufficient conditions for a Pareto-optimum in convex programming.

Utilizing (3.2) we can now define and construct, im(or ex-)plicitly the Pareto-optimal (or "Pareto-efficient") empirical (frontier) production function. Other usages of (3.2) to generalizations such as the "functional efficiency" of Charnes and Cooper [5] will not be developed here.

First, by (3.2), the Pareto-optimal points (inputs!) among our n empirical points can be determined. The empirical Pareto-optimal function is then defined on the convex hull of these points by convex combinations of the "output" values. Note that the convex hull of the Pareto-optimal points might not include all of $\mathcal{D}_E$ since only the doubled line portion of the frontier is Pareto-optimal.

Since for efficient production we wish to maximize on outputs while minimizing on inputs, our relevant $g_k(x)$ include both outputs and inputs, e.g.

$$(3.3)\qquad -g_k(x) \triangleq \begin{cases} y_k\ , & 1 \leqslant k \leqslant s \\ -x_i\ , & k = s+i,\ i=1,\dots,m \\ \text{for} & (x,y) \in Q_E \end{cases}$$

For the optimization in (3.2) we clearly need only consider $(x,y) \in P_E$ rather than Q_E. Thus the constraint inequalities in (3.2) are for a test point (x^*,y^*):

$$(3.4)\qquad y \geqslant y^*\ ,\quad x \leqslant x^*$$

and we have

Theorem 2: The envelopment constraints of Data Envelopment Analysis in production analysis are the Charnes-Cooper constraints for testing Pareto-optimality of an empirical production point.

In no way, as others, e.g. Färe [13] have mistakenly asserted, is Data Envelopment Analysis restricted to linear constant returns to scale functions or to truncated cone domains. Evidently via (3.2), Data Envelopment Analysis applies to much more general functions, function domains and other situations than the current empirical production function one.

To test an empirical "input-output" point (X_o, Y_o) for Pareto-optimality, the C^2 (Charnes and Cooper) test of (3.2) becomes

$$(3.5)\qquad \begin{array}{llll} \min & -e^T Y\lambda + e^T X\lambda & & \\ \text{subject to} & Y\lambda \ - \ s^+ & & = Y_o \\ & -X\lambda & -s^- & = -X_o \\ & e^T\lambda & & = 1 \\ & \lambda,\ s^+,\ s^- \geqslant 0 & & \end{array}$$

where $X \triangleq [X_1,\dots,X_n]$, $Y \triangleq [Y_1,\dots,Y_n]$.

Since $-e^T(Y\lambda - Y_o) + e^T(X\lambda - X_o)$ is an equivalent functional (it differs from the above one only by a constant), we can rewrite the problem for convenience in later comparisons as:

$$
\begin{array}{llll}
 & \min & -e^T s^+ - e^T s^- & \\
 & \text{subject to} & Y\lambda - s^+ & = Y_o \\
(3.6) & & -X\lambda \quad -s^- & = -X_o \\
 & & e^T\lambda & = 1 \\
 & \text{with } \lambda, s^+, s^- \geqslant 0 & &
\end{array}
$$

This is the new DEA form for the production possibility set Q_E via P_E. As we shall see later, other variations of Q_E can be accomodated easily by simple modifications of or additions to the constraints on λ. Its informatic and software involve only minor modification from that of the Charnes, Cooper, Seiford and Stutz paper [11] as developed by I. Ali and J. Stutz for the Center for Cybernetic Studies of The University of Texas at Austin.

EFFICIENCY ANALYSIS

As mentioned, managerial and program comparison aspects of efficiency analysis were initiated by Charnes, Cooper and Rhodes in [6], [8], and [9], through a generalization of the single input, single output absolute efficiency determination of classical engineering and science to multi-input, multi-output relative efficiencies of a finite number of decision-making units "DMU's" (sometimes called "productive" units or "response" units). The multi-input, multi-output situations were reduced to the "virtual" single input single output ones through use of virtual multipliers and sums. Explicitly, the CCR ratio measure of efficiency of the DMU designated "o" is given by the non-linear, non-convex, non-Archimedean fractional program (see [7]).

$$
\begin{aligned}
&\text{Max} && \frac{\eta^T Y_o}{\xi^T X_o} && \\
&\text{subject to} && \frac{\eta^T Y_j}{\xi^T X_j} \leqslant 1 && , j = 1, \ldots, n \\
(4.1)\quad & && -\frac{\eta^T}{\xi^T X_o} \leqslant -\varepsilon e^T && \\
& && -\frac{\xi^T}{\xi^T X_o} \leqslant -\varepsilon e^T &&
\end{aligned}
$$

where the entries of the X_j and Y_j are assumed positive, ε is a non-Archimedean infinitesimal, e^T is a row vector of ones and, by abuse of notation, has s entries for η^T, m entries for ξ^T. (X_o, Y_o) is one of the n input-output pairs.

Employing the Charnes-Cooper transformation of fractional programming

(4.2) $\mu^T \triangleq \eta^T/\xi^T X_o$, $\nu^T \triangleq \xi^T/\xi^T X_o$, $\nu^T X_o = 1$

we obtain the dual non-Archimedean linear programs

$$
\begin{array}{llll}
 & \max & \mu^T Y_o & \\
 & \text{subject to} & \nu^T X_o & = 1 \\
(4.3) & & \mu^T Y - \nu^T X & \leq 0 \\
 & & -\mu^T & \leq \varepsilon e^T \\
 & & -\nu^T & \leq \varepsilon e^T
\end{array}
\qquad
\begin{array}{ll}
\min \theta & -\varepsilon e^T s^+ - \varepsilon e^T s^- \\
 & Y\lambda - s^+ = Y_o \\
 & \theta X_o - X\lambda - s^- = 0 \\
 & \lambda,\ s^+,\ s^- \geq 0
\end{array}
$$

where $X \triangleq [X_1,\ldots,X_n]$, $Y \triangleq [Y_1,\ldots,Y_n]$.

Although, clearly, no assumptions have been made concerning the type of functional relations for the input-output pairs (X_j,Y_j), the dual program may be recognized as having the Data Envelopment Analysis constraints for an empirical production possibility set of Farrell, Shephard, etc. cone type $\mathcal{Q}_E \cup \mathcal{B} \cup C$, and, since

(4.4) $\theta - \varepsilon[e^T Y\lambda - e^T X\lambda]$

is an equivalent form for the functional, as being a Charnes-Cooper Pareto-optimality test for $(\theta X_o,Y_o)$ over the cone on the (X_j,Y_j), $j=1,\ldots,n$, with pre-emption on the intensity θ of input X_o. As shown, for example, in [7], DMU_o is efficient iff $\theta^* = 1$, $s^{*+} = 0$, $s^{*-} = 0$.

Re informatics, which are particularly important since all n efficiency evaluations must be made (i.e., n linear programs must be solved), the dual problem can be computed exactly (in the base field) as shown in [5], e.g., with the code NONARC of Dr. I. Ali (Center ⁝ r Cybernetic Studies, The University of Texas at Austin), or approximately by using a sufficiently small numerical value for ε. A typical efficient point is designated by $(\hat{x},\hat{y})$ in Figure 1.

If a DMU is inefficient, the optimal $\lambda^*_j > 0$ in its DEA problem (=Charnes-Cooper test) designate efficient DMU's. Thus, a "proper" subset of the efficient DMU's determines the efficiency value of an inefficient DMU. The convex combinations of this subset are also efficient. Thereby to each inefficient DMU a "facet" of efficient DMU's is associated. The transformation

$$(4.5) \qquad X_o \rightarrow \theta^* X_o - s^{*-} \;, \quad Y_o \rightarrow Y_o + s^{*+}$$

where the asterisk designates optimality, projects DMU_o, i.e., (X_o,Y_o), onto its efficiency facet.

This projection was employed by Charnes, Cooper and Rhodes [9] to correct for differences in managerial ability in their analysis of programs Follow-Through and non-Follow-Through. It also shows quantitatively what improvements in inputs and outputs will (ceteris paribus) bring a DMU to efficient operation. Thus, although the relative efficiency measure of an inefficient DMU will involve the infinitesimal ε, non-infinitesimal changes for improvement are suggested.

Both Farrell and Shepard knew that ratio measures required adjustments to correctly exhibit inefficiency of the second DMU in examples like the following 2 input, 1 output, 2 DMU case:

DMU	x_1	x_2	y
1	1	2	1
2	1	4	1

Farrell added geometric points at infinity; Shephard simply excluded such cases without giving a method for their exclusion. The non-Archimedean extension in the CCR formulation is necessary to have an algebraically closed system of linear programming type. Linear programming theory holds for non-Archimedean as well as Archimedean entries in the vector and matrix problem data.

Our new Pareto-optimal DEA method like C^2S^2 [11] associates facets with non-optimal (=non-Pareto-efficient) DMU's. Clearly, by the C^2-test, DMU_o is Pareto-efficient (Pareto-optimal) iff $-e^T s^{*+} - e^T s^{*-} = 0$, i.e., iff the ℓ_1-distance from (X_o, Y_o) to the farthest "northwesterly" (X_j, Y_j) point is zero. The CCR efficient DMU's are also among the new Pareto-optimal DMU's. Projection of a non-optimal DMU onto its Pareto-efficient facet is rendered by

$$(4.6) \qquad X_o \rightarrow X_o - s^{*-} , \quad Y_o \rightarrow Y_o + s^{*+}$$

To achieve a convenient efficiency measure, we modify the functional by multiplying it by a $\delta > 0$ and consider

$$(4.7) \qquad -\delta e^T s^{*+} - \delta e^T s^{*-} ,$$

where the asterisk denotes optimality, as the logarithm of the efficiency measure. When the data in X and Y are scaled to lie between 0 and 100, a $\delta = 1/10(m+s)$ will yield a logarithm between 0 and -10. This measure might then be called the "efficiency pH" by analogy with the pH of chemistry.

Our new measure relates to the units invariant multiplicative measure of Charnes, Cooper, Seiford and Stutz [12], which as shown there is necessary and sufficient that the DEA envelopments be piecewise Cobb-Douglas, by considering the entries in the X_j, Y_j to be logarithms of the entries in $\tilde{X}_j$, $\tilde{Y}_j$ which we employ in the multiplicative formulation.

INFORMATICS AND FUNCTION PROPERTIES

(A) Partial Derivatives:

The guidance provided by the CCR, BCC, C^2S^2 formulations does not include convenient access to the rates of change of the outputs with change in the inputs. The optimal dual variables in the DEA side linear programming problems give rates of change of the efficiency measure with changes in inputs or outputs. The non-Archimedean formulations further may give infinitesimal rates, which are not easily employed. And, for most of the efficient points one has non-differentiability because they are extreme points rather than (relative) interior points. Nevertheless, because of the informatics, e.g., computational tactics, we employ in testing via C^2 for Pareto-optimality, the following constructive method can be employed.

On reaching a non-Pareto optimal point, our software discovers all the optimal points in its facet, hence, implicitly, all the convex combinations which form the facet. Since the Pareto-optimal facet is a linear surface it is not only differentiable everywhere in its relative interior but all its partial derivatives are constant throughout the facet. Thus, we need only obtain these for any relative interior point of the facet to have them for the whole facet. Such a point is the average of the Pareto-optimal points of the facet.

Let

$$(5.1) \qquad F(x_1,\ldots,x_m, y_1,\ldots,y_s) = 0$$

be the linear equation of the facet. Since we have sufficient differentiability in the neighborhood of the average point $(\bar{x},\bar{y})$, we know

$$(5.2) \qquad \left.\frac{\partial y_r}{\partial x_i}\right|_{\bar{x},\bar{y}} = -\left(\frac{\partial F}{\partial x_i}\right)\Bigg/\left(\frac{\partial F}{\partial y_r}\right)$$

where the right side partial derivatives are also evaluated at $(\bar{x},\bar{y})$.

Suppose we run the C^2-test with $(\bar{x},\bar{y})$ as the point being tested. Then the optimal dual variables corresponding to input $\bar{x}_i$ and $\bar{y}_r$ are respectively $\left(\frac{\partial F}{\partial x_i}\right)\Big|_{\bar{x},\bar{y}}$ and $\left(\frac{\partial F}{\partial y_r}\right)\Big|_{\bar{x},\bar{y}}$. Thus, the rate of change of output y_r with respect to input x_i is simply the negative of the ratio of the optimal dual x_i constraint variable to the optimal dual y_r constraint variable!

More specifically, all Pareto-optimal (X_j,Y_j) of the facet for the barycenter $(\bar{x},\bar{y})$ satisfy

$$(5.4)\quad \mu^{*T}y - \nu^{*T}x - \varphi^* = 0$$

where $(\mu^{*T}, \nu^{*T}, \varphi^*)$ are the dual evaluators at an optimal basic solution, since they do not depend on the C^2-test right hand sides. Thereby our

$$(5.5)\quad F(x,y) = \mu^{*T}y - \nu^{*T}x - \varphi^* = 0$$

Clearly, $\mu_r^* = \partial F/\partial y_r$, $-\nu_i^* = \partial F/\partial x_i$ as already stated.

(B) Isotonicity and Economies of Scale:

Theorem 1 shows that every component of the empirical frontier production function is a concave function.

Suppose x^1 and x^2 are the inputs of two Pareto-optimal DMU's in the same facet and $x^1 \geqslant x^2$. Since $x^1 = X\lambda^*(x^1)$ and $x^2 = X\lambda^*(x^2)$ we must have $e^T X\lambda^*(x^1) \geqslant e^T X\lambda^*(x^2)$. But for Pareto-optimality, $e^T Y\lambda^*(x^i) = e^T X\lambda^*(x^i)$, i=1,2 so that $e^T Y\lambda^*(x^1) \geqslant e^T Y\lambda^*(x^2)$. Then, letting $f^P(x)$ denote the empirical Pareto-optimal (vector) function we have

(5.3) $$e^T f^P(x^1) \geqslant e^T f^P(x^2)$$

Further, if $x^\mu \triangleq \mu x^1 + (1-\mu)x^2$, $0 \leqslant \mu \leqslant 1$, $f^P(x^\mu) \triangleq \mu f^P(x^1) + (1-\mu) f^P(x^2)$ by construction of the empirical frontier function and we have

$$e^T f^P(x^1) \geqslant \mu e^T f^P(x^\mu) \geqslant e^T f^P(x^2).$$

For the single output case of Farrell, etc., then

Theorem 3: If there is only a single output, the empirical Pareto-optimal production function is isotonic in every facet (regardless of what underlying production function we have sampled from).

Proof: A function $f(x)$ is "isotonic" iff $x^a \geqslant x^b$ implies $f(x^a) \geqslant f(x^b)$. Also $e^T f^P(x) = f^P(x)$ with a single output.

∎

Possibly because of ignorance of standard mathematical terminology, the isotonic property has been called "strong disposability" in the economics literature. The name "weak disposability" has also been used for the weaker property $f(\rho x) \geqslant f(x)$ whenever $\rho \geqslant 1$. A better name might be "ray isotonic."

Our arguments preceding Theorem 3 establish a "sum isotonic" property on facets for the empirical Pareto-optimal function with multiple output components (regardless of the underlying production function set we have sampled from), namely,

Theorem 4: $e^T f^p(x^a) \geq e^T f^p(x^b)$ whenever $e^T x^a \geq e^T x^b$ with x^a, x^b in the same facet.

Classically in economics, production functions studied have usually been assumed to be homogeneous and defined on the non-negative orthant. Thereby, whether or not a function for which $f(\rho x) = \rho^{\alpha} f(x)$, with $\rho \geq 0$, had economies of scale would be decided by the value of the exponent α. More generally, increasing or decreasing "return to scale" would be present respectively, at $\bar{x}$ if $f(\rho\bar{x}) > \rho f(\bar{x})$ or $f(\rho\bar{x}) < \rho f(\bar{x})$ for $\rho > 1$ at points $\rho\bar{x}$ in a small neighborhood of $\bar{x}$. The BCC paper [3] gives a criterion for deciding this (with production possibility set $\mathcal{Q}_E \cup \mathcal{B} \cup \mathcal{C}$ or $\mathcal{Q}_E \cup \mathcal{B}$) but does not give us the rates of change.

Because of our preceding theorems, however, we know that empirical Pareto-optimal functions are sum-isotonic on facets and concave in each component function regardless of the nature of the underlying production possibility set. Thereby, we automatically anticipate lower and lower returns to scale in going from facet to facet with increasing $e^T x$. And our partial derivatives can give us explicitly the rates of change in each observed facet.

Practically, our choices of inputs are generally made with the expectation that the underlying Pareto-optimal function is isotonic, i.e., we choose the form of the inputs so that an increase in an input should not decrease the outputs. But even here we need still more to determine the non-concave portions of an isotonic function. For example, in Figure 2 an isotonic function is plotted together with the resulting concave cap (large dashed lines) obtained as the empirical function:

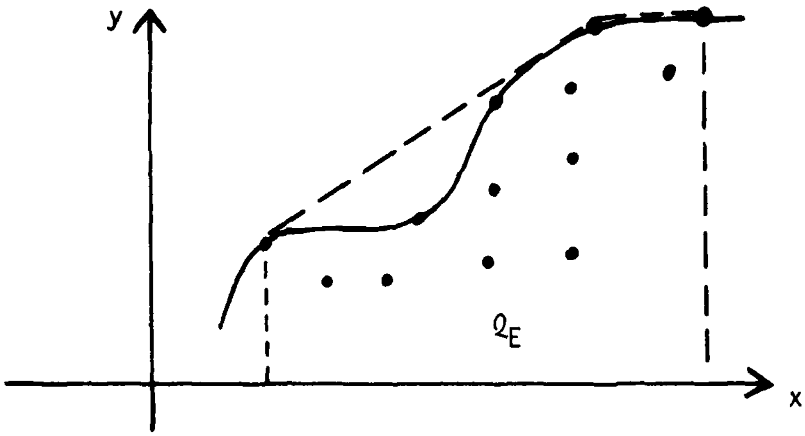

Figure 2

As suggested in our original (1981) paper, non-concavity can be explored by applying (output) component by component strictly concave transformations g_r to obtain $g_r(y_r)$ instead of y_r so that $g_r(y_r(x))$ would be concave and our plot would look like

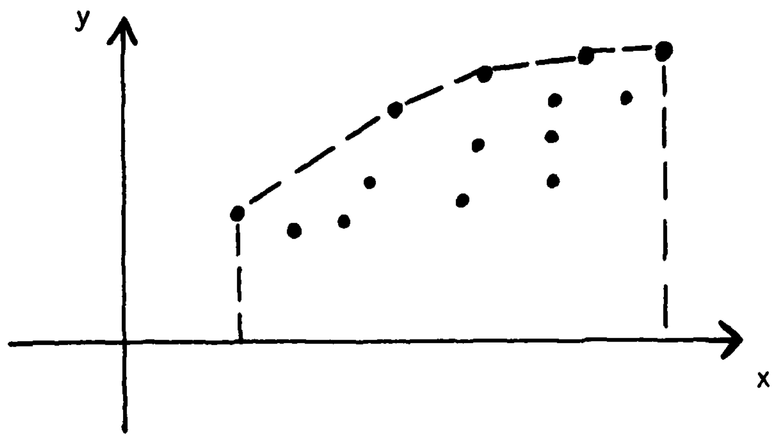

Figure 3

Informatically, we are doing this by applying transformations of form $g_\ell\,(y_\ell) = \bar{y}_\ell + (y_\ell - \bar{y}_\ell)^{\frac{1}{\beta}}$ with $\beta \approx 20$ to obtain possible new facets in the $g_\ell(y_\ell)$.

Problems do arise, of course, on whether one gets spurious empirical frontier portions in this manner for empirical points which should "really" be inefficient. Evidently such non-concave portions are portions of increasing returns to scale if they are truly on the frontier.

(C) <u>Discretionary and Non-Discretionary Inputs</u>:

In a number of practical applications, certain relevant inputs, e.g., unemployment rate, population, median income, are not subject to "discretionary" change by the decision-makers of decision-making units. These are called "non-discretionary" inputs. They are important in influencing the outputs and in furnishing the <u>reference</u> background in terms of which units' efficiency is rated. Not infrequently the facet associated with an inefficient unit has the same values for the non-discretionary inputs, in which case there is no problem with the rating assigned. If not, however, to obtain more meaningful ratings we can add constraints on λ to those in (3.5) which require the non-discretionary inputs to be the same as that of the unit being evaluated. Thereby, a more meaningful rating will be attained.

CONCLUSIONS

We have shown how direct application of the Charnes-Cooper test for Pareto optimality leads to a simpler and more robust method, efficiency pH, encompassing all previous ones for ascertaining "efficiency." Further, Pareto-optimal characterizations and constructions of empirical production functions restrict us methodologically to exploration of such functions by means of concave sum-isotonic caps. Economies of scale from these thereby expectedly decrease with increase in the magnitude of the input vectors. Use of transformations of outputs, as we suggest, can uncover non-concave regions of the underlying production function where substantial economies of scale may prevail. Our new informatics device and theory of the use of the facet average (or barycenter) also constructively furnishes quantitative estimates of *the rates of change* of outputs with respect to inputs which have not been available previously. These new devices, as with other usages of empirical functions, suggest important new areas for development of statistical theory to distinguish between true properties and sampling "accidents." The vital importance of further development of the informatics of solution of systems of adaptively developed linear programming problems for Pareto-optimal constructions should also be clear.

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