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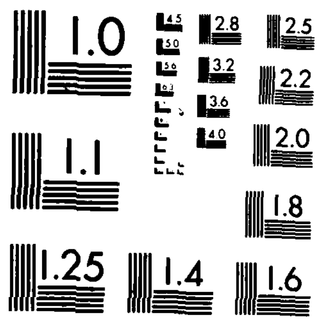

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LOCAL LINEAR INDEPENDENCE OF THE TRANSLATES OF A BOX SPLINE

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ABSTRACT

Let $\Xi = (\xi_i)_1^n$ be a sequence of vectors in $\mathbb{R}^m$. The box spline M_Ξ is defined as the distribution given by

$$M_\Xi : \phi \to \int_{[0,1]^n} \phi\Big(\sum_{i=1}^n \lambda(i)\xi_i\Big)d\lambda, \quad \phi \in C_c^\infty(\mathbb{R}^m).$$

Suppose that Ξ contains a basis for $\mathbb{R}^m$. Then $M_\Xi \in L_\infty(\mathbb{R}^m)$. Let $V = \mathbb{Z}^m$. Consider the translates $M_v := M_\Xi(\cdot - v)$, $v \in V$. Is is known that $(M_v)_V$ is linearly dependent unless

$$(*) \qquad |\det Z| = 1 \quad \text{for all bases } Z \subset \Xi.$$

This report demonstrates that, under condition (*), $(M_v)_V$ is locally linearly independent, i.e.,

$$\{M_v ; \operatorname{supp} M_v \cap A \neq \emptyset\}$$

is linearly independent over any open set A contained in some component of $\mathbb{R}^m \backslash K(\Xi)$, where

$$K(\Xi) := \bigcup_{\langle \Xi \backslash Z\rangle \neq \mathbb{R}^m} [\Xi \backslash Z] + \sum_{u \in \mathbb{Z}^n} \sum_{i=1}^n u(i)\xi_i .$$

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SIGNIFICANCE AND EXPLANATION

In early 1970's, the finite element analysts were interested in the space spanned by certain translates of one element. Recently, with the appearance of box splines, people are interested in the space spanned by certain translates of one box spline. Such a space ~~has been~~ was proved useful in certain approximation problems. This report studies some properties of this space.

The responsibility for the wording and views expressed in this descriptive summary lies with MRC, and not with the author of this report.

LOCAL LINEAR INDEPENDENCE OF THE TRANSLATES OF A BOX SPLINE

Rong-Qing Jia

Let us begin with some notations. For a set S, we denote by $|S|$ the cardinality of S. For a function f defined on a topological space X, its support is denoted by supp f. Let $\mathbb{R}^m$ denote the m-dimensional real vector space. We identify $\mathbb{R}^{m-1}$ with $\mathbb{R}^{m-1} \times \{0\} \subset \mathbb{R}^m$. We use $x(r)$ for the r-th entry of the vector $x \in \mathbb{R}^m$; i.e.

$$x = \big(x(1),\dots,x(r),\dots,x(m)\big).$$

With the norm

$$\|x\| = \sup_{1\le r\le m} \{|x(r)|\},$$

$\mathbb{R}^m$ becomes a normed vector space. By $B_r(y)$ we mean the ball $\{x \in \mathbb{R}^m;\ \|x-y\| < r\}$. If A and B are two sets in $\mathbb{R}^m$, then

$$A + B := \{a + b;\ a \in A,\ b \in B\}.$$

The set $A-B$ is defined in the same fashion. We emphasize that the set $\{x \in A;\ x \notin B\}$ is denoted by $A\backslash B$ rather than $A-B$. With $A \subset \mathbb{R}^m$, we denote by $\langle A\rangle$ the affine span of A. Let $e_i\,(i=1,\dots,m)$ be the unit coordinator vectors in $\mathbb{R}^m$; that is, $e_i(j) = \delta_i^j$, where δ_i^j are the Kronecker signs. For a function f defined on a domain in $\mathbb{R}^m$, we use the notation $D_i f$ for the partial derivative with respect to its i-th argument of the function f. We also use the notation $D_y := \sum_1^m y(i)D_i f$. Also, we define

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the difference operator ∇_y by the rule $\nabla_y f = f-f(\cdot-y)$. Finally, we denote by $C_c^\infty(R^m)$ the space of all C^∞-functions with compact support in $\mathbb{R}^m$.

Box splines are introduced by [BD] and [BH1]. Here we follow [BH1], and give a brief description for them. Let $\xi_1,\ldots,\xi_n$ be n vectors in $\mathbb{R}^m$. Let $\Xi = (\xi_i)_1^n$. Then the box spline M_Ξ is defined as the distribution

$$M_\Xi: \phi \to \int_{[0,1]^n} \phi\Big(\sum_{i=1}^n \lambda(i)\xi_i\Big)d\lambda, \ \phi \in C_c^\infty(R^m).$$

This M_Ξ is nonnegative and

$$\operatorname{supp} M_\Xi = [\Xi] := \Big\{\sum \lambda(\xi)\xi : \lambda \in [0,1]^\Xi\Big\}. \tag{1}$$

Moreover,

$$M_\Xi \in L_\infty^{(d)} \subset C^{(d-1)},$$

where

$$d := \max\,\{r: \langle \Xi\backslash Z\rangle = \mathbb{R}^m \ \text{ for all } \ Z \subset \Xi \ \text{ with } \ |Z| = r\}.$$

If $|\Xi| > m$, then M_Ξ agrees with some polynomial of degree $\leq |\Xi|-m$ on each connected component of the complement of

$$\{[\Xi\backslash Z] + \sum_H \eta; \ \ H \subset Z,\ \langle \Xi\backslash Z\rangle \neq \mathbb{R}^m\}.$$

For the derivatives of box splines, we have the following formula: for $\xi \in \Xi$,

$$D_\xi M_\Xi = M_{\Xi\backslash\xi} - M_{\Xi\backslash\xi}(\cdot-\xi) = \nabla_\xi M_{\Xi\backslash\xi}.$$

Let a be a mapping from $\mathbb{Z}^m$ to $\mathbb{R}$. Then the above formula together with summation by parts gives

$$D_\xi\Big(\sum a(j)M_\Xi(\cdot-j)\Big) = \sum (\nabla_\xi a)(j)M_{\Xi\backslash\xi}(\cdot-j). \tag{2}$$

If Ξ contains a basis for $\mathbb{R}^m$, then M_Ξ is a function in L_∞. We consider the collection of translates $M_v := M(\cdot - v)$, $v \in V = \mathbb{Z}^m$, for the box spline $M := M_\Xi$. It was shown by [BH_1] that $(M_v)_V$ is linearly dependent unless

$$|\det Z| = 1 \quad \text{for all bases} \quad Z \subset \Xi. \tag{3}$$

Later, [DM] showed that condition (3) is also sufficient for $(M_v)_V$ to be linearly independent. Independently, [J] gave a more elementary proof for this fact. When $m = 2$ and $\Xi = \{e_1, e_2, e_1+e_2\}$, [$BH_2$] got a stronger result that $(M_v)_V$ is <u>locally</u> linearly independent, i.e., $\{M_v;\ \text{supp}\, M_v \cap A \neq \emptyset\}$ is linearly independent over any open set A contained in some component of $\mathbb{R}^2\backslash\{(x_1,x_2);\ x_1,\ x_2 \text{ or } x_1-x_2 \in \mathbb{Z}\}$. A question naturally arises: Whether can this result be extended to general Ξ? The purpose of this paper is to give an affirmative answer to this question. Our result is

<u>Theorem</u>. <u>Let</u> $\Xi = (\xi_i)_1^n$ <u>be a sequence of vectors in</u> $\mathbb{R}^m$. <u>Suppose that</u>

(i) Ξ <u>contains a basis for</u> $\mathbb{R}^m$,

(ii) $|\det Z| = 1$ <u>for all bases</u> $Z \subset \Xi$.

<u>Let</u> $V := \mathbb{Z}^m$. <u>Also, for any</u> $v \in V$, <u>let</u> $M_v := M_\Xi(\cdot - v)$. <u>Then</u> $(M_v)_V$ <u>is locally linearly independent</u>, i.e.,

$$\{M_v;\ \text{supp}\, M_v \cap A \neq \emptyset\}$$

<u>is linearly independent over any open set A contained in some component of</u> $\mathbb{R}^m\backslash K(\Xi)$, <u>where</u>

$$K(\Xi) := \bigcup_{\langle \Xi\backslash Z\rangle \neq \mathbb{R}^m} [\Xi\backslash Z] + \sum_{u \in \mathbb{Z}^n} \sum_{i=1}^n u(i)\xi_i.$$

Proof. The proof proceeds by induction on $|\Xi|$. The case $|\Xi| = 1$ is trivial (see, e.g., [B; Lemma 5.1.]). Suppose that the theorem has been proved for any Ξ' with $|\Xi'| < |\Xi|$.

Without loss of any generality, we may assume that Ξ containes all the unit coordinate vectors, i.e.,

$$\{e_1,\ldots,e_m\} \subset \Xi.$$

(see [J]). There are two possible cases:

Case 1. There exists some e_k such that $\langle e_k\rangle \cap \langle \Xi\backslash e_k\rangle = 0$.

Case 2. The complement of Case 1; i.e., $\langle \Xi\backslash e_k\rangle = \mathbb{R}^m$ for every $k=1,\ldots,m$.

In case 1, we may assume

$$\langle e_m\rangle \cap \langle \Xi\backslash e_m\rangle = 0.$$

Then $(\Xi\backslash e_m) = \mathbb{R}^{m-1}$. By the definition of $K(\Xi)$, we have

$$\mathbb{R}^{m-1} + je_m \subset K(\Xi), \quad \text{for any } j \in \mathbb{Z}. \tag{4}$$

Also,

$$K(\Xi) = K(\Xi\backslash e_m) \times I_m + \mathbb{Z}e_m, \tag{5}$$

where $I_m := \{te_m;\ 0 < t < 1\}$.

Let A be an open set contained in a component of $\mathbb{R}^m \backslash K(\Xi)$. Let

$$V_A = V_{A,\Xi} := \{v \in \mathbb{Z}^m;\ \operatorname{supp} M_\Xi(\cdot - v) \cap A \neq \emptyset\}.$$

Suppose

$$\sum_{v\in V_A} a(v)M_\Xi(x-v) = 0 \quad \text{for all } x \in A. \tag{6}$$

We want to prove that

$$a(v) = 0 \quad \text{for all } v \in V_A. \tag{7}$$

Pick a point $x \in A$. Let $x' := x - x(m)e_m$. Then $x' \in \mathbb{R}^{m-1}$ and $x = x' + x(m)e_m$. Since $x \notin K(\Xi)$, (4) tells us that $x(m) \notin \mathbb{Z}$. Assume $i < x(m) < i + 1$. Similarly, let $v' := v - v(m)e_m$. Then $v = v' + v(m)e_m$. It is easily seen from the definition of M_Ξ that

$$M_\Xi(x-v) = M_{\Xi\backslash e_m}(x'-v')M_{e_m}\big(x(m)-v(m)\big). \tag{8}$$

Since A is an open set, (5) tells us that there exists an open set A' such that $A' \times \{x(m)\} \subset A$ and

$$x' \in A' \subset \text{some component of } \mathbb{R}^{m-1}\backslash K(\Xi\backslash e_m).$$

By (8), $v \in V_A$ implies that $v(m) = i$, and that

$$\operatorname{supp} M_{\Xi\backslash e_m}(\cdot - v') \cap A' \neq 0.$$

This is to say that $v \in V_A$ implies that $v(m) = i$ and $v' \in V_{A',\Xi\backslash e_m}$. Thus (6) yields

$$\sum_{v' \in V_{A',\Xi\backslash e_m}} a(v' + ie_m)M_{\Xi\backslash e_m}(x'-v') = 0, \quad \text{for all } v' \in A'.$$

By induction hypothesis

$$a(v' + ie_m) = 0 \quad \text{for all} \quad v' \in V_{A',\Xi\backslash e_m}.$$

This proves (7).

<u>Case 2.</u> $(\Xi\backslash e_k) = \mathbb{R}^m$ for any $k = 1,\ldots,m$.

This case is more complicated. We need several lemmas.

Lemma 1. Let A be an open set contained in some component of $\mathbb{R}^m\backslash K(\Xi)$. If $\operatorname{supp} M_\Xi \cap A \neq \emptyset$, then $\operatorname{supp} M_\Xi \supset A$. In particular, $v \in V_A$ implies that $\operatorname{supp} M_\Xi(\cdot - v) \supset A$.

Proof. Since A is open, $\operatorname{supp} M_\Xi \cap A \neq \emptyset$ implies that $M_\Xi(x) > 0$ for some $x \in A$. Let C be the component of $\mathbb{R}^m\backslash K(\Xi)$ which contains A. Then C is open and connected. If $\operatorname{supp} M_\Xi \not\supset A$, then $A \cap \big(\mathbb{R}^m\backslash \operatorname{supp} M_\Xi\big)$ is a nonempty open set. There exists $y \in \mathbb{R}^m$ and $\nu > 0$ such that

$$B_r(y) \subset A \cap (\mathbb{R}^m \setminus \operatorname{supp} M_\Xi) \subset C.$$

Since M_Ξ is a polynomial on C, and since M_Ξ vanishes on $B_r(y)$, hence M_Ξ vanishes on the whole C. This contradicts the fact that $x \in A \subset C$ and $M_\Xi(x) > 0$. Thus Lemma 1 is proved.

Lemma 2. If $y \in \operatorname{supp} M_\Xi$, and if $y + \xi_i \in \operatorname{supp} M_\Xi$, then $y \in \operatorname{supp} M_{\Xi\setminus\xi_i}$.

Proof. Without loss of any generality, we may assume $i = n$. By (1), there exist λ and $\mu \in [0,1]^n$ such that

$$y + \xi_n = \sum_{i=1}^{n} \lambda(i)\xi_i \tag{9}$$

$$y = \sum_{i=1}^{n} \mu(i)\xi_i. \tag{10}$$

If $\mu_n = 0$, then

$$y = \sum_{i=1}^{n-1} \mu(i)\xi_i \in \operatorname{supp} M_{\Xi\setminus\xi_n}.$$

Thus we may assume $\mu_n > 0$ in the following arguments. Subtracting (10) from (9) gives

$$\xi_n = \sum_{i=1}^{n} \big(\lambda(i)-\mu(i)\big)\xi_i.$$

It follows that

$$(1 + \mu_n - \lambda_n)\xi_n = \sum_{i=1}^{n-1} \big(\lambda(i)-\mu(i)\big)\xi_i.$$

Since $1 + \mu_n - \lambda_n > 0$, we obtain

$$\xi_n = \sum_{i=1}^{n} \frac{\lambda_i-\mu_i}{1+\mu_n-\lambda_n}\,\xi_i.$$

Substitute the above expression to (10):

$$y = \sum_{i=1}^{n-1} \mu_i\xi_i + \sum_{i=1}^{n-1} \mu_n \frac{\lambda_i-\mu_i}{1+\mu_n-\lambda_n}\,\xi_i = \sum_{i=1}^{n-1} \nu_i\xi_i,$$

where

$$\nu_i = \frac{\mu_i(1-\lambda_n)+\lambda_i\mu_n}{1-\lambda_n+\mu_n}.$$

It is clear that

$$\nu_i > 0 \quad \text{and} \quad \nu_i < \frac{(1-\lambda_n)+\mu_n}{1-\lambda_n+\mu_n} = 1.$$

Therefore $y \in \operatorname{supp} M_{\Xi\backslash\xi_n}$. This finishes the proof of Lemma 2.

Before stating Lemma 3, we make some conventions. As before, A is an open set contained in a component of $\mathbb{R}^m\backslash K(\Xi)$. For $v_1, v_2 \in V_A = \{v \in \mathbb{Z}^m; \operatorname{supp} M_\Xi(\cdot-v) \cap A \neq \emptyset\}$, we write

$$v_1 \sim v_2$$

If and only if

$$v_1 = v_2, \quad \text{or} \quad v_1-v_2 \in \Xi, \quad \text{or} \quad v_2-v_1 \in \Xi.$$

For $u, w \in V_A$, we write

$$u \approx w$$

if and only if there exist $v_1,\ldots,v_j \in V_A$ such that

$$u = v_1,\ w = v_j \quad \text{and} \quad v_i \sim v_{i+1} (i=1,\ldots,j-1).$$

Clearly, $\approx$ is an equivalence relation on V_A.

<u>Lemma 3</u>. <u>For any</u> $u, w \in V_A$,

$$u \approx w.$$

Proof. The proof proceeds by induction on $|\Xi|$. The case $|\Xi| = m$ is trivial. Suppose that $|\Xi| > m$, and that this lemma is true for any Ξ' with $|\Xi'| = |\Xi| - 1$.

Since Ξ contains a basis for $\mathbb{R}^m$, without loss of any generality, we may assume $\xi_i = e_i (i=1,\ldots,m)$. Suppose

$$\xi_{m+1} = b_1 e_1 + \dots + b_m e_m .$$

Since $\det Z = 1$ for any basis $Z \subset \Xi$, we must have

$$b_i = -1,\ 0 \ \text{ or } \ 1 \quad (i=1,\dots,m).$$

After an appropriate coordinate transform, we may assume

$$\xi_{m+1} = e_1 + \dots + e_k \ (k \le m). \tag{11}$$

Since $\langle \Xi\backslash\xi_{m+1}\rangle = R^m$, the set $V_{A,\Xi\backslash\xi_{m+1}}$ is nonempty. Also, A is contained in some component of $R^m\backslash K(\Xi\backslash\xi_{m+1})$. Furthermore, $\operatorname{supp} M_{\Xi\backslash\xi_{m+1}} \subset \operatorname{supp} M_\Xi$ shows that $V_{A,\Xi\backslash\xi_{m+1}} \subset V_A$.

Pick an element $v_0 \in V_{A,\Xi\backslash\xi_{m+1}}$. By induction hypothesis,

$$v \approx v_0 \quad \text{for any} \quad u \in V_{A,\Xi\backslash\xi_{m+1}} .$$

We want to prove

$$u \approx v_0 \quad \text{for any} \quad v \in V_A .$$

Let $u \in V_A$. Then there exist some $x_0 \in A$ and $r > 0$ such that $B_r(x_0) \subset A$ and $x - u \in \operatorname{supp} M_\Xi$ for any $x \in B_r(x_0)$. Hence there exists $x \in B_r(x_0)$ such that

$$x - u = \sum_{i=1}^{n} \lambda_i \xi_i$$

with

$$0 < \lambda_i < 1 \ \text{ all } \ i, \quad \lambda_i + \lambda_j \ne 1 \ \text{ all } \ i,j,$$
$$\text{and } \ \lambda_i \ne \lambda_j \ \text{ whenever } \ i \ne j.$$

Without loss of any generality, we may assume

$$1 > \lambda_1 > \dots > \lambda_k > 0.$$

<u>Subcase 1</u>. $\lambda_k + \lambda_{m+1} > 1$.

In this case,

$$x - u - \xi_{m+1} = \sum_{i=1}^{k} (\lambda_i + \lambda_{m+1} - 1) e_i + \sum_{\substack{k+1 \le i \le n \\ i \ne m+1}} \lambda_i \xi_i \in \operatorname{supp} M_{\Xi\backslash\xi_{m+1}} .$$

Hence $u + \xi_{m+1} \in V_{A,\Xi\setminus\xi_{m+1}}$, and therefore $u + \xi_{m+1} \approx v_0$. But $u \sim u + \xi_{m+1}$. We conclude that $u \approx v_0$.

<u>Subcase 2</u>. $\lambda_1 + \lambda_{m+1} < 1$.

In this case,

$$x - u = \sum_{i=1}^{k} (\lambda_i + \lambda_{m+1})e_i + \sum_{\substack{k+1 \le i \le n \\ i \ne m+1}} \lambda_i \xi_i \in \operatorname{supp} M_{\Xi\setminus\xi_{m+1}}.$$

It follows that $u \in V_{A,\Xi\setminus\xi_{m+1}}$. Therefore $u \approx v_0$.

<u>Subcase 3</u>. $\lambda_1 + \lambda_{m+1} > 1$ and $\lambda_k + \lambda_{m+1} < 1$.

Let j be the largest integer such that $\lambda_j + \lambda_{m+1} > 1$. Let

$$y_i := \sum_{r=1}^{i} (\lambda_r + \lambda_{m+1} - 1)e_r + \sum_{r=i+1}^{k} (\lambda_r + \lambda_{m+1})e_r + z, \quad (i=0,1,\ldots,j), \tag{12}$$

where

$$z = \sum_{\substack{k+1 \le r \le n \\ r \ne m+1}} \lambda_r \xi_r.$$

We claim that

$$y_i \in \operatorname{supp} M_\Xi, \quad i = 0,1,\ldots,j. \tag{13}$$

Indeed, it follows from (11) that

$$e_i = \xi_{m+1} - \sum_{\substack{1 \le r \le m \\ r \ne i}} e_r.$$

Substituting the above expression into (12), we obtain

$$y_i = \sum_{r=1}^{i-1} (\lambda_r - \lambda_i)e_r + \sum_{r=i+1}^{k} \big(1-(\lambda_i - \lambda_r)\big)e_r + (\lambda_i + \lambda_{m+1} - 1)\xi_{m+1} + z.$$

This proves (13). Furthermore, it is obvious that

$$y_j \in \operatorname{supp} M_{\Xi\setminus\xi_{m+1}}. \tag{14}$$

Also, we have

$$x - u = y_0$$

$$\text{and}\quad y_{i-1} = y_i + e_i \quad (i=1,\dots,j).$$

Let

$$w_i := x - y_i \quad (i=0,1,\dots,j).$$

Then $u = w_0$ and $w_i = w_{i-1} + e_i (i=1,\dots,j)$ Hence $w_i \in \mathbb{Z}^m$. By (13), $x - w_i \in \operatorname{supp} M_{\Xi}$. Therefore $w_i \in V_A$. Moreover, by (14), we have $x - w_j \in \operatorname{supp} M_{\Xi\setminus\xi_{m+1}}$ and $w_j \in V_{A,\Xi\setminus\xi_{m+1}}$. Thus $w_j \sim v_0$, and therefore $u \sim v_0$. The proof of Lemma 3 is complete.

We are now in a position to prove our theorem in Case 2. Let A be an open set contained in some component of $\mathbb{R}^m\setminus K(\Xi)$. Suppose that

$$\sum_{v\in V_A} a(v)M_{\Xi}(x-v) = 0 \quad \text{for all} \quad x \in A.$$

Pick $v_0 \in V_A$. We claim

$$a(v) = a(v_0) \quad \text{for all} \quad v \in V_A \tag{15}$$

Since $v \sim v_0$ by Lemma 3, there exist $v_1,\dots,v_k \in V_A$ such that $v = v_k$ and $v_i \sim v_{i-1}$ $(i=1,\dots,k)$. Thus proving (15) reduces to proving the following statement: If $v_1, v_2 \in V_A$ and $v_2-v_1 = \xi_i$ for some i, then $a(v_1) = a(v_2)$. To this end, we let

$$a(v) := a(v_0) \quad \text{for} \quad v \in \mathbb{R}^m\setminus V_A.$$

Then

$$\sum_{v\in V} a(v)M_{\Xi}(x-v) = 0 \quad \text{for all} \quad x \in A.$$

It follows that

$$D_{\xi_i}\Big(\sum_{v\in V} a(v)M_{\Xi}(\cdot-v)\Big) = 0 \quad \text{on} \quad A;$$

that is

$$\sum_{v\in V} \nabla_{\xi_i} a(v)M_{\Xi\setminus\xi_i}(x-v) = 0 \quad \text{for all} \quad x \in A.$$

By induction hypothesis,

$$\nabla_{\xi_i} a(v) = 0 \quad \text{for all} \quad v \in V_{A,\Xi\backslash\xi_i}. \tag{16}$$

By Lemma 1, v_1 and $v_2 \in V_A$ imply that

$$\text{supp } M_\Xi(\cdot - v_k) \supset A \quad (k=1,2).$$

It follows that

$$\forall\, x \in A, \quad M_\Xi(x-v_k) > 0 \quad (k=1,2).$$

This is to say

$$x - v_2 \in \text{supp } M_\Xi \quad \text{and} \quad x - v_2 + \xi_i \in \text{supp } M_\Xi.$$

By Lemma 2, we have

$$\forall\, x \in A, \quad x - v_2 \in \text{supp } M_{\Xi\backslash\xi_i}.$$

Therefore $v_2 \in V_{A,\Xi\backslash\xi_i}$. By (16),

$$a(v_2) - a(v_2 - \xi_i) = 0.$$

This shows that $a(v_2) = a(v_1)$, and proves our claim (15).

Now we have $a(v) = a(v_0)$ for all $v \in V$. Thus, for $x \in A$,

$$0 = \sum_{v \in V} a(v) M_\Xi(x-v) = a(v_0) \sum_{v \in V} M_\Xi(x-v) = a(v_0),$$

using the fact $\sum_{v \in V} M_\Xi(x-v) = 1$. Finally, we get the desired result:

$$a(v) = a(v_0) = 0 \quad \text{for all} \quad v \in V_A.$$

Postscript. This work was done in the summer of 1983. Since then I have become aware of the research announcement "Some Results on Box Splines" by W. Dahmen and C. A. Micchelli in which they state a result which covers the main result of this paper. However, the proof presented here seems more elementary and simple than theirs.

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20. ABSTRACT (Continue on reverse side if necessary and identify by block number)

Let $\Xi = (\xi_i)_1^n$ be a sequence of vectors in $\mathbb{R}^m$. The box spline M_Ξ is defined as the distribution given by

$$M_\Xi : \phi \mapsto \int_{[0,1]^n} \phi\Big(\sum_{i=1}^n \lambda(i)\xi_i\Big)d\lambda, \quad \phi \in C_c^\infty(\mathbb{R}^m).$$

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20. Abstract (cont.)

Suppose that Ξ contains a basis for $\mathbb{R}^m$. Then $M_\Xi \in L_\infty(\mathbb{R}^m)$. Let $V = \mathbb{Z}^m$. Consider the translates $M_v := M_\Xi(\cdot - v)$, $v \in V$. Is is known that $(M_v)_V$ is linearly dependent unless

$$(*) \qquad |\det Z| = 1 \quad \text{for all bases } Z \subset \Xi.$$

This report demonstrates that, under condition (*), $(M_v)_V$ is locally linearly independent, i.e.,

$$\{M_v;\ \operatorname{supp} M_v \cap A \neq \emptyset\}$$

is linearly independent over any open set A contained in some component of $\mathbb{R}^m \setminus K(\Xi)$, where

$$K(\Xi) := \bigcup_{\langle \Xi \setminus Z \rangle \neq \mathbb{R}^m} [\Xi \setminus Z] + \sum_{u \in \mathbb{Z}^n} \sum_{i=1}^{n} u(i)\xi_i .$$

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