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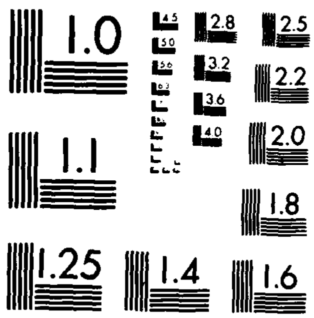
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RADC-TR-84-9, Pt VI (of six)
Final Technical Report
April 1984

AD-A141 748

ON THE SCATTERING OF ELECTROMAGNETIC WAVES BY PERFECTLY CONDUCTING BODIES MOVING IN VACUUM
Manifolds in Euclidean Spaces, Regularity Properties of Domains

University of Delaware

Allan G. Dallas

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ROME AIR DEVELOPMENT CENTER
Air Force Systems Command
Griffiss Air Force Base, NY 13441

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Various standard results concerning manifolds in euclidean spaces, coordinate systems, and functions defined on such manifolds are developed and organized. For example, conditions are identified under which the image of a manifold is again a manifold.

A development of Lebesgue measure and integration on a manifold is presented. Included is a "change-of-variables" formula for the transformation of an integral over a manifold to integration over a second manifold suitably (over

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Classes of regular domains are defined. Special attention is given to those regular domains possessing a Holder-continuous exterior unit normal field, or "Lyapunov domains." Slightly modifying the standard presentations, geometric and analytic properties of the boundary of a Lyapunov domain are derived, including the identification of certain canonical "tangent-plane" coordinate systems.

ORIENTATION

This is Part VI of a six-part report on the results of an investigation into the problem of determining the scattered field resulting from the interaction of a given electromagnetic incident wave with a perfectly conducting body executing specified motion and deformation in vacuum. Part I presents the principal results of the study of the case of a general motion, while Part II contains the specialization and completion of the general reasoning in the situation in which the scattering body is stationary. Part III is devoted to the derivation of a boundary-integral-type representation for the scattered field, in a form involving scalar and vector potentials. Parts IV, V, and VI are of the nature of appendices, containing the proofs of numerous auxiliary technical assertions utilized in the first three parts. Certain of the chapters of Part I are sufficient preparation for studying each of Parts III through VI. Specifically, the entire report is organized as follows:

The section- and equation-numbering scheme is fairly self-explanatory. For example, "[I.5.4]" designates the fourth section of Chapter 5 of Part I, while "(I.5.4.1)" refers to the equation numbered (1) in that section; when the reference is made within Part I, however, these are shortened to "[5.4]" and "(5.4.1)," respectively. Note that Parts II-VI contain no chapter-subdivisions. "[IV.14]" indicates the fourteenth section of Part IV, "(IV.14.6)" the equation numbered (6) within that section; the Roman-numeral designations are never dropped in Parts II-VI.

A more detailed outline of the contents of the entire report appears in [I.1.2]. An index of notations and the bibliography are also to be found in Part I. References to the bibliography are made by citing, for example, "Mikhlin [34]." Finally, it should be pointed out that notations connected with the more common mathematical concepts are standarized for all parts of the report in [I.1.4].

PART VI

MANIFOLDS IN EUCLIDEAN SPACES.

REGULARITY PROPERTIES OF DOMAINS

The major portion of Chapter [I.2] comprises just those definitions and bare statements of technical results concerning manifolds in Euclidean spaces, Lebesgue measure and integration on such manifolds, and the implications of various regularity hypotheses for open sets in a Euclidean space, which are needed in the subsequent study of the scattering problem. This essentially self-contained Part VI is an expanded version of that same material, providing the requisite auxiliary concepts and complete proofs. The development draws freely upon, and modifies, presentations appearing in Fleming [14, 15], Munkres [40], Günter [19], and Mikhlin [34].

We begin with two standard results.

[VI.1] INVERSE FUNCTION THEOREM. *Let* $n \in \mathbb{N}$ *with* $n \geq 2$, *and* $q \in \mathbb{N} \cup \{\infty\}$. *Let* $U \subset \mathbb{R}^n$ *be open and* $f^q \in C^q(U;\mathbb{R}^n)$ *and suppose that* $x_0 \in U$ *with* $Jf(x_0) \neq 0$. *Then there exists an open neighborhood of* x_0, $U_0 \subset U$, *such that*

(*i*) $f_0 := f | U_0$ *is an injection;*

(*ii*) $f(U_0)$ *is open in* $\mathbb{R}^n$*;*

(iii) $f_0^{-1} \in C^q(f(U_0);\mathbb{R}^n)$;

(iv) *for each* $x \in U_0$, $Jf(x) \neq 0$, *and* $\{Df(x)\}^{-1} = (Df_0^{-1})(f(x))$.

P R O O F. Cf., Fleming [15]. □.

[VI.2] I M P L I C I T F U N C T I O N T H E O R E M. *Let* $n,m \in \mathbb{N}$ *with* $m < n$, *and* $q \in \mathbb{N}\cup\{\infty\}$. *Suppose that* $U \subset \mathbb{R}^n$ *is open,* $\Phi \in C^q(U;\mathbb{R}^m)$, *and* $x_0 \in U$ *is such that* $\Phi(x_0) = 0$ *and* $D\Phi(x_0)$ *has (maximum) rank* m. *Then there exist an open neighborhood* $U_0 \subset U$ *of* x_0, *an open set* $V_0 \subset \mathbb{R}^{n-m}$, *an increasing* $(n-m)$*-tuple* $\lambda = (i_1,\ldots,i_{n-m})$ *of integers in* $\{1,\ldots,n\}$, *and a unique function* $\phi \in C^q(V_0;\mathbb{R}^m)$ *such that, with* $(j_1,\ldots,j_m)$ *denoting the increasing* m*-tuple of integers in* $\{1,\ldots,n\}$ *complementary to* λ,

(i) $\det\,(\Phi^i_{,j_k}(x))_{1\le i,k\le m} \neq 0$ *for each* $x \in U_0$;

(ii) $x_0^\lambda = (x_0^{i_1},\ldots,x_0^{i_{n-m}}) \in V_0$;

(iii) $\{x \in U_0 \mid \Phi(x) = 0\} =$
$\{x \in \mathbb{R}^n \mid x^\lambda = (x^{i_1},\ldots,x^{i_{n-m}}) \in V_0,\ x^{j_k} = \phi^k(x^\lambda),$
$k = 1,\ldots,m\}$.

We shall give an outline of the proof, in order to point out the construction of an auxiliary function which turns out to be of later use.

P R O O F (S K E T C H). Since $D\Phi(x_0)$ has rank m, there is an increasing m-tuple $(j_1,\ldots,j_m)$ of integers in $\{1,\ldots,n\}$ such that $\det(\phi^i_{,j_k}(x_0))_{1\le i,k\le m} \neq 0$. Let $\lambda = (i_1,\ldots,i_{n-m})$ denote the increasing $(n-m)$-tuple of integers in $\{1,\ldots,n\}$ which is complementary to $(j_1,\ldots,j_m)$. Define a function $f\colon U \to \mathbb{R}^n$ according to

$$f^k(x) := x^{i_k}, \quad k = 1,\ldots,n-m,$$

$$f^{n-m+k}(x) := \phi^k(x), \quad k = 1,\ldots,m,$$

for each $x \in U$. Clearly, $f \in C^q(U;\mathbb{R}^n)$. A short computation and use of the properties of determinants produce the equality $|Jf(x)| = |\det(\phi^i_{,j_k}(x))_{1\le i,k\le m}|$, for each $x \in U$. In particular, we find then that $Jf(x_0) \neq 0$. According to the Inverse Function Theorem [VI.1], there exists an open neighborhood $U_0 \subset U$ of x_0 such that $Jf(x) \neq 0$ for each $x \in U_0$, $f_0 := f|\ U_0\colon\ U_0 \to f(U_0)$ is a homeomorphism of U_0 onto the open set $f(U_0) \subset \mathbb{R}^n$, and $f_0^{-1} \in C^q(f(U_0);\mathbb{R}^n)$.

Noting that $f(x_0) = (x_0^{i_1},\ldots,x_0^{i_{n-m}},0,\ldots,0) \in f(U_0)$, we see that the set

$$V_0 := \{\hat{x} \in \mathbb{R}^{n-m} |\ (\hat{x},0) := (\hat{x}^1,\ldots,\hat{x}^{n-m},0,\ldots,0) \in f(U_0)\}$$

contains x_0^λ and is open in $\mathbb{R}^{n-m}$, since $f(U_0)$ is open in $\mathbb{R}^n$. Define $\phi\colon V_0 \to \mathbb{R}^m$ by

$$\phi^k(\hat{x}) := (f_0^{-1})^{j_k}(\hat{x}^1,\ldots,\hat{x}^{n-m},0,\ldots,0), \quad \text{for each} \quad \hat{x} \in V_0,$$
$$k \in \{1,\ldots,m\}.$$

Then it is routine to check that $\phi \in C^q(V_0;\mathbb{R}^m)$ and that (iii) holds. To show that ϕ is unique, let $\tilde{\phi}$: $V_0 \to \mathbb{R}^m$ be any function satisfying (iii): choosing $\hat{x} \in V_0$, define x and $\tilde{x} \in \mathbb{R}^n$ by $x^{i_k} = \tilde{x}^{i_k} := \hat{x}^k$, for $k = 1,\ldots,n-m$, and $x^{j_k} := \phi^k(\hat{x})$, $\tilde{x}^{j_k} := \tilde{\phi}^k(\hat{x})$, for $k = 1,\ldots,m$. Then $x^\lambda = \tilde{x}^\lambda = \hat{x}$, and (iii) gives x, $\tilde{x} \in U_0$, with $\Phi(x) = \Phi(\tilde{x}) = 0$. Obviously, it follows that $f_0(x) = f_0(\tilde{x})$, whence $x = \tilde{x}$, since f_0 is injective. Consequently, $\phi^k(\hat{x}) = \tilde{\phi}^k(\hat{x})$ for $k = 1,\ldots,m$. Thus, $\phi = \tilde{\phi}$. □.

We proceed to the definition of "manifold" in a Euclidean space. We shall not need the idea of a "manifold with boundary" (cf. Munkres [40]), and so we can avoid introducing this more inclusive concept.

[VI.3] D E F I N I T I O N. Let $n \in \mathbb{N}$ and $q \in \mathbb{N}\cup\{\infty\}$. A non-void open set in $\mathbb{R}^n$ shall be referred to as an $(n,n;q)$-*manifold*, whenever it is convenient to do so. Now suppose that $r \in \mathbb{N}$ and $r < n$ (so $n \geq 2$): a non-void set $M \subset \mathbb{R}^n$ is a *manifold of dimension* r *and class* C^q, or $(r,n;q)$-*manifold*, iff whenever $x \in M$, there exist an open neighborhood $U_x \subset \mathbb{R}^n$ of x and a function $\Phi_x \in C^q(U_x;\mathbb{R}^{n-r})$ such that rank $D\Phi_x(y) = n-r$ for each $y \in U_x$, and

$$M \cap U_x = \{y \in U_x \mid \Phi_x(y) = 0\}. \qquad \blacksquare.$$

[VI.4] R E M A R K S. (a) It is clear that if M is an $(r,n;q)$-manifold $(r \leq n)$ and $\tilde{M}$ is a relatively open subset of M, then

$\tilde{M}$ is also an $(r,n;q)$-manifold.

(b) Just as obvious is the fact that a non-void set $M \subset \mathbb{R}^n$ is an $(r,n;q)$-manifold iff each $x \in M$ possesses a relatively open neighborhood $M_x \subset M$ such that M_x is an $(r,n;q)$-manifold.

(c) Let M be an $(r,n;q)$-manifold: then $M\times\mathbb{R}$ is an $(r+1,n+1;q)$-manifold. To see this, suppose first that $r < n$. Let $(x,t) \in M\times\mathbb{R}$, then $U_x \subset \mathbb{R}^n$, $\Phi_x \in C^q(U_x;\mathbb{R}^{n-r})$ be as in [VI.3]. $U_x\times\mathbb{R} \subset \mathbb{R}^{n+1}$ is an open neighborhood of (x,t). Define $\Phi_{(x,t)}$: $U_x\times\mathbb{R} \to \mathbb{R}^{n-r} = \mathbb{R}^{(n+1)-(r+1)}$ by $\Phi_{(x,t)}(y,s) := \Phi_x(y)$, $(y,s) \in U_x\times\mathbb{R}$. For each $(y,s) \in U_x\times\mathbb{R}$, the matrix of $D\Phi_{(x,t)}(y,s)$: $\mathbb{R}^{n+1} \to \mathbb{R}^{n-r}$ relative to the standard bases is

$$\begin{pmatrix} & 0 \\ (\Phi^i_{x,j}(y))_{\substack{1\le i\le n-r \\ 1\le j\le n}} & \vdots \\ & 0 \end{pmatrix}. \tag{1}$$

Clearly, rank $D\Phi_{(x,t)}(y,s) = n-r = (n+1)-(r+1)$ for each $(y,s) \in U_x\times\mathbb{R}$, while $\{M\times\mathbb{R}\}\cap\{U_x\times\mathbb{R}\} = \{(y,s) \in U_x\times\mathbb{R} \mid \Phi_{(x,t)}(y,s) = 0\}$. The case $r = n$ is even more trivial.

For any $(r,n;q)$-manifold M and any $x \in M$, we shall define associated "tangent" and "normal" spaces to M at x, as follows:

[VI.5] D E F I N I T I O N S. Let M be an $(r,n;q)$-manifold, and $x \in M$. $\beta \in \mathbb{R}^n$ is a *tangent vector to* M *at* x iff there is a $\delta > 0$ and a function $\psi \in C^1((-\delta,\delta);\mathbb{R}^n)$ such that $\psi(s) \in M$

for $|s| < \delta$, $\psi(0) = x$, and $\psi'(0) = \beta$. The set of all tangent vectors to M at x is called the *tangent space to* M *at* x, and denoted by $T_M(x)$. The orthogonal complement of $T_M(x)$ in $\mathbb{R}^n$ shall be referred to as the *normal space to* M *at* x, and denoted by $N_M(x)$. ■.

It is easy to show that $T_M(x) = \mathbb{R}^n$ and $N_M(x) = \{0\}$ for any $(n,n;q)$-manifold (non-void open set $\subset \mathbb{R}^n$) M and any $x \in M$. In the general case, it is clear that $0 \in T_M(x)$, and it can be proven directly that $T_M(x)$ is a subspace of $\mathbb{R}^n$. The implicit function theorem allows us to show that $T_M(x)$ is non-trivial, by showing that it has dimension r; as noted, we shall consider this fact proven for the case $r = n$.

[VI.6] P R O P O S I T I O N. *Let* M *be an* $(r,n;q)$*-manifold, with* $r < n$; *let* $x \in M$. *Then* $T_M(x)$ *is an* r*-dimensional subspace of* $\mathbb{R}^n$. *In fact, if* $U_x \subset \mathbb{R}^n$ *and* $\Phi_x \in C^q(U_x;\mathbb{R}^{n-r})$ *are as in* [VI.3], *then*

$$T_M(x) = \ker D\Phi_x(x).$$

P R O O F. Noting that $\dim \ker D\Phi_x(x) = r$, since $D\Phi_x(x)$: $\mathbb{R}^n \to \mathbb{R}^{n-r}$ and $\operatorname{rank} D\Phi_x(x) = n-r$, the first statement will follow once the second has been proven.

Suppose then that $\beta \in T_M(x)$; $\beta = \psi'(0)$ for some $\psi \in C^1((-\delta,\delta);\mathbb{R}^n)$ as in [VI.5]. Since ψ is continuous, and U_x is a neighborhood of $x = \psi(0)$, there is some $\delta' \in (0,\delta]$ for which

$\psi(s) \in U_x \cap M$ whenever $|s| < \delta'$, so also $\Phi_x(\psi(s)) = 0$ for $|s| < \delta'$. Differentiating, by the composite function theorem, and setting $s = 0$ gives $D\Phi_x(\psi(0))\psi'(0) = 0$, or $D\Phi_x(x)\beta = 0$. Thus, $\beta \in \ker D\Phi_x(x)$.

Now, let $\beta \in \ker D\Phi_x(x)$: we show that there exists a $\delta > 0$ and a function ψ as in [VI.5]. For this, observe that, since $\Phi_x \in C^q(U_x;\mathbb{R}^{n-r})$, with rank $D\Phi_x(x) = n-r$, $x \in U_x$, and $\Phi_x(x) = 0$, we can appeal to the construction carried out in the proof of the implicit function theorem: there exist an increasing r-tuple $\lambda = (i_1,\ldots,i_r)$ of integers in $\{1,\ldots,n\}$ and an open neighborhood $U_0 \subset U_x$ of x such that the function f_0: $U_0 \to \mathbb{R}^n$ given by

$$f_0^k(y) := y^{i_k}, \qquad k = 1,\ldots,r,$$

$$f_0^{r+k}(y) := \Phi_x^k(y), \qquad k = 1,\ldots,n-r,$$

for each $y \in U_0$, is in $C^q(U_0;\mathbb{R}^n)$, is a homeomorphism of U_0 onto the open set $f_0(U_0)$, and for which $f_0^{-1} \in C^q(f_0(U_0);\mathbb{R}^n)$. As in the proof of [VI.2], the set

$$V_0 := \{\hat{y} \in \mathbb{R}^r \mid \ (\hat{y},0) := (\hat{y}^1,\ldots,\hat{y}^r,0,\ldots,0) \in f_0(U_0)\}$$

is an open neighborhood of $x^\lambda = (x^{i_1},\ldots,x^{i_r})$ in $\mathbb{R}^r$. Hence, there is a $\delta > 0$ such that $x^\lambda + s\beta^\lambda \in V_0$ whenever $|s| < \delta$, so it is permissible to define ψ: $(-\delta,\delta) \to \mathbb{R}^n$ according to

$$\psi(s) := f_0^{-1}(x^\lambda + s\beta^\lambda, 0), \qquad \text{for each} \quad s \in (-\delta,\delta).$$

We claim that this ψ fulfills the requirements of [VI.5]. Since $f_0^{-1} \in C^q(f_0(U_0);\mathbb{R}^n)$, certainly $\psi \in C^1((-\delta,\delta);\mathbb{R}^n)$. Whenever $|s| < \delta$, $f_0^{-1}(x^\lambda+s\beta^\lambda,0) \in U_0 \subset U_x$ and

$$\begin{aligned}\Phi_x^k(f_0^{-1}(x^\lambda+s\beta^\lambda,0)) &= f_0^{r+k}(f_0^{-1}(x^\lambda+s\beta^\lambda,0)) \\ &= (x^\lambda+s\beta^\lambda,0)^{r+k} \\ &= 0, \quad \text{for} \quad k = 1,\ldots,n-r,\end{aligned}$$

showing that $\psi(s) \in U_x$ and $\Phi_x(\psi(s)) = 0$, so $\psi(s) \in M$. Also, $\psi(0) = f_0^{-1}(x^\lambda,0) = f_0^{-1}(f_0(x))$. Finally, we must show that $\psi'(0) = \beta$: since $\psi'(s) = \{(Df_0^{-1})(x^\lambda+s\beta^\lambda,0)\}(\beta^\lambda,0)$ if $|s| < \delta$, we have $\psi'(0) = \{(Df_0^{-1})(x^\lambda,0)\}(\beta^\lambda,0) = \{Df_0^{-1}(f_0(x))\}(\beta^\lambda,0) = \{Df_0(x)\}^{-1}(\beta^\lambda,0)$, the latter equality following from [VI.1.iv], in view of the manner in which f_0 was constructed. Now, $\sum_{j=1}^n f_{0,j}^k(x)\cdot\beta^j = \sum_{j=1}^n \delta_{ji_k}\cdot\beta^j = \beta^{i_k}$ for $k = 1,\ldots,r$, and $\sum_{j=1}^n f_{0,j}^{r+\ell}(x)\beta^j = \sum_{j=1}^n \Phi_{x,j}^\ell(x)\beta^j = 0$ for $\ell = 1,\ldots,n-r$, since $D\Phi_x(x)\beta = 0$. These facts show that $Df_0(x)\beta = (\beta^\lambda,0)$, whence $\beta = \{Df_0(x)\}^{-1}(\beta^\lambda,0)$. Thus, $\psi'(0) = \beta$. We conclude that $\beta \in T_M(x)$. □.

[VI.7] C O R O L L A R Y. *Let* M *be an* $(r,n;q)$*-manifold with* $r < n$. *Let* $x \in M$. *Let* $U_x \subset \mathbb{R}^n$ *and* $\Phi_x \in C^q(U_x;\mathbb{R}^{n-r})$ *be as in* [VI.3]. *Then the set* $\{\text{grad}\ \Phi_x^k(x)\}_{k=1}^{n-r}$ *provides a basis for* $N_M(x)$.

P R O O F. $N_M(x)$ is the orthogonal complement of $T_M(x)$. Clearly, then, $N_M(x)$ has dimension $n-r$. Since $D\phi_x(x)$ has rank $n-r$, the set $\{\text{grad } \phi_x^k(x)\}_{k=1}^{n-r}$ is linearly independent. Whenever $\beta \in T_M(x) = \ker D\phi_x(x)$, $D\phi_x(x)\beta = 0$, implying that $\text{grad } \phi_x^k(x) \bullet \beta = 0$, for $k = 1,\ldots,n-r$. Thus, $\{\text{grad } \phi_x^k(x)\}_{k=1}^{n-r} \subset N_M(x)$. The statement of the corollary obviously follows from these facts. □.

[VI.8] R E M A R K. Consider, as in [VI.4.c], the $(r+1,n+1;q)$-manifold $M\times\mathbb{R}$, where M is an $(r,n;q)$-manifold. Let $x \in M$, and U_x, ϕ_x be as in [VI.3]. Choosing $t \in \mathbb{R}$ and constructing $U_{(x,t)} := U_x\times\mathbb{R}$ and $\phi_{(x,t)} \in C^q(U_{(x,t)};\mathbb{R}^{n-r})$ as in [VI.4.c], it is clear from the form of the matrix (VI.4.1) that $\ker D\phi_{(x,t)}(x,t) = T_M(x)\times\mathbb{R}$, since $\ker D\phi_x(x) = T_M(x)$. Consequently, $T_{M\times\mathbb{R}}(x,t) = T_M(x)\times\mathbb{R}$, for each $(x,t) \in M\times\mathbb{R}$.

The next objective is the study of functions on a manifold into a Euclidean space; for this, the idea of a *coordinate system* in a manifold is an indispensable tool. Such coordinate systems also provide the means for defining Lebesgue measure on a manifold. Before discussing these topics, we consider *regular transformations* (cf., Fleming [15]).

[VI.9] D E F I N I T I O N S. Let $r,n \in \mathbb{N}$, with $r \leq n$, and $q \in \mathbb{N}\cup\{\infty\}$. Let Ω be an open set in $\mathbb{R}^r$, M an $(r,n;q)$-manifold, and $g\colon\ \Omega \to \mathbb{R}^n$.

(i) If $g \in C^1(\Omega;\mathbb{R}^n)$, we define $Jg\colon\ \Omega \to [0,\infty)$ by

$$\mathcal{J}g(x) := \left|\bigwedge_{i=1}^{r} g_{,i}(x)\right| = \left|\bigwedge_{i=1}^{r} Dg(x)e_i^{(r)}\right| \tag{1}$$

$$\text{for each} \quad x \in \Omega.$$

(ii) Suppose $g(\Omega) \subset M$. Then g is q-*regular* iff

(1) $g \in C^q(\Omega;\mathbb{R}^n)$,

(2) g is injective,

(3) $\operatorname{rank} Dg(x) = r$, for each $x \in \Omega$. ∎.

[VI.10] R E M A R K S. (Notation as in [VI.9])

(a) Suppose $g \in C^1(\Omega;\mathbb{R}^n)$. The (r,n)-vector $g_{,1}(x)\wedge\ldots\wedge g_{,r}(x)$ is non-zero, *i.e.*, $\mathcal{J}g(x) > 0$, for some $x \in \Omega$, iff the set $\{g_{,k}(x)\}_{k=1}^{r} \subset \mathbb{R}^n$ is linearly independent, which, in turn, is true iff $\operatorname{rank} Dg(x) = r$ (since $\{g_{,k}(x)\}_{k=1}^{r}$ is just the collection of column vectors of the matrix of $Dg(x)\colon \mathbb{R}^r \to \mathbb{R}^n$ with respect to the standard bases). Thus, condition [VI.9.ii.3] holds iff $\mathcal{J}g(x) > 0$ for each $x \in \Omega$.

(b) Consider the case $r = n$, and suppose $g\colon \Omega \to \mathbb{R}^n$ is q-regular. Then M is an open set in $\mathbb{R}^n$, which we can take to be just $\mathbb{R}^n$ itself. Now, $\operatorname{rank} Dg(x) = n$, so $\mathcal{J}g(x) \neq 0$, for each $x \in \Omega$. Since $g \in C^q(\Omega;\mathbb{R}^n)$ is injective, it follows from the inverse function theorem that $g(\Omega) \subset \mathbb{R}^n$ is open, g is a homeomorphism of Ω onto $g(\Omega)$, and $g^{-1} \in C^q(g(\Omega);\mathbb{R}^n)$. g is sometimes referred to as a *flat transformation* in this case.

(c) Again supposing $r = n$, so $\Omega \subset \mathbb{R}^n$, let $g \in C^1(\Omega;\mathbb{R}^n)$.

Now it can be shown that $Jg(x) = |Jg(x)|$, for each $x \in \Omega$, cf., the proof of [VI.13], *infra*, which appears in Fleming [14]. In any case, *i.e.*, whenever $r \le n$ and $g \in C^1(\Omega;\mathbb{R}^n)$, it can be shown that

$$Jg(x) = \frac{|Dg(x)T_1\wedge\ldots\wedge Dg(x)T_r|}{|T_1\wedge\ldots\wedge T_r|}, \qquad x \in \Omega,$$

for any choice of basis $\{T_i\}_{i=1}^r$ for $\mathbb{R}^r$; if $T_i = e_i^{(r)}$, $i = 1,\ldots,r$, this expression reduces to (VI.9.1), since $|e_1^{(r)}\wedge\ldots\wedge e_r^{(r)}| = 1$. Once again, we refer to Fleming [14] for the proofs of these statements; cf., also, [VI.24.c], *infra.*

We proceed to provide several basic facts concerning regular transformations.

[VI.11] P R O P O S I T I O N. *Let* Ω *be open in* $\mathbb{R}^r$, M *an* $(r,n;q)$-*manifold, and* $g\colon\ \Omega \to \mathbb{R}^n$ *be* p-*regular* $(p \in \mathbb{N}\cup\{\infty\})$, *with* $g(\Omega) \subset M$. *Let* $x \in \Omega$. *Then*

(*i*) $Dg(x)\colon \mathbb{R}^r \to \mathbb{R}^n$ *is an injection taking* $\mathbb{R}^r$ *onto* $T_M(g(x))$;

(*ii*) $\{g,_i(x)\}_{i=1}^r$ *is a basis for* $T_M(g(x))$.

P R O O F. $Dg(x)$ is linear, defined on $\mathbb{R}^r$, and has rank r (in particular, it is injective), while $T_M(g(x))$ is an r-dimensional subspace of $\mathbb{R}^n$. Therefore, we need only demonstrate that $Dg(x)\mathbb{R}^r \subset T_M(g(x))$ in order to prove (i). If $r = n$, then

$T_M(g(x)) = \mathbb{R}^n$, and the result is obvious, so we may suppose that $r < n$. Let $U_{g(x)} \subset \mathbb{R}^n$ and $\Phi_{g(x)} \in C^q(U_{g(x)};\mathbb{R}^{n-r})$ be as in [VI.3]. Let $\Omega_x \subset \Omega$ be an open neighborhood of x such that $g(\Omega_x) \subset U_{g(x)}$. Since $g(\Omega) \subset M$, we have $g(\Omega_x) \subset M \cap U_{g(x)}$, whence $\Phi_{g(x)}(g(y)) = 0$ for each $y \in \Omega_x$. The composite function theorem then shows that $D\Phi_{g(x)}(g(x)) \circ Dg(x)\colon \mathbb{R}^r \to \mathbb{R}^{n-r}$ is the zero operator, so $D\Phi_{g(x)}(g(x))(Dg(x)\alpha) = 0$ for each $\alpha \in \mathbb{R}^r$. By [VI.6], $Dg(x)\alpha \in \ker D\Phi_{g(x)}(g(x)) = T_M(g(x))$ for each $\alpha \in \mathbb{R}^r$, *i.e.*, $Dg(x)\mathbb{R}^r \subset T_M(g(x))$, as required.

To prove (ii), simply note that, by (i), $Dg(x)$ takes any basis for $\mathbb{R}^r$ to a basis for $T_M(g(x))$, and that $g,_i(x) = Dg(x)e_i^{(r)}$, for $i = 1,\dots,r$. □.

[VI.12] N O T A T I O N. In the setting and notation of Proposition [VI.11], we shall denote the inverse of $Dg(x)\colon \mathbb{R}^r \to T_M(g(x))$ by $\{Dg(x)\}^{-1}\colon T_M(g(x)) \to \mathbb{R}^r$.

[VI.13] P R O P O S I T I O N. *Let* $\Omega, \tilde{\Omega} \subset \mathbb{R}^r$ *be open sets,* $f\colon \Omega \to \tilde{\Omega}$ *be in* $C^p(\Omega;\mathbb{R}^r)$, M *an* $(r,n;q)$*-manifold, and* $\tilde{g}\colon \tilde{\Omega} \to M$ *be* p*-regular. Set* $g := \tilde{g} \circ f\colon \Omega \to M$. *Then*

(*i*) $Jf = |Jf|$;

(*ii*) *if* f *is* p*-regular, then* g *is* p*-regular, and the equality* $Jg(x) = J\tilde{g}(f(x)) \cdot |Jf(x)|$ *holds for each* $x \in \Omega$.

P R O O F. See Fleming [14]; although his proof is for the case $p = 1$, its extension to the case of an arbitrary positive integer p or $p = \infty$ is trivial, under the hypotheses given above. □.

We next present an important method for constructing manifolds, by formalizing an example appearing in Fleming [14].

[VI.14] L E M M A. *Let* $r,n \in \mathbb{N}$ *with* $r < n$, *and* $q \in \mathbb{N}\cup\{\infty\}$. *Let* $\lambda = (i_1,\ldots,i_r)$ *be an increasing* r-*tuple of integers chosen from* $\{1,\ldots,n\}$, *and* $(j_1,\ldots,j_{n-r})$ *the increasing* $(n-r)$-*tuple complementary to* λ *in* $\{1,\ldots,n\}$. *Let* $\Omega \subset \mathbb{R}^r$ *be an open set, and* $\phi \in C^q(\Omega;\mathbb{R}^{n-r})$. *Define* $G\colon \Omega \to \mathbb{R}^n$ *by setting*

$$\left.\begin{array}{ll} G^{i_k}(\hat{x}) := \hat{x}^k, & k = 1,\ldots,r, \\ G^{j_k}(\hat{x}) := \phi^k(\hat{x}), & k = 1,\ldots,n-r \end{array}\right\} \text{ for each } \hat{x} \in \Omega.$$

Then

(i) $G(\Omega)$ *is an* $(r,n;q)$-*manifold;*

(ii) G *is* q-*regular;*

(iii) with $\Xi^\lambda\colon \mathbb{R}^n \to \mathbb{R}^r$ *denoting the projection map* $x \mapsto x^\lambda := (x^{i_1},\ldots,x^{i_r})$, $G^{-1}\colon G(\Omega) \to \Omega$ *is just* $\Xi^\lambda|\,G(\Omega)$, *so* G *is a homeomorphism of* Ω *onto* $G(\Omega)$.

P R O O F. (i) We show that $G(\Omega) \subset \mathbb{R}^n$ fulfills all requirements of [VI.3]. Set $U := \{x \in \mathbb{R}^n \mid x^\lambda := (x^{i_1},\ldots,x^{i_r}) \in \Omega\}$, *i.e.*,

$U = \Xi^{\lambda^{-1}}(\Omega)$. Since Ω is open in $\mathbb{R}^r$ and Ξ^λ is continuous, $U \subset \mathbb{R}^n$ is open. Define Φ: $U \to \mathbb{R}^{n-r}$ by setting

$$\Phi^k(x) := x^{j_k} - \phi^k(x^\lambda), \quad \text{for each} \quad x \in U, \quad k = 1,\ldots,n-r.$$

Obviously, $\Phi \in C^q(U;\mathbb{R}^{n-r})$. We have

$$\left.\begin{array}{ll} \Phi^\ell_{,i_k}(x) = -\phi^\ell_{,k}(x^\lambda), & k = 1,\ldots,r \\ \Phi^\ell_{,j_k}(x) = \delta^\ell_k, & k = 1,\ldots,n-r \end{array}\right\} \quad x \in U, \quad \ell = 1,\ldots,n-r.$$

It follows that, for $x \in U$, the determinant of the $(n-r)\times(n-r)$ submatrix of $(\Phi^i_{,j}(x))_{1\leq i\leq n-r,1\leq j\leq n}$ consisting of the columns indexed $j_1,\ldots,j_{n-r}$ is equal to one. Thus, rank $D\Phi(x) = n-r$ for each $x \in U$. Observe next that $G(\Omega) = \{x \in U | \ \Phi(x) = 0\}$. For, suppose first that $x \in G(\Omega)$, *i.e.*, $x = G(\hat{x})$, $\hat{x} \in \Omega$. Then $x^\lambda = (G^{i_1}(\hat{x}),\ldots,G^{i_r}(\hat{x})) = \hat{x} \in \Omega$, so $x \in U$, while $\Phi^\ell(x) = x^{j_\ell} - \phi^\ell(x^\lambda) = G^{j_\ell}(\hat{x}) - \phi^\ell(x^\lambda) = \phi^\ell(\hat{x}) - \phi^\ell(\hat{x}) = 0$, for $\ell = 1,\ldots,n-r$. Thus, $x \in U$ and $\Phi(x) = 0$. On the other hand, if we begin with $x \in U$ for which $\Phi(x) = 0$, then $x^\lambda \in \Omega$ and $x^{j_k} = \phi^k(x^\lambda)$, for $\ell = 1,\ldots,n-r$. Thus, $G(x^\lambda)$ is defined, and $G^{i_k}(x^\lambda) = x^{i_k}$ for $k = 1,\ldots,r$, $G^{j_k}(x^\lambda) = \phi^k(x^\lambda) = x^{j_k}$ for $k = 1,\ldots,n-r$, so $G(x^\lambda) = x$, and $x \in G(\Omega)$.

Now, suppose $x \in G(\Omega)$. Then U is an open neighborhood of x, $\Phi \in C^q(U;\mathbb{R}^{n-r})$, $D\Phi(y)$ has rank $n-r$ for each $y \in U$, and $G(\Omega)\cap U = G(\Omega) = \{y \in U | \ \Phi(y) = 0\}$. We conclude that $G(\Omega)$ is an $(r,n;q)$-manifold.

(ii) Referring to [VI.9.ii], we have just seen that G takes Ω onto an $(r,n;q)$-manifold. It is a simple matter to check that $G \in C^q(\Omega;\mathbb{R}^n)$ and that G is injective. Further, for $\hat{x} \in \Omega$ and $\ell = 1,\ldots,r$, we find

$$G,_\ell(\hat{x}) = \sum_{k=1}^{r} G,_\ell^{i_k}(\hat{x})e_{i_k}^{(n)} + \sum_{m=1}^{n-r} G,_\ell^{j_m}(\hat{x})e_{j_m}^{(n)}$$

$$= e_{i_\ell}^{(n)} + \sum_{m=1}^{n-r} \phi^m,_\ell(\hat{x})e_{j_m}^{(n)},$$

so that the coefficient of $e_{i_1}^{(n)}\wedge\ldots\wedge e_{i_r}^{(n)}$ in the expansion of the product $G,_1(\hat{x})\wedge\ldots\wedge G,_r(\hat{x})$, *i.e.*, the λ^{th} component of the latter, is just 1. This implies that $JG(\hat{x}) := |G,_1(\hat{x})\wedge\ldots\wedge G,_r(\hat{x})| \neq 0$, so, by [VI.10.a], rank $DG(\hat{x}) = r$ for each $\hat{x} \in \Omega$. G is q-regular.

(iii) If $\hat{x} \in \Omega$, $(\Xi^\lambda|\ G(\Omega))(G(\hat{x})) = (G^{i_1}(\hat{x}),\ldots,G^{i_r}(\hat{x})) = \hat{x}$. If $x \in G(\Omega)$, it has been shown that $x^\lambda \in \Omega$ and $\Phi(x) = 0$, whence $G(x^\lambda) = x$. Therefore, $G((\Xi^\lambda|\ G(\Omega))(x)) = G(x^\lambda) = x$. This proves that $G^{-1} = \Xi^\lambda|\ G(\Omega)$, so G^{-1}: $G(\Omega) \to \Omega$ is continuous. $\square$.

Regular transformations generated as in [VI.14] are homeomorphisms (into). In fact, any regular transformation possesses this property (we already know this to be true in the case $r = n$; cf., [VI.10.b]):

[VI.15] P R O P O S I T I O N. *Let* Ω *be open in* $\mathbb{R}^r$, M *an*

$(r,n;q)$*-manifold with* $r < n$, *and* $g\colon\ \Omega \to M$ *be* p*-regular. Then*

(i) $g\colon\ \Omega \to M$ *is an open mapping;*

(ii) $g\colon\ \Omega \to g(\Omega)$ *is a homeomorphism.*

P R O O F. (i) Let us show first that $g(\Omega)$ is open in M. For this, select $\hat{x} \in \Omega$: we prove that $\hat{x}$ possesses an open neighborhood $\Omega_{\hat{x}} \subset \Omega$ such that $g(\Omega_{\hat{x}})$ is open in M, whence the fact that $g(\Omega)$ is open in M shall follow immediately. Let $U_{g(\hat{x})} \subset \mathbb{R}^n$, $\Phi_{g(\hat{x})} \in C^q(U_{g(\hat{x})};\mathbb{R}^{n-r})$ be as in [VI.3]. By the implicit function theorem, there exist an open neighborhood $U_0 \subset U_{g(\hat{x})}$ of $g(\hat{x})$, an increasing r-tuple $\lambda = (i_1,\dots,i_r)$ of integers in $\{1,\dots,n\}$, an open neighborhood $V_0 \subset \mathbb{R}^r$ of $g(\hat{x})^\lambda$, and a function $\phi \in C^q(V_0;\mathbb{R}^{n-r})$ such that

$$\{x \in U_0 |\ \Phi_{g(\hat{x})}(x) = 0\} = \{x \in \mathbb{R}^n |\ x^\lambda \in V_0,\ x^{j_k} = \phi^k(x^\lambda),\ k = 1,\dots,n-r\}, \tag{1}$$

where $(j_1,\dots,j_{n-r})$ is the increasing $(n-r)$-tuple complementary to λ in $\{1,\dots,n\}$. Note that, since $M \cap U_{g(\hat{x})} = \{x \in U_{g(\hat{x})} |\ \Phi_{g(\hat{x})}(x) = 0\}$,

$$M \cap U_0 = \{x \in U_0 |\ \Phi_{g(\hat{x})}(x) = 0\}. \tag{2}$$

Define, as in [VI.14], $G\colon\ V_0 \to \mathbb{R}^n$ according to

$$\left.\begin{aligned} G^{i_k}(\hat{y}) &:= \hat{y}^k, \quad k = 1,\dots,r, \\ G^{j_k}(\hat{y}) &:= \phi^k(\hat{y}), \quad k = 1,\dots,n-r, \end{aligned}\right\} \text{ for each } \hat{y} \in V_0. \tag{3}$$

It is easy to verify, using (1) and (2), that $G(V_0) = M \cap U_0$. Clearly, with the necessary changes in notation, the hypotheses of Lemma [VI.14] are fulfilled in the present setting, so we can assert that G is a q-regular homeomorphism of V_0 onto $M \cap U_0$, with inverse $G^{-1} = \Xi^\lambda | (M \cap U_0): M \cap U_0 \to V_0$. Now, $M \cap U_0$ is a relatively open neighborhood of $g(\hat{x})$ in M, so there is an open neighborhood $\Omega_{\hat{x}} \subset \Omega$ of x for which $g(\Omega_{\hat{x}}) \subset M \cap U_0$. We can write $g(\Omega_{\hat{x}}) = G(G^{-1}(g(\Omega_{\hat{x}}))) = G(\Xi^\lambda(g(\Omega_{\hat{x}})))$, so if we prove that $\Xi^\lambda(g(\Omega_{\hat{x}})) \subset V_0$ is open (in $\mathbb{R}^r$ or V_0), we shall have $g(\Omega_{\hat{x}})$ open in $M \cap U_0$ (since G is a homeomorphism), hence open in M (since $U_0 \subset \mathbb{R}^n$ is open), which is the desired result. To show then that $\Xi^\lambda(g(\Omega_{\hat{x}}))$ is open in $\mathbb{R}^r$, first define $f: \Omega_{\hat{x}} \to \mathbb{R}^r$ by $f := \Xi^\lambda \circ (g | \Omega_{\hat{x}})$. Since $g(\Omega_{\hat{x}}) \subset M \cap U_0$ and $\Xi^\lambda(M \cap U_0) \subset V_0$, $f(\Omega_{\hat{x}}) \subset V_0$, and it is easy to see that $g| \Omega_{\hat{x}} = G \circ f$, in view of the fact that $G^{-1} = \Xi^\lambda | (M \cap U_0)$. The injectiveness of g and $\Xi^\lambda | (M \cap U_0)$ imply that f is injective, while it is clear that $f \in C^p(\Omega_{\hat{x}}; \mathbb{R}^r)$. Whenever $\hat{y} \in \Omega_{\hat{x}}$, we find $Dg(\hat{y}) = DG(f(\hat{y})) \circ Df(\hat{y})$; since $\operatorname{rank} Dg(\hat{y}) = r$, we infer that $\operatorname{rank} Df(\hat{y}) = r$. Thus, $f: \Omega_{\hat{x}} \to \mathbb{R}^r$ is p-regular, so (cf., [VI.10.b]) $\Xi^\lambda(g(\Omega_{\hat{x}})) = f(\Omega_{\hat{x}})$ is open in $\mathbb{R}^r$. As noted, the proof that $g(\Omega)$ is open in M is complete.

Now, to prove (i), let $\tilde{\Omega} \subset \Omega$ be open. Obviously, $g| \tilde{\Omega}$ is p-regular, so the reasoning just concluded, with $\tilde{\Omega}$ replacing Ω and $g| \tilde{\Omega}$ in place of g, shows that $g(\tilde{\Omega})$ is open in M. Thus, $g: \Omega \to M$ is open.

(ii) We need only verify $g^{-1}: g(\Omega) \to \Omega$ is continuous, *i.e.*, that $g: \Omega \to g(\Omega)$ is open. But the latter fact follows directly from (i). $\square$.

We should like to identify conditions sufficient to ensure that the image $g(\Omega) \subset \mathbb{R}^n$ of an open set $\Omega \subset \mathbb{R}^r$ under a function $g: \Omega \to \mathbb{R}^n$ be an $(r,n;q)$-manifold (for $r < n$; the case $r = n$ is already taken care of). The following fact allows us to do this.

[VI.16] L E M M A. *Let* $r,n \in \mathbb{N}$ *with* $r \leq n$, *and* $q \in \mathbb{N}\cup\{\infty\}$. *Suppose that* Ω *is open in* $\mathbb{R}^r$, *and* $g \in C^q(\Omega;\mathbb{R}^n)$. *Let* $x \in \Omega$ *with* rank $Dg(x) = r$. *Then there exists an open neighborhood of* x, $\Omega_x \subset \Omega$, *such that* $g(\Omega_x)$ *is an* $(r,n;q)$-*manifold.*

P R O O F. If $r = n$, the proof follows from the inverse function theorem, so we suppose $r < n$. Since $Dg(x)$ has rank r, there is an increasing r-tuple $\lambda = (i_1,\ldots,i_r)$ of integers in $\{1,\ldots,n\}$ such that $Jg^\lambda(x) \neq 0$, where $g^\lambda \in C^q(\Omega;\mathbb{R}^r)$ is the function $x \mapsto (g^{i_1}(x),\ldots,g^{i_r}(x))$, $x \in \Omega$. By the inverse function theorem, there exists an open neighborhood $\Omega_x \subset \Omega$ of x such that $g^\lambda(\Omega_x)$ is open in $\mathbb{R}^r$, $g_0^\lambda := g^\lambda|\,\Omega_x$ is a homeomorphism of Ω_x onto $g^\lambda(\Omega_x)$, and $g_0^{\lambda^{-1}} \in C^q(g^\lambda(\Omega_x);\mathbb{R}^r)$. Let $(j_1,\ldots,j_{n-r})$ be the increasing $(n-r)$-tuple of integers complementary to λ in $\{1,\ldots,n\}$, and define $G: g^\lambda(\Omega_x) \to \mathbb{R}^n$ by

$$\left.\begin{aligned} G^{i_k}(y) &:= y^k, \quad k = 1,\ldots,r, \\ G^{j_k}(y) &:= g^{j_k}\circ(g_0^{\lambda^{-1}})(y), \quad k = 1,\ldots,n-r \end{aligned}\right\} \text{ for each } \quad y \in g^\lambda(\Omega_x).$$

Since $g^{j_k}\circ(g_0^{\lambda})^{-1} \in C^q(g^\lambda(\Omega_x))$, for $k = 1,\ldots,n-r$, and $g^\lambda(\Omega_x)$ is open in $\mathbb{R}^r$, we can apply Lemma [VI.14] to assert that $G(g^\lambda(\Omega))$ is an $(r,n;q)$-manifold.

We claim that $g|\,\Omega_x = G\circ g_0^\lambda$; once this has been substantiated, there shall follow $g(\Omega_x) = G\circ g_0^\lambda(\Omega_x) = G(g^\lambda(\Omega))$, and the proof shall be complete. Suppose, then, that $y \in \Omega_x$. For $k = 1,\ldots,r$,

$$g^{i_k}(y) = (g^\lambda)^k(y) = (g_0^\lambda)^k(y) = G^{i_k}(g_0^\lambda(y)) = (G\circ g_0^\lambda)^{i_k}(y),$$

while, for $k = 1,\ldots,n-r$,

$$g^{j_k}(y) = g^{j_k}\circ(g_0^{\lambda})^{-1}\circ g_0^\lambda(y) = G^{j_k}\circ g_0^\lambda(y) = (G\circ g_0^\lambda)^{j_k}(y),$$

whence it does follow that $g|\,\Omega_x = G\circ g_0^\lambda$. □.

[VI.17] P R O P O S I T I O N. *Let* $r,n \in \mathbb{N}$ *with* $r \leq n$, *and* $q \in \mathbb{N}\cup\{\infty\}$. *Let* Ω *be non-void and open in* $\mathbb{R}^r$. *Suppose that* $g\colon\ \Omega \to \mathbb{R}^n$ *and*

(*i*) $g\colon\ \Omega \to g(\Omega)$ *is a homeomorphism,*

(*ii*) $g \in C^q(\Omega;\mathbb{R}^n)$,

and

(*iii*) $\operatorname{rank} Dg(x) = r$ *for each* $x \in \Omega$.

Then $g(\Omega)$ *is an* $(r,n;q)$-*manifold, and* g *is* q-*regular.*

P R O O F. In view of conditions (i)-(iii) on g, the q-regularity of g will follow as soon as it is known that $g(\Omega)$ is an

$(r,n;q)$-manifold. To prove the latter, choose $x \in \Omega$. According to [VI.16], there exists some open neighborhood $\Omega_x \subset \Omega$ of x such that $g(\Omega_x)$ is an $(r,n;q)$-manifold. Since $g\colon \Omega \to g(\Omega)$ is a homeomorphism, $g(\Omega_x)$ is relatively open in $g(\Omega)$ and, of course, contains $g(x)$. Thus, each point of $g(\Omega)$ possesses a relatively open neighborhood in $g(\Omega)$ which is an $(r,n;q)$-manifold. As we pointed out in [VI.4.b], this implies that $g(\Omega)$ itself is an $(r,n;q)$-manifold. $\square$.

As promised, we introduce the idea of a *coordinate system* in a manifold.

[VI.18] D E F I N I T I O N S. Let M be an $(r,n;q)$-manifold $(r \leq n)$. A non-void relatively open subset U of M is called a *coordinate patch on* M iff there exists a function $h\colon U \to \mathbb{R}^r$ such that

(i) $h(U)$ is open in $\mathbb{R}^r$,

(ii) $h\colon U \to h(U)$ is a homeomorphism,

(iii) $h^{-1} \in C^q(h(U);\mathbb{R}^n)$,

and

(iv) rank $Dh^{-1}(x) = r$ for each $x \in h(U)$.

Whenever U is a coordinate patch on M, and $h\colon U \to \mathbb{R}^r$ satisfies (i)-(iv), h is called a *coordinate function for* U, and the pair (U,h) is called a *coordinate system in* M. $\blacksquare$.

The inverses of q-regular transformations generate the coordinate systems in an $(r,n;q)$-manifold:

[VI.19] P R O P O S I T I O N. *Let* M *be an* $(r,n;q)$*-manifold. A non-void subset* $U \subset M$ *is a coordinate patch on* M *iff* $U = g(\Omega)$ *for some* q*-regular transformation* $g: \Omega \to M$, *where* Ω *is open in* $\mathbb{R}^r$. *In the latter case,* (U, g^{-1}) *is a coordinate system in* M.

P R O O F. If U is a coordinate patch on M, then $U = h^{-1}(h(U))$ for some coordinate function for U. $h(U)$ is open in $\mathbb{R}^r$, and it is clear that h^{-1}: $h(U) \to M$ is q-regular.

Conversely, suppose $U \subset M$ is non-void, and there exist an open set $\Omega \subset \mathbb{R}^r$ and a q-regular transformation $g: \Omega \to M$ such that $U = g(\Omega)$. By [VI.15] (or [VI.10.b], in case $r = n$), $g: \Omega \to M$ is open, and $g: \Omega \to g(\Omega)$ is a homeomorphism. Thus, $U = g(\Omega)$ is open in M, and it is a simple matter to check that g^{-1}: $U \to \Omega = g^{-1}(U)$ is a coordinate function for U, *i.e.*, that (U, g^{-1}) is a coordinate system in M. $\square$.

In addition to providing another criterion which can be used to identify an appropriate subset of some $\mathbb{R}^n$ as a manifold, the following theorem asserts that any manifold has sufficiently many coordinate patches to form a covering of the manifold. In fact, the latter property characterizes the manifolds amongst all subsets of a given Euclidean space.

[VI.20] T H E O R E M. *Let* $r, n \in \mathbb{N}$ *with* $r \leq n$, *and* $q \in \mathbb{N} \cup \{\infty\}$.

A non-void set $M \subset \mathbb{R}^n$ *is an* $(r,n;q)$*-manifold iff there exists a family of pairs* $\{(U_\iota, h_\iota)\}_{\iota \in I}$ *such that*

(*i*) $U_\iota \subset M$ *is non-void and relatively open, for each* $\iota \in I$, *and* $\{U_\iota\}_{\iota \in I}$ *is a covering of* M,

(*ii*) *for each* $\iota \in I$, $h_\iota \colon U_\iota \to \mathbb{R}^r$ *is a homeomorphism of* U_ι *onto an open set in* $\mathbb{R}^r$, *such that* $h_\iota^{-1} \in C^q(h_\iota(U_\iota);\mathbb{R}^n)$, *with* $\operatorname{rank} Dh_\iota^{-1}(\hat{x}) = r$ *for each* $\hat{x} \in h_\iota(U_\iota)$.

P R O O F. Suppose first that M is an $(r,n;q)$-manifold. If $r = n$, then $\{(M,i)\}$, where $i\colon M \to \mathbb{R}^n$ is the identity on M, fulfills the requirements of (i) and (ii), so we may suppose $r < n$. We shall show that each point of M lies in a coordinate patch on M, *i.e.*, whenever $x \in M$, there is a coordinate system $(\tilde{U}_x, h_x)$ with $x \in \tilde{U}_x$. The collection $\{(\tilde{U}_x, h_x)\}_{x \in M}$ shall then fulfill the requirements. Then choose $x \in M$. Let $U_x \subset \mathbb{R}^n$ and $\Phi_x \in C^q(U_x;\mathbb{R}^{n-r})$ be as in Definition [VI.3]. We repeat the construction carried out at the beginning of the proof of [VI.15]: according to the implicit function theorem, there exist an open neighborhood $U_0 \subset U_x$ of x in $\mathbb{R}^n$, an increasing r-tuple $\lambda = (i_1,\ldots,i_r)$ of integers in $\{1,\ldots,n\}$, an open neighborhood $V_0 \subset \mathbb{R}^r$ of x^λ, and a function $\phi \in C^q(V_0;\mathbb{R}^{n-r})$ such that

$$\{y \in U_0 \mid \Phi_x(y) = 0\} = \{y \in \mathbb{R}^n \mid y^\lambda \in V_0, \quad y^{j_k} = \phi(y^\lambda), \quad k = 1,\ldots,n-r\},$$

where $(j_1,\ldots,j_{n-r})$ is the increasing $(n-r)$-tuple complementary to λ in $\{1,\ldots,n\}$. Once again, we have $M\cap U_0 = \{y \in U_0 | \ \Phi_x(y) = 0\}$, and $G(V_0) = M\cap U_0$, where $G\colon\ V_0 \to \mathbb{R}^n$ is defined by

$$\left.\begin{array}{ll} G^{i_k}(\hat{y}) := \hat{y}^k, & k = 1,\ldots,r, \\ G^{j_k}(\hat{y}) := \phi^k(\hat{y}), & k = 1,\ldots,n-r \end{array}\right\} \text{ for each } \quad \hat{y} \in V_0.$$

According to Lemma [VI.14], $G\colon\ V_0 \to M$ is a q-regular homeomorphism of the open set $V_0 \subset \mathbb{R}^r$ onto the relatively open neighborhood $M\cap U_0$ of x in M, with $G^{-1}\colon\ M\cap U_0 \to \mathbb{R}^r$ being just $\Xi^\lambda|\ (M\cap U_0)$. Clearly, $(M\cap U_0,\ \Xi^\lambda|\ (M\cap U_0))$ is then a coordinate system in M, with $x \in M\cap U_0$. As noted, this implies the necessity of the stated condition.

To prove the sufficiency (now, $r \leq n$), suppose that there exists $\{(U_\iota,h_\iota)\}_{\iota\in I}$ with properties (i) and (ii). Choose $x \in M$, then $\iota \in I$ with $x \in U_\iota$. Then $h_\iota(U_\iota)$ is open in $\mathbb{R}^r$, h_ι^{-1}: $h_\iota(U_\iota) \to U_\iota$ is a homeomorphism, with $h_\iota^{-1} \in C^q(h_\iota(U_\iota);\mathbb{R}^n)$ and rank $Dh_\iota^{-1}(\hat{y}) = r$ for each $\hat{y} \in h_\iota(U_\iota)$. Proposition [VI.17] allows us to conclude that $U_\iota = h_\iota^{-1}(h_\iota(U_\iota))$ is an $(r,n;q)$-manifold. By (i), U_ι is open in M. Thus, each point of M lies in a relatively open subset of M which is an $(r,n;q)$-manifold, whence M itself is an $(r,n;q)$-manifold (cf., [VI.4.b]). $\square$.

It is important to point out the necessary relationship between coordinate systems with "overlapping" coordinate patches. We shall consider only the case $r < n$, since the inverse function theorem can be used to prove the corresponding statement for $r = n$.

[VI.21] PROPOSITION. *Let* M *be an* $(r,n;q)$*-manifold,* $r < n$, *and* (U_1,h_1), (U_2,h_2) *coordinate systems in* M *with* $U_1 \cap U_2 \neq \phi$. *Define*

$$\phi_{12} := h_1 \circ (h_2^{-1} | \; h_2(U_1 \cap U_2)): \; h_2(U_1 \cap U_2) \to \mathbb{R}^r,$$

$$\phi_{21} := h_2 \circ (h_1^{-1} | \; h_1(U_1 \cap U_2)): \; h_1(U_1 \cap U_2) \to \mathbb{R}^r.$$

Then

(*i*) $h_1(U_1 \cap U_2)$ *and* $h_2(U_1 \cap U_2)$ *are open in* $\mathbb{R}^r$,

(*ii*) ϕ_{12} *is a homeomorphism of* $h_2(U_1 \cap U_2)$ *onto* $h_1(U_1 \cap U_2)$, *and* $\phi_{12}^{-1} = \phi_{21}$,

(*iii*) ϕ_{12} *and* ϕ_{21} *are* q*-regular,*

and

(*iv*)
$$h_1 | \; (U_1 \cap U_2) = \phi_{12} \circ (h_2 | \; (U_1 \cap U_2)),$$
$$h_1^{-1} | \; h_1(U_1 \cap U_2) = h_2^{-1} \circ \phi_{21},$$
$$h_2 | \; (U_1 \cap U_2) = \phi_{21} \circ (h_1 | \; (U_1 \cap U_2)),$$
$$h_2^{-1} | \; h_2(U_1 \cap U_2) = h_1^{-1} \circ \phi_{12}.$$

PROOF. (i) Since U_2 is open in M, $U_1 \cap U_2$ is open in U_1. Thus, $h_1(U_1 \cap U_2)$ is open in $\mathbb{R}^r$, since $h_1: U_1 \to h_1(U_1)$ is a homeomorphism and $h_1(U_1)$ is open in $\mathbb{R}^r$. Similarly, $h_2(U_1 \cap U_2)$ is open in $\mathbb{R}^r$.

(ii) This is obvious.

(iii) Let us show that ϕ_{12} is q-regular: since we know that ϕ_{12} is injective, it suffices to show that each $\hat{x} \in h_2(U_1 \cap U_2)$ has an open neighborhood $V_{\hat{x}} \subset h_2(U_1 \cap U_2)$ such that $\phi_{12}| \; V_{\hat{x}} \in C^q(V_{\hat{x}};\mathbb{R}^r)$, with rank $D\phi_{12}(\hat{y}) = r$ for each $\hat{y} \in V_{\hat{x}}$. Then, choose $\hat{x} \in h_2(U_1 \cap U_2)$. Set $x = h_2^{-1}(\hat{x})$. Since $\phi_{12}(\hat{x}) = h_1(h_2^{-1}(\hat{x}))$, we also have $x = h_1^{-1}(\phi_{12}(\hat{x}))$. Let $U_x \subset \mathbb{R}^n$ be an open neighborhood of x, and $\Phi_x \in C^q(U_x;\mathbb{R}^{n-r})$ as in Definition [VI.3]. Let $U_0 \subset U_x$ be an open neighborhood of x, $\lambda = (i_1,\ldots,i_r)$ an increasing r-tuple of integers in $\{1,\ldots,n\}$, $V_0 \subset \mathbb{R}^r$ an open neighborhood of x^λ, and $\phi \in C^q(V_0;\mathbb{R}^{n-r})$ such that

$$\begin{aligned} M \cap U_0 &= \{y \in U_0 | \; \Phi_x(y) = 0\} \\ &= \{y \in \mathbb{R}^n | \; y^\lambda \in V_0, \quad y^{j_k} = \phi(y^\lambda), \quad k = 1,\ldots,n-r\}, \end{aligned}$$

where $(j_1,\ldots,j_{n-r})$ is the increasing $(n-r)$-tuple complementary to λ in $\{1,\ldots,n\}$. We can find an open neighborhood $\Omega_1 \subset h_1(U_1 \cap U_2)$ of $\phi_{12}(\hat{x})$ such that $h_1^{-1}(\Omega_1) \subset M \cap U_0$, and an open neighborhood $\Omega_2 \subset h_2(U_1 \cap U_2)$ of $\hat{x}$ such that $h_2^{-1}(\Omega_2) \subset M \cap U_0$, since h_1^{-1} and h_2^{-1} are continuous, $M \cap U_0$ is a neighborhood of x in M, and $x = h_2^{-1}(\hat{x}) = h_1^{-1}(\phi_{12}(\hat{x}))$. We define $f_1\colon \Omega_1 \to \mathbb{R}^r$, $f_2\colon \Omega_2 \to \mathbb{R}^r$ by

$$f_1 := \Xi^\lambda \circ (h_1^{-1} | \; \Omega_1),$$

$$f_2 := \Xi^\lambda \circ (h_2^{-1} | \; \Omega_2).$$

Just as in the proof of [VI.15], using the auxiliary function

$G \in C^q(V_0;\mathbb{R}^n)$ given by (VI.15.3), we can show that $h_1^{-1}|\ \Omega_1 = G\circ f_1$ and $h_2^{-1}|\ \Omega_2 = G\circ f_2$, and so that f_1 and f_2 are q-regular; the details can be easily supplied, so we omit them here. Using the inverse function theorem (cf., [VI.10.b]), it follows that $f_i(\Omega_i)$ is open in $\mathbb{R}^r$, and f_i^{-1}: $f_i(\Omega_i) \to \mathbb{R}^r$ is q-regular, for $i = 1,2$. Now, $f_2(\hat{x}) = \Xi^\lambda(h_2^{-1}(\hat{x})) = \Xi^\lambda(h_1^{-1}(\phi_{12}(\hat{x}))) = f_1(\phi_{12}(\hat{x})) \in f_1(\Omega_1)$, so $f_1(\Omega_1)$ is an open neighborhood of $f_2(\hat{x})$, showing that we can choose an open neighborhood $V_{\hat{x}} \subset \Omega_2$ of $\hat{x}$ such that $f_2(V_{\hat{x}}) \subset f_1(\Omega_1)$. Then $f_1^{-1}(f_2(\hat{y}))$ is defined whenever $\hat{y} \in V_{\hat{x}}$, and it is a simple matter to check that $\phi_{12}(\hat{y}) = f_1^{-1}(f_2(\hat{y}))$ for each $\hat{y} \in V_{\hat{x}}$. Since f_2 and f_1^{-1} are q-regular, we can conclude that $\phi_{12}|\ V_{\hat{x}} \in C^q(V_{\hat{x}};\mathbb{R}^r)$ and rank $D\phi_{12}(\hat{y}) = r$ for each $\hat{y} \in V_{\hat{x}}$. As noted, this completes the proof that ϕ_{12} is q-regular. The proof that ϕ_{21} is q-regular can be given in a similar manner or by simply noting that $\phi_{21} = \phi_{12}^{-1}$.

(iv) These equalities are easy to check. □.

[VI.22] R E M A R K. Suppose that M is an $(r,n;q)$-manifold, and (U,h) is a coordinate system in M. It is clear that if $\tilde{U} \subset U$ is open in M, then $(\tilde{U}, h|\ \tilde{U})$ is also a coordinate system in M. Also, whenever ϕ: $h(U) \to \mathbb{R}^r$ is q-regular, then $(U,\phi\circ h)$ is another coordinate system in M.

We turn next to the definitions, and certain elementary properties of, classes of smooth functions on a manifold into a Euclidean space.

[VI.23] D E F I N I T I O N S. Let M be an $(r,n;q)$-manifold, $m \in \mathbb{N}$, $k \in \mathbb{N}\cup\{\infty\}$, and $f\colon M \to \mathbb{R}^m$.

(i) $f \in C^k(M;\mathbb{R}^m)$ iff the function $f\circ h^{-1}$ is in $C^k(h(U);\mathbb{R}^m)$ whenever (U,h) is a coordinate system in M.

(ii) Let $f \in C^k(M;\mathbb{R}^m)$ and $x \in M$. Then we define *the rank of* f *at* x to be $\operatorname{rank} D(f\circ h^{-1})(h(x))$, where (U,h) is a coordinate system in M with $x \in U$. Further, we define *the differential of* f *at* x to be the linear operator $Df(x)\colon T_M(x) \to \mathbb{R}^m$ given by

$$Df(x) := D(f\circ h^{-1})(h(x))\circ\{Dh^{-1}(h(x))\}^{-1}, \tag{1}$$

where (U,h) is a coordinate system in M with $x \in U$ (recall that we established, in [VI.12], the notation $\{Dh^{-1}(h(x))\}^{-1}\colon T_M(x) \to \mathbb{R}^r$ for the inverse of the operator $Dh^{-1}(h(x))$ taking $\mathbb{R}^r$ onto $T_M(x)$, since $h^{-1}\colon h(U) \to M$ is q-regular; cf., [VI.11]).

(iii) Let $f \in C^k(M;\mathbb{R}^m)$. We define $\mathcal{J}f\colon M \to [0,\infty)$ *via*

$$\mathcal{J}f(x) := \frac{|Df(x)T_1(x)\wedge\ldots\wedge Df(x)T_r(x)|}{|T_1(x)\wedge\ldots\wedge T_r(x)|} \quad \text{for each} \quad x \in M, \tag{2}$$

where $\{T_1(x),\ldots,T_r(x)\}$ is a basis for $T_M(x)$, for each $x \in M$.

(iv) $f\colon M \to \mathbb{R}^m$ is a k-*imbedding* iff

(1) $f\colon M \to f(M)$ is a homeomorphism,

(2) $f \in C^k(M;\mathbb{R}^m)$,

and

(3) the rank of f at x is r, for each $x \in M$. ∎.

[VI.24] R E M A R K S. Maintain the notation of [VI.23].

(a) Suppose that $f \in C^k(M;\mathbb{R}^m)$ and $x \in M$. Let (U_1,h_1), (U_2,h_2) be coordinate systems in M, with $x \in U_1 \cap U_2$. Then $(f \circ h_2^{-1})(h_2(y)) = (f \circ h_1^{-1}) \circ \phi_{12}(h_2(y))$, for each $y \in U_1 \cap U_2$, with ϕ_{12} as in [VI.21]. Since $\phi_{12} \in C^q(h_2(U_1 \cap U_2);\mathbb{R}^r)$,

$$D(f \circ h_2^{-1})(h_2(x)) = D(f \circ h_1^{-1})(h_1(x)) \circ D\phi_{12}(h_2(x)), \qquad (1)$$

since $\phi_{12}(h_2(x)) = h_1(x)$. Again by [VI.21], $\operatorname{rank} D\phi_{12}(h_2(x)) = r$, so we conclude that $\operatorname{rank} D(f \circ h_2^{-1})(h_2(x)) = \operatorname{rank} D(f \circ h_1^{-1})(h_1(x))$. Thus, *the rank of* f *at* x is well-defined in [VI.23.ii].

(b) Continuing the setting introduced in (a), we can write

$$\begin{aligned} D\phi_{12}(h_2(x)) &= \{Dh_1^{-1}(h_1(x))\}^{-1} \circ Dh_1^{-1}(h_1(x)) \circ D\phi_{12}(h_2(x)) \\ &= \{Dh_1^{-1}(h_1(x))\}^{-1} \circ D(h_1^{-1} \circ \phi_{12})(h_2(x)) \\ &= \{Dh_1^{-1}(h_1(x))\}^{-1} \circ Dh_2^{-1}(h_2(x)), \end{aligned}$$

so (1) gives

$$D(f\circ h_2^{-1})(h_2(x))\circ\{Dh_2^{-1}(h_2(x))\}^{-1}$$

$$= D(f\circ h_1^{-1})(h_1(x))\circ D\phi_{12}(h_2(x))\circ\{Dh_2^{-1}(h_2(x))\}^{-1}$$

$$= D(f\circ h_1^{-1})(h_1(x))\circ\{Dh_1^{-1}(h_1(x))\}^{-1},$$

which shows that $Df(x)$ is well-defined by (VI.23.1). Observe that, since $\{Dh^{-1}(h(x))\}^{-1}$ takes $T_M(x)$ onto $\mathbb{R}^r$, it follows from (VI.23.1) that the rank of f at x is simply equal to rank $Df(x)$.

(c) Let $\{T_{1i}(x)\}_{i=1}^r$ and $\{T_{2i}(x)\}_{i=1}^r$ be bases for $T_M(x)$. It is shown in Fleming [15] that there exists a constant $a \in \mathbb{R}$ such that $T_{21}(x)\wedge\ldots\wedge T_{2r}(x) = aT_{11}(x)\wedge\ldots\wedge T_{1r}(x)$ and $Df(x)T_{21}(x)\wedge\ldots\wedge Df(x)T_{2r}(x) = aDf(x)T_{11}(x)\wedge\ldots\wedge Df(x)T_{1r}(x)$. From this, it is clear that $\mathcal{J}f(x)$ is independent of the basis chosen to compute it by (VI.23.2), hence that $\mathcal{J}f$ is well-defined.

(d) There is a consistency question which should be resolved: when M is an $(n,n;q)$-manifold, *i.e.*, a non-void open set in $\mathbb{R}^n$, and $f \in C^q(M;\mathbb{R}^m)$, then $\mathcal{J}f\colon M \to [0,\infty)$ has already been defined, in [VI.9.i]. It turns out, cf., [VI.28], *infra*, that $C^q(M;\mathbb{R}^m) = C^q(M;\mathbb{R}^m)$ in this case, from which it is easy to see that the definitions [VI.9.i] and [VI.23.iii] are in fact consistent.

(e) Observe that $\mathcal{J}f(x) > 0$ iff rank $Df(x) = r$, iff the rank of f at x is r. For, if $\mathcal{J}f(x) > 0$, then $Df(x)T_1(x)\wedge\ldots\wedge Df(x)T_r(x) \neq 0$, so $\{Df(x)T_i(x)\}_{i=1}^r$ is linearly independent, by the properties of the exterior product. Thus, rank $Df(x) \geq r$. But $\dim T_M(x) = r$, so we always have $\operatorname{rank} Df(x) \leq r$, so equality must hold. Conversely, if $\operatorname{rank} Df(x) = r$, then $Df(x)\colon T_M(x) \to \mathbb{R}^m$ is an injection, since $\dim T_M(x) = r$. The linear independence of $\{T_i(x)\}_{i=1}^r$ then implies the linear independence of $\{Df(x)T_i(x)\}_{i=1}^r$, so $Df(x)T_1(x)\wedge\ldots\wedge Df(x)T_r(x) \neq 0$, and $\mathcal{J}f(x) > 0$.

(f) Let (U,h) be a coordinate system in M. $h^{-1}\colon h(U) \to \mathbb{R}^n$ is q-regular, with $h^{-1}(h(U)) = U \subset M$. From [VI.11], the collection $\{h^{-1}_{,i}(h(x)) = Dh^{-1}(h(x))e_i^{(r)}\}_{i=1}^r$ forms a basis for $T_M(x)$, for each $x \in U$. We find, from (VI.23.1), the especially simple form

$$Df(x)h^{-1}_{,i}(h(x)) = D(f\circ h^{-1})(h(x))e_i^{(r)} = (f\circ h^{-1})_{,i}(h(x)), \tag{2}$$

for the images of $\mathbb{R}^m$ of these particular basis vectors, for $f \in C^q(M;\mathbb{R}^m)$, $i = 1,\ldots,r$, and $x \in U$. Consequently, we have the representation

$$\mathcal{J}f(x) = \frac{|(f\circ h^{-1})_{,1}(h(x))\wedge\ldots\wedge(f\circ h^{-1})_{,r}(h(x))|}{|h^{-1}_{,1}(h(x))\wedge\ldots\wedge h^{-1}_{,r}(h(x))|}, \quad \text{for each} \quad x \in U, \tag{3}$$

valid whenever M is an $(r,n;q)$-manifold, (U,h) is a coordinate system in M, and $f \in C^k(M;\mathbb{R}^m)$. Of course, $\{h^{-1}_{,i}(h(x))\}_{i=1}^r$ is linearly independent, so the denominator in (3) is non-zero, for each $x \in U$.

[VI.25] P R O P O S I T I O N. *Let* M *be an* $(r,n;q)$*-manifold, and* $f \in C^k(M;\mathbb{R}^m)$. *Then* *(i)* f *is continuous;*

(ii) Jf *is continuous.*

P R O O F. (i) Let $(x_n)_1^\infty$ be a sequence in M, converging to some $x \in M$. Let (U,h) be a coordinate system in M, with $x \in U$. Then U is a relatively open neighborhood of x in M, so $x_n \in U$ for all n greater than some $n_0 \in \mathbb{N}$. We have $\lim_{n\to\infty} h(x_n) = h(x)$, and so, since $f \circ h^{-1} \in C(h(U);\mathbb{R}^m)$,

$$\lim_{n\to\infty} f(x_n) = \lim_{n\to\infty} (f\circ h^{-1})(h(x_n)) = (f\circ h^{-1})(h(x)) = f(x).$$

Thus, f is sequentially continuous, hence continuous, on M.

(ii) Choose $x_0 \in M$. Let (U,h) be a coordinate system in M, with $x_0 \in U$. For each $x \in U$, $Jf(x)$ is given by (VI.24.3). Now, $x \mapsto h^{-1}_{,i}(h(x))$ is in $C(U;\mathbb{R}^r)$, $x \mapsto (f\circ h^{-1})_{,i}(h(x))$ is in $C(U;\mathbb{R}^m)$, for $i = 1,\ldots,r$, while $(\alpha_1,\ldots,\alpha_r) \mapsto \alpha_1\wedge\ldots\wedge\alpha_r$ is continuous on either $(\mathbb{R}^r)^r$ into $\mathbb{R}^r_r$ or $(\mathbb{R}^m)^r$ into $\mathbb{R}^m_r$, as the case may be, and the norm on any $\mathbb{R}^\ell_p$ into $[0,\infty)$ is also continuous. Since rank $Dh^{-1}(h(x)) = r$ for each $x \in U$, $h^{-1}_{,1}(h(x))\wedge\ldots\wedge h^{-1}_{,r}(h(x)) \neq 0$ for each $x \in U$. These facts, coupled with (VI.24.3), show that Jf is continuous on U, hence, in particular, at x_0. Then Jf is continuous on M. □.

The following improvement of [VI.25.i] is important.

[VI.26] P R O P O S I T I O N. *Let* M *be an* $(r,n;q)$*-manifold, and* $f \in C^1(M;\mathbb{R}^m)$, *for some* $m \in \mathbb{N}$. *Then, whenever* K *is a* *compact subset of* M, $f|K$ *is Lipschitz continuous, i.e., there exists an* $a_{K,f} > 0$ *such that*

$$|f(y_2)-f(y_1)|_m \leq a_{K,f}\cdot|y_2-y_1|_n, \qquad \text{for} \quad y_1,y_2 \in K. \tag{1}$$

P R O O F. Select any $x \in M$. As in the proof of Theorem [VI.20], we can find an open neighborhood U_{0x} of x in $\mathbb{R}^n$ and an increasing r-tuple $\lambda_x = (i_1^x,\ldots,i_r^x)$ from $\{1,\ldots,n\}$ such that (W_x,k_x) is a coordinate system in M, where

$$W_k := M\cap U_{0x}, \tag{2}$$

$$k_x := \Xi^{\lambda_x}|\ M\cap U_{0x}. \tag{3}$$

Thus, $k_x(W_x) = \Xi^{\lambda_x}(M\cap U_{0x})$ is an open neighborhood of $x^{\lambda_x} := \Xi^{\lambda_x}(x)$ in $\mathbb{R}^r$; let ε_x denote a positive number such that $B^r_{\varepsilon_x}(x^{\lambda_x})^- \subset k_x(W_x)$, and then let $\delta_x > 0$ be such that both $B^n_{\delta_x}(x) \subset U_{0x}$ and $k_x(M\cap B^n_{\delta_x}(x)) \subset B^r_{\varepsilon_x}(x^{\lambda_x})$ hold. Note that k_x is Lipschitz continuous on W_x: indeed, whenever $y_1,y_2 \in W_x$,

$$|k_x(y_2)-k_x(y_1)|_r = |\Xi^{\lambda_x}(y_2)-\Xi^{\lambda_x}(y_1)|_r \leq |y_2-y_1|_n. \tag{4}$$

Now, since $f \in C^1(M;\mathbb{R}^m)$, we know that $f\circ k_x^{-1} \in C^1(k_x(W_x);\mathbb{R}^m)$. In particular, the partial derivatives of $f\circ k_x^{-1}$ are bounded on the compact subset $B^r_{\varepsilon_x}(x^{\lambda_x})^-$ of $k_x(W_x)$.

Let K be any compact subset of M, and choose a finite set $\{x_i\}_{i=1}^N \subset K$ such that the collection $\{M\cap B^n_{\delta_{x_i}/2}(x_i)\}_{i=1}^N$ of open subsets of M covers K. Write $\delta := \min \{\delta_{x_i}/2\}_{i=1}^{N}$. Suppose that $y_1, y_2 \in K$. Assuming first that $|y_2 - y_1|_n \geq \delta$, we have

$$\begin{aligned} |f(y_2)-f(y_1)|_m &= (|f(y_2)-f(y_1)|_m / |y_2-y_1|_n)\cdot|y_2-y_1|_n \\ &\leq \frac{1}{\delta}\cdot 2 \sup_{y \in K} |f(y)|_m \cdot |y_2-y_1|_n, \end{aligned} \tag{5}$$

in which $\sup_{y \in K} |f(y)|_m < \infty$, since $f \in C(M;\mathbb{R}^m)$, by [VI.25.i]. Next, assume that $0 < |y_2-y_1|_n < \delta$. Denoting by j an element of $\{1,\ldots,N\}$ such that $y_1 \in M \cap B^n_{\delta_{x_j}/2}(x_j) \subset W_{x_j}$, it is clear that we also have $y_2 \in M \cap B^n_{\delta_{x_j}}(x_j) \subset W_{x_j}$, since $|y_2 - x_j|_n \leq |y_2-y_1|_n + |y_1-x_j|_n < \delta + \delta_{x_j}/2 \leq \delta_{x_j}$. Moreover, we then have $k_{x_j}(y_1),\ k_{x_j}(y_2) \in B^r_{\varepsilon_{x_j}}(x_j^{\lambda_{x_j}})$, since $k_{x_j}(M \cap B^n_{\delta_{x_j}}(x_j)) \subset B^r_{\varepsilon_{x_j}}(x_j^{\lambda_{x_j}})$, by the first part of the proof. Consequently, we can apply the mean-value theorem to write

$$\begin{aligned} |f(y_2)-f(y_1)|_m &= |f\circ k_{x_j}^{-1}(k_{x_j}(y_2)) - f\circ k_{x_j}^{-1}(k_{x_j}(y_1))|_m \\ &\leq |D(f\circ k_{x_j}^{-1})(\hat{z})(k_{x_j}(y_2)-k_{x_j}(y_1))|_m \\ &\leq \left\{ \sum_{i=1}^{m} \sum_{\ell=1}^{r} \{(f\circ k_{x_j}^{-1})^i_{;\ell}(\hat{z})\}^2 \right\}^{1/2} \\ &\quad \cdot |k_{x_j}(y_2)-k_{x_j}(y_1)|_r \\ &\leq A_{K,f}\cdot|y_2-y_1|_n, \end{aligned} \tag{6}$$

having used (4), where $\hat{z}$ is some point on the line segment joining $k_{x_j}(y_1)$ and $k_{x_j}(y_2)$ (which are distinct, since $y_1 \neq y_2$), and

$$\Lambda_{K,f} := \max_{1 \leq p \leq N} \left\{ \max_{\hat{z} \in B^r_{\varepsilon_{x_p}}(x_p^{\lambda_{x_p}})^-} \left\{ \sum_{i=1}^{m} \sum_{\ell=1}^{r} \{(f\circ k_{x_p}^{-1})^i_{,\ell}(\hat{z})\}^2 \right\}^{1/2} \right\}.$$

In any case, (1) holds with $a_{K,f} := \max\left\{\frac{2}{\delta} \max_{y \in K} |f(y)|_m, \Lambda_{K,f}\right\}$, as (5) and (6) show. $\square$.

[VI.27] P R O P O S I T I O N. *Let* M *be an* $(r,n;q)$*-manifold,* $m \in \mathbb{N}$, *and* $k \in \mathbb{N}\cup\{\infty\}$ *with* $k \leq q$. *Let* $f\colon M \to \mathbb{R}^m$. *Then* $f \in C^k(M;\mathbb{R}^m)$ *iff there exists a family of coordinate systems* $\{(U_\iota,h_\iota)\}_{\iota\in I}$ *in* M *such that* $\{U_\iota\}_{\iota\in I}$ *covers* M, *and* $f\circ h_2^{-1} \in C^k(h_\iota(U_\iota);\mathbb{R}^m)$ *for each* $\iota \in I$.

P R O O F. From Definition [VI.23.i] and Theorem [VI.20], the condition is obviously necessary.

Now, suppose $\{(U_\iota,h_\iota)\}_{\iota\in I}$ is a family of coordinate systems in M possessing the stated properties. Let (U,h) be any coordinate system in M: we must show that $f\circ h^{-1} \in C^k(h(U);\mathbb{R}^m)$. Choose $\hat{x} \in h(U)$, then $\iota \in I$ such that $h^{-1}(\hat{x}) \in U_\iota$. Then $U_\iota \cap U \neq \emptyset$, open in U, and $h(U_\iota\cap U)$ is an open neighborhood of $\hat{x}$ in $\mathbb{R}^r$. Set $\phi := h_\iota\circ(h^{-1}|\ h(U_\iota\cap U))$. Then $\phi(h(U_\iota\cap U)) \subset h_\iota(U_\iota)$, and, just as in [VI.21], $\phi \in C^q(h(U_\iota\cap U);\mathbb{R}^r)$. Clearly, $(f\circ h^{-1})|\ h(U_\iota\cap U) = (f\circ h_\iota^{-1})\circ\phi$. Since it is known that $f\circ h_\iota^{-1} \in C^k(h_\iota(U_\iota);\mathbb{R}^m)$ and

$k \leq q$, we conclude that $(f \circ h^{-1})|\ h(U_1 \cap U) \in C^k(h(U_1 \cap U);\mathbb{R}^m)$. Thus, $f \circ h^{-1}$ is of class C^k in a neighborhood of each point of $h(U)$, so that $f \circ h^{-1} \in C^k(h(U);\mathbb{R}^m)$. $\square$.

[VI.28] R E M A R K. Let $\Omega \subset \mathbb{R}^n$ be non-void and open, *i.e.*, an $(n,n;q)$-manifold (for any $q \in \mathbb{N} \cup \{\infty\}$). The single coordinate system (Ω, i_Ω), where i_Ω: $\Omega \to \mathbb{R}^n$ is the identity on Ω, covers Ω. Let f: $\Omega \to \mathbb{R}^m$. Directly from [VI.27], $f \in C^k(\Omega;\mathbb{R}^m)$ for some $k \in \mathbb{N} \cup \{\infty\}$ iff $f \circ i_\Omega^{-1} \in C^k(i_\Omega(\Omega);\mathbb{R}^m)$, *i.e.*, iff $f \in C^k(\Omega;\mathbb{R}^m)$. Thus $C^k(\Omega;\mathbb{R}^m) = C^k(\Omega;\mathbb{R}^m)$, in this case, for each $k \in \mathbb{N} \cup \{\infty\}$.

[VI.29] P R O P O S I T I O N. *Let* M *be an* $(r,n;q)$*-manifold,* N *a* $(p,m;s)$*-manifold, and* $f \in C^k(M;\mathbb{R}^m)$, *with* $f(M) \subset N$. *Then*

(*i*) *for each* $x \in M$, $Df(x)T_M(x) \subset T_N(f(x))$;

(*ii*) *if* $p = r$ *and the rank of* f *at* $x \in M$ *is* r, *or* $Jf(x) > 0$, *then* $Df(x)T_M(x) = T_N(f(x))$, *and* $Df(x)$ *is injective.*

P R O O F. (i) If $p = m$, then N is a non-void open set in $\mathbb{R}^m$ and the result is trivially true, since, for $x \in M$, $Df(x)$: $T_M(x) \to \mathbb{R}^m = T_N(f(x))$. Suppose then that $p < m$. Choose $x \in M$. Let $U_{f(x)} \subset \mathbb{R}^m$ be an open neighborhood of $f(x)$ and $\Phi_{f(x)} \in C^s(U_{f(x)};\mathbb{R}^{m-p})$ be as in Definition [VI.3] for the $(p,m;s)$-manifold N. Let (U,h) be a coordinate system in M with $x \in M$. Since f is continuous ([VI.25.i]), $f^{-1}(U_{f(x)})$ is an open neighborhood of x in M, while

$$\Phi_{f(x)}((f\circ h^{-1})(h(y))) = \Phi_{f(x)}(f(y)) = 0 \quad \text{whenever} \quad y \in U\cap f^{-1}(U_{f(x)}).$$

Since $h(U\cap f^{-1}(U_{f(x)}))$ is an open neighborhood of $h(x)$ in $\mathbb{R}^r$, $f\circ h^{-1} \in C^k(h(U);\mathbb{R}^m)$, and $\Phi_{f(x)} \in C^s(U_{f(x)};\mathbb{R}^{m-p})$, the composite function theorem shows that $D\Phi_{f(x)}(f(x))\circ D(f\circ h^{-1})(h(x))$ is the zero operator on $\mathbb{R}^r$ into $\mathbb{R}^{m-p}$. Thus, for each $T \in T_M(x)$,

$$D\Phi_{f(x)}(f(x))\circ D(f\circ h^{-1})(h(x))\circ\{Dh^{-1}(h(x))\}^{-1}T = 0,$$

or

$$D\Phi_{f(x)}(f(x))\circ Df(x)T = 0.$$

Since $T_N(f(x)) = \ker D\Phi_{f(x)}(f(x))$, by [VI.6], we conclude that $Df(x)T \in T_N(f(x))$ for each $T \in T_M(x)$. This proves (i).

(ii) Now we know that the rank of $Df(x)\colon T_M(x) \to \mathbb{R}^m$ is r, $\dim T_N(f(x)) = r$, and, by (i), $Df(x)T_M(x) \subset T_N(f(x))$, so we must have equality: $Df(x)T_M(x) = T_N(f(x))$. Since $\dim T_M(x) = r$ and $\operatorname{rank} Df(x) = r$, $Df(x)$ is injective. □.

It is essential to have a reasonable condition under which the image of a manifold is also a manifold. More precisely, we have the following statement.

[VI.30] T H E O R E M. *Let* M *be an* $(r,n;q)$*-manifold,* $m \in \mathbb{N}$, *and* $k \in \mathbb{N}\cup\{\infty\}$ *with* $k \le q$. *Suppose that* $f\colon M \to \mathbb{R}^m$ *is a* k*-imbedding. Then* (*i*) $f(M)$ *is an* $(r,m;k)$*-manifold;*

(*ii*) *for each* $x \in M$, $Df(x)$: $T_M(x) \to \mathbb{R}^m$ *is an injection taking* $T_M(x)$ *onto* $T_{f(M)}(f(x))$;

(*iii*) f^{-1}: $f(M) \to \mathbb{R}^n$ *is a* k*-imbedding;*

(*iv*) *for each* $x \in M$, *the inverse of the bijection* $Df(x)$: $T_M(x) \to T_{f(M)}(f(x))$ *is given by* $Df^{-1}(f(x))$;

(*v*) *for each* $x \in M$, $Jf^{-1}(f(x)) = \{Jf(x)\}^{-1}$.

P R O O F. (i) We shall use Theorem [VI.20] to show that $f(M)$ is an $(r,m;k)$-manifold. Choose a collection of coordinate systems $\{(U_\iota,h_\iota)\}_{\iota\in I}$ in M such that $\{U_\iota\}_{\iota\in I}$ covers M. Consider the family of pairs $\{(\tilde{U}_\iota,\tilde{h}_\iota)\}_{\iota\in I}$, where $\tilde{U}_\iota := f(U_\iota)$ and $\tilde{h}_\iota := (f\circ h_\iota^{-1})^{-1}$. Note that $(f\circ h_\iota^{-1})^{-1}$ takes $f(U_\iota)$ onto $h_\iota(U_\iota)$, and is a homeomorphism between these two sets, since h_ι: $U_\iota \to h_\iota(U_\iota)$ and f: $M \to f(M)$ are homeomorphisms. Since U_ι is open in M, $\tilde{U}_\iota = f(U_\iota)$ is open in $f(M)$. Thus, for each $\iota \in I$, $\tilde{h}_\iota$: $\tilde{U}_\iota \to \mathbb{R}^r$ is a homeomorphism of the relatively open subset $\tilde{U}_\iota$ onto $\tilde{h}_\iota(\tilde{U}_\iota) = h_\iota(U_\iota)$; the latter is open in $\mathbb{R}^r$, by the properties of h_ι. We can also write $\tilde{h}_\iota = h_\iota\circ(f^{-1}|\ f(U_\iota))$, for $\iota \in I$. Clearly, the collection $\{\tilde{U}_\iota\}_{\iota\in I}$ is a covering of $f(M)$.

Now, choose $\iota \in I$ and consider $\tilde{h}_\iota^{-1} = f\circ h_\iota^{-1}$: $\tilde{h}_\iota(\tilde{U}_\iota) = h_\iota(U_\iota) \to \mathbb{R}^m$. We see immediately that $\tilde{h}_\iota^{-1} \in C^k(\tilde{h}_\iota(\tilde{U}_\iota);\mathbb{R}^m)$, since $f \in C^k(M;\mathbb{R}^m)$ requires that $f\circ h_\iota^{-1} \in C^k(h_\iota(U_\iota);\mathbb{R}^m)$. Suppose that $\hat{x} \in \tilde{h}_\iota(\tilde{U}_\iota) = h_\iota(U_\iota)$; because the rank of f at $h_\iota^{-1}(\hat{x}) \in M$ is r, we have rank $D(f \circ h_\iota^{-1})(\hat{x}) = r$, *i.e.*, rank $D\tilde{h}_\iota^{-1}(\hat{x}) = r$.

The existence of the collection $\{(\tilde{U}_\iota, \tilde{h}_\iota)\}_{\iota \in I}$ with these properties then shows, *via* Theorem [VI.20], that $f(M)$ is an $(r,m;k)$-manifold; (i) has been proven.

It is clear that we have also shown that whenever $\{(U_\iota, h_\iota)\}_{\iota \in I}$ is a family of coordinate systems in M, with $\{U_\iota\}_{\iota \in I}$ covering M, then $\{(f(U_\iota), (f \circ h_\iota^{-1})^{-1}\}_{\iota \in I}$ is a collection of coordinate systems in $f(M)$ such that $\{f(U_\iota)\}_{\iota \in I}$ covers $f(M)$.

(ii) Now, we know that $f(M)$ is an $(r,m;k)$-manifold. Since $f \in C^k(M;\mathbb{R}^m)$ and the rank of f at each $x \in M$ is r, (ii) follows from [VI.29.ii].

(iii) We already know that f^{-1}: $f(M) \to f^{-1}(f(M)) = M$ is a homeomorphism. To show that $f^{-1} \in C^k(f(M);\mathbb{R}^n)$, select a family of coordinate systems in M, $\{(U_\iota, h_\iota)\}_{\iota \in I}$, such that $\{U_\iota\}_{\iota \in I}$ covers M. Then $\{(f(U_\iota), (f \circ h_\iota^{-1})^{-1}\}_{\iota \in I}$ is a covering collection of coordinate systems for $f(M)$. According to [VI.27], the inclusion $f^{-1} \in C^k(f(M);\mathbb{R}^n)$ shall follow once it has been shown that $f^{-1} \circ (f \circ h_\iota^{-1}) \in C^k((f \circ h_\iota^{-1})^{-1}(f(U_\iota));\mathbb{R}^n)$ for each $\iota \in I$. But the latter is clear, from $f^{-1} \circ (f \circ h_\iota^{-1}) = h_\iota^{-1}$, $(f \circ h_\iota^{-1})^{-1}(f(U_\iota)) = h_\iota(U_\iota)$, $h_\iota^{-1} \in C^q(h_\iota(U_\iota);\mathbb{R}^n)$, for each $\iota \in I$, and $k \leq q$. Finally, we must verify that the rank of f^{-1} at each point of $f(M)$ is r. For this, choose $x \in f(M)$, and let $\iota \in I$ be such that $x \in f(U_\iota)$, where $\{(U_\iota, h_\iota)\}_{\iota \in I}$ is as before. Then $(f(U_\iota), (f \circ h_\iota^{-1})^{-1})$ is a coordinate system in $f(M)$ with $x \in f(U_\iota)$, so it suffices to show that $D(f^{-1}\circ(f\circ h_\iota^{-1}))((f\circ h_\iota^{-1})^{-1}(x)) = Dh_\iota^{-1}((f\circ h_\iota^{-1})^{-1}(x))$ has rank r. But $(f\circ h_\iota^{-1})^{-1}(x) \in h_\iota(U_\iota)$, while $Dh_\iota^{-1}(\hat{y})$ has rank r for each $\hat{y} \in h_\iota(U_\iota)$. Thus, f^{-1} is a k-imbedding.

(iv) Choose $x \in M$. We know that $Df(x)$ is an injection taking $T_M(x)$ onto $T_{f(M)}(f(x))$, and (because of (iii) and [VI.29.ii]) $Df^{-1}(f(x))$ is an injection of $T_{f(M)}(f(x))$ onto $T_M(x)$. *Consequently*, it suffices, for the proof of (iv), to show that, say, $Df^{-1}(f(x))\circ Df(x) = i_{T_M(x)}$, the identity operator on $T_M(x)$. Let (U,h) be a coordinate system in M, with $x \in U$; $(f(U),(f\circ h^{-1})^{-1})$ is a coordinate system in $f(M)$, with $f(x) \in f(U)$. According to the definition in [VI.23.ii], we have, on $T_M(x)$, using $(f\circ h^{-1})^{-1} = h\circ(f^{-1}|\ f(U))$,

$$Df^{-1}(f(x))\circ Df(x)$$

$$= D(f^{-1}\circ(f\circ h^{-1}))((f\circ h^{-1})^{-1}(f(x)))\circ\{D(f\circ h^{-1})((f\circ h^{-1})^{-1}(f(x)))\}^{-1}$$

$$\circ D(f\circ h^{-1})(h(x))\circ\{Dh^{-1}(h(x))\}^{-1}$$

$$= Dh^{-1}(h(x))\circ\{D(f\circ h^{-1})(h(x))\}^{-1}\circ D(f\circ h^{-1})(h(x))\circ\{Dh^{-1}(h(x))\}^{-1}$$

$$= i_{T_M(x)},$$

as required.

(v) Choose $x \in M$. Let $\{T_1(x),\dots,T_r(x)\}$ be a basis for $T_M(x)$. By (iv), $Df^{-1}(f(x))\circ Df(x)T_i(x) = T_i(x)$, for $i = 1,\dots,r$, whence

$$\frac{|Df(x)T_1(x)\wedge\ldots\wedge Df(x)T_r(x)|}{|T_1(x)\wedge\ldots\wedge T_r(x)|}$$

$$\cdot\frac{|Df^{-1}(f(x))\circ Df(x)T_1(x)\wedge\ldots\wedge Df^{-1}(f(x))\circ Df(x)T_r(x)|}{|Df(x)T_1(x)\wedge\ldots\wedge Df(x)T_r(x)|} = 1.$$

But $Df(x)$ is an injection carrying $T_M(x)$ onto $T_{f(M)}(f(x))$, so $\{Df(x)T_1(x),\ldots,Df(x)T_r(x)\}$ is a basis for $T_{f(M)}(f(x))$ (*which also* shows that $|Df(x)T_1(x)\wedge\ldots\wedge Df(x)T_r(x)| \neq 0$). In view of the definition in [VI.23.iii], the preceding equality is just $\mathcal{J}f(x)\cdot\mathcal{J}f^{-1}(f(x)) = 1$. □.

[VI.31] R E M A R K. Maintain the setting and notation of [VI.30]. Let $x \in M$, and (U,h) be a coordinate system in M, with $x \in U$. Since $\{h^{-1}_{,i}(h(x))\}_{i=1}^{r}$ forms a basis for $T_M(x)$, it is clear that $\{(f\circ h^{-1})_{,i}(h(x))\}_{i=1}^{r}$ forms a basis for $T_{f(M)}(f(x))$, since $Df(x)h^{-1}_{,i}(h(x)) = (f\circ h^{-1})_{,i}(h(x))$, for $i = 1,\ldots,r$; cf., [VI.24.f]. Also, $\mathcal{J}f(x)$ can be computed from (VI.24.3).

We shall prepare a statement concerning composite functions in a somewhat restricted setting; as it turns out, this is all that we require.

[VI.32] P R O P O S I T I O N. *Let* M *be an* $(r,n;q)$-*manifold*, $f\colon M \to \mathbb{R}^m$ *a* q-*imbedding, and* $g \in C^{\ell}(f(M);\mathbb{R}^k)$, *where* $\ell \in \mathbb{N}\cup\{\infty\}$, $\ell \leq q$. *Then*

(*i*) $g\circ f \in C^{\ell}(M;\mathbb{R}^k)$;

(ii) $D(g \circ f)(x) = Dg(f(x)) \circ Df(x)$, *for each* $x \in M$;

(iii) $J(g \circ f)(x) = Jg(f(x)) \circ Jf(x)$, *for each* $x \in M$;

(iv) *if* g *is an* ℓ*-imbedding, then* $g \circ f$ *is an* ℓ*-imbedding.*

P R O O F. Note that $f(M)$ is an $(r,m;q)$-manifold (and, if g is an ℓ-imbedding, then $g(f(M))$ is an $(r,k;\ell)$-manifold), by [VI.30.i].

(i) Let (U,h) be a coordinate system in M. Setting $\tilde{U} := f(U)$ and $\tilde{h} := (f \circ h^{-1})^{-1} = h \circ (f^{-1}| \ f(U))$, it is easy to see, as in the proof of [VI.30.i], that $(\tilde{U},\tilde{h})$ is a coordinate system in $f(M)$, since f is a q-imbedding of M into $\mathbb{R}^m$. We know that $g \in C^{\ell}(f(M);\mathbb{R}^k)$, so $g \circ \tilde{h}^{-1} \in C^{\ell}(\tilde{h}(\tilde{U});\mathbb{R}^k)$, *i.e.*, $(g \circ f) \circ h^{-1} \in C^{\ell}(h(U);\mathbb{R}^k)$ (obviously, $\tilde{h}(\tilde{U}) = h(U)$). Thus, $g \circ f \in C^{\ell}(M;\mathbb{R}^k)$.

(ii) Let $x \in M$. Choose any coordinate system (U,h) in M such that $x \in U$. Recalling [VI.23.ii], we have

$$D(g \circ f)(x) := D(g \circ f \circ h^{-1})(h(x)) \circ \{Dh^{-1}(h(x))\}^{-1}, \tag{1}$$

and

$$Df(x) := D(f \circ h^{-1})(h(x)) \circ \{Dh^{-1}(h(x))\}^{-1}. \tag{2}$$

Define the coordinate system $(\tilde{U},\tilde{h})$ in M as in the proof of (i); then $f(x) \in \tilde{U}$, and so

$$\begin{aligned} Dg(f(x)) &:= D(g\circ\tilde{h}^{-1})(\tilde{h}(f(x)))\circ\{D\tilde{h}^{-1}(\tilde{h}(f(x)))\}^{-1} \\ &= D(g\circ f\circ h^{-1})(h(x))\circ\{D(f\circ h^{-1})(h(x))\}^{-1} \end{aligned} \tag{3}$$

(according to [VI.12], $\{Dh^{-1}(h(x))\}^{-1}$ denotes the inverse of $Dh^{-1}(h(x))$ taking $\mathbb{R}^r$ onto $T_M(x)$, while $\{D(f\circ h^{-1})(h(x))\}^{-1}$ denotes the inverse of $D(f\circ h^{-1})(h(x))$ taking $\mathbb{R}^r$ onto $T_{f(M)}(f(x))$. Now, the equality $D(g\circ f)(x) = Dg(f(x))\circ Df(x)$ clearly follows from (1), (2), and (3).

(iii) Let $x \in M$; choose any basis $\{T_i(x)\}_{i=1}^r$ for $T_M(x)$. Then $\{Df(x)T_i(x)\}_{i=1}^r$ is a basis for $T_{f(M)}(f(x))$ (from [VI.30.ii]), so the definition [VI.23.iii], with (ii), shows that

$$\begin{aligned} J(g\circ f)(x) &:= \frac{|D(g\circ f)(x)T_1(x)\wedge\ldots\wedge D(g\circ f)(x)T_r(x)|}{|T_1(x)\wedge\ldots\wedge T_r(x)|} \\ &= \frac{|Dg(f(x))Df(x)T_1(x)\wedge\ldots\wedge Dg(f(x))Df(x)T_r(x)|}{|Df(x)T_1(x)\wedge\ldots\wedge Df(x)T_r(x)|} \\ &\quad\cdot \frac{|Df(x)T_1(x)\wedge\ldots\wedge Df(x)T_r(x)|}{|T_1(x)\wedge\ldots\wedge T_r(x)|} \\ &= Jg(f(x))\cdot Jf(x). \end{aligned}$$

Thus, (iii) is proven.

(iv) Now, suppose that g is known to be an ℓ-imbedding. Then $f\colon\ M \to f(M)$ and $g\colon\ f(M) \to g(f(M))$ are homeomorphisms, so $g\circ f\colon\ M \to g(f(M))$ is a homeomorphism as well. We have already seen that $g \circ f \in C^{\ell}(M;\mathbb{R}^k)$. Finally, since $\mathcal{J}g > 0$ on $f(M)$ and $\mathcal{J}f > 0$ on M, (iii) shows that $\mathcal{J}(g \circ f) > 0$ on M, whence the rank of $g \circ f$ at x is r, for each $x \in M$ (cf., [VI.24.e]). These facts show that $g \circ f$ is an ℓ-imbedding. $\square$.

The following geometric fact is in accord with one's intuition.

[VI.33] PROPOSITION. *Let* M *be an* $(r,n;q)$*-manifold which is closed in* $\mathbb{R}^n$. *Let* $x \in \mathbb{R}^n$. *Then there exists at least one* $z_x \in M$ *for which*

$$r_x(z_x) = \text{dist } (x,M) := \inf \{r_x(z) \mid z \in M\}. \tag{1}$$

Moreover, whenever $z_x \in M$ *satisfies* (1), *then*

$$(x-z_x) \in N_M(z_x). \tag{2}$$

PROOF. The first statement is, of course, well known (and holds for any closed subset of $\mathbb{R}^n$): we can find a sequence in M, $(z_i)_{i=1}^{\infty}$, such that $r_x(z_i) \to \text{dist } (x,M)$. It is easy to see that $(z_i)_{i=1}^{\infty}$ is Cauchy in $\mathbb{R}^n$, hence converges to some z_x, which must then be in the closed set M. Finally, $r_x(z_x) = \lim_{i \to \infty} r_x(z_i) = \text{dist } (x,M)$.

Now, suppose $z_x \in M$ and (1) holds. Consider any $z \in M$. We compute

$$|x-z|_n^2 = |(x-z_x)-(z-z_x)|_n^2 = |x-z_x|_n^2+|z-z_x|_n^2-2(x-z_x)\bullet(z-z_x),$$

giving, since $|x-z_x|_n \leq |x-z|_n$, by (1),

$$2(x-z_x)\bullet(z-z_x) = \{|x-z_x|_n^2-|x-z|_n^2\}+|z-z_x|_n^2 \leq |z-z_x|_n^2. \tag{3}$$

Now, choose any $\alpha \in T_M(z_x)$. By Definition [VI.5], there exists a $\delta > 0$ and a function $\psi \in C^1((-\delta,\delta);\mathbb{R}^n)$ such that $\psi(s) \in M$ if $|s| < \delta$, $\psi(0) = z_x$, and $\psi'(0) = \alpha$. Whenever $0 < s < \delta$, (3) shows that

$$2(x-z_x)\bullet(\psi(s)-\psi(0)) \leq |\psi(s)-\psi(0)|_n^2,$$

$$2(x-z_x)\bullet\{-(\psi(-s)-\psi(0))\} \geq -|\psi(-s)-\psi(0)|_n^2,$$

so

$$2(x-z_x)\bullet\left\{\frac{\psi(s)-\psi(0)}{s}\right\} \leq |\psi(s)-\psi(0)|_n\cdot\left|\frac{\psi(s)-\psi(0)}{s}\right|_n, \tag{4}$$

and

$$2(x-z_x)\bullet\left\{\frac{\psi(-s)-\psi(0)}{(-s)}\right\} \geq -|\psi(-s)-\psi(0)|_n\cdot\left|\frac{\psi(-s)-\psi(0)}{(-s)}\right|_n . \tag{5}$$

Letting $s \to 0^+$ in (4) and (5) results in $0 \leq 2(x-z_x)\bullet\alpha \leq 0$. Thus, $(x-z_x)$ is in the orthogonal complement of $T_M(z_x)$, *i.e.*, is in $N_M(z_x)$. □.

There are, of course, standard techniques for constructing extensions to $\mathbb{R}^n$ for smooth functions on an $(n-1,n;q)$-manifold. We have need of a special result of this sort. It is essentially no more work to consider manifolds of lower dimension, as well.

[VI.34] P R O P O S I T I O N. *Let* M *be an* $(r,n;1)$*-manifold,*

$r < n$, *and* $g \in C^1(M;\mathbb{R}^m)$, *with* $\sup_{y \in M} |g(y)|_m < \infty$. *Let* Γ *be a non-void compact subset of* M. *Then there exists a* $\tilde{g} \in C_0^1(\mathbb{R}^n;\mathbb{R}^m)$ *such that* $\tilde{g}|\,\Gamma = g|\,\Gamma$ *and*

$$\max_{y \in \mathbb{R}^n} |\tilde{g}(y)|_m \leq \sup_{y \in M} |g(y)|_m. \tag{1}$$

P R O O F. Choose $x \in M$. Just as in the proof of Theorem [VI.20], we can find an increasing r-tuple of integers in $\{1,\ldots,n\}$, $\lambda(x)$, and an open neighborhood U_x of x in $\mathbb{R}^n$ such that $(M \cap U_x, \Xi^{\lambda(x)}|\, M \cap U_x)$ is a coordinate system in M, with $x \in M \cap U_x$. Then $\Xi^{\lambda(x)}(M \cap U_x)$ is an open neighborhood of $x^{\lambda(x)} := \Xi^{\lambda(x)}(x)$ in $\mathbb{R}^r$, so we can choose $\rho_x > 0$ such that $B^r_{\rho_x}(x^{\lambda(x)}) \subset \Xi^{\lambda(x)}(M \cap U_x)$ as well as $B^n_{\rho_x}(x) \subset U_x$. Now, whenever $y \in B^n_{\rho_x}(x)$,

$$|\Xi^{\lambda(x)}(y) - \Xi^{\lambda(x)}(x)|_r = |\Xi^{\lambda(x)}(y) - x^{\lambda(x)}|_r \leq |y-x|_n < \rho_x,$$

so $\Xi^{\lambda(x)}(y) \in B^r_{\rho_x}(x^{\lambda(x)})$. Therefore, we can define $g_x\colon B^n_{\rho_x}(x) \to \mathbb{R}^m$ according to

$$g_x(y) := g \circ (\Xi^{\lambda(x)}|\; M \cap U_x)^{-1} \circ \Xi^{\lambda(x)}(y), \quad \text{for each} \quad y \in B^n_{\rho_x}(x). \tag{2}$$

Now, $g \in C^1(M;\mathbb{R}^m)$, so $g \circ (\Xi^{\lambda(x)}|\; M \cap U_x)^{-1} \in C^1(\Xi^{\lambda(x)}(M \cap U_x);\mathbb{R}^m)$, whence $g_x \in C^1(B^n_{\rho_x}(x);\mathbb{R}^m)$. Since $M \cap B^n_{\rho_x}(x) \subset M \cap U_x$, directly from (2) we have

$$g_x(y) = g(y) \quad \text{for each} \quad y \in M \cap B^n_{\rho_x}(x), \tag{3}$$

i.e., g_x is an extension of $g|_{M\cap B^n_{\rho_x}(x)}$ to $B^n_{\rho_x}(x)$. Obviously, (2) also shows that

$$\sup_{y \in B^n_{\rho_x}(x)} |g_x(y)|_m \leq \sup_{y \in M\cap U_x} |g(y)|_m \leq \sup_{y \in M} |g(y)|_m. \tag{4}$$

Now, choose a finite set $\{x_i\}_{i=1}^N \subset \Gamma$ such that the collection $\{B^n_{\rho_{x_i}}(x_i)\}_{i=1}^N$ provides a covering of Γ. For brevity, write $\mathcal{O}_i := B^n_{\rho_{x_i}}(x_i)$ and $g_i := g_{x_i}$, for each $i \in \{1,\dots,N\}$. For convenience, we may, and shall, suppose that the covering $\{\mathcal{O}_i\}_{i=1}^N$ is minimal, *i.e.*, no proper sub-collection of $\{\mathcal{O}_i\}_{i=1}^N$ provides a cover for Γ, since it is clear that we can extract such a minimal subcover from the original cover, if the latter does not already possess this property. Thus, for each $i \in \{1,\dots,N\}$, $\mathcal{O}_i\cap\Gamma \neq \phi$, and there exists some $z_i \in \mathcal{O}_i\cap\Gamma$ such that $z_i \in \mathcal{O}_j'$ if $j \in \{1,\dots,N\}$ and $j \neq i$. Now, let Ψ be a locally finite C^∞-partition of unity for $\cup_{i=1}^N \mathcal{O}_i$, subordinate to $\{\mathcal{O}_i\}_{i=1}^N$ (cf., Lemma [VI.49], *infra*). Thus,

(i) $\Psi \subset C_0^\infty(\mathbb{R}^n)$,

(ii) $0 \leq \psi \leq 1$, for each $\psi \in \Psi$,

(iii) for each $\psi \in \Psi$, there exists $i_\psi \in \{1,\dots,N\}$ with $\operatorname{supp} \psi \subset \mathcal{O}_{i_\psi}$,

(iv) whenever $K \subset \cup_{i=1}^N \mathcal{O}_i$ is compact, there exists an open set $W \subset \cup_{i=1}^{N} \mathcal{O}_i$ such that $K \subset W$ and all but a finite number of elements of Ψ vanish in W,

and

(v) $$\sum_{\psi\in\Psi} \psi(x) = 1 \quad \text{for each} \quad x \in \cup_{i=1}^{N} \mathcal{O}_i.$$

Set

$$\Psi_\Gamma := \{\psi \in \Psi \mid \psi(\Gamma) \neq \{0\}\};$$

since Γ is a non-void compact subset of $\cup_{i=1}^{N} \mathcal{O}_i$, properties (iv) and (v) show that Ψ_Γ is non-void and finite, and

$$\sum_{\psi\in\Psi_\Gamma} \psi(x) = \sum_{\psi\in\Psi} \psi(x) = 1 \quad \text{for each} \quad x \in \Gamma. \tag{5}$$

Define

$$\Psi_1 := \{\psi \in \Psi_\Gamma \mid \operatorname{supp} \psi \subset \mathcal{O}_1\},$$

$$\Psi_j := \{\psi \in \Psi_\Gamma \mid \operatorname{supp} \psi \subset \mathcal{O}_j\} \cap \{\cup_{k=1}^{j-1} \Psi_k\}', \quad \text{for} \quad j = 2,\dots,N.$$

Obviously, the collection $\{\Psi_i\}_{i=1}^{N}$ is pairwise disjoint, and we have $\cup_{i=1}^{N} \Psi_i \subset \Psi_\Gamma$. But if $\psi \in \Psi_\Gamma$, there exists a smallest integer in $\{1,\dots,N\}$, j_ψ, such that $\operatorname{supp} \psi \subset \mathcal{O}_{j_\psi}$, and it is clear that we must have, therefore, $\psi \in \Psi_{j_\psi}$. Consequently, we conclude that

$$\Psi_\Gamma = \cup_{i=1}^{N} \Psi_i. \tag{6}$$

It is also easy to see that, if $i \in \{1,\dots,N\}$, $\Psi_i \neq \phi$. For, recalling the properties of $z_i \in \mathcal{O}_i \cap \Gamma$, introduced previously, we can choose $\psi_{z_i} \in \Psi_\Gamma$ such that $\psi_{z_i}(z_i) \neq 0$, by (5); if $j \in \{1,\ldots,N\}$ and $j \neq i$, then $z_i \in \mathcal{O}_j'$, so supp ψ_{z_i} cannot lie in $\mathcal{O}_j$. Thus, supp $\psi_{z_i} \subset \mathcal{O}_i$, and $\psi_{z_i} \in \Psi_i$, verifying our claim. Each Ψ_i is clearly finite. Defining

$$\psi_i := \sum_{\psi \in \Psi_i} \psi \quad \text{for each} \quad i \in \{1,\ldots,N\}, \tag{7}$$

it is easy to see that $\psi_i \in C_0^\infty(\mathbb{R}^n)$ and supp $\psi_i \subset \mathcal{O}_i$ for each i, while

$$\sum_{i=1}^N \psi_i(x) = \sum_{i=1}^N \sum_{\psi \in \Psi_i} \psi(x) = \sum_{\psi \in \Psi_\Gamma} \psi(x) = 1 \quad \text{for} \quad x \in \Gamma, \tag{8}$$

by (5) and the properties proven for $\{\Psi_i\}_{i=1}^N$. Next, for each $i \in \{1,\ldots,N\}$, define $\hat{g}_i\colon \mathbb{R}^n \to \mathbb{R}^m$ by

$$\hat{g}_i(x) := \begin{cases} \psi_i(x) g_i(x) & \text{if} \quad x \in \mathcal{O}_i, \\ 0 & \text{if} \quad x \in \mathcal{O}_i'. \end{cases} \tag{9}$$

Since $g_i \in C^1(\mathcal{O}_i;\mathbb{R}^m)$, $\psi_i \in C_0^\infty(\mathbb{R}^n)$, and supp $\psi_i \subset \mathcal{O}_i$, we must have $\hat{g}_i \in C_0^1(\mathbb{R}^n;\mathbb{R}^m)$ with supp $\hat{g}_i \subset \mathcal{O}_i$, for each $i \in \{1,\ldots,N\}$. Finally, set

$$\tilde{g} := \sum_{i=1}^N \hat{g}_i; \tag{10}$$

we claim that $\tilde{g}$ has each of the desired properties. The inclusion $\tilde{g} \in C_0^1(\mathbb{R}^n;\mathbb{R}^m)$ is plain enough. To see that $\tilde{g}$ is an extension of g, observe first that

$$\hat{g}_i(x) = \psi_i(x)\cdot g(x) \quad \text{for each} \quad x \in \Gamma \quad \text{and} \quad i \in \{1,\dots,N\}. \qquad (11)$$

Indeed, suppose that $i \in \{1,\dots,N\}$ and $x \in \Gamma$: if $x \in \mathcal{O}_i'$, then $\hat{g}_i(x) = 0$ and $\psi_i(x) = 0$, while if $x \in \mathcal{O}_i$, then (3) shows that $\psi_i(x)\cdot g(x) = \psi_i(x)\cdot g_i(x) := \hat{g}_i(x)$. Thus, (11) is true. But then, whenever $x \in \Gamma$, in view of (8) we can write

$$\tilde{g}(x) := \sum_{i=1}^{N} \hat{g}_i(x) = \left\{\sum_{i=1}^{N} \psi_i(x)\right\} g(x) = g(x),$$

whence $\tilde{g}|\,\Gamma = g|\,\Gamma$. Finally, to verify inequality (1), let $x \in \mathbb{R}^n$: if $x \in \bigcup_{i=1}^N \mathcal{O}_i$, then, using (4),

$$\begin{aligned} |\tilde{g}(x)| &\le \sum_{i=1}^{N} |\hat{g}_i(x)|_m \\ &= \sum_{\{i|\ x\in\mathcal{O}_i\}} \psi_i(x)\cdot|g_i(x)|_m \\ &\le \sup_{y \in M} |g(y)|_m \cdot \sum_{\{i|\ x\in\mathcal{O}_i\}} \psi_i(x) \\ &\le \sup_{y \in M} |g(y)|_m \cdot \sum_{i=1}^{N} \sum_{\psi\in\Psi_i} \psi(x) \\ &\le \sup_{y \in M} |g(y)|_m ; \end{aligned}$$

on the other hand, if $x \in (\bigcup_{i=1}^N \mathcal{O}_i)' = \bigcap_{i=1}^N \mathcal{O}_i'$, then

$$\tilde{g}(x) = \sum_{i=1}^{N} \hat{g}_i(x) = 0.$$

Thus, (1) holds. $\square$.

The development to this point provides sufficient preparation for the definition and study of the Lebesgue measure and integral on a manifold in some Euclidean space. Fleming [15] gives some discussion of these topics, but his presentation is inadequate for our purposes; a precise formulation is required here in order to meet the exigencies of a number of lines of reasoning in Parts I-V.

We begin by citing certain measure-theoretic facts, the principal references being Hewitt and Stromberg [20], and Rudin [46]. The definition of a common measure-theoretic term will be set down here only if these sources employ distinct definitions for that term; otherwise, such basic terms will be used without preliminary comment. In general, we shall adhere to the definitions of Hewitt and Stromberg [20].

[VI.35] D E F I N I T I O N. Let X be a locally compact Hausdorff space, and denote the σ-algebra of Borel sets of X by $\mathcal{B}(X)$. Let μ be a measure defined on a σ-algebra $\mathcal{A}$ of subsets of X, such that $\mathcal{B}(X) \subset \mathcal{A}$. Then μ is called a *regular measure* iff

(i) $\mu(K) < \infty$, for each compact $K \subset X$,

(ii) $\mu(A) = \inf\{\mu(U) \mid U \text{ is open in } X,\ A \subset U\}$, for each $A \in \mathcal{A}$,

and

(iii) $\mu(U) = \sup\{\mu(K) \mid K \text{ is compact in } X,\ K \subset A\}$, for each open set $U \subset X$. ■.

It turns out that a regular measure possesses a property stronger than [VI.35.iii].

[VI.36] PROPOSITION. *Let* μ *be a regular measure defined on a* σ*-algebra* A *of subsets of a locally compact Hausdorff space* X *(so* $B(X) \subset A$*). Then*

$$\mu(A) = \sup \{\mu(K) \mid K \text{ is compact in } X, \quad K \subset A\},$$

for each $A \in A$ *which is* σ*-finite with respect to* μ.

PROOF. Cf., Hewitt and Stromberg [20]. □.

[VI.37] RECAPITULATION: THE EXPLICIT CONSTRUCTION OF A REPRESENTING MEASURE CORRESPONDING TO A GIVEN RADON MEASURE. Let X be a locally compact Hausdorff space. Let $C_0(X)$ denote the complex linear space composed of all complex-valued continuous functions of compact support on X. Recall that a *Radon measure*, or *nonnegative linear functional*, on X is a linear functional $I: C_0(X) \to \mathbb{K}$ such that $I(f) \geq 0$ whenever $f \in C_0(X)$ and $f \geq 0$. Let I be a Radon measure on X. Hewitt and Stromberg [20] contains the explicit construction, from I, of a σ-algebra M_ι of subsets of X, and a measure ι on M_ι such that

$$(1) \qquad I(f) = \int_X f \, d\iota, \quad \text{for each } f \in C_0(X),$$

(ii) $\mathcal{B}(X) \subset \mathcal{M}_\iota$,

(iii) ι is regular,

(iv) $(X, \mathcal{M}_\iota, \iota)$ is a complete measure space,

(v) if $A \subset X$, then $A \in \mathcal{M}_\iota$ iff $A \cap K \in \mathcal{M}_\iota$ for each compact $K \subset X$,

and

(vi) if μ is any regular measure on $\mathcal{M}_\iota$ such that $I(f) = \int_X f \, d\mu$ for each $f \in C_0(X)$, then $\mu = \iota$.

There are certain other technical results of the construction, which we shall not give here; these results shall be used implicitly, in the sense that they are used to prove other statements which we shall later provide explicitly. In all such cases, we shall refer to the work of Hewitt and Stromberg for the proofs.

In order to have a precise nomenclature, we shall call any measure generated from a nonnegative linear functional on $C_0(X)$, where X is a locally compact Hausdorff space, by the particular construction cited above, *a measure in the sense of* [Hewitt and Stromberg, §9].

Of course, the well-known representation theorem of F. Riesz is an immediate consequence of the facts given above. Since we shall need a number of other properties of the representing measure whose existence is the assertion of the Riesz theorem, we have chosen the more detailed presentation of Hewitt and Stromberg as our primary source, rather than merely stating Riesz' theorem.

It is important to note that in our terminology, the usual Lebesgue measure on $\mathbb{R}^n$ $(n \in \mathbb{N})$ is a measure in the sense of [Hewitt and Stromberg, §9]. Indeed, one of the standard ways of defining Lebesgue measure is *via* the Riemann integral on $C_0(\mathbb{R}^n)$, clearly a Radon measure. We shall denote Lebesgue measure on $\mathbb{R}^n$ by λ_n (so that M_{λ_n} denotes the σ-algebra of Lebesgue-measurable subsets of $\mathbb{R}^n$).

We next recount some facts concerning another familiar method for constructing measures.

[VI.38] PROPOSITION. *Let* (X,A,μ) *be a measure space, and* $\psi\colon X \to [0,\infty]$ *an* A*-measurable function. Define* $\mu_\psi\colon A \to [0,\infty]$ *by*

$$\mu_\psi(A) := \int_A \psi \, d\mu := \int_X \Xi_A \psi \, d\mu, \quad \textit{for each} \quad A \in A. \tag{1}$$

Then

(i) μ_ψ *is a measure on* A, *and*

$$\int_X f \, d\mu_\psi = \int_X f\psi \, d\mu, \tag{2}$$

for each A*-measurable function* $f\colon X \to [0,\infty]$.

(ii) Whenever $A \in A$ *and* $\mu(A) = 0$, *then* $\mu_\psi(A) = 0$.

If $\psi > 0$, *then* $\mu(A) = 0$ *if* $A \in \mathcal{A}$ *with* $\mu_\psi(A) = 0$. *Thus, if* $\psi > 0$, $(X,\mathcal{A},\mu_\psi)$ *is complete iff* $(X,\mathcal{A},\mu)$ *is complete.*

(iii) *If* f *is defined* μ*-a.e. on* X *and is* $\mathcal{A}$*-measurable, then* $f \in L_1(X,\mathcal{A},\mu_\psi)$ *iff* $f\psi \in L_1(X,\mathcal{A},\mu)$; *in either case,* (2) *holds.*

(iv) *Suppose* $\psi > 0$. *Then* $f \in L_1(X,\mathcal{A},\mu_\psi)$ *iff* $f\psi \in L_1(X,\mathcal{A},\mu)$; *in either case,* (2) *holds.*

(v) *Suppose that*

(1) X *is a locally compact* σ*-compact Hausdorff space with* $\mathcal{B}(X) \subset \mathcal{A}$,

(2) μ *is regular,*

and

(3) $\psi \in L_1^{\mathrm{loc}}(X,\mathcal{A},\mu)$, *i.e.,* $\psi\Xi_K \in L_1(X,\mathcal{A},\mu)$ *for each compact* $K \subset X$.

Then μ_ψ *is regular and* σ*-finite.*

PROOF. (i) Cf., Rudin [46], Theorem 1.29.

(ii) The first statement is obvious. Suppose $\psi > 0$, $A \in \mathcal{A}$, and $\mu_\psi(A) = 0$. Then, since $\Xi_A\psi \geq 0$, $\Xi_A\psi$ must vanish μ-*a.e.* on X. Since $\psi > 0$, $\Xi_A = 0$ μ-*a.e.* on X, *i.e.*, $\mu(A) = 0$, and the second statement is proven. For the third, let $\psi > 0$ and $(X,\mathcal{A},\mu)$ be complete. Suppose $A \in \mathcal{A}$, $\mu_\psi(A) = 0$, and $B \subset A$.

Then $\mu(A) = 0$, by what was just proven, so $B \in A$, since (X,A,μ) is complete. Thus, (X,A,μ_ψ) is complete. The proof of the converse is just as simple (and goes through even if ψ is only nonnegative).

(iii) We may suppose that f is defined on X and is A-measurable. Then $f\psi$ is A-measurable on X. From (2),

$$\int_X |f|\, d\mu_\psi = \int_X |f|\cdot\psi\, d\mu, \quad \text{so} \quad \int_X |f|\, d\mu_\psi < \infty \quad \text{iff} \quad \int_X |f|\cdot\psi\, d\mu < \infty,$$

and the first statement of (iii) follows. If, say, $f\psi \in L_1(X,A,\mu)$, the equality $\int_X f\, d\mu_\psi = \int_X f\psi\, d\mu$ follows from (i) and the definition of the integral of a complex-valued function in terms of integrals of nonnegative functions.

(iv) Observe that, since $\psi > 0$, a function f is defined μ_ψ-*a.e.* on X iff it is defined μ-*a.e.* on X, and is A-measurable iff $f\psi$ is A-measurable (if $f\psi$ is A-measurable, the equality $f = \frac{1}{\psi} f\psi$ shows that f is A-measurable). The proof of (iv) can now be completed by using reasoning similar to that employed in the proof of (iii).

(v) In passing, note that μ is σ-finite under the hypotheses given, since $\mu(K) < \infty$ for each compact $K \subset X$, and X is the union of a countable family of compact sets.

For any compact $K \subset X$, we have $\mu_\psi(K) = \int_X \xi_K\psi\, d\mu < \infty$, since $\Xi_K\psi \in L_1(X,A,\mu)$. This shows at once that μ_ψ fulfills requirement [VI.35.i] and that μ_ψ is σ-finite, since X is σ-compact.

We must show that, whenever $A \in A$,

$$\mu_\psi(A) = \inf\{\mu_\psi(V) \mid V \text{ is open in } X,\ A \subset V\}; \tag{3}$$

we know that this holds with μ replacing μ_ψ. Suppose first that A^- is compact. Let $(\tilde{U}_n)_1^\infty$ be a sequence of open sets in X with $A \subset \tilde{U}_n$ for each $n \in \mathbb{N}$ and $\lim_{n\to\infty} \mu(\tilde{U}_n) = \mu(A)$. Since A^- is compact and X is a locally compact Hausdorff space, we know that there exists an open set $U_0 \subset X$ such that $A^- \subset U_0$ and U_0^- is compact (cf., *e.g.*, Hewitt and Stromberg [20], Theorem (6.79)), so that, replacing each $\tilde{U}_n$ by $\tilde{U}_n \cap U_0$, if necessary, we may suppose that $\tilde{U}_n^-$ is compact, for each $n \in \mathbb{N}$. Setting $U_n := \cap_{j=1}^n \tilde{U}_n$ for each $n \in \mathbb{N}$, we obtain a sequence $(U_n)_1^\infty$ of open sets in X such that $A \subset U_{n+1} \subset U_n$ and U_n^- is compact, for each $n \in \mathbb{N}$; since $0 \le \mu(U_n)-\mu(A) \le \mu(\tilde{U}_n)-\mu(A)$, $n \in \mathbb{N}$, it is clear that $\mu(U_n) \downarrow \mu(A)$. The fact that $(U_n)_1^\infty$ is decreasing, with $\mu(U_1) < \infty$, gives $\mu(\cap_1^\infty U_n) = \lim_{n\to\infty} \mu(U_n)$. Thus, setting $A_0 := \cap_1^\infty U_n$, we have $\mu(A_0) = \mu(A)$, and $A \subset A_0$. Consequently, $\Xi_{A_0} = \Xi_A$ μ-*a.e.* on X. For, $\Xi_{A_0}-\Xi_A$ is non-zero only on the set $A_0 \cap A'$, while $\mu(A_0\cap A') = \mu(A_0)-\mu(A_0\cap A) = \mu(A_0)-\mu(A) = 0$. It is easy to show that the sequence $(\Xi_{U_n})_1^\infty$ is non-increasing, and converges pointwise on X to Ξ_{A_0}. Thus, since $\psi \geq 0$, $(-\Xi_{U_n}\psi)_1^\infty$ is non-decreasing, non-positive, and converges pointwise to $-\Xi_{A_0}\psi$. Further, $\int_X \Xi_{U_1}\psi\, d\mu < \infty$, since U_1^- is compact. Using B. Levi's theorem (Hewitt and Stromberg [20], Theorem (12.22)) to justify the second equality, we can then write

$$\begin{aligned}
\lim_{n\to\infty} \mu_\psi(U_n) &= \lim_{n\to\infty} \int_X \Xi_{U_n}\psi\, d\mu \\
&= \int_X \lim_{n\to\infty} \Xi_{U_n}\psi\, d\mu \\
&= \int_X \Xi_{A_0}\psi\, d\mu \\
&= \int_X \Xi_A\psi\, d\mu + \int_X (\Xi_{A_0}-\Xi_A)\psi\, d\mu \\
&= \int_X \Xi_A\psi\, d\mu \\
&= \mu_\psi(A).
\end{aligned}$$

This clearly suffices to prove that (3) is true in this case, in which $A \in \mathcal{A}$, A^- compact.

To prove (3) in the general case, first note that there exists a collection $\{F_n\}_1^\infty$ of pairwise disjoint relatively compact sets in $\mathcal{A}$ such that $X = \cup_1^\infty F_n$: simply choose a family $\{\tilde{F}_n\}_1^\infty$ of compact subsets of X such that $X = \cup_1^\infty \tilde{F}_n$ (X is σ-compact), and set $F_1 := \tilde{F}_1$, $F_n := \tilde{F}_n \cap (\cup_{j=1}^{n-1} F_j)'$ for $n = 2,3,\ldots$; it is routine to show that $\{F_n\}_1^\infty$ possesses the requisite properties. Now, choose $A \in \mathcal{A}$. Clearly, $\{A \cap F_n\}_1^\infty$ is a collection of pairwise disjoint relatively compact sets in $\mathcal{A}$, with $A = \cup_1^\infty (A \cap F_n)$. Let $\varepsilon > 0$. By what was just proven, we can select, for each $n \in \mathbb{N}$, an open set $U_{\varepsilon n} \subset X$ such that $A \cap F_n \subset U_{\varepsilon n}$ and $\mu_\psi(U_{\varepsilon n}) < \mu_\psi(A \cap F_n) + (\varepsilon/2^n)$. Let $U_\varepsilon := \cup_1^\infty U_{\varepsilon n}$; U_ε is open, contains A, and we find that

$$\begin{aligned}\mu_\psi(U_\varepsilon) &= \mu_\psi(\cup_1^\infty U_{\varepsilon n}) \\ &\leq \sum_1^\infty \mu_\psi(U_{\varepsilon n}) \\ &< \sum_1^\infty \left\{\mu_\psi(A \cap F_n) + \frac{\varepsilon}{2^n}\right\} \\ &= \mu_\psi(A) + \varepsilon,\end{aligned}$$

since we know that μ_ψ is a measure on $\mathcal{A}$. The desired equality (3) is an immediate consequence of this reasoning.

Finally, let U be an open subset of X. We must show that

$$\mu_\psi(U) = \sup\{\mu_\psi(K) \mid K \text{ compact in } X,\ K \subset U\}. \tag{4}$$

Once again, we already know that (4) holds with μ replacing μ_ψ, so we can find a sequence $(\tilde{K}_n)_1^\infty$ of compact subsets of X such that $\tilde{K}_n \subset U$ for each n, and $\lim_{n \to \infty} \mu(\tilde{K}_n) = \mu(U)$. Setting $K_n := \cup_{j=1}^n \tilde{K}_j$, for each $n \in \mathbb{N}$, we find that K_n is compact, $K_n \subset K_{n+1} \subset U$, and $\mu(K_n) \uparrow \mu(U)$. Further, with $U_0 := \cup_1^\infty K_n$, we have $\mu(U_0) = \lim_{n \to \infty} \mu(K_n) = \mu(U)$, and $U_0 \subset U$. Just as before, it is shown that $\Xi_{U_0} = \Xi_U$ μ-*a.e.* on X, whence $\mu_\psi(U_0) = \mu_\psi(U)$. Now, clearly, $(\Xi_{K_n} \psi)_1^\infty$ is a nondecreasing sequence of nonnegative functions, converging pointwise on X to $\Xi_{U_0} \psi$. Once again using B. Levi's theorem (or Lebesgue's monotone convergence theorem, cf., Rudin [46], §1.26) to justify the second equality,

$$\begin{aligned}\lim_{n \to \infty} \mu_\psi(K_n) &= \lim_{n \to \infty} \int_X \Xi_{K_n} \psi \, d\mu \\ &= \int_X \Xi_{U_0} \psi \, d\mu \\ &= \mu_\psi(U_0) \\ &= \mu_\psi(U);\end{aligned}$$

the required equality (4) follows directly.

Thus, μ_ψ is regular. □.

We should note that the regularity assertion of [VI.38.v] appears as an exercise in Hewitt and Stromberg [20].

Still another method for constructing a measure, of which we shall also make use presently, appears in Hewitt and Stromberg [20] (cf., §(12.45)); we describe it next.

[VI.39] C O N S T R U C T I O N: T H E I M A G E O F A M E A S U R E S P A C E U N D E R A C O N T I N U O U S

M A P P I N G. Let X and Y be locally compact Hausdorff spaces, and $\phi: X \to Y$ a continuous surjection. Let μ be a measure on a σ-algebra M_μ of subsets of X in the sense of [Hewitt and Stromberg, §9]. Suppose that either (i) $\phi^{-1}(K)$ is compact in X whenever K is compact in Y or (ii) (X, M_μ, μ) is a finite measure space. Under either hypothesis, $f \circ \phi \in L_1(X, M_\mu, \mu)$ for each $f \in C_0(Y)$. For, if (i) should hold, $f \circ \phi \in C_0(X)$, while $f \circ \phi \in L_1(X, M_\mu, \mu)$ if $\mu(X) < \infty$, as in (ii), since $f \circ \phi$ is bounded and continuous on X. Consequently, the map $f \mapsto \int_X f \circ \phi \, d\mu$ is a Radon measure on $C_0(Y)$, with which there is associated the measure μ_ϕ on a σ-algebra M_{μ_ϕ} of subsets of Y as in [Hewitt and Stromberg, §9], such that $\int_X f \circ \phi \, d\mu = \int_Y f \, d\mu_\phi$ for each $f \in C_0(Y)$. This measure space $(Y, M_{\mu_\phi}, \mu_\phi)$ is called *the image of* (X, M_μ, μ) *under the continuous mapping* ϕ.

[VI.40] P R O P O S I T I O N. *Maintain the setting and notation of* [VI.39].

(*i*) *Whenever* $B \in M_{\mu_\phi}$, *then* $\phi^{-1}(B) \in M_\mu$. *Thus,* $g \circ \phi$ *on* X *is* M_μ*-measurable whenever* g *on* Y *is* M_{μ_ϕ}*-measurable.*

(*ii*) *For each* σ*-finite* $B \in M_{\mu_\phi}$, *we have*

$$\mu_\phi(B) = \mu(\phi^{-1}(B)) = \int_X \xi_B \circ \phi \, d\mu.$$

(iii) If $g \in L_1(Y, M_{\mu_\phi}, \mu_\phi)$, *then* $g \circ \phi \in L_1(X, M_\mu, \mu)$, *and*

$$\int_Y g \, d\mu_\phi = \int_X g \circ \phi \, d\mu.$$

P R O O F. Cf., Hewitt and Stromberg [20], Theorem (12.46). □.

We shall return shortly, in [VI.42], *infra*, to make further observations concerning continuous images of measures.

The following technical facts shall be called upon later.

[VI.41] P R O P O S I T I O N. *Let* X *be a locally compact Hausdorff space. Suppose* (X, M_μ, μ) *and* (X, M_ν, ν) *are measure spaces with* $B(X) \subset M_\mu \cap M_\nu$, *and* μ *and* ν *are regular. Suppose further that*

$$\int_X f \, d\mu = \int_X f \, d\nu, \quad \text{for each} \quad f \in C_0(X) \quad \text{with} \quad f \geq 0. \tag{1}$$

Then

(i) $\mu(E) = \nu(E)$ *for each* $E \in M_\mu \cap M_\nu$.

Now, in addition, assume that (X, M_μ, μ) *and* (X, M_ν, ν) *are complete, with* M_μ *and* M_ν *each possessing the property of* [VI.37.v], *i.e., if* $E \subset X$, *then* $E \in M_\mu (M_\nu)$ *iff* $E \cap K \in M_\mu (M_\nu)$ *for each compact* $K \subset X$. *Then, also,*

(ii) $M_\mu = M_\nu$, *so the measure spaces* (X, M_μ, μ) *and*

$(X, \mathcal{M}_\nu, \nu)$ *are identical.*

P R O O F. (i) This is Theorem (12.41) of Hewitt and Stromberg [20]. Note that this statement implies [VI.37.vi].

(ii) We shall provide a proof of this statement along the lines of the proof given for Theorem (12.42) of Hewitt and Stromberg [20]. First, let $E \in \mathcal{M}_\mu$, with $\mu(E) < \infty$. Since μ is regular, there can be found, using [VI.36], an increasing sequence $(K_n)_1^\infty$ of compact sets, and, using [VI.35.ii], a decreasing sequence $(U_n)_1^\infty$ of open sets such that $K_n \subset E \subset U_n$ for each n, $\mu(K_n) \uparrow \mu(E)$, and $\mu(U_n) \downarrow \mu(E)$. Since $\mu(E) < \infty$, we may suppose that $\mu(U_1) < \infty$. Set $A := \cup_1^\infty K_n$ and $B := \cap_1^\infty U_n$. Then $A, B \in \mathcal{B}(X)$, $A \subset E \subset B$, and $\mu(A) = \lim_{n \to \infty} \mu(K_n) = \mu(E) = \lim_{n \to \infty} \mu(U_n) = \mu(B)$. Thus, $\mu(B \cap A') = \mu(B) - \mu(B \cap A) = \mu(B) - \mu(A) = 0$. Since, by (i), $\mu = \nu$ on $\mathcal{B}(X)$, and $B \cap A' \in \mathcal{B}(X)$, we have also $\nu(B \cap A') = 0$. Now, write

$$\begin{aligned} E &= (E \cap A) \cup (E \cap A') \\ &= A \cup (E \cap A') \\ &= A \cup \{E \cap \{(A' \cap B) \cup (A' \cap B')\}\} \\ &= A \cup \{E \cap (A' \cap B)\}. \end{aligned}$$

But $(X, \mathcal{M}_\nu, \nu)$ is complete, $\nu(A' \cap B) = 0$, and $E \cap (A' \cap B) \subset A' \cap B$, whence it follows that $E \cap (A' \cap B) \in \mathcal{M}_\nu$. Since $A \in \mathcal{M}_\nu$, we conclude that $E \in \mathcal{M}_\nu$.

Thus, whenever $E \in \mathcal{M}_\mu$ and $\mu(E) < \infty$, we have $E \in \mathcal{M}_\nu$.

Switching the roles of μ and ν in the preceding argument shows that, whenever $E \in M_\nu$ and $\nu(E) < \infty$, there follows $E \in M_\mu$.

Next, consider any $E \in M_\mu$: if K is compact in X, then $E \cap K \in M_\mu$ and $\mu(E \cap K) < \infty$ (since μ is regular), so $E \cap K \in M_\nu$, by the result already obtained. Since M_ν has the property of [VI.37.v], we must have $E \in M_\nu$. Thus, $M_\mu \subset M_\nu$. Switching the roles of μ and ν in this argument secures the reversed inclusion, $M_\nu \subset M_\mu$. □.

We next study the continuous image of a measure under a homeomorphism.

[VI.42] P R O P O S I T I O N. *Let* X *and* Y *be locally compact Hausdorff spaces, and* ϕ: $X \to Y$ *a homeomorphism. Let* (X, M_μ, μ) *be a measure space in the sense of* [*Hewitt and Stromberg*, §9], *and* $(Y, M_{\mu_\phi}, \mu_\phi)$ *its image under* ϕ: *cf.*, [VI.39]. *Then*

(i) (X, M_μ, μ) *is the image of* $(Y, M_{\mu_\phi}, \mu_\phi)$ *under* ϕ^{-1};

(ii) *a subset* B *of* Y *is in* M_{μ_ϕ} *iff* $\phi^{-1}(B) \in M_\mu$;

(iii) *a complex function* g *on* Y *is* M_{μ_ϕ}*-measurable iff* $g \circ \phi$ *on* X *is* M_μ*-measurable;*

(iv) *a function* g *(more precisely, an equivalence class of functions whose members are pairwise equal* μ_ϕ*-a.e. on* Y*) is in* $L_1(Y, M_{\mu_\phi}, \mu_\phi)$ *iff* $g \circ \phi \in L_1(X, M_\mu, \mu)$; *in either case, we have*

$$\int_Y g\, d\mu_\phi = \int_X g \circ \phi\, d\mu. \tag{1}$$

P R O O F. Obviously, μ_ϕ is well-defined, for ϕ is a continuous surjection, and $\phi^{-1}(K)$ is compact in X whenever $K \subset Y$ is compact.

(i) Since ϕ is a homeomorphism, it is clear that the image $(\mu_\phi)_{\phi^{-1}}$ of μ_ϕ under ϕ^{-1} is well-defined; writing $\nu := (\mu_\phi)_{\phi^{-1}}$, we have the measure space $(X, \mathcal{M}_\nu, \nu)$ constructed from $(Y, \mathcal{M}_{\mu_\phi}, \mu_\phi)$ and ϕ^{-1} as in [VI.39] (note that μ_ϕ is a measure in the sense of [Hewitt and Stromberg, §9], as is ν). We have, by the manner in which μ_ϕ and ν are constructed, $\int_Y f\, d\mu_\phi = \int_X f \circ \phi\, d\mu$ for each $f \in C_0(Y)$, and $\int_X g\, d\nu = \int_Y g \circ \phi^{-1}\, d\mu_\phi$ for each $g \in C_0(X)$. Now, if $g \in C_0(X)$, then $g \circ \phi^{-1} \in C_0(Y)$, so

$$\int_X g\, d\nu = \int_Y g \circ \phi^{-1}\, d\mu_\phi = \int_X g \circ \phi^{-1} \circ \phi\, d\mu = \int_X g\, d\mu.$$

Consequently, in view of the properties listed in [VI.37], which are possessed by μ and ν, all hypotheses of [VI.41] are fulfilled, and we can assert that $\mathcal{M}_\mu = \mathcal{M}_\nu$, with $\mu = \nu$ on $\mathcal{M}_\mu$, *i.e.*, that $\mu = \nu$.

(ii) Let $B \subset Y$. From [VI.40.i], the inclusion $B \in \mathcal{M}_{\mu_\phi}$ implies the inclusion $\phi^{-1}(B) \in \mathcal{M}_\mu$. Now, suppose $\phi^{-1}(B) \in \mathcal{M}_\mu$. Since $\mathcal{M}_\mu = \mathcal{M}_\nu$, $\nu := (\mu_\phi)_{\phi^{-1}}$, we can again apply [VI.40.i] (for

μ_ϕ and ϕ^{-1}) to conclude that $B = (\phi^{-1})^{-1}(\phi^{-1}(B)) \in \mathcal{M}_{\mu_\phi}$.

(iii) Let $U \subset \mathbb{K}$ be open. By (ii), $g^{-1}(U) \in \mathcal{M}_{\mu_\phi}$ iff $(g\circ\phi)^{-1}(U) = \phi^{-1}(g^{-1}(U)) \in \mathcal{M}_\mu$. Statement (iii) follows from this observation.

(iv) Applying [VI.40.iii], we can assert that, if $g \in L_1(Y,\mathcal{M}_{\mu_\phi},\mu_\phi)$, then $g\circ\phi \in L_1(X,\mathcal{M}_\mu,\mu)$, with $\int_Y g\, d\mu_\phi = \int_X g\circ\phi\, d\mu$. Conversely, suppose g is defined μ_ϕ-*a.e.* on Y (whence it is easy to see, from [VI.40.ii], that $g\circ\phi$ is defined μ-*a.e.* on X), with $g\circ\phi \in L_1(X,\mathcal{M}_\mu,\mu)$. Then, by (i), $g\circ\phi \in L_1(X,\mathcal{M}_\nu,\nu)$, $\nu := (\mu_\phi)_{\phi^{-1}}$, so we can apply [VI.40.iii] once again (for μ_ϕ and ϕ^{-1}), finding that $g = g\circ\phi\circ\phi^{-1} \in L_1(Y,\mathcal{M}_{\mu_\phi},\mu_\phi)$, and $\int_X g\circ\phi\, d\mu = \int_X g\circ\phi\, d\nu =$ $\int_Y g\circ\phi\circ\phi^{-1}\, d\mu_\phi = \int_Y g\, d\mu_\phi$. □.

[VI.43] R E M A R K S. We recall here certain facts concerning the measure space generated by restricting a measure to one of its measurable sets. Let (X,A,μ) be a measure space, and choose $E \in A$. Set $A_E := \{A \in A \mid A \subset E\}$, then $\mu_E := \mu \mid A_E$. It is easy to show that A_E is a σ-algebra of subsets of E and that μ_E is a measure on A_E, so there results the measure space (E,A_E,μ_E). Whenever f on X is A-measurable, then $f\mid E$ is A_E-measurable; whenever $\int_E f\, d\mu := \int_X \Xi_E f\, d\mu$ is defined, then $\int_E (f\mid E)\, d\mu_E$ is defined, and these integrals are equal. If (X,A,μ) is complete, then (E,A_E,μ_E) is complete. The proofs of these statements are routine.

Now, suppose that, in addition, X is a locally compact Hausdorff space, $\mathcal{B}(X) \subset \mathcal{A}$, μ is regular, and E is *open* in X. Let $x \in E$: then we can find an open neighborhood V of x such that V^- is compact and $V^- \subset E$ (cf., *e.g.*, Hewitt and Stromberg [20], Theorem (6.78)), whence it follows that E is locally compact in its relative topology (E is also Hausdorff, of course). Since $\mathcal{A}$ contains E along with each open subset of X, $\mathcal{A}_E$ contains each open subset of E, so $\mathcal{A}_E$ also contains $\mathcal{B}(E)$. It is a simple matter to prove now that μ_E is regular; we omit the details. Next, impose the additional hypotheses that E is σ-compact and that $\mathcal{A}$ possesses the property of [VI.37.v] (*i.e.*, if $A \subset X$, then $A \in \mathcal{A}$ iff $A \cap K \in \mathcal{A}$ for each compact $K \subset X$): then $\mathcal{A}_E$ inherits this property. For, suppose that $A \subset E$. Let $A \in \mathcal{A}_E$, and K be compact in E. Then $A \in \mathcal{A}$ and K is compact in X, so $A \cap K \in \mathcal{A}$, whence $A \cap K \in \mathcal{A}_E$. Conversely, suppose that $A \cap K \in \mathcal{A}_E$ whenever K is compact in E. Writing $E = \cup_1^\infty K_n$, where each K_n is compact in X and contained in E, we have $A = \cup_1^\infty (A \cap K_n)$. Since K_n is compact in E, $A \cap K_n \in \mathcal{A}_E$ for each $n \in \mathbb{N}$, which shows that $A \in \mathcal{A}_E$. Thus, if $A \subset E$, then $A \in \mathcal{A}_E$ iff $A \cap K \in \mathcal{A}_E$ for each K compact in E.

As an example, suppose $n \in \mathbb{N}$, $X = \mathbb{R}^n$, $\mathcal{A} = \mathcal{M}_{\lambda_n}$, and $\mu = \lambda_n$. Let $E \subset \mathbb{R}^n$ be open. Recalling (cf., the remark in [VI.37]) that λ_n is a measure in the sense of [Hewitt and Stromberg, §9], and since any open subset of $\mathbb{R}^n$ is σ-compact, we see that each of the conditions imposed above is in fact fulfilled by these particular choices, so we can make the corresponding assertions concerning the measure space $(E,(M_{\lambda_n})_E,(\lambda_n)_E)$. In this instance, we shall usually denote $(\lambda_n)_E$ again by λ_n; no confusion should result from this practice.

Let us cite the following familiar result:

[VI.44] P R O P O S I T I O N. *Let* Ω *be an open subset of* $\mathbb{R}^n$, *and* $g\colon \Omega \to \mathbb{R}^n$ *a* q-*regular transformation, for some* $q \in \mathbb{N}$. *Suppose* $f \in C_0(g(\Omega))$. *Then*

$$\int_{g(\Omega)} f \, d\lambda_n = \int_{\Omega} f\circ g \cdot |Jg| \, d\lambda_n.$$

P R O O F. Cf., *e.g.*, Fleming [15], Theorem 5.8. □.

Of course, the transformation formula of [VI.44] is true under much less stringent hypotheses on the integrand. The more general statement is obtained as a particular case of a result to be proven later (cf., Theorem [VI.52] and the remark following).

We wish to describe next the manner in which Lebesgue measure on $\mathbb{R}^n$ $(n \geq 2)$ induces a measure on a σ-algebra of subsets of an $(r,n;q)$-manifold, called the *Lebesgue measure* on the manifold. Essentially, the idea is to first construct the measure for any coordinate patch and show that the measures on overlapping patches agree on their intersection. These local measures are then used to construct a measure on the manifold, which, as it turns out, coincides on each coordinate patch with the original measure on that patch.

The following simple observations must be made.

[VI.45] L E M M A. *Let* M *be an* (r,n;q)*-manifold. Then, with their respective relative topologies inherited from* $\mathbb{R}^n$, M *and each coordinate patch on* M *are locally compact* σ*-compact Hausdorff spaces.*

P R O O F. Let U be a coordinate patch on M; choose any coordinate function h for U. Then h: $U \to h(U)$ is a homeomorphism, where, of course h(U) has its relative topology as a subset of $\mathbb{R}^r$. Then h(U) is locally compact and Hausdorff, since h(U) is an open subset of the locally compact Hausdorff space $\mathbb{R}^r$. Any open subset of $\mathbb{R}^r$ is σ-compact as a subset, hence also as a subspace, since it is clear that any subset of a topological subspace is compact in the subspace iff it is compact in the containing space. The existence of the homeomorphism h then shows that U is locally compact, σ-compact, and, of course, Hausdorff. Note that the topology which U inherits from the subspace M coincides with that which it inherits from the space $\mathbb{R}^n$.

Now, suppose $x \in M$. Then x is in some coordinate patch U on M. We have just seen that there is an open U-neighborhood

V of x such that V^{-U} is compact in U. From the last remark in the preceding paragraph, V^{-U} is also compact in M. M is obviously Hausdorff, so V^{-U} is closed in M. From this, and the equality $V^{-U} = V^{-M} \cap M$, it is easy to see that $V^{-M} = V^{-U}$. Observing that V is also an open M-neighborhood of x, since U is open in M, we conclude finally that x possesses an open neighborhood in M with compact closure in M (in passing, note that we can easily show also that V^{-U} is compact in $\mathbb{R}^n$, with $V^{-U} = V^{-}$). Thus, M is locally compact and, of course, Hausdorff.

To see that M is σ-compact, first choose a covering collection $\{U_\iota\}_{\iota \in I}$ of coordinate patches on M, then a corresponding collection $\{\hat{U}_\iota\}_{\iota \in I}$ of open sets in $\mathbb{R}^n$ such that $U_\iota = \hat{U}_\iota \cap M$ for each $\iota \in I$ (each U_ι is open in M). Clearly, $\{\hat{U}_\iota\}_{\iota \in I}$ is an open cover for M in $\mathbb{R}^n$, whence the Lindelöf covering theorem shows that there exists a countable set $I_0 \subset I$ such that $\{\hat{U}_\iota\}_{\iota \in I_0}$ also covers M. Thus, $\{U_\iota\}_{\iota \in I_0}$ is a covering of M by a countable collection of coordinate patches. Each U_ι is σ-compact as a subspace, hence also as a subset of M; we conclude that M is σ-compact. □.

[VI.46] CONSTRUCTION: MEASURE SPACE $(h(U), (\mathcal{M}_{\lambda_r})_{h(U)}, \lambda_h)$. Let M be an $(r,n;q)$-manifold, and (U,h) a coordinate system in M. Then $h(U)$ is open in $\mathbb{R}^r$, and Jh^{-1} is continuous, hence λ_r-measurable and λ_r-locally integrable, as well as positive, on $h(U)$. All hypotheses of Proposition [VI.38] are clearly satisfied in this setting, so we obtain the measure $\lambda_h := (\lambda_r)_{Jh^{-1}}$ on $(\mathcal{M}_{\lambda_r})_{h(U)}$ given by

$$\lambda_h(E) := \int_E Jh^{-1}\, d\lambda_r, \qquad \text{for each} \qquad E \in (\mathcal{M}_{\lambda_r})_{h(U)}. \tag{1}$$

Proposition [VI.38] also provides a number of properties of the measure space $(h(U),(\mathcal{M}_{\lambda_r})_{h(U)},\lambda_h)$: it is complete, regular, and σ-finite, while, if f is a complex function defined λ_r-*a.e.* on $h(U)$, then (the equivalence class containing) $f \in L_1(h(U),(\mathcal{M}_{\lambda_r})_{h(U)},\lambda_h)$ iff (the equivalence class containing) $f\cdot Jh^{-1} \in L_1(h(U),(\mathcal{M}_{\lambda_r})_{h(U)},\lambda_r)$, and in either case, we have

$$\int_{h(U)} f\, d\lambda_h = \int_{h(U)} f\cdot Jh^{-1}\, d\lambda_r. \tag{2}$$

Let us show that λ_h is a measure in the sense of [Hewitt and Stromberg, §9]: the map $f \mapsto \int_{h(U)} f\cdot Jh^{-1}\, d\lambda_r$ is clearly a Radon measure on $C_0(h(U))$; let $\mathcal{M}_{\lambda_h'}$ denote the σ-algebra of subsets of $h(U)$, and λ_h' the measure on $\mathcal{M}_{\lambda_h'}$ associated with this Radon measure as in [VI.37]. Then

$$\int_{h(U)} f\, d\lambda_h' = \int_{h(U)} f\cdot Jh^{-1}\, d\lambda_r = \int_{h(U)} f\, d\lambda_h,$$

$$\text{for each} \qquad f \in C_0(h(U)).$$

Thus, it is clear that $(h(U),\mathcal{M}_{\lambda_h'},\lambda_h')$ and $(h(U),(\mathcal{M}_{\lambda_r})_{h(U)},\lambda_h)$

fulfill all requirements of [VI.41] (note that $(\mathcal{M}_{\lambda_r})_{h(U)}$ possesses the property of [VI.37.v]; cf., [VI.43]), and we conclude that these measure spaces are identical.

[VI.47] CONSTRUCTION: MEASURE SPACE $(U,\mathcal{M}_{\lambda_U},\lambda_U)$. Let M be an (r,n;q)-manifold, and U a coordinate patch on M; let h be a coordinate function for U. Then we have the homeomorphism $h^{-1}: h(U) \to U$ between locally compact σ-compact Hausdorff spaces, and the measure space $(h(U),(\mathcal{M}_{\lambda_r})_{h(U)},\lambda_h)$; we showed that the latter is generated by a Radon measure on $C_0(h(U))$, as in [Hewitt and Stromberg, §9]. Consequently, we can specialize to this setting the general construction and results of [VI.39, 40, and 42]. *We define the measure space* $(U,\mathcal{M}_{\lambda_U},\lambda_U)$ *to be the image of the measure space* $(h(U),(\mathcal{M}_{\lambda_r})_{h(U)},\lambda_h)$ *under the mapping* h^{-1}. Note that $(U,\mathcal{M}_{\lambda_U},\lambda_U)$ is a measure in the sense of [Hewitt and Stromberg, §9], generated by the Radon measure $f \mapsto \int_{h(U)} f\circ h^{-1}\, d\lambda_h = \int_{h(U)} f\circ h^{-1}\cdot Jh^{-1}\, d\lambda_r$ on $C_0(U)$. We provide a list of the properties of this measure space; if the origin of a particular property is sufficiently clear, we shall state it without further comment.

(i) $(U,\mathcal{M}_{\lambda_U},\lambda_U)$ is a complete measure space.

(ii) $\mathcal{B}(U) \subset \mathcal{M}_{\lambda_U}$, and λ_U is regular.

(iii) λ_U is σ-finite. [For, U is σ-compact, and λ_U

is regular, from which the σ-finiteness of λ_U follows.]

(iv) If $A \subset U$, then $A \in M_{\lambda_U}$ iff $A \cap K \in M_{\lambda_U}$ for each compact $K \subset U$.

(v) If μ is any regular measure on M_{λ_U} such that $\int_U f\, d\mu = \int_U f\, d\lambda_U$ for each $f \in C_0(U)$, then $\mu = \lambda_U$.

(vi) If $\tilde{h}$ is any coordinate function for U, then the image $(U, M_{\tilde{\lambda}_U}, \tilde{\lambda}_U)$ of the measure space $(\tilde{h}(U), (M_{\lambda_r})_{\tilde{h}(U)}, \lambda_{\tilde{h}})$ under $\tilde{h}^{-1}$ coincides with $(U, M_{\lambda_U}, \lambda_U)$. Thus, the latter is intrinsic to the coordinate patch U, *i.e.*, is independent of the particular coordinate function used to construct it. We are then justified in calling M_{λ_U} *the σ-algebra of Lebesgue-measurable subsets of* U, and λ_U *Lebesgue measure on* U. [To see that the first statement is correct, we need only show that $\int_U f\, d\lambda_U = \int_U f\, d\tilde{\lambda}_U$, whenever $f \in C_0(U)$. For then, in view of (i), (ii), and (iv), *supra* (which remain true when $M_{\tilde{\lambda}_U}$ and $\tilde{\lambda}_U$ replace M_{λ_U} and λ_U, respectively), and the fact that U is a locally compact Hausdorff space, we shall be able to apply [VI.41] to deduce that $(U, M_{\lambda_U}, \lambda_U)$ and $(U, M_{\tilde{\lambda}_U}, \tilde{\lambda}_U)$ are, in fact, identical. Consider, then, the q-regular transformation $\phi := \tilde{h} \circ h^{-1}$: $h(U) \to \mathbb{R}^r$, with $\phi(h(U)) = \tilde{h}(U)$, cf., [VI.21]. Clearly, ϕ^{-1} is q-regular from $\tilde{h}(U)$ onto $h(U)$, while $\tilde{h}^{-1} = h^{-1} \circ \phi^{-1}$, on $\tilde{h}(U)$. Applying Proposition [VI.13] yields

$$J\tilde{h}^{-1} = \{(Jh^{-1})\circ\phi^{-1}\}|J\phi^{-1}|, \quad \text{on} \quad \tilde{h}(U). \tag{1}$$

Now, choose $f \in C_0(U)$. By the manner in which λ_U, $\tilde{\lambda}_U$, λ_h, and $\lambda_{\tilde{h}}$ were constructed, we know that $\int_U f\, d\lambda_U = \int_{h(U)} f\circ h^{-1}\, d\lambda_h = \int_{h(U)} f\circ h^{-1}\cdot Jh^{-1}\, d\lambda_r$, and $\int_U f\, d\tilde{\lambda}_U = \int_{\tilde{h}(U)} f\circ\tilde{h}^{-1}\, d\lambda_{\tilde{h}} = \int_{\tilde{h}(U)} f\circ\tilde{h}^{-1}\cdot J\tilde{h}^{-1}\, d\lambda_r$. Thus, using [VI.44] to obtain the second equality, and (1) to obtain the third, we can write

$$\begin{aligned}\int_U f\, d\lambda_U &= \int_{h(U)} f\circ h^{-1}\cdot Jh^{-1}\, d\lambda_r \\ &= \int_{\phi\{h(U)\}} \{f\circ h^{-1}\circ\phi^{-1}\}\cdot\{(Jh^{-1})\circ\phi^{-1}\}\cdot|J\phi^{-1}|\, d\lambda_r \\ &= \int_{\tilde{h}(U)} f\circ\tilde{h}^{-1}\cdot J\tilde{h}^{-1}\, d\lambda_r \\ &= \int_U f\, d\tilde{\lambda}_U.\end{aligned}$$

As noted, the first statement of (vi) is hereby proven; the remaining statements are self-explanatory.]

(vii) $(h(U),(M_{\lambda_r})_{h(U)},\lambda_r)$ is the image of $(U,M_{\lambda_U},\lambda_U)$ under h. [This is just [VI.42.i], re-phrased for the present context.]

(viii) A subset $A \subset U$ is in M_{λ_U} iff $h(A) \in (M_{\lambda_r})_{h(U)}$, *i.e.*, iff $h(A)$ is a Lebesgue-measurable subset of $\mathbb{R}^r$. Whenever $A \in \mathcal{M}_{\lambda_U}$, its Lebesgue measure is given by

$$\begin{aligned} \lambda_U(A) &= \lambda_h(h(A)) \\ &= \int_{h(U)} \Xi_A \circ h^{-1}\, d\lambda_h \\ &= \int_{h(U)} \Xi_{h(A)} \cdot Jh^{-1}\, d\lambda_r \\ &= \int_{h(A)} Jh^{-1}\, d\lambda_r . \end{aligned} \tag{2}$$

[The first statement is [VI.42.ii]. For the second, we refer to [VI.40.ii] (noting that every set in $\mathcal{M}_{\lambda_U}$ is λ_U-σ-finite), (VI.46.1), and the obvious equality $\Xi_A \circ h^{-1} = \Xi_{h(A)}$ on $h(U)$.]

(ix) A complex function f on U is $\mathcal{M}_{\lambda_U}$-measurable iff $f \circ h^{-1}$ on $h(U)$ is $(\mathcal{M}_{\lambda_r})_{h(U)}$-measurable. [Cf., [VI.42.iii].]

(x) $f \in \mathcal{L}_1(U, \mathcal{M}_{\lambda_U}, \lambda_U)$ iff $f \circ h^{-1} \in \mathcal{L}_1(h(U), (\mathcal{M}_{\lambda_r})_{h(U)}, \lambda_h)$ iff $f \circ h^{-1} \cdot Jh^{-1} \in \mathcal{L}_1(h(U), (\mathcal{M}_{\lambda_r})_{h(U)}, \lambda_r)$: if any of these inclusions should obtain, then

$$\int_U f\, d\lambda_U = \int_{h(U)} f \circ h^{-1}\, d\lambda_h = \int_{h(U)} f \circ h^{-1} \cdot Jh^{-1}\, d\lambda_r . \tag{3}$$

[Combine [VI.42.iv] and the property of λ_h described by (VI.46.2).]

Apropos of (x), note that whenever f is an $\mathcal{M}_{\lambda_U}$-measurable function on U (which holds iff $f \circ h^{-1}$ is $(M_{\lambda_r})_{h(U)}$-measurable), and f is non-negative, then equality (3) must hold. For, if one integral is finite, then all three must be finite, and equality must hold, by (x); if one integral is infinite, then all must be infinite (by (x)), and the equality again holds.

(xi) Let U_0 be any non-void open subset of M, with $U_0 \subset U$ (*i.e.*, U_0 is any coordinate patch contained in U). Then

$$M_{\lambda_{U_0}} = (M_{\lambda_U})_{U_0} \qquad (:= \{A \in M_{\lambda_U} \mid A \subset U_0\}), \tag{4}$$

and

$$\lambda_{U_0} = (\lambda_U)_{U_0} \qquad (:= \lambda_U \mid (M_{\lambda_U})_{U_0}); \tag{5}$$

recall the notations and remarks of [VI.43]. Thus, if $\tilde{U}$ is any coordinate patch on M such that $U \cap \tilde{U} \neq \phi$, then (noting that $U \cap \tilde{U}$ is a coordinate patch)

$$(M_{\lambda_U})_{U\cap\tilde{U}} = M_{\lambda_{U\cap\tilde{U}}} = (M_{\lambda_{\tilde{U}}})_{U\cap\tilde{U}}, \tag{6}$$

and

$$(\lambda_U)_{U\cap\tilde{U}} = \lambda_{U\cap\tilde{U}} = (\lambda_{\tilde{U}})_{U\cap\tilde{U}}, \tag{7}$$

i.e., "the Lebesgue measures on overlapping coordinate patches agree on their intersection." [The second assertion is clearly an immediate consequence of the first. To prove the latter, we wish to use [VI.41]. In view of property (vi), any coordinate function for

U_0 can be used to construct λ_{U_0}; let us use $h|U_0$. Then, for any $f \in C_0(U_0)$, we can regard f as in $C_0(U)$, and write

$$\int_{U_0} f\, d\lambda_{U_0} = \int_{(h|U_0)(U_0)} f\circ(h|U_0)^{-1}\cdot J\{(h|U_0)^{-1}\}\, d\lambda_r$$

$$= \int_{h(U_0)} f\circ h^{-1}\cdot Jh^{-1}\, d\lambda_r$$

$$= \int_{h(U)} \Xi_{h(U_0)}\cdot f\circ h^{-1}\cdot Jh^{-1}\, d\lambda_r$$

$$= \int_{h(U)} \Xi_{U_0}\circ h^{-1}\cdot f\circ h^{-1}\cdot Jh^{-1}\, d\lambda_r$$

$$= \int_{U} \Xi_{U_0} f\, d\lambda_U$$

$$= \int_{U_0} f\, d(\lambda_U)_{U_0}.$$

In order to assert that $(U_0, M_{\lambda_{U_0}}, \lambda_{U_0})$ and $(U_0, (M_{\lambda_U})_{U_0}, (\lambda_U)_{U_0})$ are identical, *via* [VI.41], we now need verify only that these spaces are complete, that $\mathcal{B}(U_0) \subset M_{\lambda_0} \cap (M_{\lambda_U})_{U_0}$, that λ_{U_0} and $(\lambda_U)_{U_0}$ are regular, and that $M_{\lambda_{U_0}}$ and $(M_{\lambda_U})_{U_0}$ possess the property of [VI.37.v]. The requisite facts concerning $M_{\lambda_{U_0}}$ and λ_{U_0} are contained in (i), (ii), and (iv), *supra*; those concerning $(M_{\lambda_U})_{U_0}$ and $(\lambda_U)_{U_0}$ are implications of (i), (ii), (iv), the reasoning in [VI.43], the fact that U_0 is open in U, and the σ-compactness of U_0 (cf., [VI.45]). Thus, (4) and (5) are correct. We point out that $(\mathcal{M}_{\lambda_U})_{U_0}$ and $(\lambda_U)_{U_0}$ are well-defined, since U_0 is open in U, as well as in M, so that $U_0 \in \mathcal{B}(U) \subset \mathcal{M}_{\lambda_U}$.]

In passing, we note that there is no inconsistency problem in the case of an $(n,n;q)$-manifold M, on which we already have the Lebesgue measure $(\lambda_n)_M$. For, in this case, the single coordinate system (M,i_M), where i_M is the identity on M, serves for the construction of $(M,\mathcal{M}_{\lambda_M},\lambda_M)$, and from (viii) it is immediately evident that $\mathcal{M}_{\lambda_M} = (\mathcal{M}_{\lambda_n})_M$ and $\lambda_M = (\lambda_n)_M$, since $Ji_M^{-1} = Ji_M^{-1} = 1$ on M.

Property [VI.47.xi] leads one to suspect that there exists a measure on the whole of an $(r,n;q)$-manifold, the restriction of which to any coordinate patch coincides there with the Lebesgue measure for the coordinate patch. We shall presently show that this is indeed the case. The construction of this measure is most easily accomplished *via* the device of a partition of unity for a manifold (cf., Fleming [15]), so we prepare certain facts in this direction.

[VI.48] D E F I N I T I O N. Let M be an $(r,n;q)$-manifold, $p \in \mathbb{N}\cup\{\infty\}$, and $\{U_\iota\}_{\iota\in I}$ a covering collection of coordinate patches in M. A family of functions $\{\pi_\iota\}_{\iota\in I}$ is a *locally finite* *p-partition of unity for* M, *subordinate to the cover* $\{U_\iota\}_{\iota\in I}$ iff

(i) for each $\iota \in \tilde{I}$, $\pi_\iota \in C^p(M)$, $\pi_\iota \geq 0$, and there exists $\gamma_\iota \in I$ such that supp π_ι is compact and contained in U_{γ_ι},

(ii) whenever $K \subset M$ is compact, there exists an open neighborhood W of K in M and a finite set $\tilde{I}_W \subset \tilde{I}$ such that $\pi_\iota(x) = 0$ for each $\iota \in \tilde{I} \cap \tilde{I}_W'$ and $x \in W$,

and

(iii) $\sum_{\iota\in\tilde{I}} \pi_\iota(x) = 1$, for each $x \in M$. ∎.

The existence of such partitions of unity is easy to prove, from the familiar fact that there exists a smooth locally finite partition of unity subordinate to an open cover of a subset of $\mathbb{R}^n$:

[VI.49] L E M M A. *Let* Γ *be a family of open sets in* $\mathbb{R}^n$*; write* $\Omega = \cup\,\Gamma$. *Then there exists a (countable) collection* $\{\psi_i\}_{i=1}^\infty \subset C_0^\infty(\Omega)$ *such that*

(i) $\psi_i \geq 0$ *for each* $i \in \mathbb{N}$,

(ii) supp ψ_i *is contained in some member of* Γ, *for each* $i \in \mathbb{N}$,

(iii) $\sum_{i=1}^\infty \psi_i(x) = 1$, *for each* $x \in \Omega$,

and

(*iv*) *whenever* $K \subset \Omega$ *is compact, there exists an* $m \in \mathbb{N}$ *and an open set* W, *with* $K \subset W \subset \Omega$, *such that* $\psi_i(x) = 0$ *whenever* $i > m$ *and* $x \in W$.

P R O O F. This is Theorem 6.20 of Rudin [47]. $\square$.

[VI.50] P R O P O S I T I O N. *Let* M *be an* $(r,n;q)$-*manifold and* $\{U_\iota\}_{\iota \in I}$ *a covering collection of coordinate patches on* M. *Then there exists a countable family* $\{\pi_i\}_{i=1}^{\infty}$ *forming a locally finite* q-*partition of unity for* M, *subordinate to the cover* $\{U_\iota\}_{\iota \in I}$.

P R O O F. For any $\iota \in I$, U_ι is open in M, while M is a locally compact Hausdorff space ([VI.45]), so we can find, for each $x \in U_\iota$, an M-open neighborhood $U_{\iota x}$ of x such that $\bar{U}^{M}_{\iota x} \subset U_\iota$ and $\bar{U}^{M}_{\iota x}$ is M-compact (cf., Hewitt and Stromberg [20], Theorem (6.78)). Since the topology on M is that inherited from $\mathbb{R}^n$, it is clear that $\bar{U}^{M}_{\iota x}$ is compact, hence also closed, in $\mathbb{R}^n$. From this, and the equality $\bar{U}^{M}_{\iota x} = \bar{U}_{\iota x} \cap M$, it follows that $\bar{U}^{M}_{\iota x} = \bar{U}_{\iota x}$. The resulting collection $\{U_{\iota x} \mid \iota \in I, \ x \in U_\iota\}$ is then an M-open cover of M. For each $\iota \in I$ and $x \in U_\iota$, select an open set $\hat{U}_{\iota x} \subset \mathbb{R}^n$ for which $U_{\iota x} = \hat{U}_{\iota x} \cap M$. Set $\Gamma := \{\hat{U}_{\iota x} \mid \iota \in I, \ x \in U_\iota\}$: Γ is a covering of M by open subsets of $\mathbb{R}^n$. Let $\{\psi_i\}_{i=1}^{\infty} \subset C_0^{\infty}(\cup \Gamma)$ be as in [VI.49], and define $\pi_i := \psi_i \mid M$, for each $i \in \mathbb{N}$: we claim that $\{\pi_i\}_{i=1}^{\infty}$ fulfills all requirements of [VI.48]. First, $\pi_i \geq 0$, and $\sum_{i=1}^{\infty} \pi_i(x) = \sum_{i=1}^{\infty} \psi_i(x) = 1$ for each $x \in M$, from [VI.49.i] and [VI.49.iii], respectively. Next, suppose K is compact in M. Then $K \subset \cup\,\Gamma$ and is compact in $\mathbb{R}^n$, so, by [VI.49.iv], there are an open set $\hat{W} \subset \mathbb{R}^n$ such that $K \subset \hat{W} \subset \cup\,\Gamma$, and an $m \in \mathbb{N}$ such that $\psi_i(x) = 0$ for $i > m$ and $x \in \hat{W}$. Set $W := \hat{W} \cap M$: W is an open neighborhood of K in M, and $\pi_i(x) = \psi_i(x) = 0$ for $i > m$ and $x \in W$. Now, choose $i \in \mathbb{N}$: we must show that $\operatorname{supp} \pi_i$ is compact and contained in some U_ι $(\iota \in I)$. According to [VI.49.ii], $\operatorname{supp} \psi_i \subset \hat{U}_{\iota_i x_i}$ for some $\iota_i \in I$, $x_i \in U_{\iota_i}$. Thus,

$$\begin{aligned}\operatorname{supp} \pi_i &:= \{x \in M \mid \pi_i(x) \neq 0\}^{-M} \\ &= \{x \in M \mid \psi_i(x) \neq 0\}^{-} \cap M \subset \operatorname{supp} \psi_i \cap M \subset \hat{U}_{\iota_i x_i} \cap M \\ &= U_{\iota_i x_i} \subset U^{-}_{\iota_i x_i} \subset U_{\iota_i}.\end{aligned}$$

At once, we see that $\operatorname{supp} \pi_i \subset U_{\iota_i}$, and $\operatorname{supp} \pi_i$ is compact in M, since $\operatorname{supp} \pi_i$ is closed in M and contained in the compact subset $U^{-}_{\iota_i x_i}$ (then $\operatorname{supp} \pi_i$ is also compact in $\mathbb{R}^n$). Finally, we verify that $\pi_i \in C^q(M)$: let (U,h) be any coordinate system in M. Since $h^{-1} \in C^q(h(U);\mathbb{R}^n)$, with $h^{-1}(h(U)) = U \subset M \subset \cup\,\Gamma$, and $\psi_i \in C_0^\infty(\cup\,\Gamma)$, it is obvious that $\psi_i \circ h^{-1} \in C^q(h(U))$. But $\pi_i \circ h^{-1} = \psi_i \circ h^{-1}$ on $h(U)$, so $\pi_i \circ h^{-1} \in C^q(h(U))$. It follows that $\pi_i \in C^q(M)$. □.

We can now produce the Lebesgue measure on a manifold.

[VI.51] C O N S T R U C T I O N: M E A S U R E S P A C E

$(M, M_{\lambda_M}, \lambda_M)$. As usual, let M be an $(r,n;q)$-manifold. We define a Radon measure on $C_0(M)$: select a covering collection $\{(U_\iota, h_\iota)\}_{\iota \in I}$ of coordinate systems in M, and let $\{\pi_i\}_{i=1}^{\infty}$ be a (countable) locally finite q-partition of unity for M, subordinate to the covering $\{U_\iota\}_{\iota \in I}$. For each $i \in \mathbb{N}$, select $\iota(i) \in I$ such that $\operatorname{supp} \pi_i \subset U_{\iota(i)}$. Now, suppose $f \in C_0(M)$. For each $i \in \mathbb{N}$, $\pi_i f$ is continuous and has compact support in $U_{\iota(i)}$, so $(\pi_i f)| U_{\iota(i)} \in L_1(U_{\iota(i)}, M_{\lambda_{U_{\iota(i)}}}, \lambda_{U_{\iota(i)}})$, with

$$\int_{U_{\iota(i)}} \pi_i f \, d\lambda_{U_{\iota(i)}} = \int_{h_{\iota(i)}(U_{\iota(i)})} (\pi_i f) \circ h_{\iota(i)}^{-1} \cdot Jh_{\iota(i)}^{-1} \, d\lambda_r. \tag{1}$$

In fact, in view of the compactness of $\operatorname{supp} f$, and [VI.48.ii], there exists an $m_f \in \mathbb{N}$ for which $\pi_i f = 0$ whenever $i > m_f$. Consequently, setting

$$\Lambda_M f := \sum_{i=1}^{\infty} \int_{U_{\iota(i)}} \pi_i f \, d\lambda_{U_{\iota(i)}}$$

$$= \sum_{i=1}^{\infty} \int_{h_{\iota(i)}(U_{\iota(i)})} (\pi_i f) \circ h_{\iota(i)}^{-1} \cdot Jh_{\iota(i)}^{-1} \, d\lambda_r, \tag{2}$$

we see that the sum is actually finite. This process clearly defines a non-negative linear functional Λ_M on $C_0(M)$. Let us convince ourselves that Λ_M is "intrinsic" to M, *i.e.*, that it does not depend upon the particular auxiliary objects chosen for its construction. Let $\{V_\gamma\}_{\gamma \in J}$ be a covering collection of coordinate patches in M, $\{\Pi_i\}_{i=1}^{\infty}$ a locally finite q-partition of unity for

M, subordinate to $\{V_\gamma\}_{\gamma\in J}$, and select, for each $i \in \mathbb{N}$, $\gamma(i) \in J$ such that $\operatorname{supp} \Pi_i \subset V_{\gamma(i)}$. Define Λ on $C_0(M)$ by $\Lambda f :=$ $\sum_{i=1}^{\infty} \int_{V_{\gamma(i)}} \Pi_i f \, d\lambda_{V_{\gamma(i)}}$ for each $f \in C_0(M)$; as before, the sum is finite, and Λ is a Radon measure (note that the selection of coordinate functions for the patches is in no way essential to the construction). Let $f \in C_0(M)$. For each $i \in \mathbb{N}$,

$$\int_{V_{\gamma(i)}} \Pi_i f \, d\lambda_{V_{\gamma(i)}} = \int_{V_{\gamma(i)}} \sum_{j=1}^{\infty} \pi_j \cdot \Pi_i f \, d\lambda_{V_{\gamma(i)}}$$

$$= \sum_{j=1}^{\infty} \int_{V_{\gamma(i)}} \pi_j \Pi_i f \, d\lambda_{V_{\gamma(i)}}$$

($\operatorname{supp} \Pi_i$ is compact in M, so $\pi_j \Pi_i = 0$ on M for all $j >$ some $m_f \in \mathbb{N}$, and the sum is finite); since $\operatorname{supp} \pi_j \subset U_{\iota(j)}$ for each $j \in \mathbb{N}$, and in view of [VI.47.xi], we can write further

$$\int_{V_{\gamma(i)}} \Pi_i f \, d\lambda_{V_{\gamma(i)}} = \sum_{j=1}^{\infty} \int_{V_{\gamma(i)} \cap U_{\iota(j)}} \pi_j \Pi_i f \, d\lambda_{V_{\gamma(i)}}$$

$$= \sum_{j=1}^{\infty} \int_{V_{\gamma(i)} \cap U_{\iota(j)}} \pi_j \Pi_i f \, d\lambda_{V_{\gamma(i)} \cap U_{\iota(j)}} .$$

Finally, we arrive at the equality

$$\Lambda f := \sum_{i=1}^{\infty} \int_{V_{\gamma(i)}} \Pi_i f \, d\lambda_{V_{\gamma(i)}}$$

$$= \sum_{i=1}^{\infty} \sum_{j=1}^{\infty} \int_{V_{\gamma(i)} \cap U_{\iota(j)}} \pi_j \Pi_i f \, d\lambda_{V_{\gamma(i)} \cap U_{\iota(j)}} . \tag{3}$$

Retracing the reasoning with the roles of $\{\pi_i\}_{i=1}^{\infty}$ and $\{\Pi_i\}_{i=1}^{\infty}$, $\{U_{\iota(i)}\}_{i=1}^{\infty}$ and $\{V_{\gamma(i)}\}_{i=1}^{\infty}$ reversed, and keeping in mind the finiteness of the sums, we come to the same expression for $\Lambda_M f$ as that displayed in (3) for Λf. Thus, $\Lambda = \Lambda_M$.

We define the measure space $(M, M_{\lambda_M}, \lambda_M)$ *to be that associated with the Radon measure* Λ_M *in the sense of [Hewitt and Stromberg,* §9]; M_{λ_M} *is the* σ*-algebra of Lebesgue-measurable subsets of* M, *and* λ_M *is the Lebesgue measure on* M. Clearly, by what was just proven, λ_M is intrinsic to M, *i.e.*, it does not depend on the particular auxiliary objects (covering coordinate patches, partition of unity) chosen for its explicit construction. Perhaps we should point out that it is legitimate to invoke the construction outlined in [VI.37], since M is locally compact and Hausdorff ([VI.45]).

We proceed to a listing of the more immediate properties of λ_M; throughout, $\{(U_\iota, h_\iota)\}_{\iota \in I}$, $\{\pi_i\}_{i=1}^{\infty}$, and $\{\iota(i)\}_{i=1}^{\infty}$ retain their meanings as set down already.

(i) $$\int_M f \, d\lambda_M = \Lambda_M f = \sum_{i=1}^{\infty} \int_{U_{\iota(i)}} \pi_i f \, d\lambda_{U_{\iota(i)}}$$

$$= \sum_{i=1}^{\infty} \int_{h_{\iota(i)}(U_{\iota(i)})} (\pi_i f) \circ h_{\iota(i)}^{-1} \cdot J h_{\iota(i)}^{-1} \, d\lambda_r , \tag{4}$$

for $f \in C_0(M)$.

(ii) $(M, \mathcal{M}_{\lambda_M}, \lambda_M)$ is complete.

(iii) $\mathcal{B}(M) \subset \mathcal{M}_{\lambda_M}$, and λ_M is regular.

(iv) If $A \subset M$, then $A \in \mathcal{M}_{\lambda_M}$ iff $A \cap K \in \mathcal{M}_{\lambda_M}$ for each compact $K \subset M$.

(v) λ_M is σ-finite. [For, M is σ-compact, and λ_M is regular.]

(vi) Let U be any coordinate patch on M. Then $(U \in \mathcal{B}(M) \subset \mathcal{M}_{\lambda_M})$

$$\mathcal{M}_{\lambda_U} = (\mathcal{M}_{\lambda_M})_U, \tag{5}$$

and

$$\lambda_U = (\lambda_M)_U \tag{6}$$

(recall the notations established in [VI.43]). [To see that this is so, first let $f \in C_0(M)$, with $\operatorname{supp} f \subset U$. Then

$$\begin{aligned} \int_U f \, d(\lambda_M)_U &= \int_U f \, d\lambda_M \\ &= \int_M f \, d\lambda_M \\ &= \sum_{i=1}^{\infty} \int_{U_{\iota(i)}} \pi_i f \, d\lambda_{U_{\iota(i)}} \end{aligned}$$

$$= \sum_{i=1}^{\infty} \int_{U_{\iota(i)}\cap U} \pi_i f \, d\lambda_{U_{\iota(i)}}$$

$$= \sum_{i=1}^{\infty} \int_{U_{\iota(i)}\cap U} \pi_i f \, d\lambda_{U_{\iota(i)}\cap U}$$

$$= \sum_{i=1}^{\infty} \int_{U} \pi_i f \, d\lambda_U$$

$$= \int_{U} \sum_{i=1}^{\infty} \pi_i f \, d\lambda_U$$

$$= \int_{U} f \, d\lambda_U,$$

having made use of [VI.47.xi], the vanishing of all but a finite number of the π_i on supp f, and the fact that supp $\pi_i \subset U_{\iota(i)}$ for each $i \in \mathbb{N}$. Using [VI.47.i, ii, and iv], (ii), (iii), (iv), and [VI.43], recalling that U is open and σ-compact in M, it follows that all hypotheses of [VI.41] are fulfilled by $(U, \mathcal{M}_{\lambda_U}, \lambda_U)$ and $(U, (\mathcal{M}_{\lambda_M})_U, (\lambda_M)_U)$. These measure spaces are therefore identical.]

(vii) Let (U,h) be any coordinate system in M. A subset $A \subset U$ is in $\mathcal{M}_{\lambda_M}$ iff $h(A)$ is a Lebesgue-measurable subset of $\mathbb{R}^r$, in which case we have

$$\lambda_M(A) = \lambda_U(A) = \lambda_h(h(A)) = \int_{h(A)} Jh^{-1} \, d\lambda_r. \tag{7}$$

[Simply combine (vi) and [VI.47.viii].]

(viii) Let (U,h) be any coordinate system in M. A complex-valued function f on U is $(\mathcal{M}_{\lambda_M})_U$-measurable iff $f\circ h^{-1}$ on $h(U)$ is $(\mathcal{M}_{\lambda_r})_{h(U)}$-measurable; if f is $(\mathcal{M}_{\lambda_M})_U$-measurable and non-negative, then

$$\int_U f\,d\lambda_M = \int_U f\,d(\lambda_M)_U = \int_{h(U)} f\circ h^{-1}\,d\lambda_h = \int_{h(U)} f\circ h^{-1}\cdot Jh^{-1}\,d\lambda_r. \tag{8}$$

Moreover, $f \in \mathcal{L}_1(U,(\mathcal{M}_{\lambda_M})_U,(\lambda_M)_U)$ iff $f\circ h^{-1}\cdot Jh^{-1} \in \mathcal{L}_1(h(U),(\mathcal{M}_{\lambda_r})_{h(U)},\lambda_r)$; in either case, equality (8) holds. [Combine (vi) with [VI.47.ix, x] and the remark following [VI.47.x].]

(ix) Let f be a complex-valued function on M. Then f is $\mathcal{M}_{\lambda_M}$-measurable iff $f|U$ is $(\mathcal{M}_{\lambda_M})_U$-measurable for each coordinate patch U on M, which holds, in turn, iff $(f|U)\circ h^{-1}$ is $(\mathcal{M}_{\lambda_r})_{h(U)}$-measurable for each coordinate system (U,h) in M. To show that f is $\mathcal{M}_{\lambda_M}$-measurable, it suffices to show that $(f|U_\iota)\circ h_\iota^{-1}$ is $(\mathcal{M}_{\lambda_r})_{h_\iota(U_\iota)}$-measurable for each $\iota \in I$ (*i.e.*, it suffices to consider any fixed covering collection $\{(U_\iota,h_\iota)\}_{\iota\in I}$ of coordinate systems in M).

[Let f be $\mathcal{M}_{\lambda_M}$-measurable. Choose any coordinate patch U on M. If V is open in $\mathbb{K}$, then $(f|U)^{-1}(V) = U\cap f^{-1}(V) \in \mathcal{M}_{\lambda_M}$ and $\subset U$. Thus, $f|U$ is $(\mathcal{M}_{\lambda_M})_U$-measurable. Now, suppose that $f|U$ is $(\mathcal{M}_{\lambda_M})_U$-measurable for each coordinate patch U on M; in particular, this is true for each U_ι, $\iota \in I$. We can extract from the cover

$\{U_\iota\}_{\iota\in I}$ a subcover $\{U_\iota\}_{\iota\in I_0}$, where $I_0 \subset I$ is countable (cf., the proof of [VI.45]). Let $V \subset \mathbb{K}$ be open. Then $U_\iota \cap f^{-1}(V) = (f|U_\iota)^{-1}(V) \in (\mathcal{M}_{\lambda_M})_{U_\iota} \subset \mathcal{M}_{\lambda_M}$ for each $\iota \in I_0$, so also $f^{-1}(V) = \cup_{\iota\in I_0} \{U_\iota \cap f^{-1}(V)\} \in \mathcal{M}_{\lambda_M}$, since $\mathcal{M}_{\lambda_M}$ is a σ-algebra. Thus, f is $\mathcal{M}_{\lambda_M}$-measurable. By (viii), $f|U$ is $(\mathcal{M}_{\lambda_M})_U$-measurable iff $(f|U)\circ h^{-1}$ is $(\mathcal{M}_{\lambda_r})_{h(U)}$-measurable, where (U,h) is any coordinate system in M. These facts show that the first statement of (ix) is correct. The second statement is a corollary of the reasoning just completed.]

(x) Let f be a non-negative $\mathcal{M}_{\lambda_M}$-measurable function on M. Then

$$\begin{aligned}\int_M f\, d\lambda_M &= \sum_{i=1}^{\infty} \int_M \pi_i f\, d\lambda_M \\ &= \sum_{i=1}^{\infty} \int_{U_{\iota(i)}} \pi_i f\, d\lambda_M \\ &= \sum_{i=1}^{\infty} \int_{h_{\iota(i)}(U_{\iota(i)})} (\pi_i f)\circ h_{\iota(i)}^{-1} \cdot J h_{\iota(i)}^{-1}\, d\lambda_r .\end{aligned} \tag{9}$$

[A well-known theorem of Lebesgue (Hewitt and Stromberg [20], Theorem (12.21)) permits us to write $\int_M \left(\sum_{i=1}^{\infty} g_i\right) d\lambda_M = \sum_{i=1}^{\infty} \int_M g_i\, d\lambda_M$ for any sequence $(g_i)_{i=1}^{\infty}$ of non-negative $\mathcal{M}_{\lambda_M}$-measurable functions on M; the sequence $(\pi_i f)_{i=1}^{\infty}$ fulfills these requirements. Using this in conjunction with (viii) and (ix), we obtain

$$\int_M f\, d\lambda_M = \int_M \left(\sum_{i=1}^{\infty} \pi_i f\right) d\lambda_M$$

$$= \sum_{i=1}^{\infty} \int_M \pi_i f\, d\lambda_M$$

$$= \sum_{i=1}^{\infty} \int_{U_{\iota(i)}} \pi_i f\, d\lambda_M$$

$$= \sum_{i=1}^{\infty} \int_{h_{\iota(i)}(U_{\iota(i)})} (\pi_i f)\circ h_{\iota(i)}^{-1} \cdot Jh_{\iota(i)}^{-1}\, d\lambda_r . \;]$$

(xi) Let f be a complex M_{λ_M}-measurable function on (or defined λ_M-*a.e.* on) M. Then $f \in L_1(M, M_{\lambda_M}, \lambda_M)$ iff

$$\sum_{i=1}^{\infty} \int_{U_{\iota(i)}} \pi_i |f|\, d\lambda_M < \infty, \tag{10}$$

i.e. (by (x)), iff

$$\sum_{i=1}^{\infty} \int_{h_{\iota(i)}(U_{\iota(i)})} (\pi_i |f|)\circ h_{\iota(i)}^{-1} \cdot Jh_{\iota(i)}^{-1}\, d\lambda_r < \infty, \tag{11}$$

in which case we have

$$\begin{aligned} \int_M f\, d\lambda_M &= \sum_{i=1}^{\infty} \int_{U_{\iota(i)}} \pi_i f\, d\lambda_M \\ &= \sum_{i=1}^{\infty} \int_{h_{\iota(i)}(U_{\iota(i)})} (\pi_i f)\circ h_{\iota(i)}^{-1} \cdot Jh_{\iota(i)}^{-1}\, d\lambda_r . \end{aligned} \tag{12}$$

[We may suppose that f is defined on M. Since f is $\mathcal{M}_{\lambda_M}$-measurable, $f \in \mathcal{L}_1(M, \mathcal{M}_{\lambda_M}, \lambda_M)$ iff $|f| \in \mathcal{L}_1(M, \mathcal{M}_{\lambda_M}, \lambda_M)$, *i.e.*, iff $\int_M |f| \, d\lambda_M < \infty$. The latter obtains, in view of (x), iff (10) or, equivalently, (11) should hold. If $f \in \mathcal{L}_1(M, \mathcal{M}_{\lambda_M}, \lambda_M)$, the equalities in (12) follow from the definition of the integral of a complex function in terms of integrals of non-negative functions, and the fact that (12) holds for a non-negative integrand, by (9).]

For our purposes, the following "transformation of integrals" result proves to be quite a useful by-product of the development of the Lebesgue measure on a manifold.

[VI.52] THEOREM. *Let* M *be an* $(r,n;q)$*-manifold. Let* $m \in \mathbb{N}$, *and suppose that* $g: M \to \mathbb{R}^m$ *is a* q*-imbedding; the assertions of Theorem* [VI.30] *hold, so that, in particular,* $g(M)$ *is an* $(r,m;q)$*-manifold.*

(*i*) *A subset* $E \subset g(M)$ *is in* $\mathcal{M}_{\lambda_{g(M)}}$ *iff* $g^{-1}(E) \in \mathcal{M}_{\lambda_M}$. *If* $E \in \mathcal{M}_{\lambda_{g(M)}}$, *then*

$$\lambda_{g(M)}(E) = \int_M \Xi_E \circ g \cdot Jg \, d\lambda_M = \int_{g^{-1}(E)} Jg \, d\lambda_M. \qquad (1)$$

(*ii*) *A complex function* f *on* $g(M)$ *is* $\mathcal{M}_{\lambda_{g(M)}}$*-measurable iff* $f \circ g$ *is* $\mathcal{M}_{\lambda_M}$*-measurable.*

(*iii*) *A complex function* f *defined* $\lambda_{g(M)}$*-a.e. on* $g(M)$

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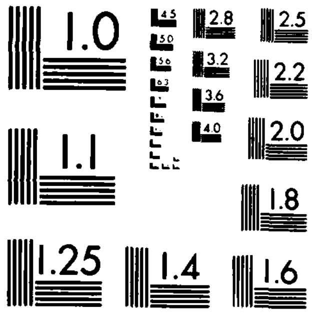

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is in $L_1(g(M), M_{\lambda_{g(M)}}, \lambda_{g(M)})$ *iff* $f \circ g \cdot Jg \in L_1(M, M_{\lambda_M}, \lambda_M)$; *if either inclusion should hold, then*

$$\int_{g(M)} f \, d\lambda_{g(M)} = \int_M f \circ g \cdot Jg \, d\lambda_M. \tag{2}$$

(iv) *If* f *is non-negative and* $M_{\lambda_{g(M)}}$*-measurable on* $g(M)$, *then equality* (2) *holds.*

P R O O F. Since M is a locally compact σ-compact Hausdorff space, $(M, M_{\lambda_M}, \lambda_M)$ is complete, $\mathcal{B}(M) \subset M_{\lambda_M}$, λ_M is regular, $Jg > 0$, and $Jg \in L_1^{loc}(M, M_{\lambda_M}, \lambda_M)$ (Jg is continuous; cf., [VI.25]), we can construct the complete measure space $(M, M_{\lambda_M}, (\lambda_M)_{Jg})$ as in Proposition [VI.38], where $(\lambda_M)_{Jg}(A) := \int_A Jg \, d\lambda_M$, for each $A \in M_{\lambda_M}$. $(\lambda_M)_{Jg}$ is regular and σ-finite. In fact, $(M, M_{\lambda_M}, (\lambda_M)_{Jg})$ is the measure space associated with the Radon measure $f \mapsto \int_M fJg \, d\lambda_M$ on $C_0(M)$ as in [VI.37]; this follows from [VI.41], once we take into account the properties just cited, those of [VI.37], [VI.51.iv], and the equality $\int_M f \, d(\lambda_M)_{Jg} = \int_M fJg \, d\lambda_M$ for $f \in C_0(M)$ ([VI.38.iii]).

Now, it is a simple matter to check that the prerequisites of the construction effected in [VI.39] are met by $(M, M_{\lambda_M}, (\lambda_M)_{Jg})$ and $g: M \to g(M)$. Thus, the image $(g(M), M_{((\lambda_M)_{Jg})_g}, ((\lambda_M)_{Jg})_g)$

of $(M, \mathcal{M}_{\lambda_M}, (\lambda_M)_{Jg})$ under g is defined; for brevity, let us write $\mathcal{M}_{\lambda^g_M} := \mathcal{M}_{((\lambda_M)_{Jg})_g}$ and $\lambda^g_M := ((\lambda_M)_{Jg})_g$. This image is a measure space in the sense of [Hewitt and Stromberg, §9], *viz.*, that generated by the Radon measure $f \mapsto \int_M f\circ g \; d(\lambda_M)_{Jg} = \int_M f\circ g Jg \; d\lambda_M$ on $C_0(g(M))$, and so possesses the properties of [VI.37]. The conclusions of both [VI.40] and [VI.42] can be applied in the present setting, the mapping $g\colon M \to g(M)$ being a homeomorphism, since $g\colon M \to \mathbb{R}^m$ is a q-imbedding ([VI.23.iv.1]).

We claim that the measure spaces $(g(M), \mathcal{M}_{\lambda_{g(M)}}, \lambda_{g(M)})$ and $(g(M), \mathcal{M}_{\lambda^g_M}, \lambda^g_M)$ coincide. Let us suppose for the moment that this has been proven, and check that all conclusions of the theorem follow thereby:

(i) By [VI.42.ii], a subset $E \subset g(M)$ is in $\mathcal{M}_{\lambda^g_M}$ iff $g^{-1}(E) \in \mathcal{M}_{\lambda_M}$. Since $\mathcal{M}_{\lambda^g_M} = \mathcal{M}_{\lambda_{g(M)}}$, the first part of (i) follows. Since $\lambda_{g(M)} = \lambda^g_M$ and $\lambda_{g(M)}$ is σ-finite, [VI.40.ii] gives, whenever $E \in \mathcal{M}_{\lambda_{g(M)}} = \mathcal{M}_{\lambda^g_M}$, using the definition of $(\lambda_M)_{Jg}$,

$$\lambda_{g(M)}(E) = \lambda^g_M(E) = (\lambda_M)_{Jg}(g^{-1}(E)) = \int_{g^{-1}(E)} Jg \; d\lambda_M .$$

Thus, (i) is correct.

(ii) From the equality $\mathcal{M}_{\lambda^g_M} = \mathcal{M}_{\lambda_{g(M)}}$, (ii) is an immediate consequence of [VI.42.iii].

(iii) From [VI.42.iv], a complex function f, defined λ^g_M-*a.e.*, *i.e.*, $\lambda_{g(M)}$-*a.e.*, on g(M), is in $L_1(g(M),\mathcal{M}_{\lambda^g_M},\lambda^g_M) = L_1(g(M),\mathcal{M}_{\lambda_{g(M)}},\lambda_{g(M)})$ iff $f \circ g \in L_1(M,\mathcal{M}_{\lambda_M},(\lambda_M)_{Jg})$, but [VI.38.iv] says that the latter inclusion is valid iff $f \circ g \cdot Jg \in L_1(M,\mathcal{M}_{\lambda_M},\lambda_M)$. If any one of these inclusions should hold, (VI.42.1) and [VI.38.iv] give

$$\int_{g(M)} f \, d\lambda_{g(M)} = \int_{g(M)} f \, d\lambda^g_M = \int_M f \circ g \; d(\lambda_M)_{Jg} = \int_M f \circ g \cdot Jg \; d\lambda_M,$$

which is just (2). This completes the proof of (iii).

(iv) Statement (iv) is a simple consequence of (iii). For, let f be non-negative and $\mathcal{M}_{\lambda_{g(M)}}$-measurable on g(M). If one of the integrals appearing in (2) is finite, then both must be finite, and (2) holds, by (iii). On the other hand, if one of the integrals in (2) is infinite, both must be infinite, so (2) holds, again by (iii). Note here that $f \circ g \cdot Jg$ is $\mathcal{M}_{\lambda_M}$-measurable, by (ii) and the continuity of Jg.

Thus, the proof of the theorem is reduced to verifying that $(g(M),\mathcal{M}_{\lambda_{g(M)}},\lambda_{g(M)}) = (g(M),\mathcal{M}_{\lambda^g_M},\lambda^g_M)$, for which we shall appeal, as usual, to [VI.41]. A quick check of the properties of these two measure spaces (cf., [VI.37.ii-v]) shows that it is enough to prove that $\int_{g(M)} f\, d\lambda_{g(M)} = \int_{g(M)} f\, d\lambda_M^g$, for each $f \in C_0(g(M))$, considering the hypotheses of [VI.41]. We already know, however, by the manner in which $\lambda_M^g := ((\lambda_M)_{Jg})_g$ and $(\lambda_M)_{Jg}$ are constructed, that

$$\int_{g(M)} f\, d\lambda_M^g = \int_M f\circ g\; d(\lambda_M)_{Jg} = \int_M f\circ g\cdot Jg\; d\lambda_M$$

whenever $f \in C_0(g(M))$. Therefore, we wish to demonstrate that

$$\int_{g(M)} f\, d\lambda_{g(M)} = \int_M f\circ g\cdot Jg\; d\lambda_M, \quad \text{for each} \quad f \in C_0(g(M)). \tag{3}$$

Suppose, first, that $f \in C_0(g(M))$ with $\operatorname{supp} f \subset U$, where U is any coordinate patch on $g(M)$. Choose a coordinate function h for U. If we set $\tilde{U} := g^{-1}(U)$ and $\tilde{h} := h\circ(g|\tilde{U})$, then $\tilde{U}$ is open in M, $\tilde{h}(\tilde{U}) = h(U)$ is open in $\mathbb{R}^r$, and $\tilde{h}$ is clearly a homeomorphism of $\tilde{U}$ onto $\tilde{h}(\tilde{U})$. Since $\tilde{h}^{-1} = g^{-1}\circ h^{-1}$ and $g^{-1}\colon g(M) \to \mathbb{R}^n$ is a q-imbedding ([VI.30.iii]), it follows that $\tilde{h}^{-1} \in C^q(\tilde{h}(\tilde{U});\mathbb{R}^n)$ and $\operatorname{rank} D\tilde{h}^{-1}(x) = r$ for each $x \in \tilde{h}(\tilde{U})$. Thus, $(\tilde{U},\tilde{h})$ is a coordinate system in M. Note that $h^{-1} = g\circ\tilde{h}^{-1}$. Recalling (VI.24.3), we have

$$\begin{aligned}
\{(Jg)\circ\tilde{h}^{-1}\}\cdot J\tilde{h}^{-1} &= \frac{|(g\circ\tilde{h}^{-1})_{,1}\wedge\ldots\wedge(g\circ\tilde{h}^{-1})_{,r}|}{|\tilde{h}^{-1}_{,1}\wedge\ldots\wedge\tilde{h}^{-1}_{,r}|}\cdot|\tilde{h}^{-1}_{,1}\wedge\ldots\wedge\tilde{h}^{-1}_{,r}| \\
&= |(g\circ\tilde{h}^{-1})_{,1}\wedge\ldots\wedge(g\circ\tilde{h}^{-1})_{,r}| \\
&= J(g\circ\tilde{h}^{-1}) = Jh^{-1}
\end{aligned}$$

on $\tilde{h}(\tilde{U}) = h(U)$. Then, using [VI.51.vi] and (VI.47.3) gives, since $\operatorname{supp} f\circ g \subset g^{-1}(U) = \tilde{U}$,

$$\int_M f\circ g\cdot Jg\, d\lambda_M = \int_{\tilde{U}} f\circ g\cdot Jg\, d\lambda_{\tilde{U}}$$

$$= \int_{\tilde{h}(\tilde{U})} f\circ g\circ \tilde{h}^{-1}\cdot (Jg)\circ \tilde{h}^{-1}\cdot J\tilde{h}^{-1}\, d\lambda_r$$

$$= \int_{h(U)} f\circ h^{-1}\cdot Jh^{-1}\, d\lambda_r .$$

But also, again from [VI.51.vi] and (VI.47.3),

$$\int_{g(M)} f\, d\lambda_{g(M)} = \int_U f\, d\lambda_U = \int_{h(U)} f\circ h^{-1}\cdot Jh^{-1}\, d\lambda_r .$$

This establishes (3) for the case in which $\operatorname{supp} f$ is contained in a coordinate patch on M.

Now, considering the general case, let $f \in C_0(g(M))$. Let $\{U_\iota\}_{\iota\in I}$ be a covering collection of coordinate patches on $g(M)$, $\{\pi_i\}_{i=1}^{\infty}$ a locally finite q-partition of unity subordinate to $\{U_\iota\}_{\iota\in I}$, and, for each $i \in \mathbb{N}$, choose $\iota(i) \in I$ such that $\operatorname{supp} \pi_i \subset U_{\iota(i)}$. For each $\iota \in I$, set $\tilde{U}_\iota := g^{-1}(U_\iota)$, and for each $i \in \mathbb{N}$, set $\tilde{\pi}_i := \pi_i\circ g$. Then it is easy to check that $\{\tilde{U}_\iota\}_{\iota\in I}$ is a covering collection of coordinate patches on M, $\{\tilde{\pi}_i\}_{i=1}^{\infty}$ is a locally finite q-partition of unity for M, subordinate to $\{\tilde{U}_\iota\}_{\iota\in I}$, and $\operatorname{supp} \tilde{\pi}_i \subset \tilde{U}_{\iota(i)}$ for each $i \in \mathbb{N}$. For example, to see that $\tilde{\pi}_i \in C^q(M)$, let (V,k) be any coordinate system in M. $(g(V),(g\circ k^{-1})^{-1})$ is then a coordinate system in $g(M)$ (cf., the proof of [VI.30]), $(g\circ k^{-1})^{-1}(g(V)) = k(V)$, and the inverse of $(g\circ k^{-1})^{-1}$ is $g\circ k^{-1}$. Since $\pi_i \in C^q(g(M))$, we have $\tilde{\pi}_i\circ k^{-1} = \pi_i\circ g\circ k^{-1} \in C^q(k(V))$: it follows that $\tilde{\pi}_i \in C^q(M)$. We omit the details required to verify the remainder of the assertions made above. Now, certainly $f \in L_1(g(M), M_{\lambda_{g(M)}}, \lambda_{g(M)})$, since supp f is compact and $\lambda_{g(M)}$ is regular, and $\operatorname{supp} \pi_i f \subset U_{\iota(i)}$ for each $i \in \mathbb{N}$. Then, using [VI.51.xi] and the preliminary result for continuous functions with support in a coordinate patch,

$$\int_{g(M)} f\, d\lambda_{g(M)} = \sum_{i=1}^{\infty} \int_{g(M)} \pi_i f\, d\lambda_{g(M)}$$

$$= \sum_{i=1}^{\infty} \int_M (\pi_i f)\circ g\cdot Jg\, d\lambda_M$$

$$= \sum_{i=1}^{\infty} \int_M \tilde{\pi}_i\cdot f\circ g\cdot Jg\, d\lambda_M$$

$$= \int_M f\circ g\cdot Jg\, d\lambda_M,$$

the last equality holding, again, by [VI.51.xi], since we obviously have $f\circ g\cdot Jg \in L_1(M, M_{\lambda_M}, \lambda_M)$. Thus, (3) has been proven, and, with it, the theorem. □.

Let us observe that [VI.52] holds in the case $r = n = m$, *i.e.*, when M is open in $\mathbb{R}^n$, and $g\colon M \to \mathbb{R}^n$ is q-regular. Then $Jg = |Jg|$ ([VI.13]), and λ_M, $\lambda_{g(M)}$ become restrictions of the usual Lebesgue measure λ_n, so (VI.52.2) reduces to the more familiar formula for the transformation of a Lebesgue integral over an open subset of $\mathbb{R}^n$; note the remark following [VI.47]. In this case, [VI.52] complements [VI.44].

In the next group of sections, we present and examine various regularity hypotheses for open subsets of $\mathbb{R}^n$.

[VI.53] D E F I N I T I O N S. Let Ω be an open subset of $\mathbb{R}^n$ $(n \geq 2)$, and $q \in \mathbb{N}\cup\{\infty\}$.

(a) Let $x \in \partial\Omega$: Ω is q-*regular at* x iff there exist an open neighborhood, U_x, of x in $\mathbb{R}^n$ and a function $\Phi_x \in C^q(U_x)$ such that

(i) $\operatorname{grad} \Phi_x(y) \neq 0$ for each $y \in U_x$,

(ii) $\partial\Omega\cap U_x = \{y \in U_x \mid \Phi_x(y) = 0\}$,

(iii) $\Omega\cap U_x = \{y \in U_x \mid \Phi_x(y) < 0\}$.

(b) Ω is a q-*regular domain* iff Ω is q-regular at each $x \in \partial\Omega$. ■.

[VI.54] R E M A R K S. Suppose that $\Omega \subset \mathbb{R}^n$ is open. (a) If Ω is q-regular at $x \in \partial\Omega$, then x lies in a relatively open subset of $\partial\Omega$ which is an $(n-1,n;q)$-manifold (*e.g.*, the set $\partial\Omega\cap U_x$ of [VI.53.a]), and Ω is in fact q-regular at each point of this open subset. Consequently, any non-void set $\Gamma \subset \partial\Omega$ such that Ω is q-regular at each $x \in \Gamma$ must be contained in an open subset of $\partial\Omega$ which is an $(n-1,n;q)$-manifold and at each point of which Ω is q-regular (*e.g.*, the set $\cup_{y\in\Gamma}\{\partial\Omega\cap U_y\}$; cf., also, Remark [VI.4.b]). If Ω is q-regular at $x \in \partial\Omega$, then Ω "lies on one side of its boundary in a neighborhood of x."

(b) If Ω is a q-regular domain with $\partial\Omega \neq \emptyset$, clearly $\partial\Omega$ is an $(n-1,n;q)$-manifold. A q-regular domain need not be connected.

[VI.55] PROPOSITION. *Let* Ω *be a regularly open subset of* $\mathbb{R}^n$.

(*i*) *Suppose that* M *is an* $(n-1,n;q)$*-manifold which is relatively open in* $\partial\Omega$*. Then* Ω *and* $\Omega^{-'}$ *are q-regular at each point of* M.

(*ii*) *If* $\partial\Omega$ *is an* $(n-1,n;q)$*-manifold, then* Ω *and* $\Omega^{-'}$ *are q-regular domains.*

PROOF. (i) Choose any $x \in M$. Since M is an $(n-1,n;q)$-manifold, there exist an open neighborhood, $\hat{U}_x$, of x in $\mathbb{R}^n$ and a function $\hat{\Phi}_x \in C^q(\hat{U}_x)$ such that $\operatorname{grad}\hat{\Phi}_x(y) \neq 0$ for each $y \in \hat{U}_x$, and $M\cap\hat{U}_x = \{y \in \hat{U}_x \mid \hat{\Phi}_x(y) = 0\}$. M is open in $\partial\Omega$, so we can find an open neighborhood, $\tilde{U}_x$, of x in $\mathbb{R}^n$ such that $\tilde{U}_x \subset \hat{U}_x$ and $\partial\Omega\cap\tilde{U}_x \subset M$. Let $\tilde{\Phi}_x := \hat{\Phi}_x| \tilde{U}_x$: then it is obvious that $\tilde{\Phi}_x \in C^q(\tilde{U}_x)$, $\operatorname{grad}\tilde{\Phi}_x(y) \neq 0$ for each $y \in \tilde{U}_x$, and $\partial\Omega\cap\tilde{U}_x = \{y \in \tilde{U}_x \mid \tilde{\Phi}_x(y) = 0\}$. If $i \in \{1,\dots,n\}$, and $y \in \mathbb{R}^n$, let us denote by $\lambda(i)$ the increasing $(n-1)$-tuple which is obtained from $\{1,\dots,n\}$ by deleting i, and write $y_{(i)} := \Xi^{\lambda(i)}(y)$.

Then, by the implicit function theorem, there exist an $i_x \in \{1,\ldots,n\}$, an open neighborhood of x, $\tilde{W}_x \subset \tilde{U}_x$, an open neighborhood of $x_{(i_x)}$, $\tilde{V}_x \subset \mathbb{R}^{n-1}$, and a function $\phi \in C^q(\tilde{V}_x)$ for which

$$\{y \in \tilde{W}_x \mid \tilde{\Phi}_x(y) = 0\} = \{y \in \mathbb{R}^n \mid y_{(i_x)} \in \tilde{V}_x, \quad y^{i_x} = \phi(y_{(i_x)})\}.$$

Clearly, since $\tilde{W}_x \subset \tilde{U}_x$,

$$\{y \in \tilde{W}_x \mid \tilde{\Phi}_x(y) = 0\} = \partial\Omega \cap \tilde{W}_x.$$

Now, choose $\varepsilon > 0$ such that $B^n_\varepsilon(x) \subset \tilde{W}_x$, then choose $\delta \in (0, \varepsilon/2]$ such that $B^{n-1}_\delta(x_{(i_x)}) \subset \tilde{V}_x$ and $|\phi(\hat{z})-\phi(x_{(i_x)})| < \varepsilon/2$ whenever $\hat{z} \in B^{n-1}_\delta(x_{(i_x)})$. Set

$$U_x := B^n_\varepsilon(x) \cap \{y \in \mathbb{R}^n \mid y_{(i_x)} \in B^{n-1}_\delta(x_{(i_x)})\},$$

and define $\Phi_x\colon U_x \to \mathbb{R}$ according to

$$\Phi_x(y) := y^{i_x} - \phi(y_{(i_x)}) \qquad \text{for each} \qquad y \in U_x.$$

Obviously, U_x is an open neighborhood of x in $\mathbb{R}^n$, $\Phi_x \in C^q(U_x)$, and grad Φ_x does not vanish on U_x. Moreover,

$$\partial\Omega \cap U_x = \{y \in U_x \mid \Phi_x(y) = 0\}:$$

for, if $y \in \partial\Omega \cap U_x$, then $y \in \partial\Omega \cap \tilde{W}_x$, whence $y_{(i_x)} \in \tilde{V}_x$ and $y^{i_x} = \phi(y_{(i_x)})$, so $\Phi_x(y) = 0$; if $y \in U_x$ and $\Phi_x(y) = 0$, then $y_{(i_x)} \in B^{n-1}_\delta(x_{(i_x)}) \subset \tilde{V}_x$ and $y^{i_x} = \phi(y_{(i_x)})$, so $y \in \partial\Omega$. Next, define

$U_{x-} := \{y \in U_x | \ \phi_x(y) < 0\}$ and $U_{x+} := \{y \in U_x | \ \phi_x(y) > 0\}$;

note that $U_x = U_{x-} \cup U_{x+} \cup \{\partial\Omega \cap U_x\}$. We claim that (exactly) one of the equalities $\Omega \cap U_x = U_{x-}$, $\Omega \cap U_x = U_{x+}$ is true; if the former holds, then U_x and ϕ_x fulfill all requirements of [VI.53.a], while if the latter equality is valid, then U_x and $-\phi_x$ fulfill those requirements. Thus, the q-regularity of Ω at x will follow once the claim has been substantiated.

To see, then, that (precisely) one of $\Omega \cap U_x = U_{x-}$, $\Omega \cap U_x = U_{x+}$ is true, suppose that U_{x-} and U_{x+} have been shown to be connected. Obviously, $\partial\Omega \cap U_{x-} = \emptyset$, $\partial\Omega \cap U_{x+} = \emptyset$, and $\Omega^{-'} = \Omega^{'o}$, so

$$U_{x-} = (U_{x-} \cap \Omega) \cup (U_{x-} \cap \Omega^{'o}), \quad \text{and} \quad U_{x+} = (U_{x+} \cap \Omega) \cup (U_{x+} \cap \Omega^{'o}).$$

The connectedness of U_{x-} implies that not both $U_{x-} \cap \Omega \neq \emptyset$ and $U_{x-} \cap \Omega^{'o} \neq \emptyset$ can hold, for Ω and $\Omega^{'o}$ are separated. Similarly, $U_{x+} \cap \Omega \neq \emptyset$ and $U_{x+} \cap \Omega^{'o} \neq \emptyset$ cannot both be true. We do know that at least one of $U_{x-} \cap \Omega \neq \emptyset$, $U_{x+} \cap \Omega \neq \emptyset$ must be true, for otherwise we should have $U_x \cap \Omega = \Omega \cap \{U_{x-} \cup U_{x+} \cup \{\partial\Omega \cap U_x\}\} = \emptyset$, which is impossible, since $x \in \partial\Omega$ and U_x is a neighborhood of x. In fact, exactly one of $\Omega \cap U_{x-} \neq \emptyset$, $\Omega \cap U_{x+} \neq \emptyset$ holds: if both hold, then $\Omega^{'o} \cap U_{x-} = \emptyset$, $\Omega^{'o} \cap U_{x+} = \emptyset$, by the observation made above. Since Ω is regularly open,

$$\partial(\Omega'^{o}) = \Omega'^{o-} \cap \Omega'^{o'-}$$

$$= \Omega'^{o'o'} \cap \Omega^{--}$$

$$= \Omega^{-o'} \cap \Omega^{-}$$

$$= \Omega' \cap \Omega^{-}$$

$$= \Omega'^{-} \cap \Omega^{-}$$

$$= \partial\Omega .$$

Thus, $x \in \partial(\Omega'^{o})$, U_x is a neighborhood of x, but $\Omega\cap U_x = \Omega'^{o} \cap \{U_{x-} \cup U_{x+} \cup \{\partial(\Omega'^{o}) \cap U_x\}\} = \emptyset$, which is impossible. Thus, precisely one of $\Omega \cap U_{x-} \neq \emptyset$, $\Omega \cap U_{x+} \neq \emptyset$ is true. We consider the two cases, in turn:

(i) Suppose that $\Omega \cap U_{x-} \neq \emptyset$: then $\Omega \cap U_{x+} = \emptyset$, and $\Omega'^{o} \cap U_{x-} = \emptyset$. If $y \in \Omega \cap U_x$, then $y \notin \partial\Omega$, $y \notin U_{x+}$, but $y \in U_x$, so we conclude that $y \in U_{x-}$. On the other hand, let $y \in U_{x-}$: then $y \notin \Omega'^{o}$ and $y \notin \partial\Omega$, so $y \in \Omega$, giving $y \in \Omega \cap U_x$. Therefore, $\Omega \cap U_x = U_{x-}$, in this first case. Since $\Omega \cap U_{x+} = \emptyset$, obviously we cannot have $\Omega \cap U_x = U_{x+}$.

(ii) Suppose that $\Omega \cap U_{x+} \neq \emptyset$: then $\Omega \cap U_{x-} = \emptyset$, and $\Omega'^{o} \cap U_{x+} = \emptyset$. Now we find, *via* reasoning similar to that just carried out, that $\Omega \cap U_x = U_{x+}$ (and $\Omega \cap U_x \neq U_{x-}$).

For the completion of the proof, there remains only the verification of the connectedness of U_{x-} and U_{x+}. We shall prove that U_{x-} is connected, the proof for U_{x+} being quite similar.

It suffices to show that U_{x-} is pathwise connected. Then, choose y_1 and y_2 in U_{x-}, so $y_j \in B^n_\varepsilon(x)$, $y_{j(i_x)} \in B^{n-1}_\delta(x_{(i_x)})$, and $y_j^{i_x} < \phi(y_{j(i_x)})$, for $j = 1, 2$. Set

$$B_\delta := \left\{ y \in \mathbb{R}^n \middle| \; y_{(i_x)} \in B^{n-1}_\delta(x_{(i_x)}), \quad y^{i_x} = \phi(x_{(i_x)}) - \frac{\varepsilon}{2} \right\} .$$

If $y \in B_\delta$, then $y \in U_x$, since $y_{(i_x)} \in B^{n-1}_\delta(x_{(i_x)})$, while

$$\begin{aligned}
|y-x|_n &= \sqrt{\{|y_{(i_x)}-x_{(i_x)}|^2_{n-1}+|y^{i_x}-x^{i_x}|^2\}} \\
&= \sqrt{\{|y_{(i_x)}-x_{(i_x)}|^2_{n-1}+|y^{i_x}-\phi(x_{(i_x)})|^2\}} \\
&= \sqrt{\left\{|y_{(i_x)}-x_{(i_x)}|^2_{n-1}+\frac{\varepsilon^2}{4}\right\}} \\
&< \sqrt{\left\{\delta^2+\frac{\varepsilon^2}{4}\right\}} \\
&\leq \sqrt{\left\{\frac{\varepsilon^2}{4}+\frac{\varepsilon^2}{4}\right\}} \\
&< \varepsilon ,
\end{aligned}$$

so $y \in B^n_\varepsilon(x)$. Moreover,

$$\begin{aligned}
\Phi_x(y) &= y^{i_x}-\phi(y_{(i_x)}) \\
&= \phi(x_{(i_x)})-\phi(y_{(i_x)})-\frac{\varepsilon}{2} \\
&\leq |\phi(x_{(i_x)})-\phi(y_{(i_x)})|-\frac{\varepsilon}{2} \\
&< 0,
\end{aligned}$$

so $y \in U_{x-}$. Thus, $B_\delta \subset U_{x-}$. Define, for $j = 1, 2$, $\tilde{y}_j \in \mathbb{R}^n$ by

$$\tilde{y}_j^{i_x} := \phi(x_{(i_x)}) - \frac{\varepsilon}{2},$$

$$\tilde{y}_j^k := y_j^k, \quad \text{for} \quad k \in \{1,\ldots,n\}, \quad k \neq i_x.$$

Clearly, $\tilde{y}_j \in B_\delta$ for $j = 1, 2$, and B_δ is convex, so the line segment joining $\tilde{y}_1$ and $\tilde{y}_2$ lies in B_δ, hence in U_{x-}. We shall show that the line segment joining y_j and $\tilde{y}_j$ lies in U_{x-}, for $j = 1, 2$; from this, the pathwise connectedness of U_{x-} follows easily. Then let $j = 1$ or 2, and suppose y is on the line segment joining y_j and $\tilde{y}_j$, *i.e.*, $y = y_j + s(\tilde{y}_j - y_j)$ for some $s \in [0,1]$, so $y^{i_x} = y_j^{i_x} + s(\tilde{y}_j^{i_x} - y_j^{i_x}) = y_j^{i_x} + s\left(\phi(x_{(i_x)}) - y_j^{i_x} - \frac{\varepsilon}{2}\right)$, and $y_{(i_x)} = y_{j(i_x)} \in B_\delta^{n-1}(x_{(i_x)})$ (having noted that $\tilde{y}_{j(i_x)} = y_{j(i_x)}$). Observing that $\phi(y_{(i_x)}) = \phi(y_{j(i_x)})$, $|\phi(x_{(i_x)}) - \phi(y_{j(i_x)})| < \varepsilon/2$, and recalling that $y_j^{i_x} < \phi(y_{j(i_x)})$, we find

$$\begin{aligned} y^{i_x} - \phi(y_{(i_x)}) &= y_j^{i_x} - \phi(y_{j(i_x)}) + s\left(\phi(x_{(i_x)}) - y_j^{i_x} - \frac{\varepsilon}{2}\right) \\ &= (1-s)\{y_j^{i_x} - \phi(y_{j(i_x)})\} + s\left\{\phi(x_{(i_x)}) - \phi(y_{j(i_x)}) - \frac{\varepsilon}{2}\right\} \\ &< 0, \end{aligned}$$

i.e., $\phi_x(y) < 0$. In order to prove that $y \in U_{x-}$, we now need only show that $y \in U_x$, for which it remains to be shown that $y \in B_\varepsilon^n(x)$. We consider the two possible cases:

(i)' Assume that $y_j^{i_x} \le \tilde{y}_j^{i_x} = \phi(x_{(i_x)}) - \frac{\varepsilon}{2}$: then, first,

$$y^{i_x} - \phi(x_{(i_x)}) = y_j^{i_x} - \phi(x_{(i_x)}) + s(\tilde{y}_j^{i_x} - y_j^{i_x}) \ge y_j^{i_x} - \phi(x_{(i_x)}),$$

and

$$\begin{aligned} y^{i_x} - \phi(x_{(i_x)}) &= y_j^{i_x} - \phi(x_{(i_x)}) + s\left(\phi(x_{(i_x)}) - \frac{\varepsilon}{2} - y_j^{i_x}\right) \\ &= (1-s)(y_j^{i_x} - \phi(x_{(i_x)})) - s\,\frac{\varepsilon}{2} \\ &\le -\frac{\varepsilon}{2}, \end{aligned}$$

which give

$$|y^{i_x} - \phi(x_{(i_x)})| = -(y^{i_x} - \phi(x_{(i_x)})) \le -(y_j^{i_x} - \phi(x_{(i_x)})) = |y_j^{i_x} - \phi(x_{(i_x)})|.$$

Thus,

$$\begin{aligned} |y-x|_n^2 &= |y_{(i_x)} - x_{(i_x)}|_{n-1}^2 + |y^{i_x} - x^{i_x}|^2 \\ &= |y_{j(i_x)} - x_{(i_x)}|_{n-1}^2 + |y^{i_x} - \phi(x_{(i_x)})|^2 \\ &\le |y_{j(i_x)} - x_{(i_x)}|_{n-1}^2 + |y_j^{i_x} - \phi(x_{(i_x)})|^2 \\ &= |y_j - x|_n^2 \\ &< \varepsilon^2, \end{aligned}$$

the latter inequality holding since $y_j \in U_x$.

(ii)' Assume that $y_j^{i_x} > \tilde{y}_j^{i_x} = \phi(x_{(i_x)}) - \frac{\varepsilon}{2}$: now,

$$y^{i_x}-\phi(x_{(i_x)}) = (1-s)(y_j^{i_x}-\phi(x_{(i_x)}))-s\,\frac{\varepsilon}{2} > (1-s)\left(-\frac{\varepsilon}{2}\right)-s\cdot\frac{\varepsilon}{2} = -\frac{\varepsilon}{2}\,,$$

and

$$\begin{aligned} y^{i_x}-\phi(x_{(i_x)}) &= (1-s)(y_j^{i_x}-\phi(y_{j(i_x)}))+(1-s)(\phi(y_{j(i_x)})-\phi(x_{(i_x)}))-s\cdot\frac{\varepsilon}{2} \\ &\le (1-s)(\phi(y_{j(i_x)})-\phi(x_{(i_x)})) \\ &< \frac{\varepsilon}{2}\,. \end{aligned}$$

Consequently, $|y^{i_x}-\phi(x_{(i_x)})| < \varepsilon/2$, so

$$|y-x|_n^2 = |y_{(i_x)}-x_{(i_x)}|_{n-1}^2+|y^{i_x}-\phi(x_{(i_x)})|^2 < \delta^2+\frac{\varepsilon^2}{4} < \varepsilon^2.$$

In either case, we find $y \in B_\varepsilon^n(x)$. Then, as noted, $y \in U_{x-}$, and the connectedness of U_{x-} follows.

We have now shown that Ω is q-regular at each point of M. To see that $\Omega^{-\prime}$ is also q-regular at each point of M, simply observe that $\Omega^{-\prime}$ is also regularly open (since $\Omega^{-\prime-o} = \Omega^{-\prime\prime o\prime o} = \Omega^{-o\prime o} = \Omega^{\prime o} = \Omega^{-\prime}$) and $\partial(\Omega^{-\prime}) = \partial\Omega$ (since $\partial(\Omega^{-\prime}) = \Omega^{-\prime-}\cap\Omega^{-\prime\prime-} = \Omega^{-\prime\prime o\prime}\cap\Omega^{-} = \Omega^{-o\prime}\cap\Omega^{-} = \Omega^{\prime}\cap\Omega^{-} = \partial\Omega$), so the first part of the proof may be applied with $\Omega^{-\prime}$ in place of Ω to secure the desired conclusion.

(ii) This statement is an immediate consequence of (i), in view of the definition [VI.53.b]. $\square$.

[VI.56] D E F I N I T I O N. Let $\Omega \subset \mathbb{R}^n$ be open. Suppose that

$x \in \partial\Omega$, and there exists an open neighborhood of x in $\partial\Omega$, M_x, which is an $(n-1,n;1)$-manifold. If $N \in N_{M_x}(x)$, then N is an *exterior normal for* $\partial\Omega$ *at* x iff there exists a positive δ such that

$$x+sN \in \Omega \quad \text{for} \quad -\delta < s < 0 \quad \text{and}$$

$$x+sN \in \Omega^{-'} \quad \text{for} \quad 0 < s < \delta. \qquad \blacksquare.$$

Suppose that, in the notation of the preceding definition, M_x and $\tilde{M}_x$ are $\partial\Omega$-open neighborhoods of x which are also $(n-1,n;1)$-manifolds. Directly from the definition [VI.5], it is easy to see that $T_{M_x}(x) = T_{\tilde{M}_x}(x)$, hence that $N_{M_x}(x) = N_{\tilde{M}_x}(x)$. From this, and the fact that $N_{M_x}(x)$ is a one-dimensional subspace of $\mathbb{R}^n$, it follows readily that there can exist at most one exterior *unit* normal for $\partial\Omega$ at x: if ν_1 and $\nu_2 \in N_{M_x}(x)$ are exterior unit normals for $\partial\Omega$ at x, then either $\nu_1 = \nu_2$ or $\nu_1 = -\nu_2$, but the latter implies that $x+s\nu_2 \in \Omega$ for all sufficiently small positive s, which is impossible. Thus $\nu_1 = \nu_2$.

[VI.57] P R O P O S I T I O N. *Let* Ω *be an open subset of* $\mathbb{R}^n$ $(n \geq 2)$. *Let* Γ *be a non-void relatively open subset of* $\partial\Omega$ *such that, for some* $q \in \mathbb{N}$, $\partial\Omega$ *is* q-*regular at each point of* Γ. *Then* Γ *is an* $(n-1,n;q)$-*manifold, and there exists a unique continuous function* $\nu_\Gamma\colon \Gamma \to \mathbb{R}^n$ *such that* $\nu_\Gamma(x)$ *is an exterior unit normal for* $\partial\Omega$ *at* x, *for each* $x \in \Gamma$; ν_Γ *is called the* exterior unit normal field for Γ. *Moreover, if* $q > 1$, *then*

$\nu_\Gamma \in C^{q-1}(\Gamma;\mathbb{R}^n)$ *if* $q \in \mathbb{N}$, *or* $\nu_\Gamma \in C^\infty(\Gamma;\mathbb{R}^n)$ *if* $q = \infty$.

In particular, if Ω *is a* q-*regular domain (with* $\partial\Omega \neq \emptyset$*), then these conclusions hold with* $\Gamma = \partial\Omega$.

Before presenting the proof, let us state that the notations ν_Γ and $\nu_{\partial\Omega}$ used herein shall be standard in the sequel, whenever the requisite hypotheses be fulfilled.

P R O O F. Let $x \in \Gamma$. Since $\partial\Omega$ is q-regular at x, there exist an open neighborhood, U_x, of x in $\mathbb{R}^n$ and a function $\phi_x \in C^q(U_x)$ such that grad $\phi_x(y) \neq 0$ for each $y \in U_x$, $\partial\Omega \cap U_x = \{y \in U_x \mid \phi_x(y) = 0\}$, and $\Omega \cap U_x = \{y \in U_x \mid \phi_x(y) < 0\}$ (so, also, $\Omega^{-'} \cap U_x = \{y \in U_x \mid \phi_x(y) > 0\}$). Γ is open in $\partial\Omega$, so we can find an open neighborhood, $\tilde{U}_x$, of x in $\mathbb{R}^n$ such that $\tilde{U}_x \subset U_x$ and $\partial\Omega \cap \tilde{U}_x \subset \Gamma$. With $\tilde{\Phi}_x := \phi_x | \tilde{U}_x$, it is clear that $\tilde{\Phi}_x \in C^q(\tilde{U}_x)$, grad $\tilde{\Phi}_x(y) \neq 0$ for each $y \in \tilde{U}_x$, and $\Gamma \cap \tilde{U}_x = \{y \in \tilde{U}_x \mid \tilde{\Phi}_x(y) = 0\}$. Thus, Γ is an $(n-1,n;q)$-manifold. Obviously, we also have $\Omega \cap \tilde{U}_x = \{y \in \tilde{U}_x \mid \tilde{\Phi}_x(y) < 0\}$, and $\Omega^{-'} \cap \tilde{U}_x = \{y \in \tilde{U}_x \mid \tilde{\Phi}_x(y) > 0\}$. We now know that x is contained in an $(n-1,n;q)$-manifold which is open in $\partial\Omega$, *viz.*, Γ. According to [VI.7], grad $\tilde{\Phi}_x(x) \in N_\Gamma(x)$; we shall show that grad $\tilde{\Phi}_x(x)$ is, in fact, an exterior normal for $\partial\Omega$ at x. For this, choose $\varepsilon_x' > 0$ such that $B^n_{\varepsilon_x'}(x) \subset \tilde{U}_x$ and set

$$\psi_x(s) := \tilde{\Phi}_x(x+s\cdot \text{grad } \tilde{\Phi}_x(x)) \quad \text{for} \quad |s| < \varepsilon_x := \frac{\varepsilon_x'}{|\text{grad } \tilde{\Phi}_x(x)|_n} .$$

This clearly defines a function $\psi_x \in C^1(-\varepsilon_x,\varepsilon_x)$, for which

$\psi_x(0) = 0$ and

$$\psi_x'(s) = \text{grad}\ \tilde{\Phi}_x(x+s\cdot \text{grad}\ \tilde{\Phi}_x(x))\bullet \text{grad}\ \tilde{\Phi}_x(x), \qquad \text{for} \quad |s| < \varepsilon_x.$$

Then $\psi_x'(0) = |\text{grad}\ \tilde{\Phi}_x(x)|_n^2 > 0$, whence there is a $\delta_x \in (0,\varepsilon_x)$ such that $\psi_x'(s) > 0$ if $|s| < \delta_x$. Now, if $0 < s < \delta_x$, we have, for some $\bar{s} \in (0,s)$,

$$\tilde{\Phi}_x(x+s\cdot \text{grad}\ \tilde{\Phi}_x(x)) = \psi_x(s) = \psi_x(0)+\psi_x'(\bar{s})\cdot s = \psi_x'(\bar{s})\cdot s > 0,$$

showing that $x+s\cdot \text{grad}\ \tilde{\Phi}_x(x) \in \Omega^{-\prime}$. In a similar manner, we can show $\tilde{\Phi}_x(x+s\cdot \text{grad}\ \tilde{\Phi}_x(x)) < 0$, *i.e.*, $x+s\cdot \text{grad}\ \tilde{\Phi}_x(x) \in \Omega$, for $-\delta_x < s < 0$. All requirements of Definition [VI.56] are thus fulfilled by $\text{grad}\ \tilde{\Phi}_x(x)$, and we can assert that it is an exterior normal for $\partial\Omega$ at $x \in \Gamma$.

Now, define $\nu_\Gamma\colon\ \Gamma \to \mathbb{R}^n$ by

$$\nu_\Gamma(x) := \frac{\text{grad}\ \tilde{\Phi}_x(x)}{|\text{grad}\ \tilde{\Phi}_x(x)|_n}, \qquad \text{for each} \quad x \in \Gamma.$$

For each $x \in \Gamma$, $|\nu_\Gamma(x)|_n = 1$ and $\nu_\Gamma(x)$ is an exterior normal for $\partial\Omega$ at x; by the observations following [VI.56], ν_Γ is the unique function with these properties. To see that ν_Γ is continuous, choose $x \in \Gamma$. For each $y \in \Gamma\cap\tilde{U}_x$, we may suppose that $\tilde{U}_y = \tilde{U}_x$ and $\tilde{\Phi}_y = \tilde{\Phi}_x$, whence

$$\nu_\Gamma(y) = \frac{\text{grad}\ \tilde{\Phi}_x(y)}{|\text{grad}\ \tilde{\Phi}_x(y)|_n}, \qquad \text{for each} \quad y \in \Gamma\cap\tilde{U}_x,$$

showing clearly that ν_Γ is continuous on $\Gamma\cap\tilde{U}_x$, thus, in particular, continuous at x.

Next, suppose that $q \in \mathbb{N}$ with $q > 1$. Let (V,h) be a coordinate system in Γ (recall that Γ is an $(n-1,n;q)$-manifold): to show that $\nu_\Gamma \in C^{q-1}(\Gamma;\mathbb{R}^n)$, we must demonstrate that $\nu_\Gamma \circ h^{-1} \in C^{q-1}(h(V);\mathbb{R}^n)$. Select $\hat{x} \in h(V)$, and write $x := h^{-1}(\hat{x})$. Then $V \cap \tilde{U}_x$ is an open neighborhood of x in Γ, and $h(V \cap \tilde{U}_x)$ is an open neighborhood of $\hat{x} = h(x)$ in $\mathbb{R}^{n-1}$, with $h(V \cap \tilde{U}_x) \subset h(V)$. Since

$$\nu_\Gamma(y) = \frac{\text{grad } \tilde{\Phi}_x(y)}{|\text{grad } \tilde{\Phi}_x(y)|_n} \quad \text{for each} \quad y \in V \cap \tilde{U}_x,$$

while $h^{-1}(\hat{y}) \in V \cap \tilde{U}_x$ whenever $\hat{y} \in h(V \cap \tilde{U}_x)$, we have

$$\nu_\Gamma \circ h^{-1}(\hat{y}) = \frac{\text{grad } \tilde{\Phi}_x(h^{-1}(\hat{y}))}{|\text{grad } \tilde{\Phi}_x(h^{-1}(\hat{y}))|_n} \quad \text{for each} \quad \hat{y} \in h(V \cap \tilde{U}_x).$$

Now, $h^{-1} \in C^q(h(V);\mathbb{R}^n)$, and $\text{grad } \tilde{\Phi}_x \in C^{q-1}(\tilde{U}_x;\mathbb{R}^n)$, so the latter equality implies that $(\nu_\Gamma \circ h^{-1})|\ h(V \cap \tilde{U}_x) \in C^{q-1}(h(V \cap \tilde{U}_x);\mathbb{R}^n)$. Thus, $\nu_\Gamma \circ h^{-1}$ is of class C^{q-1} in an open neighborhood of each point in $h(V)$, whence $\nu_\Gamma \circ h^{-1} \in C^{q-1}(h(V);\mathbb{R}^n)$. As noted, this implies that $\nu_\Gamma \in C^{q-1}(\Gamma;\mathbb{R}^n)$. If $q = \infty$, the proof of the inclusion $\nu_\Gamma \in C^\infty(\Gamma;\mathbb{R}^n)$ is almost identical.

The final assertion of the proposition is a simple application of the statements already proven, because of Definition [VI.53.b]. □.

It is worth isolating the following fact, essentially verified during the just-completed proof of [VI.57].

[VI.58] L E M M A. *Let* Ω *be an open subset of* $\mathbb{R}^n$. *Let* $x \in \partial\Omega$, *and suppose that* Ω *is* q-*regular at* x $(q \in \mathbb{N}\cup\{\infty\})$. *If the open neighborhood of* x, U_x, *and the function* $\phi_x \in C^q(U_x)$ *are as in the definition* [VI.53.a], *then* grad $\phi_x(y)$ *is an exterior normal for* $\partial\Omega$ *at each* $y \in \partial\Omega\cap U_x$.

P R O O F. Let U_x and ϕ_x be as in [VI.53.a]. It is first of all clear that the relatively open subset of $\partial\Omega$, $\partial\Omega\cap U_x$, is an (n-1,n;q)-manifold. In the proof of [VI.57], it was shown that grad $\phi_x(x)$ is an exterior normal for $\partial\Omega$ at x. But the same reasoning used there serves to prove also that grad $\phi_x(y)$ is an exterior normal for $\partial\Omega$ at any $y \in \partial\Omega\cap U_x$, for, if we select any such y and take $U_y = U_x$ and $\phi_y = \phi_x$, we obtain a set and a function for y fulfilling the requirements of [VI.53.a]. □.

We shall later find the following technical fact useful.

[VI.59] L E M M A. *Let* Ω *be a non-void proper subset of* $\mathbb{R}^n$ *which is a* q-*regular domain for some* $q \in \mathbb{N}\cup\{\infty\}$, *with* $\partial\Omega$ *compact. Then there exists a positive* δ_Ω *such that, whenever* $x \in \partial\Omega$,

$$x+s\nu_{\partial\Omega}(x) \in \Omega^{-'} \quad \text{if} \quad 0 < s < \delta_\Omega,$$

and

$$x+s\nu_{\partial\Omega}(x) \in \Omega \quad \text{if} \quad -\delta_\Omega < s < 0.$$

P R O O F. Fix $x \in \partial\Omega$. By the q-regularity of Ω at x, we can find an open neighborhood of x in $\mathbb{R}^n$, U_x, and a function

$\Phi_x \in C^q(U_x)$ such that

$$\text{grad } \Phi_x(y) \neq 0 \quad \text{for each} \quad y \in U_x, \tag{1}$$

$$\partial\Omega \cap U_x = \{y \in U_x | \quad \Phi_x(y) = 0\}, \tag{2}$$

and

$$\Omega \cap U_x = \{y \in U_x | \quad \Phi_x(y) < 0\}; \tag{3}$$

then grad $\Phi_x(y)$ is an exterior normal for $\partial\Omega$ at each $y \in \partial\Omega \cap U_x$ ([VI.58]). Choose $\varepsilon_x > 0$ such that $B^n_{2\varepsilon_x}(x) \subset U_x$, and set

$$M_x := \sup \{|\text{grad } \Phi_x(y)|_n| \quad y \in B^n_{\varepsilon_x}(x)^-\}.$$

Now, if $y \in B^n_{\varepsilon_x}(x)^-$ and $|s| < \varepsilon_x/M_x$, we have

$$|y+s\cdot \text{ grad } \Phi_x(y) - x|_n \leq \varepsilon_x + |s| \cdot M_x < 2\varepsilon_x,$$

so

$$y+s\cdot \text{ grad } \Phi_x(y) \in B^n_{2\varepsilon_x}(x) \subset U_x.$$

Thus, we can define ψ_y: $(-\varepsilon_x/M_x, \varepsilon_x/M_x) \to \mathbb{R}$ by

$$\left.\begin{aligned} \psi_y(s) &:= \Phi_x(y+s\cdot \text{ grad } \Phi_x(y)) \quad \text{for} \quad |s| < \varepsilon_x/M_x, \\ &\text{whenever} \quad y \in B^n_{\varepsilon_x}(x)^-; \end{aligned}\right. \tag{4}$$

for each such y, it is clear that $\psi_y \in C^q(-\varepsilon_x/M_x, \varepsilon_x/M_x)$, with

$$\psi'_y(s) = \text{grad } \Phi_x(y+s\cdot \text{ grad } \Phi_x(y)) \bullet \text{ grad } \Phi_x(y), \quad |s| < \varepsilon_x/M_x. \tag{5}$$

From the latter equality,

$$\psi_y'(0) = |\text{grad } \phi_x(y)|_n^2 > 0 \quad \text{for each} \quad y \in B_{\varepsilon_x}^n(x)^-,$$

with which we can assert that there is a positive m_x such that

$$\psi_y'(0) = |\text{grad } \phi_x(y)|_n^2 \geq m_x^2 \quad \text{for each} \quad y \in B_{\varepsilon_x}^n(x)^-. \tag{6}$$

Thus, the function $(y,s) \mapsto \psi_y'(s)$ is, in view of (5), continuous on $B_{\varepsilon_x}^n(x)^- \times [-\varepsilon_x/2M_x, \varepsilon_x/2M_x]$ (whence it is uniformly continuous there), and, by (6), positive on $B_{\varepsilon_x}^n(x)^- \times \{0\}$. These facts imply that there exists a $\delta_x \in (0, \varepsilon_x/2M_x]$ such that

$$\psi_y'(s) > 0 \quad \text{for each} \quad y \in B_{\varepsilon_x}^n(x)^- \quad \text{and for} \quad |s| < \delta_x; \tag{7}$$

indeed, by the uniform continuity, we can select $\delta_x \in (0, \varepsilon_x/2M_x]$ so that

$$|\psi_y'(s) - \psi_y'(0)| < m_x^2/2 \quad \text{whenever} \quad y \in B_{\varepsilon_x}^n(x)^-, \quad |s| < \delta_x,$$

giving, for such y and s,

$$\psi_y'(s) = \psi_y'(0) + \{\psi_y'(s) - \psi_y'(0)\} > m_x^2 - \frac{m_x^2}{2} > 0.$$

Note that, by (2) and (4),

$$\psi_y(0) = 0 \quad \text{for each} \quad y \in \partial\Omega \cap B_{\varepsilon_x}^n(x)^-. \tag{8}$$

Now, choose any $y \in B_{\varepsilon_x}^n(x) \cap \partial\Omega$ and $s \in (0, \delta_x)$: the mean-value theorem shows that there exists an $s_y \in (0,s)$ for which

$$\begin{aligned}\Phi_x(y+s\cdot \operatorname{grad} \Phi_x(y)) &= \psi_y(s)\\ &= \psi_y(0)+\psi_y'(s_y)\cdot s\\ &= \psi_y'(s_y)\cdot s\\ &> 0,\end{aligned}$$

having used (7) and (8); we have already convinced ourselves that $y+s\cdot \operatorname{grad} \Phi_x(y) \in U_x$, while (2) and (3) obviously imply that

$$\Omega^{-'}\cap U_x = \{y \in U_x \mid \Phi_x(y) > 0\}.$$

From these data, we infer that

$$y+s\cdot \operatorname{grad} \Phi_x(y) \in \Omega^{-'} \quad \text{whenever} \quad y \in \partial\Omega\cap B^n_{\varepsilon_x}(x), \quad s \in (0,\delta_x). \tag{9}$$

By reasoning in an analogous manner, one can show that

$$y+s\cdot \operatorname{grad} \Phi_x(y) \in \Omega \quad \text{whenever} \quad y \in \partial\Omega\cap B^n_{\varepsilon_x}(x), \quad s \in (-\delta_x,0). \tag{10}$$

From (9) and (10), since $\nu_{\partial\Omega} = |\operatorname{grad} \Phi_x|_n^{-1}\cdot \operatorname{grad} \Phi_x$ on $\partial\Omega\cap U_x$, by taking note of (6), we have

$$y+s\cdot\nu_{\partial\Omega}(y) \in \Omega^{-'}\ [\Omega] \quad \text{whenever} \quad y \in \partial\Omega\cap B^n_{\varepsilon_x}(x), \quad \text{and} \quad 0 < s < m_x\delta_x \quad [-m_x\delta_x < s < 0]. \tag{11}$$

Now, (11) holds for each $x \in \partial\Omega$. To complete the proof, we use the compactness of $\partial\Omega$ to select $\{x_i\}_{i=1}^{\ell}$ so that $\{B^n_{\varepsilon_{x_i}}(x_i)\}_{i=1}^{\ell}$ affords a cover for $\partial\Omega$, and take

$$\delta_\Omega := \min \{m_{x_i}\delta_{x_i}\}_{i=1}^{\ell};$$

it is easy to show that this δ_Ω possesses the desired property. □.

The properties of regular domains in the class described in the following definitions are particularly nice, as we shall presently discover.

[VI.60] D E F I N I T I O N S. Let $\Omega \subset \mathbb{R}^n$ be open. Ω is a *Lyapunov domain* iff

(i) Ω is a 1-regular domain

and

(ii) the exterior unit normal field for $\partial\Omega$, $\nu_{\partial\Omega}$, is Hölder continuous, *i.e.*, there exist an $a > 0$ and an $\alpha \in (0,1]$ for which

$$|\nu_{\partial\Omega}(x_2)-\nu_{\partial\Omega}(x_1)|_n \leq a|x_2-x_1|_n^\alpha \quad \text{whenever} \quad x_1, x_2 \in \partial\Omega. \tag{1}$$

Let $\Omega \subset \mathbb{R}^n$ be a Lyapunov domain. Any ordered triple (a,α,d), where $a > 0$ and $\alpha \in (0,1]$ are as in (ii), and $d > 0$ with $ad^\alpha < 1/2$, shall be referred to as a *set of Lyapunov constants for* Ω.

Let $\{\Omega_\iota\}_{\iota\in I}$ be a family of Lyapunov domains in $\mathbb{R}^n$. The family is said to be *uniformly Lyapunov* iff there exists an ordered triple (a,α,d) which is a set of Lyapunov constants for Ω_ι, for each $\iota \in I$. ∎.

[A.1.61] R E M A R K S. (a) Let $\Omega \subset \mathbb{R}^n$ be a 1-regular domain such that $\partial\Omega$ is compact. Suppose that, for each $x \in \partial\Omega$, there can be found an open neighborhood, U_x, of x in $\mathbb{R}^n$ and a function $\Phi_x \in C^1(U_x)$ as in Definition [VI.53] which is also such that $(\text{grad } \Phi_x)|\ \partial\Omega\cap U_x$ is Hölder continuous: there exist an $a_x > 0$ and $\alpha_x \in (0,1]$ with

$$|\text{grad } \Phi_x(\tilde{x}_2) - \text{grad } \Phi_x(\tilde{x}_1)|_n \leq a_x|\tilde{x}_2-\tilde{x}_1|_n^{\alpha_x}, \quad \text{for} \quad \tilde{x}_1,\ \tilde{x}_2 \in \partial\Omega\cap U_x.$$

Then Ω is a Lyapunov domain. For, let $\varepsilon_x > 0$, with $B^n_{\varepsilon_x}(x)^- \subset U_x$, and $a'_x > 0$, with $|\text{grad } \Phi_x(y)|_n \geq a'_x$ for each $y \in B^n_{\varepsilon_x}(x)^-$, for each $x \in \partial\Omega$. If $x \in \partial\Omega$, it is then easy to see that

$$|\nu_{\partial\Omega}(\tilde{x}_2)-\nu_{\partial\Omega}(\tilde{x}_1)|_n = \left|\frac{\text{grad } \Phi_x(\tilde{x}_2)}{|\text{grad } \Phi_x(\tilde{x}_2)|_n} - \frac{\text{grad } \Phi_x(\tilde{x}_1)}{|\text{grad } \Phi_x(\tilde{x}_1)|_n}\right|_n$$

$$\leq \frac{2a_x}{a'_x}|\tilde{x}_2-\tilde{x}_1|_n^{\alpha_x}, \quad \text{for} \quad \tilde{x}_1,\ \tilde{x}_2 \in \partial\Omega\cap B^n_{\varepsilon_x}(x).$$

Choosing a finite set $\{x_i\}_{i=1}^m \subset \partial\Omega$ such that $\{\partial\Omega\cap B^n_{\varepsilon_{x_i}/2}(x_i)\}_{i=1}^m$ covers $\partial\Omega$, it follows that

$$|\nu_{\partial\Omega}(y_2)-\nu_{\partial\Omega}(y_1)|_n \leq a|y_2-y_1|_n^{\alpha}, \quad \text{for} \quad y_1,\ y_2 \in \partial\Omega,$$

where $\alpha := \min_{1 \leq i \leq m} \{\alpha_{x_i}\}$, $a := \max \left\{ \max_{1 \leq i \leq m} \left\{\frac{2a_{x_i}}{a'_{x_i}}\right\}, 2 \Big/ \min_{1 \leq i \leq m} \left\{\left(\frac{\varepsilon_{x_i}}{2}\right)^{\alpha}\right\}\right\}$. Thus, $\nu_{\partial\Omega}$ is Hölder continuous, whence Ω is indeed a Lyapunov domain.

(b) Suppose that $\Omega \subset \mathbb{R}^n$ is a q-regular domain, with $q \geq 2$, and let $\partial\Omega$ be compact. Then Ω is a Lyapunov domain. To see this, for each $x \in \partial\Omega$, let $U_x \subset \mathbb{R}^n$ and $\phi_x \in C^q(U_x)$ be as in [VI.53], and choose $\varepsilon_x > 0$ such that $B^n_{\varepsilon_x}(x)^- \subset U_x$; by the mean-value theorem, grad ϕ_x is Lipschitz continuous on $B^n_{\varepsilon_x}(x)$. Following reasoning similar to that employed in (a), we even find that $\nu_{\partial\Omega}$ is Lipschitz continuous on $\partial\Omega$.

(c) In view of [VI.55], we can replace the hypothesis of (a) [(b)] that Ω be 1-regular [q-regular, with $q \geq 2$] with the hypotheses that Ω be regularly open and $\partial\Omega$ be an (n-1,n;1)-manifold [(n-1,n;q)-manifold]. Maintaining the other hypotheses, we can conclude in this case that Ω and $\Omega^{-'}$ are Lyapunov domains.

[VI.62] S T A N D A R D N O T A T I O N S A N D C O N S T R U C T I O N S. It is convenient to introduce here certain notations and simple facts relating to the geometry of the boundary of a Lyapunov domain. Throughout this section, $\Omega \subset \mathbb{R}^n$ is a Lyapunov domain, and $x \in \partial\Omega$. See Figure 1.

(i) We denote by $\pi_x : \mathbb{R}^n \to x+T_{\partial\Omega}(x)$ the orthogonal projection map of $\mathbb{R}^n$ onto the tangent hyperplane to $\partial\Omega$ at x, $x+T_{\partial\Omega}(x)$, so that, for each $\xi \in \mathbb{R}^n$, $\pi_x(\xi)$ is the unique element of $x+T_{\partial\Omega}(x)$ such that

$$|\xi-\pi_x(\xi)|_n = \inf\{|\xi-\tilde{\xi}|_n \mid \tilde{\xi} \in x+T_{\partial\Omega}(x)\}. \tag{1}$$

Letting $P_x : \mathbb{R}^n \to T_{\partial\Omega}(x)$ denote the orthogonal projection onto the $(n-1)$-dimensional tangent (sub)space $T_{\partial\Omega}(x) \subset \mathbb{R}^n$, it is a simple matter to check that

$$\pi_x(\xi) = x+P_x(\xi-x), \qquad \text{for each} \qquad \xi \in \mathbb{R}^n; \tag{2}$$

indeed, (2) is a consequence of the fact that (1) characterizes $\pi_x(\xi)$ in $x+T_{\partial\Omega}(x)$ and the equality $|\xi-\pi_x(\xi)|_n = |\xi-(x+P_x(\xi-x))|_n$ for $\xi \in \mathbb{R}^n$, which follows, in turn, from (1) and the familiar property

$$|(\xi-x)-P_x(\xi-x)|_n = \inf\{|(\xi-x)-\tilde{\xi}|_n \mid \tilde{\xi} \in T_{\partial\Omega}(x)\}.$$

We shall denote by $\{\hat{\varepsilon}_i^x\}_{i=1}^{n-1}$ an orthonormal basis for $T_{\partial\Omega}(x)$; then $\{\hat{\varepsilon}_1^x,\ldots,\hat{\varepsilon}_{n-1}^x,\nu_{\partial\Omega}(x)\}$ constitutes an orthonormal basis for $\mathbb{R}^n$. Let us show that

$$\xi-\pi_x(\xi) = \{(\xi-\pi_x(\xi))\bullet\nu_{\partial\Omega}(x)\}\nu_{\partial\Omega}(x), \qquad \text{for each} \qquad \xi \in \mathbb{R}^n: \tag{3}$$

if $\xi \in \mathbb{R}^n$, then $\xi-\pi_x(\xi) = \{(\xi-\pi_x(\xi))\bullet\nu_{\partial\Omega}(x)\}\nu_{\partial\Omega}(x)$ $+ \sum_{i=1}^{n-1} (\xi-\pi_x(\xi))\cdot\hat{\varepsilon}_i^x\hat{\varepsilon}_i^x$, but $\xi-\pi_x(\xi) = (\xi-x)-P_x(\xi-x)$ is in the orthogonal complement $N_{\partial\Omega}(x)$ of $T_{\partial\Omega}(x)$, so $(\xi-\pi_x(\xi))\bullet\hat{\varepsilon}_i^x = 0$, for each $i \in \{1,\ldots,n-1\}$. Thus, (3) is true.

(ii) If $d > 0$, we define $\Pi_x := \pi_x|(\partial\Omega\cap B_d^n(x))$. Even though Π_x depends upon d, we omit any indication of this dependence in the notation, which should cause no confusion.

(iii) Let $A_x: \mathbb{R}^n \to \mathbb{R}^n$ denote the linear isometry such that

$$\left.\begin{aligned} A_x\hat{\varepsilon}_i^x &= e_i^{(n)}, \qquad i = 1,\ldots,n-1, \\ A_x\nu_{\partial\Omega}(x) &= e_n^{(n)}, \end{aligned}\right\} \tag{4}$$

and then define $\mathcal{H}_x: \mathbb{R}^n \to \mathbb{R}^n$ according to

$$\mathcal{H}_x(\xi) := A_x(\xi-x), \qquad \text{for each} \qquad \xi \in \mathbb{R}^n. \tag{5}$$

Clearly, $\mathcal{H}_x \in C^\infty(\mathbb{R}^n;\mathbb{R}^n)$ is an (affine) isometry, hence a homeomorphism, of $\mathbb{R}^n$ onto itself; we have

$$\mathcal{H}_x^{-1}(\xi) = A_x^{-1}\xi+x, \qquad \text{for each} \qquad \xi \in \mathbb{R}^n. \tag{6}$$

Since A_x preserves inner products,

$$\mathcal{H}_x(\xi_1)\bullet\mathcal{H}_x(\xi_2) = (\xi_1-x)\bullet(\xi_2-x), \qquad \text{for} \qquad \xi_1,\ \xi_2 \in \mathbb{R}^n. \tag{7}$$

Consider the open set $\mathcal{H}_x(\Omega) \subset \mathbb{R}^n$, which is at most a translated and rotated copy of Ω. Indeed, it is quite simple to see that

$$\partial\{\mathcal{H}_x(\Omega)\} = \mathcal{H}_x(\partial\Omega), \tag{8}$$

showing that $0 \in \partial\{\mathcal{K}_x(\Omega)\}$, since $\mathcal{K}_x(x) = 0$, and, for $d > 0$,

$$\mathcal{K}_x(\partial\Omega \cap B_d^n(x)) = \partial\{\mathcal{K}_x(\Omega)\} \cap B_d^n(0), \tag{9}$$

$$\mathcal{K}_x(\Omega \cap B_d^n(x)) = \mathcal{K}_x(\Omega) \cap B_d^n(0). \tag{10}$$

Next, let us choose any $y \in \partial\{\mathcal{K}_x(\Omega)\}$, and let $\tilde{U}_y$ be an open neighborhood in $\mathbb{R}^n$ of $\mathcal{K}_x^{-1}(y) \in \partial\Omega$, $\tilde{\Phi}_y \in C^1(\tilde{U}_y)$ such that grad $\tilde{\Phi}_y(\xi) \neq 0$ for $\xi \in \tilde{U}_y$, $\partial\Omega \cap \tilde{U}_y = \{\xi \in \tilde{U}_y \mid \tilde{\Phi}_y(\xi) = 0\}$, and $\Omega \cap \tilde{U}_y = \{\xi \in \tilde{U}_y \mid \tilde{\Phi}_y(\xi) < 0\}$. Setting $U_y := \mathcal{K}_x(\tilde{U}_y)$, $\Phi_y := \tilde{\Phi}_y \circ (\mathcal{K}_x^{-1} \mid U_y)$, *it is plain that* U_y *is an open neighborhood of* y in $\mathbb{R}^n$, $\Phi_y \in C^1(U_y)$, $\partial\{\mathcal{K}_x(\Omega)\} \cap U_y = \{\xi \in U_y \mid \Phi_y(\xi) = 0\}$, and $\mathcal{K}_x(\Omega) \cap U_y = \{\xi \in U_y \mid \Phi_y(\xi) < 0\}$. Further, for $\xi \in U_y$,

$$\begin{aligned} d\Phi_y(\xi) &= d\tilde{\Phi}_y(\mathcal{K}_x^{-1}(\xi)) \circ D\mathcal{K}_x^{-1}(\xi) \\ &= d\tilde{\Phi}_y(\mathcal{K}_x^{-1}(\xi)) \circ A_x^{-1} \\ &= A_x^{-1^*} d\tilde{\Phi}_y(\mathcal{K}_x^{-1}(\xi)), \end{aligned}$$

where $A_x^{-1^*}$ is the adjoint of A_x^{-1}. Since A_x is an orthogonal transformation, it follows that

$$\text{grad } \Phi_y(\xi) = A_x \text{ grad } \tilde{\Phi}_y(\mathcal{K}_x^{-1}(\xi)), \qquad \xi \in U_y. \tag{11}$$

From (11), we have $|\text{grad } \Phi_y(\xi)|_n = |\text{grad } \tilde{\Phi}_y(\mathcal{K}_x^{-1}(\xi))|_n \neq 0$, for $\xi \in U_y$. Thus, $\mathcal{K}_x(\Omega)$ is 1-regular. Moreover, grad $\Phi_y(\xi)$ is then, for $\xi \in \partial\{\mathcal{K}_x(\Omega)\} \cap U_y$, an outer normal for $\partial\{\mathcal{K}_x(\Omega)\}$ at ξ, while grad $\tilde{\Phi}_y(\mathcal{K}_x^{-1}(\xi))$ is an outer normal for $\partial\Omega$ at $\mathcal{K}_x^{-1}(\xi)$, and (11) shows also that

$$\nu_{\partial\{\mathcal{K}_x(\Omega)\}}(\xi) = A_x \nu_{\partial\Omega}(\mathcal{K}_x^{-1}(\xi)); \tag{12}$$

(12) clearly holds for each $\xi \in \partial\{\mathcal{K}_x(\Omega)\}$. In particular, note that

$$\nu_{\partial\{\mathcal{K}_x(\Omega)\}}(0) = A_x \nu_{\partial\Omega}(x) = e_n^{(n)}. \tag{13}$$

Now, if $a > 0$ and $\alpha \in (0,1]$ are such that (VI.60.1) holds, we find, whenever $\xi_1, \xi_2 \in \partial\{\mathcal{K}_x(\Omega)\}$,

$$\begin{aligned} |\nu_{\partial\{\mathcal{K}_x(\Omega)\}}(\xi_2) - \nu_{\partial\{\mathcal{K}_x(\Omega)\}}(\xi_1)|_n &= |A_x \nu_{\partial\Omega}(\mathcal{K}_x^{-1}(\xi_2)) - A_x \nu_{\partial\Omega}(\mathcal{K}_x^{-1}(\xi_1))|_n \\ &= |\nu_{\partial\Omega}(\mathcal{K}_x^{-1}(\xi_2)) - \nu_{\partial\Omega}(\mathcal{K}_x^{-1}(\xi_1))|_n \\ &\leq a|\mathcal{K}_x^{-1}(\xi_2) - \mathcal{K}_x^{-1}(\xi_1)|_n^{\alpha} \\ &= a|\xi_2 - \xi_1|_n^{\alpha}. \end{aligned}$$

Consequently, we reach the entirely expected conclusion that $\mathcal{K}_x(\Omega)$ is a Lyapunov domain; any $a > 0$ and $\alpha \in (0,1]$ as in (VI.60.1) for $\partial\Omega$ will also do for $\partial\{\mathcal{K}_x(\Omega)\}$. In view of (13), we obviously have

$$T_{\partial\{\mathcal{K}_x(\Omega)\}}(0) = \{\xi \in \mathbb{R}^n | \quad \xi^n = 0\}. \tag{14}$$

Next, we see that

$$(A_x y)^n = 0 \qquad \text{whenever} \qquad y \in T_{\partial\Omega}(x): \tag{15}$$

for, supposing that $y \in T_{\partial\Omega}(x)$, $(A_x y)\bullet e_n^{(n)} = y \bullet A_x^{-1} e_n^{(n)} = y \bullet \nu_{\partial\Omega}(x) = 0$. As an implication of (15), we find

$$\mathcal{K}_x^n(\xi) = \{\xi-\pi_x(\xi)\}\bullet\nu_{\partial\Omega}(x) = (\xi-x)\bullet\nu_{\partial\Omega}(x), \qquad \text{for} \qquad \xi \in \mathbb{R}^n. \tag{16}$$

In fact, let $\xi \in \mathbb{R}^n$. Then $\pi_x(\xi)-x = P_x(\xi-x) \in T_{\partial\Omega}(x)$, so $\{A_x(\pi_x(\xi)-x)\}^n = 0$. Writing $\xi = x+(\xi-\pi_x(\xi))+(\pi_x(\xi)-x)$, we compute, using (3) and (4),

$$\begin{aligned}\mathcal{K}_x(\xi) &= A_x(\xi-x)\\ &= A_x(\xi-\pi_x(\xi))+A_x(\pi_x(\xi)-x)\\ &= \{(\xi-\pi_x(\xi))\bullet\nu_{\partial\Omega}(x)\}e_n^{(n)}+A_x(\pi_x(\xi)-x);\end{aligned}$$

since $\mathcal{K}_x^n(\xi) = \mathcal{K}_x(\xi)\bullet e_n^{(n)}$, the first equality in (16) follows. For the second equality of (16), we can write $\xi-\pi_x(\xi) = (\xi-x)+(x-\pi_x(\xi))$ and simply note that $x-\pi_x(\xi) = -P_x(\xi-x) \in T_{\partial\Omega}(x)$. It is important to point out also that

$$\mathcal{K}_x^i = \mathcal{K}_x^i\circ\pi_x, \qquad \text{for} \qquad i = 1,\ldots,n-1. \tag{17}$$

To see that (17) is true, choose $i \in \{1,\ldots,n-1\}$ and $\xi \in \mathbb{R}^n$. Then

$$\begin{aligned}\mathcal{K}_x^i(\xi)-\mathcal{K}_x^i\circ\pi_x(\xi) &= \{\mathcal{K}_x(\xi)-\mathcal{K}_x\circ\pi_x(\xi)\}\bullet e_i^{(n)}\\ &= \{A_x(\xi-x)-A_x(\pi_x(\xi)-x)\}\bullet e_i^{(n)}\\ &= A_x(\xi-\pi_x(\xi))\bullet e_i^{(n)}\\ &= \{(\xi-\pi_x(\xi))\bullet\nu_{\partial\Omega}(x)\}e_n^{(n)}\bullet e_i^{(n)}\\ &= 0.\end{aligned}$$

Finally, it is easy to show that

$$\mathcal{K}_x(x+T_{\partial\Omega}(x)) = \{\xi \in \mathbb{R}^n \mid \xi^n = 0\}: \tag{18}$$

if $\xi \in T_{\partial\Omega}(x)$. then $\mathcal{K}_x(x+\xi) = A_x\xi$, but $(A_x\xi)^n = 0$, by (15): on the other hand, if $\xi \in \mathbb{R}^n$ and $\xi^n = 0$, then $A_x^{-1}\xi \in T_{\partial\Omega}(x)$ (*since* $A_x^{-1}\xi \bullet \nu_{\partial\Omega}(x) = \xi \bullet A_x \nu_{\partial\Omega}(x) = \xi \bullet e_n^{(n)} = 0$) *and* $\mathcal{K}_x(x+A_x^{-1}\xi) = \xi$.

(iv) We shall define $\hat{\mathcal{K}}_x$: $x+T_{\partial\Omega}(x) \to \mathbb{R}^{n-1}$ by

$$\hat{\mathcal{K}}_x(\xi) := (\mathcal{K}_x^1(\xi),\ldots,\mathcal{K}_x^{n-1}(\xi)), \quad \text{for each} \quad \xi \in x+T_{\partial\Omega}(x). \tag{19}$$

Obviously, $\hat{\mathcal{K}}_x$ is continuous. From (18) and the injectiveness of $\mathcal{K}_x$. it is routine to prove that $\hat{\mathcal{K}}_x$ is a bijection. Since the inverse $\hat{\mathcal{K}}_x^{-1}$: $\mathbb{R}^{n-1} \to x+T_{\partial\Omega}(x)$ is just the map $\hat{\xi} \mapsto \mathcal{K}_x^{-1}(\hat{\xi},0)$ on $\mathbb{R}^{n-1}$, we see that $\hat{\mathcal{K}}_x^{-1} \in C^\infty(\mathbb{R}^{n-1};\mathbb{R}^n)$. Then $\hat{\mathcal{K}}_x$ is a homeomorphism, with

$$\begin{aligned} J\hat{\mathcal{K}}_x^{-1}(\hat{\xi}) &= \Big|\bigwedge_{i=1}^{n-1} D_i\hat{\mathcal{K}}_x^{-1}(\hat{\xi})\Big| \\ &= \Big|\bigwedge_{i=1}^{n-1} D_i\mathcal{K}_x^{-1}(\hat{\xi},0)\Big| \\ &= \Big|\bigwedge_{i=1}^{n-1} D\mathcal{K}_x^{-1}(\xi,0)e_i^{(n)}\Big| \\ &= \Big|\bigwedge_{i=1}^{n-1} A_x^{-1}e_i^{(n)}\Big| \\ &= \Big|\bigwedge_{i=1}^{n-1} \hat{\varepsilon}_i^x\Big| \\ &= 1, \end{aligned} \tag{20}$$

for each $\hat{\xi} \in \mathbb{R}^{n-1}$. It is trivial to see that $x+T_{\partial\Omega}(x)$

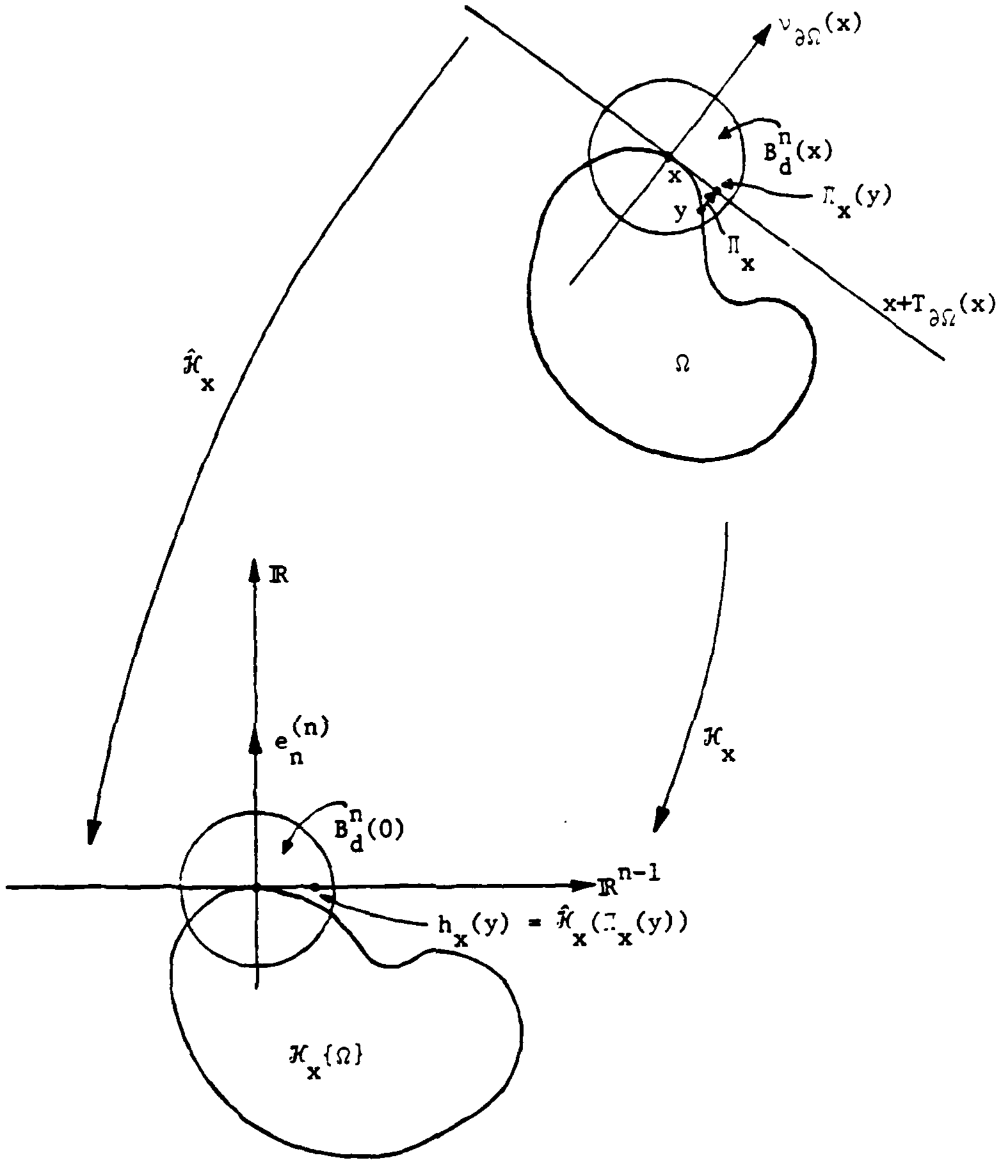

FIGURE 1. Constructions associated with Lyapunov domain Ω

$(= \{\xi \in \mathbb{R}^n | \ \mathcal{H}_x^n(\xi) = 0\})$ is an $(n-1,n;\infty)$-manifold; the preceding observations combine to show that $\hat{\mathcal{H}}_x$ is a coordinate function for $x+T_{\partial\Omega}(x)$.

From (17) and (19), we obtain $\mathcal{H}_x^i = \hat{\mathcal{H}}_x^i \circ \pi_x$, for $i \in \{1,\ldots,n-1\}$, so

$$\mathcal{H}_x = (\mathcal{H}_x^1,\ldots,\mathcal{H}_x^{n-1},\mathcal{H}_x^n) = (\hat{\mathcal{H}}_x \circ \pi_x, \mathcal{H}_x^n). \tag{21}$$

Finally, suppose that $\xi \in x+T_{\partial\Omega}(x)$. Then

$$|\xi - x|_n = |\mathcal{H}_x(\xi) - \mathcal{H}_x(x)|_n = |\mathcal{H}_x(\xi)|_n = |\hat{\mathcal{H}}_x(\xi)|_{n-1}, \tag{22}$$

since $\mathcal{H}_x^n(\xi) = 0$, in this case. Thus,

$$|\hat{\mathcal{H}}_x^{-1}(\hat{\xi}) - x|_n = |\hat{\xi}|_{n-1} \quad \text{for each} \quad \hat{\xi} \in \mathbb{R}^{n-1}. \tag{23}$$

We begin our study of Lyapunov domains by pointing out one of their most fundamental and useful properties.

[VI.63] P R O P O S I T I O N. *Let* $\Omega \subset \mathbb{R}^n$ *be a Lyapunov domain, and select* $a > 0$, $\alpha \in (0,1]$ *such that*

$$|\nu_{\partial\Omega}(x_2) - \nu_{\partial\Omega}(x_1)|_n \le a|x_2 - x_1|_n^\alpha \quad \text{for} \quad x_1,\ x_2 \in \partial\Omega.$$

Let d *be any positive number satisfying* $ad^\alpha \le \sqrt{2}$*: in particular,* (a,α,d) *may be a set of Lyapunov constants for* Ω*. Then, for each* $x \in \partial\Omega$, Π_x $(:= \pi_x|\ \partial\Omega \cap B_d^n(x))$ *is an injection of* $\partial\Omega \cap B_d^n(x)$ *into* $x+T_{\partial\Omega}(x)$.

P R O O F. Choose $x \in \partial\Omega$. We begin by showing that, whenever

$\xi \in \partial\Omega \cap B^n_d(x)$, there exists a $\delta_\xi > 0$ such that $\xi + s\nu_{\partial\Omega}(x) \in \Omega$ for $-\delta_\xi < s < 0$, and $\xi + s\nu_{\partial\Omega}(x) \in \Omega^{-'}$ for $0 < s < \delta_\xi$: since Ω is 1-regular, we can find an open neighborhood U_ξ of ξ in $\mathbb{R}^n$ and a function $\Phi_\xi \in C^1(U_\xi)$ such that $\operatorname{grad} \Phi_\xi(y) \neq 0$ for $y \in U_\xi$, $\partial\Omega \cap U_\xi = \{y \in U_\xi \mid \Phi_\xi(y) = 0\}$, and $\Omega \cap U_\xi = \{y \in U_\xi \mid \Phi_\xi(y) < 0\}$ (so also $\Omega^{-'} \cap U_\xi = \{y \in U_\xi \mid \Phi_\xi(y) > 0\}$). Choose $\varepsilon > 0$ such that $B^n_\varepsilon(\xi) \subset U_\xi$, and define $\psi\colon (-\varepsilon,\varepsilon) \to \mathbb{R}$ by

$$\psi(s) := \Phi_\xi(\xi + s\nu_{\partial\Omega}(x)), \qquad \text{for} \quad |s| < \varepsilon.$$

Clearly, $\psi \in C^1(-\varepsilon,\varepsilon)$, $\psi(0) = 0$, and $\psi'(s) = \operatorname{grad} \Phi_\xi(\xi + s\nu_{\partial\Omega}(x)) \bullet \nu_{\partial\Omega}(x)$ for $|s| < \varepsilon$. Thus,

$$\psi'(0) = \operatorname{grad} \Phi_\xi(\xi) \bullet \nu_{\partial\Omega}(x) = |\operatorname{grad} \Phi_\xi(\xi)|_n \cdot \nu_{\partial\Omega}(\xi) \bullet \nu_{\partial\Omega}(x), \tag{1}$$

the latter equality following from the fact that $\operatorname{grad} \Phi_\xi(\xi)$ is an exterior normal for $\partial\Omega$ at ξ. Now,

$$\begin{aligned} \nu_{\partial\Omega}(\xi) \bullet \nu_{\partial\Omega}(x) &= 1 - \tfrac{1}{2}\, |\nu_{\partial\Omega}(\xi) - \nu_{\partial\Omega}(x)|^2_n \\ &\geq 1 - \tfrac{1}{2}\, a^2 |\xi - x|^{2\alpha}_n \\ &> 1 - \tfrac{1}{2}\, a^2 d^{2\alpha} \\ &\geq 0, \end{aligned}$$

since $|\xi - x|_n < d$ and $ad^\alpha \leq \sqrt{2}$. In view of (1), we conclude that $\psi'(0) > 0$. Since ψ' is continuous, there exists a $\delta_\xi \in (0,\varepsilon)$ such that $\psi'(s) > 0$ if $|s| < \delta_\xi$. The mean-value theorem, coupled with the equality $\psi(0) = 0$, implies that $\Phi_\xi(\xi + s\nu_{\partial\Omega}(x)) = \psi(s) < 0$

if $-\delta_\xi < s < 0$ and $\phi_\xi(\xi+s\nu_{\partial\Omega}(x)) = \psi(s) > 0$ if $0 < s < \delta_\xi$. Therefore, $\xi+s\nu_{\partial\Omega}(x)$ lying in U_ξ whenever $|s| < \varepsilon$, we obtain the inclusions $\xi+s\nu_{\partial\Omega}(x) \in \Omega$ if $-\delta_\xi < s < 0$, $\xi+s\nu_{\partial\Omega}(x) \in \Omega^{-'}$ if $0 < s < \delta_\xi$, as required.

Now, let us suppose, contrary to the conclusion of the proposition, that there exist distinct $\xi_1, \xi_2 \in \partial\Omega\cap B^n_d(x)$ for which $\Pi_x(\xi_1) = \Pi_x(\xi_2)$. Writing $y := \Pi_x(\xi_1) = \Pi_x(\xi_2)$, we have, by (VI.62.3),

$$\xi_i = y+s_i\nu_{\partial\Omega}(x), \qquad i = 1, 2,$$

where $s_i := (\xi_i-y)\bullet\nu_{\partial\Omega}(x)$: since $\xi_1 \neq \xi_2$, it must be that $s_1 \neq s_2$, and we may suppose that $s_1 < s_2$. Set

$$(\xi_1\xi_2) := \{y+s\nu_{\partial\Omega}(x)| \quad s_1 < s < s_2\};$$

note that $(\xi_1\xi_2) \subset B^n_d(x)$. Now, whenever $-\delta_{\xi_2} < s < 0$, then $y+(s_2+s)\nu_{\partial\Omega}(x) = \xi_2+s\nu_{\partial\Omega}(x) \in \Omega$, while $0 < s < \delta_{\xi_1}$ implies that $y+(s_1+s)\nu_{\partial\Omega}(x) = \xi_1+s\nu_{\partial\Omega}(x) \in \Omega^{-'}$, whence it is clear that $(\xi_1\xi_2)$ meets both Ω and $\Omega^{-'}$. Since $(\xi_1\xi_2)$ is obviously connected, and Ω and $\Omega^{-'}$ *are separated*, $(\xi_1\xi_2)$ *must meet* $\partial\Omega$. Thus, there exists $s_3 \in (s_1,s_2)$ such that $\xi_3 := y+s_3\nu_{\partial\Omega}(x) \in \partial\Omega\cap B^n_d(x)$. Repeat the process with, in turn, ξ_1 and ξ_3, and ξ_3 and ξ_2: there exist $s_4 \in (s_1,s_3)$ and $s_5 \in (s_3,s_2)$ for which $\xi_i := y+s_i\nu_{\partial\Omega}(x) \in \partial\Omega\cap B^n_d(x)$ for $i = 4, 5$. Continuing in this manner, we generate a set of distinct points $\{s_i| \quad i \in \mathbb{N}\} \subset [s_1,s_2]$ such that $\xi_i := y+s_i\nu_{\partial\Omega}(x) \in \partial\Omega\cap B_d^n(x)$ for each $i \in \mathbb{N}$. Let $(s_{i_k})_{k=1}^{\infty}$ be a convergent sequence in $\{s_i |\ i \in \mathbb{N}\}$, converging to $s_0 \in [s_1,s_2]$. Clearly, $\lim_{k\to\infty} \xi_{i_k} = \xi_0 := y+s_0\nu_{\partial\Omega}(x)$, from which it is easy to see that $\xi_0 \in \partial\Omega\cap B_d^n(x)$, since $\partial\Omega$ and $\{y+s\nu_{\partial\Omega}(x) |\ s_1 \leq s \leq s_2\}$ are closed, the latter being contained in $B_d^n(x)$. We have $\xi_0+s\nu_{\partial\Omega}(x) \in \Omega$ if $-\delta_{\xi_0} < s < 0$, $\xi_0+s\nu_{\partial\Omega}(x) \in \Omega^{-'}$ if $0 < s < \delta_{\xi_0}$, showing that $\xi_{i_m} = \xi_0+(s_{i_m}-s_0)\nu_{\partial\Omega}(x) \in (\partial\Omega)'$ for some sufficiently large m, since the sequence $(s_{i_k})_{k=1}^{\infty}$ consists of distinct points and converges to s_0. This is impossible, contradicting the inclusion $\{\xi_i |\ i \in \mathbb{N}\} \subset \partial\Omega$. We conclude that Π_x is injective. □.

In Parts I-V we deal with a number of integrals over boundaries of Lyapunov domains. In order to facilitate those computations, we wish to derive here various estimates involving geometry quantities associated with such manifolds, as well as point out the existence of certain distinguished coordinate systems in the boundary of a Lyapunov domain. We begin with the prototype setting.

[VI.64] L E M M A. *Let* $\Omega \subset \mathbb{R}^n$ *be a Lyapunov domain. Suppose that* $0 \in \partial\Omega$, *with* $T_{\partial\Omega}(0) = \{\xi \in \mathbb{R}^n |\ \xi^n = 0\}$ *and* $\nu_{\partial\Omega}(0) = e_n^{(n)}$. *Let* (a,α,d) *be a set of Lyapunov constants for* Ω. *As usual,* Π_0: $\partial\Omega\cap B_d^n(0) \to T_{\partial\Omega}(0)$ *denotes the restriction of the orthogonal projection; in the present case, we have*

$$\Pi_0(\xi) = (\xi^1,\ldots,\xi^{n-1},0) = (\Xi^{(1,\ldots,n-1)}(\xi),0), \quad \text{for} \quad \xi \in \partial\Omega\cap B^n_d(0).$$

Then

(*i*) *the set* $\mathcal{D}_0 := \{\hat{\xi} \in \mathbb{R}^{n-1} \mid (\hat{\xi},0) \in \Pi_0(\partial\Omega\cap B^n_d(0))\} = \Xi^{(1,\ldots,n-1)}(\partial\Omega\cap B^n_d(0))$ *is an open neighborhood of* 0 *in* $\mathbb{R}^{n-1}$, *and there exists a function* $f \in C^1(\mathcal{D}_0)$ *such that*

$$\partial\Omega\cap B^n_d(0) = G(f) := \{(\hat{\xi},f(\hat{\xi})) \mid \hat{\xi} \in \mathcal{D}_0\}. \tag{1}$$

In fact, $f\colon \mathcal{D}_0 \to \mathbb{R}$ *is given explicitly by*

$$f(\hat{\xi}) = (\Pi_0^{-1})^n(\hat{\xi},0), \qquad \text{for each} \quad \hat{\xi} \in \mathcal{D}_0.$$

Thus,

$$\Pi_0 \quad \text{is the map} \quad (\hat{\xi},f(\hat{\xi})) \mapsto (\hat{\xi},0), \quad \text{for} \quad \hat{\xi} \in \mathcal{D}_0, \tag{2}$$

$$\Pi_0^{-1} \quad \text{is the map} \quad (\hat{\xi},0) \mapsto (\hat{\xi},f(\hat{\xi})), \quad \text{for} \quad \hat{\xi} \in \mathcal{D}_0. \tag{3}$$

We also have

$$\partial\Omega\cap B^n_d(0) = \Pi_0^{-1}(\mathcal{D}_0\times\{0\}). \tag{4}$$

(*ii*) $\mathcal{D}_0$ *is starlike with respect to* 0, *and*

$$B^{n-1}_{\frac{7}{9}d}(0) \subset \mathcal{D}_0 \subset B^{n-1}_d(0). \tag{5}$$

(*iii*) *The following estimates hold whenever* $\xi \in \partial\Omega\cap B^n_d(0)$, *where* $\hat{\xi} := \Xi^{(1,\ldots,n-1)}(\xi) = (\xi^1,\ldots,\xi^{n-1})$:

(1) $$\nu^n_{\partial\Omega}(\xi) := \nu_{\partial\Omega}(\xi)\bullet e^{(n)}_n \geq 1-\frac{1}{2}a^2 r_0^{2\alpha}(\xi) > 1-\frac{1}{2}a^2 d^{2\alpha} > \frac{7}{8}; \tag{6}$$

(2) $$|\text{grad } f(\hat{\xi})|_{n-1} < \frac{a r_0^\alpha(\xi)}{1-\frac{1}{2}a^2 r_0^{2\alpha}(\xi)}, \quad \textit{if} \quad \xi \neq 0; \tag{7}$$

$$|\text{grad } f(\hat{\xi})|_{n-1} \leq \frac{8}{7} a r_0^\alpha(\xi); \tag{8}$$

(3) *if* $\hat{\varepsilon} \in \mathbb{R}^n$, $|\hat{\varepsilon}|_n = 1$, *and* $\hat{\varepsilon}\bullet e^{(n)}_n = 0$, *then*

$$|\nu_{\partial\Omega}(\xi)\bullet\hat{\varepsilon}| \leq \frac{8}{7} a r_0^\alpha(\xi); \tag{9}$$

(4) $$|f(\hat{\xi})| = |\xi-\Pi_0(\xi)|_n \leq \tilde{a} r_0^{1+\alpha}(\Pi_0(\xi)) = \tilde{a}|\hat{\xi}|^{1+\alpha}_{n-1} \leq \tilde{a} r_0^{1+\alpha}(\xi), \tag{10}$$

where $\tilde{a} := \frac{8}{7} a\left(\frac{65}{49}\right)^{\alpha/2} \Big/ (1+\alpha)$;

(5) $$|\nu_{\partial\Omega}(\xi)\bullet \text{grad } r_0(\xi)| < \hat{a} r_0^\alpha(\xi), \quad \textit{if} \quad \xi \neq 0, \tag{11}$$

where $\hat{a} := \frac{8}{7}(n-1)a+\tilde{a}$;

(6) $$|\hat{\xi}|_{n-1} = r_0(\Pi_0(\xi)) \geq \frac{7}{9} r_0(\xi). \tag{12}$$

P R O O F. (i) According to [VI.63], $\Pi_0\colon\ \partial\Omega\cap B^n_d(0) \to T_{\partial\Omega}(0)$ is injective, so $\Pi_0^{-1}\colon\ \Pi_0(\partial\Omega\cap B^n_d(0)) \to \partial\Omega\cap B^n_d(0)$ is defined: specifically, if $y \in \Pi_0(\partial\Omega\cap B^n_d(0))$, then there is a unique $\xi_y \in \partial\Omega\cap B^n_d(0)$ for which $y = \Pi_0(\xi_y) = (\xi_y^1,\dots,\xi_y^{n-1},0)$, and we have $\Pi_0^{-1}(y) = \xi_y = (y^1,\dots,y^{n-1},\xi_y^n)$. Set

$$\mathcal{D}_0 := \{\hat{\xi} \in \mathbb{R}^{n-1} \mid \ (\hat{\xi},0) \in \Pi_0(\partial\Omega\cap B^n_d(0))\};$$

the equality $\mathcal{D}_0 = \Xi^{(1,\dots,n-1)}(\partial\Omega\cap B^n_d(0))$ clearly holds. Define $f\colon\ \mathcal{D}_0 \to \mathbb{R}$ according to

$$f(\hat{\xi}) := (\Pi_0^{-1})^n(\hat{\xi},0), \qquad \text{for each} \quad \hat{\xi} \in \mathcal{D}_0.$$

Now, if $\xi \in \partial\Omega\cap B^n_d(0)$, and we set $\hat{\xi} := (\xi^1,\dots,\xi^{n-1})$, then $\Pi_0(\xi) = (\hat{\xi},0)$, so $\hat{\xi} \in \mathcal{D}_0$, and $\xi = \Pi_0^{-1}(\hat{\xi},0) = (\hat{\xi},(\Pi_0^{-1})^n(\hat{\xi},0)) = (\hat{\xi},f(\hat{\xi}))$. On the other hand, if $\xi = (\hat{\xi},f(\hat{\xi}))$ for some $\hat{\xi} \in \mathcal{D}_0$, then $(\hat{\xi},0) \in \Pi_0(\partial\Omega\cap B^n_d(0))$ and $\Pi_0^{-1}(\hat{\xi},0) = (\hat{\xi},(\Pi_0^{-1})^n(\hat{\xi},0)) = (\hat{\xi},f(\hat{\xi})) = \xi$, showing that $\xi \in \partial\Omega\cap B^n_d(0)$. Thus, (1) is correct. Statements (2), (3), and (4) are sufficiently clear.

To complete the proof of (i), we must show that $\mathcal{D}_0$ is open in $\mathbb{R}^{n-1}$, and $f \in C^1(\mathcal{D}_0)$ (obviously, $\mathcal{D}_0$ contains $0 \in \mathbb{R}^{n-1}$, since $\partial\Omega\cap B^n_d(0)$ contains $0 \in \mathbb{R}^n$). Then, select $\hat{\xi} \in \mathcal{D}_0$, and set $\xi := \Pi_0^{-1}(\hat{\xi},0) = (\hat{\xi},f(\hat{\xi}))$. Let U_ξ be an open neighborhood of ξ in $\mathbb{R}^n$, and $\Phi_\xi \in C^1(U_\xi)$, with $\operatorname{grad}\Phi_\xi(y) \neq 0$ for $y \in U_\xi$, $\partial\Omega\cap U_\xi = \{y \in U_\xi \mid \ \Phi_\xi(y) = 0\}$, and $\Omega\cap U_\xi = \{y \in U_\xi \mid \ \Phi_\xi(y) < 0\}$. Set $\tilde{U}_\xi := U_\xi\cap B^n_d(0)$ and $\tilde{\Phi}_\xi := \Phi_\xi|\tilde{U}_\xi$: $\tilde{U}_\xi$ is an open neighborhood of ξ in $\mathbb{R}^n$, $\tilde{\Phi}_\xi \in C^1(\tilde{U}_\xi)$, $\partial\Omega\cap\tilde{U}_\xi = \{y \in \tilde{U}_\xi \mid \ \tilde{\Phi}_\xi(y) = 0\}$, and

$\Omega\cap\tilde{U}_\xi = \{y \in \tilde{U}_\xi \mid \tilde{\Phi}_\xi(y) < 0\}$. Observe also that $\nu_{\partial\Omega}(\xi) = \operatorname{grad} \tilde{\Phi}_\xi(\xi)/|\operatorname{grad} \tilde{\Phi}_\xi(\xi)|_n$. Let us show that $\tilde{\Phi}_{\xi,n}(\xi) \neq 0$: we have

$$\tilde{\Phi}_{\xi,n}(\xi) = |\operatorname{grad} \tilde{\Phi}_\xi(\xi)|_n \nu^n_{\partial\Omega}(\xi) = |\operatorname{grad} \tilde{\Phi}_\xi(\xi)|_n \nu_{\partial\Omega}(\xi)\bullet\nu_{\partial\Omega}(0),$$

and

$$\nu_{\partial\Omega}(\xi)\bullet\nu_{\partial\Omega}(0) = 1-\frac{1}{2}|\nu_{\partial\Omega}(\xi)-\nu_{\partial\Omega}(0)|^2_n \geq 1-\frac{1}{2}a^2r_0^{2\alpha}(\xi) > 1-\frac{1}{2}a^2d^{2\alpha} > 0,$$

since $ad^\alpha < 1/2$. Thus, $\tilde{\Phi}_{\xi,n}(\xi) > 0$. Consequently, by the Implicit Function Theorem [VI.2] (and its proof), there exist an open neighborhood $\tilde{U}_{\xi 0} \subset \tilde{U}_\xi$ of ξ in $\mathbb{R}^n$, an open neighborhood $V_{\hat{\xi}}$ of $\hat{\xi}$ in $\mathbb{R}^{n-1}$, and a unique function $\phi_\xi \in C^1(V_{\hat{\xi}})$ such that

$$\{y \in \tilde{U}_{\xi 0} \mid \tilde{\Phi}_\xi(y) = 0\} = \{y \in \mathbb{R}^n \mid \hat{y} := (y^1,\ldots,y^{n-1}) \in V_{\hat{\xi}},$$
$$y^n = \phi_\xi(\hat{y})\}.$$

Clearly, since $\tilde{U}_{\xi 0} \subset \tilde{U}_\xi$, $\partial\Omega\cap\tilde{U}_{\xi 0} = \{y \in \tilde{U}_{\xi 0} \mid \tilde{\Phi}_\xi(y) = 0\}$. We claim now that $V_{\hat{\xi}} \subset \mathcal{D}_0$ and $\phi_{\hat{\xi}} = f| V_{\hat{\xi}}$: first, if $\hat{z} \in V_{\hat{\xi}}$, then $(\hat{z},\phi_\xi(\hat{z})) \in \{y \in \tilde{U}_{\xi 0} \mid \tilde{\Phi}_\xi(y) = 0\} = \partial\Omega\cap\tilde{U}_{\xi 0} \subset \partial\Omega\cap B^n_d(0)$, whence $(\hat{z}.0) = \Pi_0(\hat{z},\phi_\xi(\hat{z})) \in \Pi_0(\partial\Omega\cap B^n_d(0))$; this says that $\hat{z} \in \mathcal{D}_0$. Moreover, since $(\hat{z},\phi_\xi(\hat{z})) = \Pi_0^{-1}(\hat{z},0)$, we also find that $\phi_\xi(\hat{z}) = (\Pi_0^{-1})^n(\hat{z},0) = f(\hat{z})$. The claim is now verified, and we can assert that each $\hat{\xi} \in \mathcal{D}_0$ possesses an open neighborhood $V_{\hat{\xi}} \subset \mathcal{D}_0$ such that $f| V_{\hat{\xi}} \in C^1(V_{\hat{\xi}})$. This shows at once that $\mathcal{D}_0$ is open in $\mathbb{R}^{n-1}$ and $f \in C^1(\mathcal{D}_0)$, completing the proof of (i).

We shall deviate here from the order in which the conclusions of the lemma are stated, proving now (iii.1-3):

(iii.1) (This has essentially already been shown, in the proof of (i).) Let $\xi \in \partial\Omega\cap B^n_d(0)$. Recalling that $\nu_{\partial\Omega}(0) = e^{(n)}_n$, and $|\nu_{\partial\Omega}(\xi)-\nu_{\partial\Omega}(0)|_n \le a|\xi-0|^\alpha_n = ar^\alpha_0(\xi)$, and noting that $|\nu_{\partial\Omega}(\xi)-\nu_{\partial\Omega}(0)|^2_n = 2-2\nu_{\partial\Omega}(\xi)\bullet\nu_{\partial\Omega}(0)$, we find

$$\begin{aligned}\nu^n_{\partial\Omega}(\xi) &= \nu_{\partial\Omega}(\xi)\bullet\nu_{\partial\Omega}(0)\\ &= 1-\frac{1}{2}\,|\nu_{\partial\Omega}(\xi)-\nu_{\partial\Omega}(0)|^2_n\\ &\ge 1-\frac{1}{2}\,a^2r^{2\alpha}_0(\xi)\\ &> 1-\frac{1}{2}\,a^2d^{2\alpha}\\ &> \frac{7}{8}\,,\end{aligned}$$

since $ad^\alpha < 1/2$.

(iii.2) Choose $\xi \in \partial\Omega\cap B^n_d(0)$. By (i), $\xi \in G(f)$, so $\xi = (\hat{\xi},f(\hat{\xi}))$, where $\hat{\xi} = (\xi^1,\ldots,\xi^{n-1})$, and it is clear that

$$\nu_{\partial\Omega}(\xi) = \frac{1}{\sqrt{\{1+|\text{grad } f(\hat{\xi})|^2_{n-1}\}}}\,(-D_1f(\hat{\xi}),\ldots,-D_{n-1}f(\hat{\xi}),1), \tag{13}$$

since $\partial\Omega\cap B^n_d(0) = \{y \in \mathcal{D}_0\times\mathbb{R}|\ \ y^n-f(y^1,\ldots,y^{n-1}) = 0\}$, while $\nu_{\partial\Omega}(0) = e^{(n)}_n$. In particular, $1+|\text{grad } f(\hat{\xi})|^2_{n-1} = \{\nu^n_{\partial\Omega}(\xi)\}^{-2}$. Use of (iii.1) yields

$$\begin{aligned}|\text{grad } f(\hat{\xi})|^2_{n-1} &= \frac{1-\{\nu^n_{\partial\Omega}(\xi)\}^2}{\{\nu^n_{\partial\Omega}(\xi)\}^2}\\ &\le \frac{a^2r^{2\alpha}_0(\xi)-\frac{1}{4}\,a^4r^{4\alpha}_0(\xi)}{\{1-\frac{1}{2}\,a^2r^{2\alpha}_0(\xi)\}^2}\end{aligned}$$

$$= \{1-\frac{1}{4}a^2 r_0^{2\alpha}(\xi)\}\cdot\left\{\frac{a r_0^{\alpha}(\xi)}{1-\frac{1}{2}a^2 r_0^{2\alpha}(\xi)}\right\}^2$$

$$\leq \left\{\frac{a r_0^{\alpha}(\xi)}{1-\frac{1}{2}a^2 r_0^{2\alpha}(\xi)}\right\}^2 .$$

Now, if $\xi \neq 0$, *i.e.*, if $r_0(\xi) > 0$, then the latter inequality is strict, so (7) follows. Likewise, (8) follows from the inequality immediately above, upon noting that $1-\frac{1}{2}a^2 r_0^{2\alpha}(\xi) > 7/8$.

(iii.3) Let $\hat{\varepsilon} \in \mathbb{R}^n$, with $|\hat{\varepsilon}|_n = 1$ and $\hat{\varepsilon}\bullet e_n^{(n)} = 0$, *i.e.*, $\hat{\varepsilon}^n = 0$. For $\xi \in \partial\Omega\cap B_d^n(0)$, we have, from (13), with $\hat{\xi} = (\xi^1,\ldots,\xi^{n-1})$,

$$\nu_{\partial\Omega}(\xi)\bullet\hat{\varepsilon} = -\frac{1}{\sqrt{\{1+|\text{grad } f(\hat{\xi})|_{n-1}^2\}}}\cdot\sum_{j=1}^{n-1} D_j f(\hat{\xi})\cdot\hat{\varepsilon}^j ,$$

from which, with the Cauchy-Schwarz inequality and (8), there follows

$$|\nu_{\partial\Omega}(\xi)\bullet\hat{\varepsilon}| \leq \left|\sum_{j=1}^{n-1} D_j f(\hat{\xi})\cdot\hat{\varepsilon}^j\right| \leq |\text{grad } f(\hat{\xi})|_{n-1} \leq \frac{8}{7}\, a r_0^{\alpha}(\xi).$$

(ii) Since $\mathcal{D}_0 = \Xi^{(1,\ldots,n-1)}(\partial\Omega\cap B_d^n(0))$, the inclusion $\mathcal{D}_0 \subset B_d^{n-1}(0)$ is plain. Note that, if $\hat{\xi} \in \mathcal{D}_0$ and $\xi := (\hat{\xi}, f(\hat{\xi}))$, then $\xi \in \partial\Omega\cap B_d^n(0)$, and so (8) gives

$$|\text{grad } f(\hat{\xi})|_{n-1} \leq \frac{8}{7}\, a r_0^{\alpha}(\xi) < \frac{8}{7}\, a d^{\alpha} < \frac{4}{7} . \tag{14}$$

Select any $\hat{\hat{\varepsilon}} \in \mathbb{R}^{n-1}$ with $|\hat{\hat{\varepsilon}}|_{n-1} = 1$, [illegible] define $(\hat{\hat{\varepsilon}}) \subset \mathbb{R}$ by

$$(\hat{\varepsilon}) := \{\zeta > 0 | \ \zeta\hat{\varepsilon} \in \mathcal{D}_0\}.$$

Since $\mathcal{D}_0$ is an open neighborhood of 0 in $\mathbb{R}^{n-1}$, and $\mathcal{D}_0 \subset B_d^{n-1}(0)$, $(\hat{\varepsilon})$ is non-void, open, contained in $(0,d)$, and contains $(0,\zeta')$ for some $\zeta' > 0$. We shall prove that, in fact, $(\hat{\varepsilon}) = (0,\zeta_{\hat{\varepsilon}})$ for some $\zeta_{\hat{\varepsilon}} > \frac{7}{9}d$; this will clearly imply that $\mathcal{D}_0$ is starlike with respect to 0, and also that $B_{\frac{7}{9}d}^{n-1}(0) \subset \mathcal{D}_0$, whence the proof of (ii) shall have been completed. To see, then, that our claim is correct, we begin by appealing to the basic structure theorem for open subsets of $\mathbb{R}$ (cf., *e.g.*, Hewitt and Stromberg [20], Theorem (6.59)), according to which there exists a countable set I, which we can take to be $\mathbb{N}$ or $\{1,\ldots,N\}$ for some $N \in \mathbb{N}$, and two sets $\{\zeta_i^1 | \ i \in I\}$ and $\{\zeta_i^2 | \ i \in I\}$ in $\mathbb{R}$ such that $\zeta_1^1 = 0$, $\zeta_i^1 < \zeta_i^2 \leq d$ for each $i \in I$, $\zeta_i^2 \leq \zeta_{i+1}^1$ for each $i \in I$ such that $i+1 \in I$, and $(\hat{\varepsilon}) = \cup_{i\in I} \ (\zeta_i^1,\zeta_i^2)$; note that the ζ_i^1 and ζ_i^2, $i \in I$, are not in $(\hat{\varepsilon})$. We wish to show that $I = \{1\}$ and $\zeta_1^2 > \frac{7}{9}d$. Define $F_{\hat{\varepsilon}}: (\hat{\varepsilon}) \to \mathbb{R}$ by

$$F_{\hat{\varepsilon}}(\zeta) := f(\zeta\hat{\varepsilon}) \quad \text{for each} \quad \zeta \in (\hat{\varepsilon}).$$

Since $(\zeta\hat{\varepsilon},f(\zeta\hat{\varepsilon})) \in B_d^n(0)$, we have $\zeta^2+\{F_{\hat{\varepsilon}}(\zeta)\}^2 = |\zeta\hat{\varepsilon}|_{n-1}^2+|f(\zeta\hat{\varepsilon})|^2 < d^2$, so

$$|F_{\hat{\varepsilon}}(\zeta)| < \sqrt{\{d^2-\zeta^2\}}, \quad \text{whenever} \quad \zeta \in (\hat{\varepsilon}). \tag{15}$$

The inclusion $f \in C^1(\mathcal{D}_0)$ implies that $F_{\hat{\varepsilon}} \in C^1((\hat{\varepsilon}))$; clearly,

$$F'_{\hat{\varepsilon}}(\zeta) = \hat{\varepsilon} \bullet \operatorname{grad} f(\zeta\hat{\varepsilon}), \quad \text{for each} \quad \zeta \in (\hat{\varepsilon}). \tag{16}$$

Using (14), we conclude that $|F'_{\hat{\varepsilon}}| < 4/7$ on $(\hat{\varepsilon})$. For each $i \in I$, the mean-value theorem shows that $F_{\hat{\varepsilon}}|\ (\zeta_i^1,\zeta_1^2)$ is then uniformly continuous, and so $F_{\hat{\varepsilon}}(\zeta_i^{1+}) := \lim_{\zeta \to \zeta_i^{1+}} F_{\hat{\varepsilon}}(\zeta)$ and $F_{\hat{\varepsilon}}(\zeta_i^{2-}) := \lim_{\zeta \to \zeta_i^{2-}} F_{\hat{\varepsilon}}(\zeta)$ exist. Letting $\zeta \to \zeta_1^{2-}$ in the inequality $\zeta^2+\{F_{\hat{\varepsilon}}(\zeta)\}^2 < d^2$, there results

$$(\zeta_1^2)^2+\{F_{\hat{\varepsilon}}(\zeta_1^{2-})\}^2 \leq d^2. \tag{17}$$

Suppose that strict inequality holds in (17): then $(\zeta_1^2\hat{\varepsilon},F_{\hat{\varepsilon}}(\zeta_1^{2-})) \in B_d^n(0)$. Let $(\zeta_i)_{i=1}^\infty$ be a sequence in $(0,\zeta_1^2)$ converging to ζ_1^2. Then the sequence $((\zeta_i\hat{\varepsilon},F_{\hat{\varepsilon}}(\zeta_i)))_{i=1}^\infty \subset G(f) = \partial\Omega\cap B_d^n(0)$ converges to $(\zeta_1^2\hat{\varepsilon},F_{\hat{\varepsilon}}(\zeta_1^{2-}))$, so $(\zeta_1^2\hat{\varepsilon},F_{\hat{\varepsilon}}(\zeta_1^{2-})) \in \partial\Omega$, since $\partial\Omega$ is closed. Thus, $(\zeta_1^2\hat{\varepsilon},F_{\hat{\varepsilon}}(\zeta_1^{2-})) \in \partial\Omega\cap B_d^n(0)$, giving $\Pi_0(\zeta_1^2\hat{\varepsilon},F_{\hat{\varepsilon}}(\zeta_1^{2-})) = (\zeta_1^2\hat{\varepsilon},0)$ and so implying that $\zeta_1^2\hat{\varepsilon} \in \mathcal{D}_0$. But this says that $\zeta_1^2 \in (\hat{\varepsilon})$, which is false. Consequently, equality must hold in (17):

$$(\zeta_1^2)^2+\{F_{\hat{\varepsilon}}(\zeta_1^{2-})\}^2 = d^2. \tag{18}$$

Note that, since $F_{\hat{\varepsilon}}(\zeta_1^{1+}) = F_{\hat{\varepsilon}}(0^+) = f(0) = 0$, we have, applying the mean-value theorem to the continuous extension of $F_{\hat{\varepsilon}}|\ (0,\zeta_1^2)$ to $[0,\zeta_1^2]$, $F_{\hat{\varepsilon}}(\zeta_1^{2-}) = F_{\hat{\varepsilon}}(\zeta_1^{2-})-F_{\hat{\varepsilon}}(0^+) = F'_{\hat{\varepsilon}}(\tilde{\zeta})\cdot\zeta_1^2$, for some $\tilde{\zeta} \in (0,\zeta_1^2)$. Since $|F'_{\hat{\varepsilon}}| < 4/7$, we obtain $|F_{\hat{\varepsilon}}(\zeta_1^{2-})| < (4/7)\cdot\zeta_1^2$, so, from (18),

$$d = \sqrt{\{(\zeta_1^2)^2+\{F_{\hat{\varepsilon}}(\zeta_1^{2-})\}^2\}} < \zeta_1^2\sqrt{\{1+(16/49)\}} = \zeta_1^2\cdot(\sqrt{65}/7). \tag{19}$$

Now, suppose that $I \supset \{1,2\}$, so there exist ζ_2^1, ζ_2^2 with

$\zeta_1^2 \le \zeta_2^1 < \zeta_2^2 \le d$ and $(\zeta_2^1, \zeta_2^2) \subset (\hat{\hat{\varepsilon}})$. By retracing the reasoning used above, *mutatis mutandis*, we conclude that

$$(\zeta_2^1)^2 + \{F_{\hat{\hat{\varepsilon}}}(\zeta_2^{1+})\}^2 = d^2. \tag{20}$$

From (20) and (15), for $\zeta > \zeta_2^1$ and $\zeta \in (\hat{\hat{\varepsilon}})$,

$$|F_{\hat{\hat{\varepsilon}}}(\zeta_2^{1+})| = \surd\{d^2 - (\zeta_2^1)^2\} > \surd\{d^2 - \zeta^2\} > |F_{\hat{\hat{\varepsilon}}}(\zeta)|. \tag{21}$$

Now, supposing that $\zeta_2^1 < \zeta < \zeta_2^2$, we may again apply the mean-value theorem to assert that there exists some $\zeta_0 \in (\zeta_2^1, \zeta)$ such that

$$\begin{aligned} |F'_{\hat{\hat{\varepsilon}}}(\zeta_0)| \cdot (\zeta - \zeta_2^1) &= |F_{\hat{\hat{\varepsilon}}}(\zeta_2^{1+}) - F_{\hat{\hat{\varepsilon}}}(\zeta)| \\ &\ge \big| |F_{\hat{\hat{\varepsilon}}}(\zeta_2^{1+})| - |F_{\hat{\hat{\varepsilon}}}(\zeta)| \big| \\ &= |F_{\hat{\hat{\varepsilon}}}(\zeta_2^{1+})| - |F_{\hat{\hat{\varepsilon}}}(\zeta)| \\ &> \surd\{d^2 - (\zeta_2^1)^2\} - \surd\{d^2 - \zeta^2\}; \end{aligned}$$

the second equality follows from (21), the second inequality from (20) and (15). Applying the mean-value theorem to the function $s \mapsto \surd\{d^2 - s^2\}$ on the interval $[\zeta_2^1, \zeta]$ (recalling that $\zeta < d$), we can write, for some $\zeta^o \in (\zeta_2^1, \zeta)$,

$$\begin{aligned} |F'_{\hat{\hat{\varepsilon}}}(\zeta_0)| \cdot (\zeta - \zeta_2^1) &> \surd\{d^2 - (\zeta_2^1)^2\} - \surd\{d^2 - \zeta^2\} \\ &= \frac{\zeta^o}{\surd\{d^2 - (\zeta^o)^2\}} (\zeta - \zeta_2^1) \\ &> \frac{\zeta^o}{d} (\zeta - \zeta_2^1) \end{aligned}$$

$$> \frac{\zeta_2^1}{d}\,(\zeta-\zeta_2^1).$$

Consequently,

$$\zeta_2^1 < |F'_{\hat{\varepsilon}}(\zeta_0)|\cdot d < \frac{4}{7}\,d,$$

so

$$\zeta_1^2 \leq \zeta_2^1 < \frac{4}{7}\,d. \tag{22}$$

Combining (19) and (22), we arrive at the impossibility

$$d < \frac{\sqrt{65}}{7}\,\zeta_1^2 < \frac{\sqrt{65}}{7}\cdot\frac{4}{7}\,d < \frac{36}{49}\,d < d.$$

Thus, we must have $I = \{1\}$, so $(\hat{\varepsilon}) = (0,\zeta_1^2)$. Further, (19) gives

$$\zeta_1^2 > \frac{7}{\sqrt{65}}\,d > \frac{7}{9}\,d.$$

As noted, statement (ii) follows.

(iii.4) Let $\xi \in \partial\Omega\cap B_d^n(0)$. Then $\hat{\xi} := (\xi^1,\ldots,\xi^{n-1}) \in \mathcal{D}_0$, and $\xi = (\hat{\xi},f(\hat{\xi})) = \Pi_0^{-1}(\hat{\xi},0)$, so

$$|\xi-\Pi_0(\xi)|_n = |\xi^n| = |(\Pi_0^{-1})^n(\hat{\xi},0)| = |f(\hat{\xi})|,$$

giving the first equality of (10). Note also that

$$r_0^2(\xi) = |\xi|_n^2 = |\hat{\xi}|_{n-1}^2+|f(\hat{\xi})|^2 \geq |\hat{\xi}|_{n-1}^2. \tag{23}$$

Now, since $f(0) = 0$, (10) clearly holds if $\xi = 0$, so we may suppose that $\xi \neq 0$ (so $\hat{\xi} \neq 0$, as well).

Choose any non-zero $\hat{z} \in \mathcal{D}_0$: $\mathcal{D}_0$ is starlike with respect to

0, so $s\hat{z} \in \mathcal{D}_0$ whenever $0 \le s \le 1$, showing that we can define $G_{\hat{z}}\colon [0,1] \to \mathbb{R}$ by

$$G_{\hat{z}}(s) := f(s\hat{z}), \quad \text{for each} \quad s \in [0,1].$$

$f \in C^1(\mathcal{D}_0)$ and $\mathcal{D}_0$ is open, so $G_{\hat{z}} \in C^1([0,1])$, with

$$G'_{\hat{z}}(s) = \text{grad } f(s\hat{z})\bullet\hat{z}, \quad \text{for} \quad 0 \le s \le 1.$$

$G_{\hat{z}}(0) = f(0) = 0$, and we can write

$$f(\hat{z}) = G_{\hat{z}}(1)-G_{\hat{z}}(0) = \int_0^1 G'_{\hat{z}}\, d\lambda_1 = \int_0^1 \text{grad } f(s\hat{z})\bullet\hat{z}\, d\lambda_1(s). \tag{24}$$

The estimate $|\text{grad } f|_{n-1} < 4/7$, following from (8), produces, with (24),

$$|f(\hat{z})| \le \int_0^1 |\text{grad } f(s\hat{z})|_{n-1}\cdot|\hat{z}|_{n-1}\, d\lambda_1(s) < \frac{4}{7}\, |\hat{z}|_{n-1}. \tag{25}$$

Re-applying (24) (with $\hat{z} = \hat{\xi}$) and (8), and using (25),

$$\begin{aligned}
|f(\hat{\xi})| &\le \int_0^1 |\text{grad } f(s\hat{\xi})|_{n-1}\cdot|\hat{\xi}|_{n-1}\, d\lambda_1(s) \\
&\le \frac{8}{7}\, a|\hat{\xi}|_{n-1} \int_0^1 |(s\hat{\xi},f(s\hat{\xi}))|_n^{\alpha}\, d\lambda_1(s) \\
&= \frac{8}{7}\, a|\hat{\xi}|_{n-1} \int_0^1 \{|s\hat{\xi}|_{n-1}^2+|f(s\hat{\xi})|^2\}^{\alpha/2}\, d\lambda_1(s) \\
&< \frac{8}{7}\, a|\hat{\xi}|_{n-1} \int_0^1 \{|s\hat{\xi}|_{n-1}^2+\frac{16}{49}\,|s\hat{\xi}|_{n-1}^2\}^{\alpha/2}\, d\lambda_1(s) \\
&= \frac{8}{7}\left(\frac{65}{49}\right)^{\alpha/2} a\cdot|\hat{\xi}|_{n-1}^{1+\alpha} \int_0^1 s^{\alpha}\, d\lambda_1(s)
\end{aligned}$$

$$= \frac{8a}{7(1+\alpha)} \left(\frac{65}{49}\right)^{\alpha/2} \cdot |\hat{\xi}|_{n-1}^{1+\alpha} .$$

Since $|\hat{\xi}|_{n-1} = |\Pi_0(\xi)|_n = r_0(\Pi_0(\xi))$, and $|\hat{\xi}|_{n-1} \le r_0(\xi)$, the proof of (10) is complete.

(iii.5) If $\xi \in \partial\Omega\cap B_d^n(0)$, $\xi \neq 0$, and $\hat{\xi} := (\xi^1,\ldots,\xi^{n-1})$, then

$$\begin{aligned} |\nu_{\partial\Omega}(\xi) \bullet \operatorname{grad} r_0(\xi)| &= \left|\sum_{k=1}^{n-1} \nu_{\partial\Omega}^k(\xi) \cdot \frac{\xi^k}{|\xi|_n} + \nu_{\partial\Omega}^n(\xi) \cdot \frac{\xi^n}{|\xi|_n}\right| \\ &\le \sum_{k=1}^{n-1} |\nu_{\partial\Omega}^k(\xi)| + \frac{|f(\hat{\xi})|}{|\xi|_n} \\ &< (n-1) \cdot \frac{8}{7} a \cdot r_0^\alpha(\xi) + \tilde{a} r_0^\alpha(\xi) \\ &= \hat{a} r_0^\alpha(\xi), \end{aligned}$$

by (9) and (10), having noted that $\xi^n = f(\hat{\xi})$ and $|\xi|_n = r_0(\xi)$. Thus, inequality (11) is correct.

(iii.6) Again with $\xi \in \partial\Omega\cap B_d^n(0)$ and $\hat{\xi} := (\xi^1,\ldots,\xi^{n-1})$, we have $|\hat{\xi}|_{n-1}^2 = |\xi|_n^2 - |f(\hat{\xi})|^2$, while (25) gives $|f(\hat{\xi})| \le \frac{4}{7} |\hat{\xi}|_{n-1}$. Thus, $|\hat{\xi}|_{n-1} \ge |\xi|_n^2 - \frac{16}{49} |\hat{\xi}|_{n-1}^2$, which leads to $|\hat{\xi}|_{n-1} \ge \frac{7}{9} |\xi|_n$, which is just (12). $\square$.

[VI.65] R E M A R K. Let us bring out several other facts concerning the setting of the preceding lemma; retain the notation introduced there. First, choose $\hat{\xi} = (\xi^1,\ldots,\xi^{n-1}) \in \mathcal{D}_0$ and consider the line segment $\{(\hat{\xi},s) \mid s > f(\hat{\xi})\}\cap B_d^n(0)$: this segment is connected and does not meet $\partial\Omega$, by (VI.64.1), so it either lies in Ω or is contained in $\Omega^{-'}$. We shall show that, in fact, there are points of the line segment in $\Omega^{-'}$, whence it shall follow that

$$\{(\hat{\xi},s)\mid \ s > f(\hat{\xi})\}\cap B_d^n(0) \subset \Omega^{-'}, \tag{1}$$

by the preceding observation. Then, setting $\xi := (\hat{\xi},f(\hat{\xi})) \in \partial\Omega$, let $U_\xi \subset \mathbb{R}^n$ be an open neighborhood of ξ, and $\Phi_\xi \in C^1(U_\xi)$ such that $\operatorname{grad} \Phi_\xi(x) \neq 0$ for each $x \in U_\xi$, $\partial\Omega\cap U_\xi = \{x \in U_\xi \mid \Phi_\xi(x) = 0\}$, and $\Omega\cap U_\xi = \{x \in U_\xi \mid \ \Phi_\xi(x) < 0\}$, so $\Omega^{-'}\cap U_\xi = \{x \in U_\xi \mid \Phi_\xi(x) > 0\}$. Then $\nu_{\partial\Omega}(\xi) = \operatorname{grad} \Phi_\xi(\xi)/|\operatorname{grad} \Phi_\xi(\xi)|_n$, so, using [VI.64.iii.1],

$$\begin{aligned} D_n\Phi_\xi(\xi) &= \operatorname{grad} \Phi_\xi(\xi)\bullet e_n^{(n)} \\ &= |\operatorname{grad} \Phi_\xi(\xi)|_n \nu_{\partial\Omega}(\xi)\bullet e_n^{(n)} \\ &> \frac{7}{8}\,|\operatorname{grad} \Phi_\xi(\xi)|_n \\ &> 0; \end{aligned}$$

since $\Phi_\xi(\xi) = 0$ and $D_n\Phi_\xi$ is continuous on U_ξ, we may assert that $\Phi_\xi(\xi+se_n^{(n)}) > 0$ for all sufficiently small positive s, so also $(\hat{\xi},f(\hat{\xi})+s) = \xi+se_n^{(n)} \in \Omega^{-'}$ for all sufficiently small positive s. As noted, this implies that (1) is true. Reasoning in a similar manner, we can also deduce that

$$\{(\hat{\xi},s)\mid \ s < f(\hat{\xi})\}\cap B_d^n(0) \subset \Omega, \tag{2}$$

whenever $\hat{\xi} \in \mathcal{D}_0$. In turn, from (1) and (2), it is easy to see that if $\xi \in B^n_d(0)$ with $(\xi^1,\ldots,\xi^{n-1}) \in \mathcal{D}_0$, then

$$\left.\begin{array}{lll} \xi \in \Omega^{-'} & \text{iff} & \xi^n > f(\xi^1,\ldots,\xi^{n-1}), \\ \xi \in \Omega & \text{iff} & \xi^n < f(\xi^1,\ldots,\xi^{n-1}) \end{array}\right\} \qquad (3)$$

(and, of course, $\xi \in \partial\Omega$ iff $\xi^n = f(\xi^1,\ldots,\xi^{n-1})$).

The more general statement in which we are interested can now be proven.

[VI.66] P R O P O S I T I O N. *Let* $\Omega \subset \mathbb{R}^n$ *be a Lyapunov domain, and* (a,α,d) *a set of Lyapunov constants for* Ω. *Let* $x \in \partial\Omega$. *Recall the notations established in* [VI.62].

(i) *Define* $h_x\colon\ \partial\Omega \cap B^n_d(x) \to \mathbb{R}^{n-1}$ *by*

$$h_x := \hat{\mathcal{K}}_x \circ \Pi_x.$$

Then $h_x(\partial\Omega \cap B^n_d(x))$ *is an open neighborhood of* 0 *in* $\mathbb{R}^{n-1}$, *which is starlike with respect to* 0 *and such that*

$$B^{n-1}_{\frac{7}{9}d}(0) \subset h_x(\partial\Omega \cap B^n_d(x)) \subset B^{n-1}_d(0).$$

(ii) $(\partial\Omega \cap B^n_d(x), h_x)$ *is a coordinate system in* $\partial\Omega$. *We have*

$$\left.\begin{aligned} h_x^{-1} &= \Pi_x^{-1}\circ(\hat{\mathcal{H}}_x^{-1}|\ \ h_x(\partial\Omega\cap B_d^n(x))), \\ Jh_x^{-1} &= \{\nu_{\partial\Omega}\circ h_x^{-1}\bullet\nu_{\partial\Omega}(x)\}^{-1}, \end{aligned}\right\} \text{ in } \ h_x(\partial\Omega\cap B_d^n(x)). \quad \begin{aligned}(1)\\(2)\end{aligned}$$

and

$$Jh_x^{-1} < \sqrt{2} \qquad (3)$$

(iii) The following estimates hold for each $\xi \in \partial\Omega\cap B_d^n(x)$:

(1) $\nu_{\partial\Omega}(\xi)\bullet\nu_{\partial\Omega}(x) \geq 1-\frac{1}{2}a^2r_x^{2\alpha}(\xi) > 1-\frac{1}{2}a^2d^{2\alpha} > \frac{7}{8}$; (4)

(2) if $\hat{\varepsilon} \in T_{\partial\Omega}(x)$ *with* $|\hat{\varepsilon}|_n = 1$, *then*

$$|\nu_{\partial\Omega}(\xi)\bullet\hat{\varepsilon}| \leq \frac{8}{7}\, ar_x^{\alpha}(\xi); \qquad (5)$$

(3) $|\xi-\Pi_x(\xi)|_n < \tilde{a}r_x^{1+\alpha}(\Pi_x(\xi)) \leq \tilde{a}r_x^{1+\alpha}(\xi)$, (6)

where $\tilde{a} := \frac{8}{7}\, a\left(\frac{65}{49}\right)^{\alpha/2}\Big/(1+\alpha)$:

(4) $|\nu_{\partial\Omega}(\xi)\bullet \text{ grad } r_x(\xi)| < \hat{a}r_x^{\alpha}(\xi)$, *if* $\xi \neq x$, (7)

where $\hat{a} := \frac{8}{7}(n-1)a+\tilde{a}$;

(5) $\frac{7}{9}\, r_x(\xi) < r_x(\Pi_x(\xi)) \leq r_x(\xi)$. (8)

P R O O F. We shall use here the notations and results of [VI.62]. We showed that $\mathcal{H}_x(\Omega)$ is a Lyapunov domain, $0 = \mathcal{H}_x(x) \in \partial\{\mathcal{H}_x(\Omega)\}$, $T_{\partial\{\mathcal{H}_x(\Omega)\}}(0) = \{\xi \in \mathbb{R}^n|\ \ \xi^n = 0\}$, and $\nu_{\partial\{\mathcal{H}_x(\Omega)\}}(0) = e_n^{(n)}$. Consequently, Lemma [VI.64] can be applied to $\mathcal{H}_x(\Omega)$. Note that (a,α,d) is a set of Lyapunov constants for $\mathcal{H}_x(\Omega)$. We denote by π_0 the orthogonal projection map $\xi \mapsto (\xi^1,\dots,\xi^{n-1},0)$ on $\mathbb{R}^n$ onto

$T_{\partial\{\mathcal{K}_x(\Omega)\}}(0)$, and $\Pi_0 := \pi_0 | \partial\{\mathcal{K}_x(\Omega)\}\cap B^n_d(0)$; π_x and Π_x have their meanings as established in [VI.62].

(i) Let us observe first that

$$\mathcal{K}_x \circ \pi_x = \pi_0 \circ \mathcal{K}_x \quad \text{on} \quad \mathbb{R}^n. \tag{9}$$

For, suppose that $\xi \in \mathbb{R}^n$: then, on the one hand, $\pi_0 \circ \mathcal{K}_x(\xi) = (\mathcal{K}^1_x(\xi), \ldots, \mathcal{K}^{n-1}_x(\xi), 0)$. Since $\pi_x(\xi) \in x + T_{\partial\Omega}(x)$, (VI.62.18) gives $\mathcal{K}^n_x(\pi_x(\xi)) = 0$, while (VI.62.17) says that $\mathcal{K}^i_x \circ \pi_x = \mathcal{K}^i_x$ on $\mathbb{R}^n$, for each $i \in \{1, \ldots, n-1\}$. Thus, on the other hand, $\mathcal{K}_x \circ \pi_x(\xi) = (\mathcal{K}^1_x \circ \pi_x(\xi), \ldots, \mathcal{K}^{n-1}_x \circ \pi_x(\xi), 0) = (\mathcal{K}^1_x(\xi), \ldots, \mathcal{K}^{n-1}_x(\xi), 0)$. This proves (9). Next, because $\mathcal{K}_x(\partial\Omega \cap B^n_d(x)) = \partial\{\mathcal{K}_x(\Omega)\}\cap B^n_d(0)$, and Π_0 is defined on the latter set, it is easy to see that (9) implies

$$\mathcal{K}_x \circ \Pi_x = \Pi_0 \circ (\mathcal{K}_x | \partial\Omega \cap B^n_d(x)), \tag{10}$$

which gives directly

$$\hat{\mathcal{K}}_x \circ \Pi_x = \Xi^{(1,\ldots,n-1)} \circ (\mathcal{K}_x | \partial\Omega \cap B^n_d(x)). \tag{11}$$

If we define, as in the statement of the proposition, $h_x := \hat{\mathcal{K}}_x \circ \Pi_x$, then (11) shows that

$$\begin{aligned} h_x(\partial\Omega \cap B^n_d(x)) &= \Xi^{(1,\ldots,n-1)}(\mathcal{K}_x(\partial\Omega \cap B^n_d(x))) \\ &= \Xi^{(1,\ldots,n-1)}(\partial\{\mathcal{K}_x(\Omega)\}\cap B^n_d(0)); \end{aligned} \tag{12}$$

according to Lemma [VI.64], the latter set is an open neighborhood of 0 in $\mathbb{R}^{n-1}$, which is starlike with respect to 0, contains

$B^{n-1}_{\frac{7}{9}d}(0)$, and is contained in $B^n_d(0)$. Statement (i) is proven.

(ii) Obviously, $\partial\Omega\cap B^n_d(x)$ is relatively open in $\partial\Omega$; we must show that h_x is a coordinate function for $\partial\Omega\cap B^n_d(x)$. Now, Π_x is continuous, since π_x is continuous, and Π_x is injective by [VI.63]; $\hat{\mathcal{K}}_x$ is also continuous and injective (cf., [VI.62.iv]), whence it follows that h_x possesses these properties, as well. We have shown in (i) that $h_x(\partial\Omega\cap B^n_d(x))$ is open in $\mathbb{R}^{n-1}$. The equality $h_x^{-1} = \Pi_x^{-1}\circ(\hat{\mathcal{K}}_x^{-1}|\ h_x(\partial\Omega\cap B^n_d(x)))$ is plain enough, from the definition $h_x := \hat{\mathcal{K}}_x\circ\Pi_x$. We claim that we also have

$$h_x^{-1}(\hat{\xi}) = \mathcal{K}_x^{-1}\circ\Pi_0^{-1}(\hat{\xi},0), \qquad \text{for each} \qquad \hat{\xi}\in h_x(\partial\Omega\cap B^n_d(x)). \tag{13}$$

To prove (13), let $\hat{\xi}\in h_x(\partial\Omega\cap B^n_d(x)) = \hat{\mathcal{K}}_x\circ\Pi_x(\partial\Omega\cap B^n_d(x))$, so $(\hat{\xi},0)\in \mathcal{K}_x\circ\Pi_x(\partial\Omega\cap B^n_d(x)) = \Pi_0\circ\mathcal{K}_x(\partial\Omega\cap B^n_d(x))$, and $\mathcal{K}_x^{-1}\circ\Pi_0^{-1}(\hat{\xi},0)$ is defined. Moreover, from (11),

$$\begin{aligned} h_x(\mathcal{K}_x^{-1}\circ\Pi_0^{-1}(\hat{\xi},0)) &= \Xi^{(1,\dots,n-1)}\circ\mathcal{K}_x(\mathcal{K}_x^{-1}\circ\Pi_0^{-1}(\hat{\xi},0)) \\ &= \Xi^{(1,\dots,n-1)}(\Pi_0^{-1}(\hat{\xi},0)) = \hat{\xi}. \end{aligned}$$

Also, if $\xi\in\partial\Omega\cap B^n_d(x)$, then $(h_x(\xi),0)\in\mathcal{K}_x\circ\Pi_x(\partial\Omega\cap B^n_d(x)) = \Pi_0\circ\mathcal{K}_x(\partial\Omega\cap B^n_d(x))$, so we can compute, using (VI.62.18) and (10),

$$\begin{aligned} \mathcal{K}_x^{-1}\circ\Pi_0^{-1}(h_x(\xi),0) &= \mathcal{K}_x^{-1}\circ\Pi_0^{-1}(\hat{\mathcal{K}}_x\circ\Pi_x(\xi),0) \\ &= \mathcal{K}_x^{-1}\circ\Pi_0^{-1}(\mathcal{K}_x\circ\Pi_x(\xi)) \\ &= \mathcal{K}_x^{-1}\circ\Pi_0^{-1}(\Pi_0\circ\mathcal{K}_x(\xi)) = \xi. \end{aligned}$$

We conclude that (13) does indeed hold. By Lemma [VI.64], we can write $\Pi_0^{-1}(\hat{\xi},0) = (\hat{\xi},(\Pi_0^{-1})^n(\hat{\xi},0))$ for $\hat{\xi} \in h_x(\partial\Omega\cap B_d^n(x))$, and we know that the function $\hat{\xi} \mapsto (\Pi_0^{-1})^n(\hat{\xi},0)$ is in $C^1(h_x(\partial\Omega\cap B_d^n(x)))$; as in [VI.64], we call this function f. Thus, with (13),

$$h_x^{-1}(\hat{\xi}) = \mathcal{H}_x^{-1}(\hat{\xi},(\Pi_0^{-1})^n(\hat{\xi},0)), \quad \text{for each} \quad \hat{\xi} \in h_x(\partial\Omega\cap B_d^n(x)). \tag{14}$$

Clearly, (14) implies that $h_x^{-1} \in C^1(h_x(\partial\Omega\cap B_d^n(x));\mathbb{R}^n)$; in particular, h_x^{-1} is continuous, so $h_x\colon \partial\Omega\cap B_d^n(x) \to h_x(\partial\Omega\cap B_d^n(x))$ is a homeomorphism. To prove that h_x is a coordinate function for $\partial\Omega\cap B_d^n(x)$, we now need demonstrate only that $Jh_x^{-1} > 0$ on $h_x(\partial\Omega\cap B_d^n(x))$: letting $\hat{\xi} \in h_x(\partial\Omega\cap B_d^n(x))$, we find, from (14),

$$Dh_x^{-1}(\hat{\xi}) = \{D\mathcal{H}_x^{-1}(F_0(\hat{\xi}))\}\circ DF_0(\hat{\xi}) = A_x^{-1}\circ DF_0(\hat{\xi}),$$

where $A_x\colon \mathbb{R}^n \to \mathbb{R}^n$ is the linear isometry introduced in [VI.62.iii], and $F_0 \in C^1(h_x(\partial\Omega\cap B_d^n(x));\mathbb{R}^n)$ is just the map $\hat{z} \mapsto \Pi_0^{-1}(\hat{z},0) = (\hat{z},(\Pi_0^{-1})^n(\hat{z},0)) = (\hat{z},f(\hat{z}))$, the latter in the notation of [VI.62]. The $n\times(n-1)$ matrix of $DF_0(\hat{\xi})$ with respect to the standard basis vectors of $\mathbb{R}^n$ and $\mathbb{R}^{n-1}$ is just

$$\begin{pmatrix} 1 & 0 & \cdots & 0 \\ 0 & 1 & \cdots & 0 \\ \vdots & \vdots & & \vdots \\ 0 & 0 & \cdots & 1 \\ f_{,1}(\hat{\xi}) & f_{,2}(\hat{\xi}) & \cdots & f_{,n-1}(\hat{\xi}) \end{pmatrix},$$

so

$$(h_x^{-1}),_i(\hat{\xi}) = Dh_x^{-1}(\hat{\xi})e_i^{(n-1)}$$
$$= A_x^{-1}(DF_0(\hat{\xi})e_i^{(n-1)})$$
$$= A_x^{-1}(e_i^{(n)}+f,_i(\hat{\xi})e_n^{(n)}), \quad \text{for each} \quad i \in \{1,\dots,n-1\}.$$

Now, because A_x^{-1} is a linear isometry, it follows from an exercise appearing in Fleming [15], p. 309, that

$$\left|\bigwedge_{i=1}^{n-1} A_x^{-1}y_i\right| = \left|\bigwedge_{i=1}^{n-1} y_i\right|$$

whenever $\{y_i\}_{i=1}^{n-1} \subset \mathbb{R}^n$. Consequently,

$$Jh_x^{-1}(\hat{\xi}) = \left|\bigwedge_{i=1}^{n-1} (h_x^{-1}),_i(\hat{\xi})\right|$$
$$= \left|\bigwedge_{i=1}^{n-1} A_x^{-1}(e_i^{(n)}+f,_i(\hat{\xi})e_n^{(n)})\right|$$
$$= \left|\bigwedge_{i=1}^{n-1} (e_i^{(n)}+f,_i(\hat{\xi})e_n^{(n)})\right| \tag{15}$$
$$= \sqrt{\{1+\sum_{i=1}^{n-1} (f,_i(\hat{\xi}))^2\}}$$
$$= \sqrt{\{1+|\text{grad } f(\hat{\xi})|_{n-1}^2\}},$$

the penultimate equality resulting from a simple computation. Clearly, (15) shows that $Jh_x^{-1}(\hat{\xi}) > 0$, which completes the proof of the assertion that $(\partial\Omega\cap B_d^n(x),h_x)$ is a coordinate system in $\partial\Omega$.

To verify the representation (2), again choose a point $\hat{\xi} \in h_x(\partial\Omega\cap B_d^n(x))$: because $f \in C^1(h_x(\partial\Omega\cap B_d^n(x)))$ and

$$\partial\{\mathcal{K}_x(\Omega)\}\cap B_d^n(0) = \{(\hat{z},f(\hat{z}))| \quad \hat{z} \in h_x(\partial\Omega\cap B_d^n(x))\},$$

with the equality $\nu_{\partial\{\mathcal{K}_x(\Omega)\}}(0) = e_n^{(n)}$, it is easy to see that

$$\nu_{\partial\{\mathcal{K}_x(\Omega)\}}(\Pi_0^{-1}(\hat{\xi},0)) = \frac{(f,_1(\hat{\xi}),\ldots,f,_{n-1}(\hat{\xi}),1)}{\sqrt{\{1+|\text{grad } f(\hat{\xi})|_{n-1}^2\}}} .$$

Thus, using the inner-product-preserving property of A_x, (13), (VI.62.12), and the fact that $A_x\nu_{\partial\Omega}(x) = e_n^{(n)}$,

$$\begin{aligned}\nu_{\partial\Omega}(h_x^{-1}(\hat{\xi}))\bullet\nu_{\partial\Omega}(x) &= A_x\nu_{\partial\Omega}(h_x^{-1}(\hat{\xi}))\bullet A_x\nu_{\partial\Omega}(x)\\ &= \nu_{\partial\{\mathcal{K}_x(\Omega)\}}(\mathcal{K}_x(h_x^{-1}(\hat{\xi})))\bullet e_n^{(n)}\\ &= \nu^n_{\partial\{\mathcal{K}_x(\Omega)\}}(\Pi_0^{-1}(\hat{\xi},0))\\ &= \frac{1}{\sqrt{\{1+|\text{grad } f(\hat{\xi})|_{n-1}^2\}}} .\end{aligned}$$

Upon comparing this result with (15), it becomes evident that (2) must hold.

To secure the estimate (3), simply use (15) in conjunction with (VI.64.8), setting $\xi := (\hat{\xi},f(\hat{\xi}))$, and noting that $ad^\alpha <$ 1/2:

$$Jh_x^{-1}(\hat{\xi}) \leq \sqrt{\left\{1+\frac{64}{49}\,a^2r_0^{2\alpha}(\xi)\right\}} < \sqrt{\left\{1+\frac{64}{49}\,a^2d^{2\alpha}\right\}} < \sqrt{\left\{1+\frac{16}{49}\right\}} < \sqrt{2} .$$

With this, (ii) is proven.

Throughout the following proofs of (iii.1-5), ξ is a point of $\partial\Omega\cap B_d^n(x)$, so $\mathcal{K}_x(\xi) \in \partial\{\mathcal{K}_x(\Omega)\}\cap B_d^n(0)$. Each statement follows from a corresponding estimate derived in Lemma [VI.64] (applied to

$\mathcal{H}_x(\Omega)$, recalling that (a,α,d) is a set of Lyapunov constants for $\mathcal{H}_x(\Omega)$, as well as for Ω).

(iii.1) From [VI.64.iii.1],

$$\nu_{\partial\{\mathcal{H}_x(\Omega)\}}(\mathcal{H}_x(\xi))\bullet e_n^{(n)} \geq 1-\frac{1}{2}a^2r_0^{2\alpha}(\mathcal{H}_x(\xi)) > 1-\frac{1}{2}a^2d^{2\alpha} > \frac{7}{8}, \tag{16}$$

but $\nu_{\partial\{\mathcal{H}_x(\Omega)\}}(\mathcal{H}_x(\xi))\bullet e_n^{(n)} = \nu_{\partial\{\mathcal{H}_x(\Omega)\}}(\mathcal{H}_x(\xi))\bullet\nu_{\partial\{\mathcal{H}_x(\Omega)\}}(\mathcal{H}_x(x)) =$ $A_x\nu_{\partial\Omega}(\xi)\bullet A_x\nu_{\partial\Omega}(x) = \nu_{\partial\Omega}(\xi)\bullet\nu_{\partial\Omega}(x)$ (by (VI.62.12) and the fact that A_x is a linear isometry), while $r_0(\mathcal{H}_x(\xi)) = |\mathcal{H}_x(\xi)-\mathcal{H}_x(x)|_n =$ $|\xi-x|_n$ (since $\mathcal{H}_x$ is an isometry). Thus, (4) results from (16).

(iii.2) Let $\hat{\varepsilon}\in T_{\partial\Omega}(x)$, with $|\hat{\varepsilon}|_n = 1$. Then $|A_x\hat{\varepsilon}|_n = 1$, and $A_x\hat{\varepsilon}\bullet e_n^{(n)} = A_x\hat{\varepsilon}\bullet A_x\nu_{\partial\Omega}(x) = \hat{\varepsilon}\bullet\nu_{\partial\Omega}(x) = 0$, showing that we may apply [VI.64.iii.3]:

$$|\nu_{\partial\{\mathcal{H}_x(\Omega)\}}(\mathcal{H}_x(\xi))\bullet A_x\hat{\varepsilon}| \leq \frac{8}{7}\,ar_0^{\alpha}(\mathcal{H}_x(\xi)). \tag{17}$$

Since $\nu_{\partial\{\mathcal{H}_x(\Omega)\}}(\mathcal{H}_x(\xi))\bullet A_x\hat{\varepsilon} = A_x\nu_{\partial\Omega}(\xi)\bullet A_x\hat{\varepsilon} = \nu_{\partial\Omega}(\xi)\bullet\hat{\varepsilon}$, and $r_0(\mathcal{H}_x(\xi)) = r_x(\xi)$, (5) follows from (17).

(iii.3) According to [VI.64.iii.4], we have the estimate

$$|\mathcal{H}_x(\xi)-\Pi_0(\mathcal{H}_x(\xi))|_n \leq \tilde{a}r_0^{1+\alpha}(\Pi_0(\mathcal{H}_x(\xi))) \leq \tilde{a}r_0^{1+\alpha}(\mathcal{H}_x(\xi)),$$

which can be written

$$|\mathcal{H}_x(\xi)-\mathcal{H}_x(\Pi_x(\xi))|_n \leq \tilde{a}r_0^{1+\alpha}(\mathcal{H}_x(\Pi_x(\xi))) \leq \tilde{a}r_0^{1+\alpha}(\mathcal{H}_x(\xi)), \tag{18}$$

upon recalling (10). (6) follows from (18), by the isometric property of $\mathcal{K}_x$, and the equality $\mathcal{K}_x(x) = 0$.

(iii.4) Suppose that $\xi \neq x$. Then $\mathcal{K}_x(\xi) \neq 0$, so [VI.64.iii.5] implies that

$$|\nu_{\partial\{\mathcal{K}_x(\Omega)\}}(\mathcal{K}_x(\xi)) \bullet \text{grad } r_0(\mathcal{K}_x(\xi))| < \hat{a}r_0^\alpha(\mathcal{K}_x(\xi)). \tag{19}$$

Now,

$$\text{grad } r_0(\mathcal{K}_x(\xi)) = \frac{\mathcal{K}_x(\xi)}{r_0(\mathcal{K}_x(\xi))} = \frac{A_x(\xi-x)}{r_x(\xi)} = A_x \frac{\xi-x}{r_x(\xi)} = A_x \text{ grad } r_x(\xi),$$

so the left-hand side of (19) is just $|A_x\nu_{\partial\Omega}(\xi)\bullet A_x \text{ grad } r_x(\xi)| = |\nu_{\partial\Omega}(\xi)\bullet \text{ grad } r_x(\xi)|$. Clearly, (11) then follows from (19).

(iii.5) From [VI.64.iii.6], $r_0(\Pi_0(\mathcal{K}_x(\xi))) \geq \frac{7}{9} r_0(\mathcal{K}_x(\xi))$, but $r_0(\Pi_0(\mathcal{K}_x(\xi))) = r_0(\mathcal{K}_x(\Pi_x(\xi))) = r_x(\Pi_x(\xi))$, and $r_0(\mathcal{K}_x(\xi)) = r_x(\xi)$, whence the first inequality in (8) follows. The second inequality is obvious. □.

Pogorzelski [42] cites a fact which is quite convenient to have available when estimating various integrals over the boundary of a Lyapunov domain. We shall formulate and prove the pertinent statement here.

[VI.67] L E M M A. *Let* Ω *be a Lyapunov domain in* $\mathbb{R}^n$. *Let* $a > 0$ *and* $\alpha \in (0,1]$ *be, respectively, a Hölder coefficient and Hölder exponent for* $\nu_{\partial\Omega}$, *i.e., such that*

$$|\nu_{\partial\Omega}(y_2)-\nu_{\partial\Omega}(y_1)|_n \leq a|y_2-y_1|_n^\alpha \quad \textit{whenever} \quad y_1,\ y_2 \in \partial\Omega.$$

Then there exists a positive number d_0, *depending only on* a *and* α, *such that for each* $d \in (0,d_0)$ *there exists a* $\gamma_d \in (0,1)$, *depending only on* a, α, *and* d, *possessing the following property:*

> *whenever* $x \in \partial\Omega$, *and then* $z \in \{x+s\nu_{\partial\Omega}(x) \mid s \in \mathbb{R}\}$ *and* $\xi \in \partial\Omega\cap B_d^n(x)\cap\{z\}'$ *are chosen, the inequalities*
>
> $$\gamma_d < \frac{r_z(\xi)}{r_z(\Pi_x(\xi))} < \frac{1}{\gamma_d} \tag{1}$$
>
> *obtain.*

P R O O F. Let $d > 0$ with $ad^\alpha < 1/2$ (so (a,α,d) *is a set of* Lyapunov constants for Ω). Select $x \in \partial\Omega$, then $z \in \{x+s\nu_{\partial\Omega}(x) \mid s \in \mathbb{R}\}$, then $\xi \in \partial\Omega\cap B_d^n(x)\cap\{z\}'$. Observe that $r_z(\Pi_x(\xi)) > 0$, for, if $z = \Pi_x(\xi)$, then $z \in x+T_{\partial\Omega}(x)$, so $z-x \in N_{\partial\Omega}(x)\cap T_{\partial\Omega}(x)$, which implies that $z = x$; thus, $\Pi_x(\xi) = x$, so $\xi = x$ (since Π_x is injective, and $\Pi_x(x) = x$), and we arrive at the equality $\xi = z$, contradicting the hypothesis on ξ, and proving the claim. Next, since $z-x \in N_{\partial\Omega}(x)$ and $\Pi_x(\xi)-x \in T_{\partial\Omega}(x)$, it is clear that

$$r_z(\Pi_x(\xi)) = \sqrt{\{r_z^2(x)+r_x^2(\Pi_x(\xi))\}} \geq r_x(\Pi_x(\xi)). \tag{2}$$

Similarly, because $\xi-\Pi_x(\xi) \in N_{\partial\Omega}(x)$ (cf., (VI.62.3)),

$$r_x(\xi) = \sqrt{\{r_x^2(\Pi_x(\xi))+r_\xi^2(\Pi_x(\xi))\}} \geq r_x(\Pi_x(\xi)). \tag{3}$$

Proceeding to the main line of reasoning of the proof, first use [VI.66.iii.3] to write

$$|r_z(\xi)-r_z(\Pi_x(\xi))| \leq r_\xi(\Pi_x(\xi)) < \tilde{a}r_x^{1+\alpha}(\Pi_x(\xi)),$$

from which there follows

$$1-\frac{\tilde{a}r_x^{1+\alpha}(\Pi_x(\xi))}{r_z(\Pi_x(\xi))} < \frac{r_z(\xi)}{r_z(\Pi_x(\xi))} < 1+\frac{\tilde{a}r_x^{1+\alpha}(\Pi_x(\xi))}{r_z(\Pi_x(\xi))}\ ; \tag{4}$$

note that $\tilde{a} > 0$ and depends only on a and α. With the inequality (2), $r_z(\Pi_x(\xi)) \geq r_x(\Pi_x(\xi))$, (4) implies that

$$1-\tilde{a}r_x^\alpha(\Pi_x(\xi)) < \frac{r_z(\xi)}{r_z(\Pi_x(\xi))} < 1+\tilde{a}r_x^\alpha(\Pi_x(\xi)),$$

and then, because $r_x(\Pi_x(\xi)) \leq r_x(\xi) < d$,

$$1-\tilde{a}d^\alpha < \frac{r_z(\xi)}{r_z(\Pi_x(\xi))} < 1+\tilde{a}d^\alpha. \tag{5}$$

Now, simply choose $d_0 > 0$ such that $\max\{2ad_0^\alpha, \tilde{a}d_0^\alpha\} < 1$, and suppose $d \in (0,d_0)$; since $ad^\alpha < 1/2$, (5) holds. Set $\gamma_d :=$ $1-\tilde{a}d^\alpha$. Since $0 < \tilde{a}d^\alpha < 1$, $\gamma_d \in (0,1)$, and $1-(\tilde{a}d^\alpha)^2 < 1$, so $1+\tilde{a}d^\alpha < 1/(1-\tilde{a}d^\alpha) = 1/\gamma_d$. Consequently, (1) follows directly from (5), with γ_d as defined. Obviously, d_0 depends only on a and α, while γ_d depends only on a, α, and d. $\square$.

The following auxiliary construction is used in conjunction with the divergence theorem to derive representations of solutions of Maxwell's equations in Part III.

[VI.68] L E M M A. *Let* Ω *be a non-void proper subset of* $\mathbb{R}^n$ *which is a* q*-regular domain for some* $q \geq 2$, *and such that* $\partial\Omega$ *is compact. For each* $\varepsilon \in \mathbb{R}$, *let the function* $G^\varepsilon\colon \partial\Omega \to \mathbb{R}^n$ *be defined by*

$$G^\varepsilon(x) := x+\varepsilon\nu_{\partial\Omega}(x) \quad \textit{for each} \quad x \in \partial\Omega, \tag{1}$$

and set

$$\Omega_\varepsilon := \{x \in \Omega \mid \operatorname{dist}(x,\partial\Omega) > -\varepsilon\} \quad \textit{if} \quad \varepsilon < 0, \tag{2}$$

$$\Omega_\varepsilon := \{x \in \Omega^{-'} \mid \operatorname{dist}(x,\partial\Omega) > \varepsilon\} \quad \textit{if} \quad \varepsilon > 0. \tag{3}$$

Then there exists an $\varepsilon_0 > 0$ *such that whenever* $0 < |\varepsilon| < \varepsilon_0$,

(*i*) G^ε *is a* (q-1)*-imbedding, taking* $\partial\Omega$ *onto* $\partial\Omega_\varepsilon$,

(*ii*) Ω_ε *is a* (q-1)*-regular domain,*

and

(*iii*) $\nu_{\partial\Omega_\varepsilon} = -\operatorname{sgn}\varepsilon \cdot \nu_{\partial\Omega}\circ(G^\varepsilon)^{-1}$ *on* $\partial\Omega_\varepsilon$.

Further,

(*iv*) $\lim_{\varepsilon \to 0} JG^\varepsilon = 1$ *uniformly on* $\partial\Omega$.

P R O O F. We note at the outset that Ω is a Lyapunov domain (by Remark [VI.61.b]), $\nu_{\partial\Omega} \in C^{q-1}(\partial\Omega;\mathbb{R}^n)$ ([VI.57]), and $\nu_{\partial\Omega}$ is Lipschitz continuous ([VI.26]). In particular, we can find a set of Lyapunov constants for Ω of the form $(a,1,d)$.

(i) First, for any real ε it is easy to see that $G^{\varepsilon} \in C^{q-1}(\partial\Omega;\mathbb{R}^n)$, for, choosing a covering collection $\{(U_\iota,h_\iota)\}_{\iota\in I}$ of coordinate systems in $\partial\Omega$, so $h_\iota^{-1} \in C^q(h_\iota(U_\iota);\mathbb{R}^n)$ for each $\iota \in I$, we have

$$G^{\varepsilon}\circ h_\iota^{-1} = h_\iota^{-1}+\varepsilon\cdot\nu_{\partial\Omega}\circ h_\iota^{-1} \quad \text{on} \quad h_\iota(U_\iota), \tag{4}$$

so $G^{\varepsilon}\circ h^{-1} \in C^{q-1}(h_\iota(U_\iota);\mathbb{R}^n)$ for each $\iota \in I$, since $\nu_{\partial\Omega}$ is in $C^{q-1}(\partial\mathbb{B};\mathbb{R}^n)$. This implies that the claim is true. Observe also that G^{ε} is continuous for any ε.

Suppose now that $|\varepsilon| < a^{-1}$: then G^{ε} is an injection. Indeed, let $x_1, x_2 \in \partial\Omega$, with $G^{\varepsilon}(x_1) = G^{\varepsilon}(x_2)$, *i.e.*, $x_1+\varepsilon\nu_{\partial\Omega}(x_1) = x_2+\varepsilon\nu_{\partial\Omega}(x_2)$. Then

$$|x_2-x_1|_n = |\varepsilon|\cdot|\nu_{\partial\Omega}(x_2)-\nu_{\partial\Omega}(x_1)|_n \le a\cdot|\varepsilon|\cdot|x_2-x_1|_n,$$

which can hold only if $x_1 = x_2$, since $a|\varepsilon| < 1$. This proves our assertion. But now, for these same ε, G^{ε}: $\partial\Omega \to G^{\varepsilon}(\partial\Omega)$ is a continuous bijection, and $\partial\Omega$ is compact, whence the map is a homeomorphism.

To show that G^{ε} is a (q-1)-imbedding whenever $|\varepsilon|$ is sufficiently small, we must verify now only that G^{ε} has rank n-1 at each point of $\partial\Omega$, or, equivalently, that $JG^{\varepsilon} > 0$ on $\partial\Omega$ whenever $|\varepsilon|$ is sufficiently small. Clearly, the latter shall follow once (iv) has been established. To prove (iv), we begin by pointing out that we can assume that the covering collection $\{(U_\iota,h_\iota)\}_{\iota\in I}$ of coordinate systems in $\partial\Omega$ has been chosen so that

I is finite, and, for each $\iota \in I$, all partial derivatives of h_ι^{-1} and $\nu_{\partial\Omega} \circ h_\iota^{-1}$ are bounded on $h_\iota(U_\iota)$, while $|(h_\iota^{-1})_{,1} \wedge \ldots \wedge (h_\iota^{-1})_{,n-1}|$ is bounded below by a positive number on $h_\iota(U_\iota)$, a consequence of the compactness of $\partial\Omega$ and the properties of coordinate systems. Then, from (4), it is clear that, for, say, $|\varepsilon| < 1$,

$$\left| \bigwedge_{i=1}^{n-1} (G^\varepsilon \circ h_\iota^{-1})_{,i} \right| \leq \left| \bigwedge_{i=1}^{n-1} (h_\iota^{-1})_{,i} \right| + M_\iota \cdot |\varepsilon| \quad \text{on} \quad h_\iota(U_\iota)$$

for each $\iota \in I$, for certain positive numbers $\{M_\iota\}_{\iota \in I}$. Using (VI.24.3), it follows that

$$JG^\varepsilon | U_\iota \leq 1 + M'_\iota \cdot |\varepsilon| \quad \text{for each} \quad \iota \in I, \tag{5}$$

for certain positive numbers $\{M'_\iota\}_{\iota \in I}$. Assertion (iv) surely follows from (5), since I is finite. As remarked, we have now proven that G^ε is a $(q-1)$-imbedding if $|\varepsilon|$ is sufficiently small; among other consequences of [VI.30], we now know that $G^\varepsilon(\partial\Omega)$ is a compact $(n-1,n;q-1)$-manifold for these same ε.

We shall next prove that, for $|\varepsilon|$ sufficiently small,

$$G^\varepsilon(\partial\Omega) = \{y \in \Omega^{-'} \mid \text{dist}(y, \partial\Omega) = \varepsilon\}, \quad \text{if} \quad \varepsilon > 0, \tag{6}$$

whereas

$$G^\varepsilon(\partial\Omega) = \{y \in \Omega \mid \text{dist}(y, \partial\Omega) = -\varepsilon\}, \quad \text{if} \quad \varepsilon < 0. \tag{7}$$

For this, we first appeal to [VI.59], which tells us that there exists a positive δ_Ω for which

$$\left.\begin{array}{ll} x+s\nu_{\partial\Omega}(x) \in \Omega^{-'} & \text{if} \quad 0 < s < \delta_\Omega, \\ \text{and} & \\ x+s\nu_{\partial\Omega}(x) \in \Omega & \text{if} \quad -\delta_\Omega < s < 0 \end{array}\right\} \text{ for each } \quad x \in \partial\Omega. \quad \begin{array}{r}(8)\\ \\ (9)\end{array}$$

Now, suppose first that $0 < \varepsilon < \min\{\delta_\Omega, d/2, 1/2\hat{a}\}$, with $\hat{a}$ ($> a$) given in [VI.66.iii.4]: we shall prove that (6) holds thereby. If we assume that $y \in \Omega^{-'}$ and $\text{dist}\,(y,\partial\Omega) = \varepsilon$, then there exists $x_y \in \partial\Omega$ such that $|y-x_y|_n = \varepsilon$ and $y-x_y \in N_{\partial\Omega}(x_y)$ (cf., [VI.33]), whence y is given by one of $x_y+\varepsilon\nu_{\partial\Omega}(x_y)$, $x_y-\varepsilon\nu_{\partial\Omega}(x_y)$; y cannot equal the latter, for otherwise we should have $y \in \Omega$, by (9). Thus, $y = x_y+\varepsilon\nu_{\partial\Omega}(x_y) = G^\varepsilon(x_y)$. To secure the opposite inclusion, choose any $x \in \partial\Omega$. Then $G^\varepsilon(x) = x+\varepsilon\nu_{\partial\Omega}(x) \in \Omega^{-'}$, by (8). We must show also that

$$\delta := \text{dist}\,(G^\varepsilon(x),\partial\Omega) := \inf_{z \in \partial\Omega} |G^\varepsilon(x)-z|_n = \varepsilon. \tag{10}$$

Since $|G^\varepsilon(x)-x|_n = \varepsilon$, the inequality $\delta \leq \varepsilon$ is plainly true. It is just as clear that $\delta > 0$. Suppose that $\delta < \varepsilon$: then $G^\varepsilon(x) = \tilde{x}+\delta\nu_{\partial\Omega}(\tilde{x})$ for some $\tilde{x} \in \partial\Omega$ with $\tilde{x} \neq x$, and

$$\begin{aligned} |x-\tilde{x}|_n &= |\delta\nu_{\partial\Omega}(\tilde{x})-\varepsilon\nu_{\partial\Omega}(x)|_n \\ &\leq (\varepsilon-\delta)+\varepsilon\cdot|\nu_{\partial\Omega}(\tilde{x})-\nu_{\partial\Omega}(x)|_n \\ &\leq (\varepsilon-\delta)+a\varepsilon\cdot|x-\tilde{x}|_n \\ &< (\varepsilon-\delta)+\frac{1}{2}\,|x-\tilde{x}|_n, \end{aligned}$$

yielding

$$|x-\tilde{x}|_n < 2(\varepsilon-\delta) < 2\varepsilon < d.$$

Thus, $x \in \partial\Omega \cap B^n_d(\tilde{x})$ and $x \neq \tilde{x}$, allowing the application of [VI.66.iii.4], giving first

$$|\nu_{\partial\Omega}(x) \bullet \text{grad } r_{\tilde{x}}(x)| < \hat{a}|x-\tilde{x}|_n,$$

then

$$2\varepsilon \cdot |x-\tilde{x}|_n \cdot \nu_{\partial\Omega}(x) \bullet \text{grad } r_{\tilde{x}}(x) > -2\hat{a}\varepsilon \cdot |x-\tilde{x}|^2_n. \tag{11}$$

We can write, further,

$$\delta^2 = |x+\varepsilon\nu_{\partial\Omega}(x)-\tilde{x}|^2_n = |x-\tilde{x}|^2_n+\varepsilon^2+2\varepsilon\nu_{\partial\Omega}(x)\bullet(x-\tilde{x}),$$

so

$$2\varepsilon \cdot |x-\tilde{x}|_n \nu_{\partial\Omega}(x) \bullet \text{grad } r_{\tilde{x}}(x) + |x-\tilde{x}|^2_n = \delta^2-\varepsilon^2,$$

and, now recalling that $\varepsilon < 1/2\hat{a}$, (11) implies that

$$0 < (1-2\hat{a}\varepsilon) \cdot |x-\tilde{x}|^2_n < \delta^2-\varepsilon^2 < 0,$$

which is impossible. Thus, $\delta = \varepsilon$. This completes the proof of (6), if ε is as specified. (7) can be verified in a similar manner, for $\varepsilon < 0$ and $|\varepsilon|$ sufficiently small.

Statement (i) will be completely proven once we have shown that, for $|\varepsilon|$ sufficiently small,

$$\partial\Omega_\varepsilon = \{y \in \Omega^{-'} | \text{ dist }(y,\partial\Omega) = \varepsilon\}, \quad \text{if} \quad \varepsilon > 0, \tag{12}$$

and

$$\partial\Omega_\varepsilon = \{y \in \Omega \mid \text{dist}\,(y,\partial\Omega) = -\varepsilon\}, \quad \text{if} \quad \varepsilon < 0, \tag{13}$$

which we shall do presently. Let us make some preliminary observations: by the continuity of the map $x \mapsto \text{dist}\,(x,\partial\Omega)$ on $\mathbb{R}^n$, we know that $\text{dist}^{-1}\,((\eta,\infty),\partial\Omega)$ is open, while $\text{dist}^{-1}\,([\eta,\infty),\partial\Omega)$ is closed in $\mathbb{R}^n$, for each $\eta > 0$. Thus, since

$$\Omega_\varepsilon = \begin{cases} \Omega^{-\prime} \cap \text{dist}^{-1}\,((\varepsilon,\infty),\partial\Omega) & \text{if} \quad \varepsilon > 0, \\ \Omega \cap \text{dist}^{-1}\,((-\varepsilon,\infty),\partial\Omega) & \text{if} \quad \varepsilon < 0, \end{cases}$$

each Ω_ε is open. Moreover, the sets given by

$$\tilde{\Omega}_\varepsilon := \begin{cases} \{x \in \Omega \mid \text{dist}\,(x,\partial\Omega) \geq -\varepsilon\} & \text{if} \quad \varepsilon < 0, \\ \{x \in \Omega^{-\prime} \mid \text{dist}\,(x,\partial\Omega) \geq \varepsilon\} & \text{if} \quad \varepsilon > 0 \end{cases}$$

must be closed; in fact, if $\varepsilon < 0$, one can easily show that

$$\begin{aligned}\{x \in \Omega \mid \text{dist}\,(x,\partial\Omega) \geq -\varepsilon\} &= \{x \in \Omega^- \mid \text{dist}\,(x,\partial\Omega) \geq -\varepsilon\} \\ &= \Omega^- \cap \text{dist}^{-1}\,([-\varepsilon,\infty),\partial\Omega),\end{aligned}$$

with similar reasoning for $\tilde{\Omega}_\varepsilon$ if $\varepsilon > 0$. We claim that

$$\tilde{\Omega}_\varepsilon = \Omega_\varepsilon^- \quad \text{whenever} \quad |\varepsilon| \quad \text{is sufficiently small (and positive).} \tag{14}$$

To see that this is so, suppose first that $\varepsilon > 0$: since $\tilde{\Omega}_\varepsilon$ is closed and certainly contains Ω_ε, the inclusion $\Omega_\varepsilon^- \subset \tilde{\Omega}_\varepsilon$ must hold. Now, assume that $x \in \tilde{\Omega}_\varepsilon$, *i.e.*, $x \in \Omega^{-\prime}$ and $\text{dist}\,(x,\partial\Omega) \geq \varepsilon$. If $\text{dist}\,(x,\partial\Omega) > \varepsilon$, then $x \in \Omega_\varepsilon \subset \Omega_\varepsilon^-$, so we must examine only the possibility that $\operatorname{dist}(x,\partial\Omega) := \inf_{z \in \partial\Omega} |x-z|_n = \varepsilon$: supposing that this holds, and that ε is small enough, it is easy to see, as before, that we must have $x = G^{\varepsilon}(\hat{x}) = \hat{x}+\varepsilon\nu_{\partial\Omega}(\hat{x})$ for some $\hat{x} \in \partial\Omega$. Since it was shown that there is some $\tilde{\varepsilon} > 0$ such that

$$G^{\eta}(\partial\Omega) = \{y \in \Omega^{-'} \mid \operatorname{dist}(y,\partial\Omega) = \eta\} \quad \text{for} \quad 0 < \eta < \tilde{\varepsilon},$$

if we select any sequence $(\varepsilon_j)_{j=1}^{\infty} \subset (0,\infty)$ with $\varepsilon_n \to 0$, and assume that $\varepsilon < \tilde{\varepsilon}$, for all sufficiently large j we find that

$$\hat{x}+(\varepsilon+\varepsilon_j)\nu_{\partial\Omega}(\hat{x}) = G^{\varepsilon+\varepsilon_j}(\hat{x}) \in \{y \in \Omega^{-'} \mid \operatorname{dist}(y,\partial\Omega) = \varepsilon+\varepsilon_j\} \subset \Omega_{\varepsilon}.$$

Since $(G^{\varepsilon+\varepsilon_j}(\hat{x}))_{j=1}^{\infty}$ converges to $\hat{x}+\varepsilon\nu_{\partial\Omega}(\hat{x}) = x$, we can conclude that $x \in \Omega_{\varepsilon}^{-}$ if $\varepsilon < \tilde{\varepsilon}$. Thus, (14) has been proven for the case $\varepsilon > 0$. The consideration of the case $\varepsilon < 0$ proceeds along similar lines, and so we omit the details. Now, having (14) available, (12) and (13) can be proven easily: if $|\varepsilon|$ is sufficiently small and $\varepsilon > 0$, then

$$\begin{aligned}
\partial\Omega_{\varepsilon} &= \Omega_{\varepsilon}^{-}\cap\Omega_{\varepsilon}^{'-} \\
&= \Omega_{\varepsilon}^{-}\cap\Omega_{\varepsilon}^{'} \\
&= \{x \in \Omega^{-'} \mid \operatorname{dist}(x,\partial\Omega) \geq \varepsilon\}\cap(\{x \in \Omega^{-'} \mid \operatorname{dist}(x,\partial\Omega) \leq \varepsilon\}\cup\Omega^{-}) \\
&= \{x \in \Omega^{-'} \mid \operatorname{dist}(x,\partial\Omega) = \varepsilon\},
\end{aligned}$$

giving (12), while if $\varepsilon < 0$, (13) follows in much the same fashion.

Since we can now state, by (6), (7), (12), and (13), that $G^{\varepsilon}(\partial\Omega) = \partial\Omega_{\varepsilon}$ for each non-zero ε with $|\varepsilon|$ sufficiently small,

(i) has been verified.

(ii) Let us begin here by showing that Ω_ε is a regularly open set, provided $|\varepsilon|$ is sufficiently small. We investigate first the case in which $\varepsilon > 0$: we are to prove that

$$\Omega_\varepsilon^{-o} = \Omega_\varepsilon \tag{15}$$

whenever ε is sufficiently small. Recalling that Ω_ε is open, the inclusion $\Omega_\varepsilon^{-o} \supset \Omega_\varepsilon$ is obvious. Now, let $x \in \Omega_\varepsilon^{-o}$ and choose a positive δ such that $B_\delta^n(x) \subset \Omega_\varepsilon^-$. If we can show that $B_\delta^n(x) \subset \Omega_\varepsilon$, then we shall have $x \in \Omega_\varepsilon^o = \Omega_\varepsilon$, completing the proof of (15). Assume, then, that there exists some $y \in B_\delta^n(x) \cap \Omega_\varepsilon'$; this implies that $y \in \Omega_\varepsilon^- \cap \Omega_\varepsilon' = \partial\Omega_\varepsilon$, hence, if ε is sufficiently small, that dist $(y,\partial\Omega) = \varepsilon$, and $y = \tilde{y}+\varepsilon\nu_{\partial\Omega}(\tilde{y})$ for some $\tilde{y} \in \partial\Omega$. Note that, again if ε is sufficiently small, $\Omega_\varepsilon^- = \{x \in \Omega^{-\prime} |\ \text{dist}\ (x,\partial\Omega) \geq \varepsilon\}$. Now, certainly we can select $\eta \in (0,\varepsilon)$ so that $y-\eta\nu_{\partial\Omega}(\tilde{y}) \in B_\delta^n(x)$, but then

$$y-\eta\nu_{\partial\Omega}(\tilde{y}) = \tilde{y}+(\varepsilon-\eta)\nu_{\partial\Omega}(\tilde{y}) \in \{x \in \Omega^{-\prime} |\ \text{dist}\ (x,\partial\Omega) = \varepsilon-\eta\},$$

implying that

$$y-\eta\nu_{\partial\Omega}(\tilde{y}) \notin \{x \in \Omega^{-\prime} |\ \text{dist}\ (x,\partial\Omega) \geq \varepsilon\} = \Omega_\varepsilon^-,$$

and so contradicting the inclusions $y-\eta\nu_{\partial\Omega}(\tilde{y}) \in B_\delta^n(x) \subset \Omega_\varepsilon^-$. Thus, if ε is sufficiently small, $B_\delta^n(x) \cap \Omega_\varepsilon' = \emptyset$, *i.e.*, $B_\delta^n(x) \subset \Omega_\varepsilon$, giving (15) for these same $\varepsilon > 0$. The proof of (15), in case $\varepsilon < 0$ and $|\varepsilon|$ is sufficiently small, is similar.

Now we know Ω_ε to be regularly open, with (by (i)) $\partial\Omega_\varepsilon = G^\varepsilon(\partial\Omega)$ an $(n-1,n;q-1)$-manifold, if $|\varepsilon|$ is small enough. But we may therefore invoke [VI.55] to conclude that Ω_ε is a $(q-1)$-regular domain for these same ε.

(iii) We suppose here that $|\varepsilon|$ is so small that (i) and (ii) hold. Choose any $x \in \partial\Omega$. We aim first to prove that

$$T_{\partial\Omega}(x) = T_{\partial\Omega_\varepsilon}(G^\varepsilon(x)). \tag{16}$$

Since each tangent space here is an $(n-1)$-dimensional subspace of $\mathbb{R}^n$, (16) shall follow once the inclusion

$$T_{\partial\Omega_\varepsilon}(G^\varepsilon(x)) \subset T_{\partial\Omega}(x) \tag{17}$$

is known. To prove (17), let $\beta \in T_{\partial\Omega_\varepsilon}(G^\varepsilon(x))$. Then there exist $\delta > 0$ and $\psi^\varepsilon \in C^1((-\delta,\delta);\mathbb{R}^n)$ with $\psi^\varepsilon(-\delta,\delta) \subset \partial\Omega_\varepsilon$, $\psi^\varepsilon(0) = G^\varepsilon(x)$, and $\psi^{\varepsilon\prime}(0) = \beta$. We define ψ: $(-\delta,\delta) \to \mathbb{R}^n$ *via*

$$\psi(\sigma) := G^{\varepsilon^{-1}} \circ \psi^\varepsilon(\sigma) \quad \text{for} \quad |\sigma| < \delta, \tag{18}$$

and f_ν: $(-\delta,\delta) \to \mathbb{R}^n$ by setting

$$f_\nu(\sigma) := \nu_{\partial\Omega} \circ \psi(\sigma) \quad \text{for} \quad |\sigma| < \delta. \tag{19}$$

Let us assume, for the moment, that

$$\psi|(-\delta_0,\delta_0) \in C^1((-\delta_0,\delta_0);\mathbb{R}^n) \quad \text{for some} \quad \delta_0 \in (0,\delta]. \tag{20}$$

Then, since $\psi((-\delta_0,\delta_0)) \subset \partial\Omega$ and $\psi(0) = x$, we see that

$\psi'(0) \in T_{\partial\Omega}(x)$. Further, since Ω is a q-regular domain, reasoning which is by now familiar allows us to assert that there exist an open neighborhood of x in $\mathbb{R}^n$, U_x, and a function $\Phi_x \in C^q(U_x)$ such that $\operatorname{grad} \Phi_x(y) \neq 0$ for each $y \in U_x$, and

$$\nu_{\partial\Omega}(y) = |\operatorname{grad} \Phi_x(y)|_n^{-1} \cdot \operatorname{grad} \Phi_x(y) \quad \text{for each} \quad y \in \partial\Omega \cap U_x. \tag{21}$$

In view of the definition (19), (20) and (21) together imply that f_ν is of class C^1 on a neighborhood of 0. Therefore, we can deduce that $f_\nu(0) \cdot f_\nu'(0) = 0$, since $|f_\nu(\sigma)|_n^2 = 1$ for $|\sigma| < \delta$, *i.e.*, $\nu_{\partial\Omega}(x) \cdot f_\nu'(0) = 0$, whence $f_\nu'(0) \in T_{\partial\Omega}(x)$. But, by writing

$$\psi^\varepsilon = G^\varepsilon \circ G^{\varepsilon^{-1}} \circ \psi^\varepsilon = G^\varepsilon \circ \psi = \psi + \varepsilon \nu_{\partial\Omega} \circ \psi = \psi + \varepsilon f_\nu \quad \text{on} \quad (-\delta, \delta), \tag{22}$$

we come to the desired result

$$\beta = \psi^{\varepsilon'}(0) = \psi'(0) + \varepsilon f_\nu'(0) \in T_{\partial\Omega}(x).$$

This implies that (17) is correct; as remarked, (16) follows.

Of course, the preceding reasoning depends upon the validity of (20) whenever $|\varepsilon|$ is sufficiently small, independently of the x chosen in $\partial\Omega$. For this, observe first that, by (22),

$$\psi(\sigma) + \varepsilon \nu_{\partial\Omega}(\psi(\sigma)) - \psi^\varepsilon(\sigma) = 0 \quad \text{for} \quad |\sigma| < \delta. \tag{23}$$

Making use of the q-regularity of Ω and the compactness of $\partial\Omega$, we can find a finite collection of open subsets of $\mathbb{R}^n$, $\{U_i\}_{i=1}^p$, which covers $\partial\Omega$, and a corresponding collection of functions $\{\Phi_i \in C^q(U_i)\}_{i=1}^p$ such that, for each $i \in \{1, \ldots, p\}$,

$$m_i \le |\text{grad } \Phi_i|_n \le M_i \quad \text{on} \quad U_i, \tag{24}$$

$$|\Phi_{i,jk}| \le M_i \quad \text{on} \quad U_i, \quad \text{for} \quad j,k = 1,\dots,n, \tag{25}$$

for certain positive numbers m_i and M_i, and

$$\nu_{\partial\Omega}(y) = |\text{grad } \Phi_i(y)|_n^{-1} \cdot \text{grad } \Phi_i(y), \quad \text{for each} \quad y \in \partial\Omega \cap U_i. \tag{26}$$

As a further restriction on ε, because of (24) and (25), we may, and shall, suppose $|\varepsilon|$ to be so small that

$$\det\left(\delta_{jk} + \varepsilon \cdot \left(\frac{\Phi_{i,j}}{|\text{grad } \Phi_i|_n}\right)_{,k}(y)\right) > 0 \quad \text{for each} \quad y \in \partial\Omega \cap U_i,$$
$$\text{for each} \quad i \in \{1,\dots,p\}. \tag{27}$$

Now choose $\ell \in \{1,\dots,p\}$ such that $x \in U_\ell$, and define a function $F\colon\ U_\ell \times (-\delta,\delta) \to \mathbb{R}^n$ by

$$F(y,\sigma) := y + \varepsilon \frac{\text{grad } \Phi_\ell(y)}{|\text{grad } \Phi_\ell(y)|_n} - \psi^\varepsilon(\sigma) \quad \text{for} \quad y \in U_\ell, \quad |\sigma| < \delta. \tag{28}$$

It is clear that $F \in C^1(U_\ell \times (-\delta,\delta);\mathbb{R}^n)$ (recalling $\psi^\varepsilon \in C^1((-\delta,\delta);\mathbb{R}^n)$), and

$$F(x,0) = x + \varepsilon\nu_{\partial\Omega}(x) - \psi^\varepsilon(0) = \psi(0) + \varepsilon\nu_{\partial\Omega}(\psi(0)) - \psi^\varepsilon(0) = 0, \tag{29}$$

by (23) and (26). Moreover, by (27), it is easy to see that

$$\det\left((F^j_{,k}(x,0))_{1\le j,k\le n}\right) > 0. \tag{30}$$

With these facts, the implicit function theorem says that there exist an open neighborhood $\mathcal{U} \subset U_\ell \times (-\delta,\delta)$ of $(x,0)$, a $\tilde{\delta} > 0$, and a *unique* function $\tilde{\psi}$: $(-\tilde{\delta},\tilde{\delta}) \to \mathbb{R}^n$ such that $(\tilde{\psi}(\sigma),\sigma) \in U$ for $|\sigma| < \tilde{\delta}$, and $F(\tilde{\psi}(\sigma),\sigma) = 0$ for $|\sigma| < \tilde{\delta}$. But also, because ψ: $(-\delta,\delta) \to \mathbb{R}^n$ is continuous (this much is clear, directly from the definition (18)) and $\psi(0) = x$, we can find a $\delta' > 0$ such that $(\psi(\sigma),\sigma) \in U$ whenever $|\sigma| < \delta'$, while it is easy to see that $F(\psi(\sigma),\sigma) = 0$ if $|\sigma| < \delta$, because of (23), (26), (28), and the fact that $\psi((-\delta,\delta)) \subset \partial\Omega$. Thus, ψ and $\tilde{\psi}$ must in fact coincide on a neighborhood of 0. Since the implicit function theorem also asserts that $\tilde{\psi} \in C^1((-\tilde{\delta},\tilde{\delta});\mathbb{R}^n)$, it follows that ψ is of class C^1 on a neighborhood of 0, *i.e.*, (20) is true. It is imperative to observe here that the uniqueness assertion of the implicit function theorem requires for its proof no smoothness properties of the implicitly defined function, as one can check (*cf.*, [VI.2]).

With the verification of (20), for $|\varepsilon|$ sufficiently small, the equality (16) is known to hold, whence

$$N_{\partial\Omega_\varepsilon}(G^\varepsilon(x)) = N_{\partial\Omega}(x), \tag{31}$$

and $\nu_{\partial\Omega_\varepsilon}(G^\varepsilon(x))$ is given by one of $\nu_{\partial\Omega}(x)$, $-\nu_{\partial\Omega}(x)$ ($\nu_{\partial\Omega_\varepsilon}$ is well-defined, since (ii) holds; cf., [VI.57]). Suppose first that $\varepsilon > 0$: assume that $\nu_{\partial\Omega_\varepsilon}(G^\varepsilon(x)) = \nu_{\partial\Omega}(x)$. We already know that there exists a positive ε_1 such that $y+s\nu_{\partial\Omega}(y) \in \Omega^{-'}$ and dist $(y+s\nu_{\partial\Omega}(y),\partial\Omega) = s$ whenever $y \in \partial\Omega$ and $0 < s < \varepsilon_1$. Thus, if $\varepsilon \in (0,\varepsilon_1)$ and $s \in (\varepsilon,\varepsilon_1)$, we have

$$x+\varepsilon\nu_{\partial\Omega}(x)+(s-\varepsilon)\cdot\nu_{\partial\Omega}(x) \in \Omega_\varepsilon,$$

so, according to our assumption,

$$G^\varepsilon(x)+(s-\varepsilon)\cdot\nu_{\partial\Omega_\varepsilon}(G^\varepsilon(x)) \in \Omega_\varepsilon,$$

which implies that

$$G^\varepsilon(x)+\tilde{s}\cdot\nu_{\partial\Omega_\varepsilon}(G^\varepsilon(x)) \in \Omega_\varepsilon \quad \text{whenever} \quad 0 < \tilde{s} < \varepsilon_1-\varepsilon;$$

this is impossible, for it violates the definition of $\nu_{\partial\Omega_\varepsilon}(G^\varepsilon(x))$. Consequently, we must have $\nu_{\partial\Omega_\varepsilon}(G^\varepsilon(x)) = -\nu_{\partial\Omega}(x)$ for $\varepsilon > 0$ and sufficiently small. Similarly, if $\varepsilon < 0$ and $|\varepsilon|$ is small enough, one can show that $\nu_{\partial\Omega_\varepsilon}(G^\varepsilon(x)) = \nu_{\partial\Omega}(x)$. We conclude, then, since all restrictions imposed on ε were independent of the particular x chosen in $\partial\Omega$, that

$$\nu_{\partial\Omega_\varepsilon} \circ G^\varepsilon = -\text{sgn}\, \varepsilon \cdot \nu_{\partial\Omega} \quad \text{on} \quad \partial\Omega,$$

if $\varepsilon \neq 0$ and $|\varepsilon|$ is sufficiently small.

Statement (iii) obviously follows from this.

(iv) This fact was verified in the course of proving (i). □.

We complete Part VI by using the divergence theorem and the estimates developed for the geometry associated with the boundary of a Lyapunov domain to derive a generalization of Gauss's formula (cf., Günter [19] or Mikhlin [34]).

[VI.69] LEMMA. *Let* Ω *be a bounded Lyapunov domain in* $\mathbb{R}^3$. *Let* $\xi \in \mathbb{R}^3$, *with* $|\xi|_3 < 1$. *For* $x \in \mathbb{R}^3$, *define* $\Gamma_x\{\xi\}$: $\mathbb{R}^3 \cap \{x\}' \to \mathbb{R}$ *by setting*

$$\Gamma_x\{\xi\} := \{(\xi^\ell r_{x,\ell})^2 + (1-|\xi|_3^2)\}^{-3/2}. \tag{1}$$

Then

$$\int_{\partial\Omega} \frac{1}{r_x^2} r_{x,i} \nu^i_{\partial\Omega} \cdot \Gamma_x\{\xi\}\, d\lambda_{\partial\Omega} = \begin{cases} 0 & , \quad \textit{if} \quad x \in \Omega^{-\prime}, \\ 2\pi(1-|\xi|_3^2)^{-1}, & \textit{if} \quad x \in \partial\Omega, \\ 4\pi(1-|\xi|_3^2)^{-1}, & \textit{if} \quad x \in \Omega. \end{cases}$$

PROOF. Throughout, x is fixed in $\mathbb{R}^3$. First, we observe that since, $r_{x,ij} = r_x^{-1}\{\delta_{ij} - r_{x,i} r_{x,j}\}$ for $i,j \in \{1,2,3\}$,

$$\begin{aligned}(r_x^{-2} r_{x,i} \cdot \Gamma_x\{\xi\})_{,i} = &-2r_x^{-3}\{(\xi^\ell r_{x,\ell})^2 + (1-|\xi|_3^2)\}^{-3/2} \\ &+2r_x^{-3}\{(\xi^\ell r_{x,\ell})^2 + (1-|\xi|_3^2)\}^{-3/2} \\ &-3r_x^{-2}\{(\xi^\ell r_{x,\ell})^2 + (1-|\xi|_3^2)\}^{-5/2} \\ &\cdot(\xi^k r_{x,k}) \cdot \xi^m(\delta_{im} - r_{x,i} r_{x,m}) r_{x,i},\end{aligned}$$

$$\text{on} \quad \mathbb{R}^3 \cap \{x\}';$$

noting that $(\delta_{im} - r_{x,i} r_{x,m}) r_{x,i} = 0$, it follows that

$$(r_x^{-2} r_{x,i} \cdot \Gamma_x\{\xi\})_{,i} = 0, \quad \text{in} \quad \mathbb{R}^3 \cap \{x\}'. \tag{3}$$

Also, we obviously have, on $\mathbb{R}^3 \cap \{x\}'$, recalling that $|\xi|_3 < 1$,

$$0 < (1-|\xi|_3^2) \le (\xi^\ell r_{x,\ell})^2 + (1-|\xi|_3^2) \le |\xi|_3^2 + (1-|\xi|_3^2) = 1,$$

whence

$$1 \le \Gamma_x\{\xi\} \le (1-|\xi|_3^2)^{-3/2}, \quad \text{on} \quad \mathbb{R}^3 \cap \{x\}'. \tag{4}$$

We shall now verify (2) by considering, in turn, each of the three possible positions for x.

(i) Suppose that $x \in \Omega^{-\prime}$: clearly, in view of the inequality preceding (4), and noting that Ω is a normal domain, we may apply the divergence theorem in Ω along with (3) to arrive at (2) in this case:

$$\int_{\partial\Omega} r_x^{-2} r_{x,i} \nu_{\partial\Omega}^i \cdot \Gamma_x\{\xi\}\, d\lambda_{\partial\Omega} = \int_{\Omega} (r_x^{-2} r_{x,i} \cdot \Gamma_x\{\xi\})_{,i}\, d\lambda_{\partial\Omega} = 0.$$

(ii) Suppose that $x \in \Omega$: now, for any $\varepsilon \in (0, \operatorname{dist}(x, \partial\Omega))$, we consider the normal domain $\Omega \cap B_\varepsilon^3(x)^{-\prime}$. Since the function $r_x^{-2}\Gamma_x\{\xi\}$ lies in $C^\infty(\mathbb{R}^3 \cap \{x\}')$, the divergence theorem and (3) produce, in this case,

$$\int_{\partial\Omega} r_x^{-2} r_{x,i} \nu_{\partial\Omega}^i \cdot \Gamma_x\{\xi\}\, d\lambda_{\partial\Omega} + \int_{\partial B_\varepsilon^3(x)} r_x^{-2} r_{x,i} \cdot (-r_{x,i}) \cdot \Gamma_x\{\xi\}\, d\lambda_{\partial B_\varepsilon^3(x)}$$

$$= \int_{\Omega \cap B_\varepsilon^3(x)^{-\prime}} (r_x^{-2} r_{x,i} \cdot \Gamma_x\{\xi\})_{,i}\, d\lambda_3$$

$$= 0,$$

so

$$\int_{\partial\Omega} r_x^{-2} r_{x,i} \nu^i_{\partial\Omega} \cdot \Gamma_x\{\xi\}\, d\lambda_{\partial\Omega}$$

$$= \int_{\partial B^3_\varepsilon(x)} \frac{1}{r_x^2\{(\xi^\ell r_{x,\ell})^2 + (1-|\xi|_3^2)\}^{3/2}}\, d\lambda_{\partial B^3_\varepsilon(x)} . \tag{5}$$

For the evaluation of the integral on the right in (5), it clearly suffices to assume that $x = 0$ and $\xi = |\xi|_3 e_3^{(3)}$, as the use of an appropriate affine isometry on $\mathbb{R}^3$ would show. To compute the integral in that case, we shall use a spherical coordinate function: let S_ε: $(0,\pi)\times(0,2\pi) \to \partial B^3_\varepsilon(0)$ be given by

$$S_\varepsilon(\omega^1,\omega^2) := (\varepsilon \sin \omega^1 \cos \omega^2, \varepsilon \sin \omega^1 \sin \omega^2, \varepsilon \cos \omega^1),$$
$$\text{for} \quad 0 < \omega^1 < \pi, \quad 0 < \omega^2 < 2\pi. \tag{6}$$

Then S_ε maps $(0,\pi)\times(0,2\pi)$ onto $\partial B^3_\varepsilon(0)\cap N'$, where N is a closed subset of $\partial B^3_\varepsilon(0)$ of $\lambda_{\partial B^3_\varepsilon(0)}$-measure zero, and is injective. Setting $\Theta_\varepsilon := S_\varepsilon^{-1}$: $\partial B^3_\varepsilon(0)\cap N' \to (0,\pi)\times(0,2\pi)$, it is easy to check that $(\partial B^3_\varepsilon(0)\cap N', \Theta_\varepsilon)$ is a coordinate system in $\partial B^3_\varepsilon(0)$, which can be used for the computation of $\int_{\partial B^3_\varepsilon(0)} f\, d\lambda_{\partial B^3_\varepsilon(0)}$ whenever $f \in L_1(\partial B^3_\varepsilon(0))$, since N has measure zero. Routine calculations give $JS_\varepsilon(\omega^1,\omega^2) = \varepsilon^2 \sin \omega^1$ and $(\xi^\ell r_{x,\ell})\circ S_\varepsilon(\omega^1,\omega^2) = |\xi|_3 \cos \omega^1$, for $0 < \omega^1 < \pi$ and $0 < \omega^2 < 2\pi$, the latter when $x = 0$ and $\xi = |\xi|_3 e_3^{(3)}$, as we are supposing. Thus, the integral on the right-hand side of (5) is just

$$\int_0^{2\pi}\int_0^{\pi} \frac{\sin\omega^1}{\{|\xi|_3^2\cos^2\omega^1+1-|\xi|_3^2\}^{3/2}}\,d\omega^1 d\omega^2$$
$$= 2\pi\int_0^{\pi} \frac{\sin\omega^1}{\{|\xi|_3^2\cos^2\omega^1+1-|\xi|_3^2\}^{3/2}}\,d\omega^1. \tag{7}$$

The integral appearing on the right in (7) is completely elementary; if $|\xi|_3 = 0$, it is just $\int_0^{\pi}\sin\omega^1\,d\omega^1 = 2$, while if $0 < |\xi|_3 < 1$, it can be rewritten as

$$\frac{2}{|\xi|_3\,(1-|\xi|_3^2)}\int_0^{|\xi|_3\cdot(1-|\xi|_3^2)^{-1/2}} \frac{du}{(1+u^2)^{3/2}},$$

the value of which is easily found to be $2(1-|\xi|_3^2)^{-1}$. It follows that (2) is correct when $x \in \Omega$.

(iii) Finally, we assume that $x \in \partial\Omega$, the most difficult case to analyze: let (a,α,d) be a set of Lyapunov constants for Ω. The function $r_x^{-2}r_{x,i}\nu^i_{\partial\Omega}\cdot\Gamma_x\{\xi\}$ is continuous on $\partial\Omega\cap\{x\}'$ and we have, by [VI.66.iii.4], the estimate

$$|r_x^{-2}(y)\cdot r_{x,i}(y)\nu^i_{\partial\Omega}(y)\cdot\Gamma_x\{\xi\}(y)| \le (1-|\xi|_3^2)^{-3/2}r_x^{-2}(y)\cdot|r_{x,i}(y)\cdot\nu^i_{\partial\Omega}(y)|$$
$$< (1-|\xi|_3^2)^{-3/2}\cdot\hat{a}r_x^{-(2-\alpha)}(y), \tag{8}$$

$$\text{for} \quad y \in \partial\Omega\cap B_d^3(x)\cap\{x\}';$$

since $\partial\Omega\cap B_d^3(x)'$ is compact ($\partial\Omega$ is compact), there then exists a positive k for which

$$|r_x^{-2} r_{x,i} \nu^i_{\partial\Omega} \cdot \Gamma_x\{\xi\}| \le k r_x^{-(2-\alpha)} \quad \text{on} \quad \partial\Omega\cap\{x\}'. \tag{9}$$

The measurability of $r_x^{-2} r_{x,i} \nu^i_{\partial\Omega} \cdot \Gamma_x\{\xi\}$ on $\partial\Omega\cap\{x\}'$ being obvious, it is easy to see that (9) implies that $r_x^{-2} r_{x,i} \nu^i_{\partial\Omega} \cdot \Gamma_x\{\xi\} \in L_1(\partial\Omega)$ (cf., also, [IV.19]). Thus, we can assert that (cf., [I.2.39])

$$\begin{aligned} &\int_{\partial\Omega} r_x^{-2} r_{x,i} \nu^i_{\partial\Omega} \cdot \Gamma_x\{\xi\}\, d\lambda_{\partial\Omega} \\ &= \lim_{\varepsilon \to 0^+} \int_{\partial\Omega\cap B^3_\varepsilon(x)'} r_x^{-2} r_{x,i} \nu^i_{\partial\Omega} \cdot \Gamma_x\{\xi\}\, d\lambda_{\partial\Omega} . \end{aligned} \tag{10}$$

For $0 < \varepsilon < d$, let us apply the divergence theorem in the normal domain $\Omega\cap B^3_\varepsilon(x)^{-\prime}$: in view of (3), we find

$$\begin{aligned} &\int_{\partial\Omega\cap B^3_\varepsilon(x)'} r_x^{-2} r_{x,i} \nu^i_{\partial\Omega} \cdot \Gamma_x\{\xi\}\, d\lambda_{\partial\Omega} \\ &+ \int_{\partial B^3_\varepsilon(x)\cap\Omega} r_x^{-2} r_{x,i} (-r_{x,i}) \cdot \Gamma_x\{\xi\}\, d\lambda_{\partial B^3_\varepsilon(x)} \\ &= \int_{\Omega\cap B^3_\varepsilon(x)^{-\prime}} (r_x^{-2} r_{x,i} \cdot \Gamma_x\{\xi\})_{,i}\, d\lambda_3 \\ &= 0, \end{aligned}$$

so, from (10),

$$\int_{\partial\Omega} r_x^{-2} r_{x,i} \nu^i_{\partial\Omega} \cdot \Gamma_x\{\xi\}\, d\lambda_{\partial\Omega} = \lim_{\varepsilon \to 0^+} \int_{\partial B^3_\varepsilon(x)\cap\Omega} r_x^{-2} \cdot \Gamma_x\{\xi\}\, d\lambda_{\partial B^3_\varepsilon(x)} . \tag{11}$$

We must now evaluate the limit appearing on the right in (11); for this, it suffices to evaluate

$$\lim_{\varepsilon \to 0^+} \int_{\partial B_\varepsilon^3(0)\cap\tilde{\Omega}} r_0^{-2}\Gamma_0\{\xi\}\, d\lambda_{\partial B_\varepsilon^3(0)}, \tag{12}$$

where $\tilde{\Omega} \subset \mathbb{R}^3$ is a Lyapunov domain, $0 \in \partial\tilde{\Omega}$, $T_{\partial\tilde{\Omega}}(0) = \{y \in \mathbb{R}^3 \mid y^3 = 0\}$, and $\nu_{\partial\tilde{\Omega}}(0) = e_3^{(3)}$, since the general case can be reduced to this by employing the affine isometry $\mathcal{K}_x$ introduced in [VI.62], or, more precisely, its restrictions to the spheres $\partial B_\varepsilon^3(x)$, to replace the integrations appearing on the right in (11) by corresponding integrals over $\mathcal{K}_x\{\partial B_\varepsilon^3(x)\cap\Omega\} = \partial B_\varepsilon^3(0)\cap\mathcal{K}_x\{\Omega\}$. In fact, from [VI.52] (cf., also, [I.2.26.a]),

$$\int_{\partial B_\varepsilon^3(x)\cap\Omega} r_x^{-2}\cdot\Gamma_x\{\xi\}\, d\lambda_{\partial B_\varepsilon^3(x)}$$

$$= \int_{\mathcal{K}_x(\partial B_\varepsilon^3(x)\cap\Omega)} (r_x^{-2}\cdot\Gamma_x\{\xi\})\circ\mathcal{K}_x^{-1}\cdot J\mathcal{K}_{x\varepsilon}^{-1}\, d\lambda_{\partial B_\varepsilon^3(0)} \tag{13}$$

$$= \int_{\partial B_\varepsilon^3(0)\cap\mathcal{K}_x(\Omega)} r_0^{-2}\cdot\Gamma_0\{A_x\xi\}\, d\lambda_{\partial B_\varepsilon^3(0)},$$

in which $\mathcal{K}_{x\varepsilon} := \mathcal{K}_x|\ \partial B_\varepsilon^3(x)\colon\ \partial B_\varepsilon^3(x) \to \mathbb{R}^3$ is an ∞-imbedding taking $\partial B_\varepsilon^3(x)$ onto $\partial B_\varepsilon^3(0)$, and A_x is the linear isometry defined in [VI.62.iii], the second equality following from a simple calculation taking into account the properties of $\mathcal{K}_x^{-1}$ and A_x, and the easily verified fact that $\mathcal{J}\mathcal{K}_{x\varepsilon}^{-1} = 1$ on $\partial B_\varepsilon^3(0)$ for $\varepsilon > 0$. We do have $0 \in \partial\{\mathcal{K}_x(\Omega)\}$, $T_{\partial\{\mathcal{K}_x(\Omega)\}}(0) = \{y \in \mathbb{R}^3 |\ y^3 = 0\}$, and $\nu_{\partial\{\mathcal{K}_x(\Omega)\}}(0) = e_3^{(3)}$, while $\mathcal{K}_x(\Omega)$ is a Lyapunov domain, any set of Lyapunov constants for Ω also being a set of Lyapunov constants for $\mathcal{K}_x(\Omega)$. Once we have verified that the limit, as $\varepsilon \to 0^+$, of the last integral in (13) depends only upon $|A_x\xi|_3$, the sufficiency of the simplifying assumptions shall become evident, since $|\xi|_3 = |A_x\xi|_3$.

Let us then consider (12), under the hypotheses listed. We intend to show that

$$\lim_{\varepsilon \to 0^+} \left\{ \int_{\partial B_\varepsilon^3(0)\cap\tilde{\Omega}} r_0^{-2}\Gamma_0\{\xi\}\, d\lambda_{\partial B_\varepsilon^3(0)} - \int_{\partial B_\varepsilon^3(0)\cap\mathbb{R}_-^3} r_0^{-2}\Gamma_0\{\xi\}\, d\lambda_{\partial B_\varepsilon^3(0)} \right\} = 0, \tag{14}$$

and

$$\int_{\partial B_\varepsilon^3(0)\cap\mathbb{R}_-^3} r_0^{-2}\Gamma_0\{\xi\}\, d\lambda_{\partial B_\varepsilon^3(0)} = 2\pi(1-|\xi|_3^2)^{-1}, \quad \text{for each } \varepsilon > 0, \tag{15}$$

wherein

$$\mathbb{R}_-^3 := \{y \in \mathbb{R}^3 |\ y^3 < 0\}. \tag{16}$$

Clearly, from (14) and (15) it shall follow that the limit in (12) is also $2\pi(1-|\xi|_3^2)^{-1}$, hence, in particular, depends only on $|\xi|_3$. By the reasoning outlined above, we shall be able then to conclude immediately that the limit in (11) is $2\pi(1-|\xi|_3^2)^{-1}$, with which the proof of (2) shall be complete for this final case.

To verify (14), write

$$\mathbb{R}^3_+ := \{y \in \mathbb{R}^3 | \quad y^3 > 0\}, \tag{17}$$

$$\tilde{\Omega}_{\varepsilon+} := \tilde{\Omega} \cap \partial B^3_\varepsilon(0) \cap \mathbb{R}^3_+, \quad \text{for} \quad 0 < \varepsilon < d, \tag{18}$$

and, for $0 < \varepsilon < d$,

$$\tilde{\Omega}_{\varepsilon-} := \{\partial B^3_\varepsilon(0) \cap \mathbb{R}^3_-\} \cap \{\tilde{\Omega} \cap \partial B^3_\varepsilon(0) \cap \mathbb{R}^3_-\}' = \tilde{\Omega}' \cap \partial B^3_\varepsilon(0) \cap \mathbb{R}^3_-. \tag{19}$$

Noting that $\partial B^3_\varepsilon(0) \cap \tilde{\Omega}$ is the union of $\tilde{\Omega} \cap \partial B^3_\varepsilon(0) \cap \mathbb{R}^3_-$, $\tilde{\Omega} \cap \partial B^3_\varepsilon(0) \cap \mathbb{R}^3_+$, and a set of $\lambda_{\partial B^3_\varepsilon(0)}$-measure zero, we have

$$\begin{aligned}
&\left| \int_{\partial B^3_\varepsilon(0) \cap \mathbb{R}^3_-} r_0^{-2} \Gamma_0\{\xi\} \, d\lambda_{\partial B^3_\varepsilon(0)} - \int_{\partial B^3_\varepsilon(0) \cap \tilde{\Omega}} r_0^{-2} \Gamma_0\{\xi\} \, d\lambda_{\partial B^3_\varepsilon(0)} \right| \\
&= \frac{1}{\varepsilon^2} \left| \int_{\tilde{\Omega}_{\varepsilon-}} \Gamma_0\{\xi\} \, d\lambda_{\partial B^3_\varepsilon(0)} - \int_{\tilde{\Omega}_{\varepsilon+}} \Gamma_0\{\xi\} \, d\lambda_{\partial B^3_\varepsilon(0)} \right| \\
&\leq \frac{1}{\varepsilon^2} \left\{ \int_{\tilde{\Omega}_{\varepsilon-}} \Gamma_0\{\xi\} \, d\lambda_{\partial B^3_\varepsilon(0)} + \int_{\tilde{\Omega}_{\varepsilon+}} \Gamma_0\{\xi\} \, d\lambda_{\partial B^3_\varepsilon(0)} \right\} \\
&\leq \frac{1}{\varepsilon^2} \cdot (1-|\xi|_3^2)^{-3/2} \cdot \int_{\tilde{\Omega}_{\varepsilon-} \cup \tilde{\Omega}_{\varepsilon+}} d\lambda_{\partial B^3_\varepsilon(0)} ,
\end{aligned} \tag{20}$$

having applied the inequalities in (4) (with $x = 0$). To estimate the integral in (20), we shall use the spherical coordinate system $(\partial B^3_\varepsilon(0) \cap N', \Theta_\varepsilon)$, introduced in part (ii). Note that the setting here coincides with that considered in Lemma [VI.64] and Remark [VI.65], so that we have available the facts proven there. Let (a, α, d) be a set of Lyapunov constants for $\tilde{\Omega}$. π_0 denotes the orthogonal projection $y \mapsto (y^1,y^2,0)$ taking $\mathbb{R}^3$ onto $T_{\partial\tilde{\Omega}}(0)$, and $\Pi_0 := \pi_0|(\partial\tilde{\Omega}\cap B_d^3(0))$. We assume now that ε is any number in the interval $(0,(7/9)d)$. Let us first study the case in which $y \in \tilde{\Omega}_{\varepsilon+}\cap N'$: then $r_0(\pi_0(y)) < r_0(y) = \varepsilon < (7/9)d$, so $\pi_0(y) \in \Pi_0(\partial\tilde{\Omega}\cap B_d^3(0))$ by [VI.64.ii], whence (y^1,y^2) is in the domain $\mathcal{D}_0$ of the function f introduced in [VI.64]. Since we also have $y \in \tilde{\Omega}$, (VI.65.3) shows that $y^3 < f(y^1,y^2)$, with which the estimate (VI.64.10) yields

$$\begin{aligned}
\cos \Theta_\varepsilon^1(y) &= \frac{1}{\varepsilon}\, y^3 \\
&< \frac{1}{\varepsilon}\, f(y^1,y^2) \\
&\leq \frac{1}{\varepsilon}\, \tilde{a} r_0^{1+\alpha}(\Pi_0(y^1,y^2,f(y^1,y^2))) \\
&= \frac{1}{\varepsilon}\, \tilde{a} r_0^{1+\alpha}(\pi_0(y)) \\
&< \frac{1}{\varepsilon}\, \tilde{a}\varepsilon^{1+\alpha} \\
&= \tilde{a}\varepsilon^{\alpha}.
\end{aligned}$$

Thus,

$$\sup_{y \in \tilde{\Omega}_{\varepsilon+}\cap N'} \cos \Theta_\varepsilon^1(y) \leq \tilde{a}\varepsilon^{\alpha}. \tag{21}$$

Suppose next that $y \in \tilde{\Omega}_{\varepsilon-}\cap N'$: again we have $r_0(\pi_0(y)) < \varepsilon < 7/9\, d$, so $(y^1,y^2) \in \mathcal{D}_0$. Now, $y \in \tilde{\Omega}'$, by (19), so $y \in \tilde{\Omega}^{-\prime}\cup\partial\tilde{\Omega}$, and (VI.65.3) implies that $y^3 \geq f(y^1,y^2)$. Again using (VI.64.10),

$$\cos \Theta_\varepsilon^1(y) = \frac{1}{\varepsilon}\, y^3$$

$$\geq \frac{1}{\varepsilon} f(y^1,y^2)$$

$$\geq -\frac{1}{\varepsilon} |f(y^1,y^2)|$$

$$\geq -\frac{1}{\varepsilon} \tilde{a} r_0^{1+\alpha}(\pi_0(y))$$

$$> -\frac{1}{\varepsilon} \tilde{a} \varepsilon^{1+\alpha}$$

$$= -\tilde{a}\varepsilon^{\alpha}.$$

Consequently,

$$\inf_{y \in \tilde{\Omega}_{\varepsilon-} \cap N'} \cos \Theta_{\varepsilon}^1(y) \geq -\tilde{a}\varepsilon^{\alpha}. \tag{22}$$

Now, set

$$\theta_{\varepsilon+} := \inf_{y \in \tilde{\Omega}_{\varepsilon+} \cap N'} \Theta_{\varepsilon}^1(y), \tag{23}$$

$$\theta_{\varepsilon-} := \sup_{y \in \tilde{\Omega}_{\varepsilon-} \cap N'} \Theta_{\varepsilon}^1(y). \tag{24}$$

Since cos is continuous and strictly decreasing on $[0,\pi]$, we see that

$$\cos \theta_{\varepsilon+} = \sup_{y \in \tilde{\Omega}_{\varepsilon+} \cap N'} \cos \Theta_{\varepsilon}^1(y) \leq \tilde{a}\varepsilon^{\alpha}, \tag{25}$$

$$\cos \theta_{\varepsilon-} = \inf_{y \in \tilde{\Omega}_{\varepsilon-} \cap N'} \cos \Theta_{\varepsilon}^1(y) \geq -\tilde{a}\varepsilon^{\alpha}, \tag{26}$$

by (21) and (22). Returning to the expression on the right in (20), we can clearly write, using (25) and (26),

$$\frac{1}{\varepsilon^2}(1-|\xi|_3^2)^{-3/2}\cdot \int_{\tilde\Omega_{\varepsilon-}\cup\tilde\Omega_{\varepsilon+}} d\lambda_{\partial B_\varepsilon^3(0)}$$

$$\le (1-|\xi|_3^2)^{-3/2} \int_0^{2\pi}\int_{\theta_{\varepsilon+}}^{\theta_{\varepsilon-}} \sin\omega^1\, d\omega^1\, d\omega^2 \tag{27}$$

$$= 2\pi(1-|\xi|_3^2)^{-3/2}\cdot\{\cos\theta_{\varepsilon+} - \cos\theta_{\varepsilon-}\}$$

$$\le 4\pi\tilde a(1-|\xi|_3^2)^{-3/2}\varepsilon^\alpha .$$

With (20), the estimate (27), holding whenever $0 < \varepsilon < (7/9)d$, implies (14).

Next, choose any $\varepsilon > 0$ and consider the integral on the left in (15): using the *spherical coordinate function*,

$$\int_{\partial B_\varepsilon^3(0)\cap\mathbb{R}^3_-} r_0^{-2}\Gamma_0\{\xi\}\, d\lambda_{\partial B_\varepsilon^3(0)}$$

$$= \int_{\partial B_\varepsilon^3(0)\cap\mathbb{R}^3_-} \frac{1}{r_0^2\{(\xi^\ell r_{0,\ell})^2+1-|\xi|_3^2\}^{3/2}}\, d\lambda_{\partial B_\varepsilon^3(0)} \tag{28}$$

$$= \int_{(\pi/2,\ \pi)\times(0,2\pi)} \frac{\sin\omega^1}{\{(\xi^\ell r_{0,\ell}\circ S_\varepsilon(\omega^1,\omega^2))^2+1-|\xi|_3^2\}^{3/2}}\, d\lambda_2(\omega^1,\omega^2).$$

It is no restriction to suppose, as we shall, that $\xi^2 = 0$ and $\xi^1 \ge 0$. Setting, say, $\theta := 0$ if $\xi = 0$, and

$$\theta := \cos^{-1}(\xi^3/|\xi|_3) \quad \text{if} \quad \xi \ne 0, \tag{29}$$

we then have

$$\xi = |\xi|_3(\sin\theta\, e_1^{(3)} + \cos\theta\, e_3^{(3)}).$$

Obviously,

$$\operatorname{grad} r_0(S_\varepsilon(\omega^1,\omega^2)) = \frac{1}{\varepsilon}\, S_\varepsilon(\omega^1,\omega^2)$$

$$= \sin\omega^1 \cos\omega^2\, e_1^{(3)} + \sin\omega^1 \sin\omega^2\, e_2^{(3)} + \cos\omega^1\, e_3^{(3)},$$

so

$$\xi^\ell r_{0,\ell}\circ S_\varepsilon(\omega^1,\omega^2) = |\xi|_3(\sin\theta \sin\omega^1 \cos\omega^2 + \cos\theta\cos\omega^1),$$

$$\text{for}\quad 0 < \omega^1 < \pi,\qquad 0 < \omega^2 < 2\pi.$$

Denoting, for brevity, the integral in (28) by I, we can now write, more explicitly,

$$I = (1-|\xi|_3^2)^{-3/2} \int_{(\pi/2,\ \pi)\times(0,2\pi)} \frac{\sin\omega^1}{\{1+\gamma^2(\cos\theta\cos\omega^1 + \sin\theta\sin\omega^1\cos\omega^2)^2\}^{3/2}}\, d\lambda_2(\omega^1,\omega^2),$$

where

$$\gamma := |\xi|_3\cdot(1-|\xi|_3^2)^{-1/2}.$$

Let us first rewrite the preceding integral, using the translation invariance of λ_2, as

$$I = (1-|\xi|_3^2)^{-3/2} \left\{ \int_{(\pi/2,\ \pi)\times(0,\pi)} \frac{\sin \omega^1}{\{1+\gamma^2(\cos\theta \cos\omega^1 + \sin\theta \sin\omega^1 \cos\omega^2)^2\}^{3/2}} \, d\lambda_2(\omega^1,\omega^2) \right.$$

$$\left. + \int_{(\pi/2,\ \pi)\times(0,\pi)} \frac{\sin \omega^1}{\{1+\gamma^2(\cos\theta \cos\omega^1 - \sin\theta \sin\omega^1 \cos\omega^2)^2\}^{3/2}} \, d\lambda_2(\omega^1,\omega^2) \right\} . \tag{30}$$

Now, consider the map g_1: $(0,1)\times(\pi/2,\ 3\pi/2) \to \mathbb{R}^2$ given by

$$\left.\begin{aligned} g_1^1(\rho,\omega) &:= \rho \cos \omega \\ g_1^2(\rho,\omega) &:= \rho \sin \omega \end{aligned}\right\} \quad \text{for} \quad 0 < \rho < 1, \quad \pi/2 < \omega < 3\pi/2, \tag{31}$$

and the map g_2: $(\pi/2,\ \pi)\times(0,\pi) \to \mathbb{R}^2$ defined according to

$$\left.\begin{aligned} g_2^1(\omega^1,\omega^2) &:= \cos \omega^1, \\ g_2^2(\omega^1,\omega^2) &:= \sin \omega^1 \cos \omega^2 \end{aligned}\right\} \quad \text{for} \quad \pi/2 < \omega^1 < \pi, \quad 0 < \omega^2 < \pi. \tag{32}$$

It is easy to check that both g_1 and g_2 are injective, with

$$\begin{aligned} g_2((\pi/2,\ \pi)\times(0,\pi)) &= \{y \in \mathbb{R}^2 |\ \ y^1 < 0, \quad |y|_2 < 1\} \\ &= g_1((0,1)\times(\pi/2,\ 3\pi/2)), \end{aligned}$$

$$Jg_1(\rho,\omega) = \rho \quad \text{for} \quad 0 < \rho < 1, \quad \pi/2 < \omega < 3\pi/2, \tag{33}$$

and

$$Jg_2(\omega^1,\omega^2) = \sin^2 \omega^1 \sin \omega^2 \quad \text{for} \quad \pi/2 < \omega^1 < \pi, \quad 0 < \omega^2 < \pi. \tag{34}$$

Consequently, defining $g := g_2^{-1} \circ g_1: \ (0,1)\times(\pi/2, 3\pi/2) \to \mathbb{R}^2$, we see that g is injective, $g((0,1)\times(\pi/2, 3\pi/2)) = (\pi/2, \pi)\times(0,\pi)$, and

$$Jg = \{(Jg_2^{-1})\circ g_1\}\cdot Jg_1 = \frac{Jg_1}{\{(Jg_2)\circ g_2^{-1}\}\circ g_1} = \frac{Jg_1}{(Jg_2)\circ g},$$

whence short computations produce, using (31)-(34),

$$Jg(\rho,\omega) = \frac{\rho}{(1-\rho^2)^{1/2}\cdot \sin g^1(\rho,\omega)}, \tag{35}$$

and

$$\cos\theta \cdot\cos g^1(\rho,\omega) \pm\sin\theta \sin g^1(\rho,\omega) \cdot\cos g^2(\rho,\omega) = \rho \cos(\omega\mp\theta), \tag{36}$$

$$\text{for} \quad 0 < \rho < 1, \quad \pi/2 < \omega < 3\pi/2.$$

Whenever $f: \ (\pi/2, \pi)\times(0,\pi) \to \mathbb{K}$ is Lebesgue-integrable, [VI.52] implies that $f\circ g\cdot|Jg|$ is Lebesgue-integrable on $(0,1)\times(\pi/2, 3\pi/2)$ and

$$\int_{(\pi/2,\ \pi)\times(0,\pi)} f \, d\lambda_2 = \int_{(0,1)\times(\pi/2,\ 3\pi/2)} f\circ g\cdot|Jg| \, d\lambda_2; \tag{37}$$

Fubini's theorem shows that the integral on the right in (37) can be rewritten as an iterated integral, using either order of integration. Applying the latter fact and (37), and accounting for (35) and (36), equality (30) can be recast as

$$I = (1-|\xi|_3^2)^{-3/2}$$

$$\cdot\left\{\int_{\pi/2}^{3\pi/2}\int_0^1 \frac{\rho}{\{1-\rho^2\}^{1/2}\{1+\gamma^2\rho^2\cos^2(\omega-\theta)\}^{3/2}}\,d\lambda_1(\rho)\,d\lambda_1(\omega)\right. \tag{38}$$

$$\left.+\int_{\pi/2}^{3\pi/2}\int_0^1 \frac{\rho}{\{1-\rho^2\}^{1/2}\{1+\gamma^2\rho^2\cos^2(\omega+\theta)\}^{3/2}}\,d\lambda_1(\rho)\,d\lambda_1(\omega)\right\}.$$

To evaluate the integrals in (38), first define

$$I(\beta;\eta) := \int_0^\beta \frac{\rho}{\{1-\rho^2\}^{1/2}\{1+\eta^2\rho^2\}^{3/2}}\,d\lambda_1(\rho),$$

$$\text{for each} \quad \beta \in [0,1], \quad \eta \in \mathbb{R};$$

note that the integrand is nonnegative and continuous on $[0,1)$, so the Lebesgue integral $I(1;\eta)$ is defined for each $\eta \in \mathbb{R}$, while if $\beta \in [0,1)$, $I(\beta;\eta)$ can also be interpreted as a Riemann integral. Choosing a non-decreasing sequence $(\beta_i)_1^\infty$ in $[0,1)$ such that $\beta_i \to 1$, define $f_i^\eta\colon [0,1] \to [0,\infty)$, for each $i \in \mathbb{N}$, by

$$f_i^\eta(\rho) := \begin{cases} \rho\{1-\rho^2\}^{-1/2}\{1+\eta^2\rho^2\}^{-3/2}, & \text{if} \quad 0 \le \rho \le \beta_i, \\ 0 & , \quad \text{if} \quad \beta_i < \rho \le 1, \end{cases}$$

wherein $\eta \in \mathbb{R}$ has been selected. Then $(f_i^\eta)_1^\infty$ is clearly a non-decreasing sequence of nonnegative measurable functions on $[0,1]$, so the B. Levi theorem on monotone convergence (cf., Hewitt and Stromberg [20], Theorem (12.22)) gives

$$\lim_{i \to \infty} \int_0^1 f_i^\eta \, d\lambda_1 = \int_0^1 \left(\lim_{i \to \infty} f_i^\eta \right) d\lambda_1,$$

i.e.,

$$\lim_{i \to \infty} I(\beta_i;\eta) = I(1;\eta). \tag{39}$$

Now, whenever $\beta \in [0,1)$, the elementary change-of-variables formula for Riemann integrals allows us to show easily that

$$I(\beta;\eta) = \frac{1}{1+\eta^2} \left\{ 1 - \left(\frac{1-\beta^2}{1+\eta^2\beta^2} \right)^{1/2} \right\},$$

so, from (39), recalling that $\beta_i \to 1$,

$$\int_0^1 \frac{\rho}{\{1-\rho^2\}^{1/2}\{1+\eta^2\rho^2\}^{3/2}} \, d\lambda_1(\rho) = I(1;\eta) = \frac{1}{1+\eta^2}. \tag{40}$$

Since (40) holds for each $\eta \in \mathbb{R}$, we can use it to reduce the equality (38) to the simpler form

$$\begin{aligned}
I &= (1-|\xi|_3^2)^{-3/2} \left\{ \int_{\pi/2}^{3\pi/2} \frac{1}{1+\gamma^2 \cos^2(\omega-\theta)} \, d\lambda_1(\omega) \right. \\
&\quad \left. + \int_{\pi/2}^{3\pi/2} \frac{1}{1+\gamma^2 \cos^2(\omega+\theta)} \, d\lambda_1(\omega) \right\} \\
&= (1-|\xi|_3^2)^{-1/2} \left\{ \int_{\frac{\pi}{2}-\theta}^{\frac{3\pi}{2}-\theta} \frac{1}{1-|\xi|_3^2 \sin^2 \omega} \, d\lambda_1(\omega) \right. \\
&\quad \left. + \int_{\frac{\pi}{2}+\theta}^{\frac{3\pi}{2}+\theta} \frac{1}{1-|\xi|_3^2 \sin^2 \omega} \, d\lambda_1(\omega) \right\},
\end{aligned} \tag{41}$$

after some simple manipulations, recalling that $\gamma := |\xi|_3(1-|\xi|_3^2)^{-1/2}$. Finally, to compute the values of the integrals appearing in (41), consider the function $\sigma\colon \mathbb{R}\cap\{(2n+1)\frac{\pi}{2} \mid n \in \mathbb{Z}\}' \to \mathbb{R}$ given by

$$\sigma(\zeta) := (1-|\xi|_3^2)^{-1/2} \cdot \tan^{-1}\{(1-|\xi|_3^2)^{1/2} \tan \zeta\}. \tag{42}$$

If $\zeta_n := (2n+1)\frac{\pi}{2}$ for some $n \in \mathbb{Z}$, we find

$$\lim_{\zeta \to \zeta_n^{\pm}} \sigma(\zeta) = \mp \frac{\pi}{2}(1-|\xi|_3^2)^{-1/2}, \tag{43}$$

while if $\zeta \in \mathbb{R}\cap\{(2n+1)\frac{\pi}{2} \mid n \in \mathbb{Z}\}'$, an easy calculation produces

$$\sigma'(\zeta) = (1-|\xi|_3^2 \sin^2 \zeta)^{-1}. \tag{44}$$

Then, suppose that $\alpha, \beta \in \mathbb{R}$ and $\alpha < \beta$: if the open interval (α,β) contains no odd-integral multiple of $\pi/2$, it follows that

$$\int_\alpha^\beta \frac{1}{1-|\xi|_3^2 \sin^2 \omega}\, d\lambda_1(\omega) = \lim_{\omega \to \beta^-} \sigma(\omega) - \lim_{\omega \to \alpha^+} \sigma(\omega); \tag{45}$$

if the open interval (α,β) contains exactly one odd-integral multiple of $\pi/2$, ω_0, then, clearly,

$$\begin{aligned}\int_\alpha^\beta \frac{1}{1-|\xi|_3^2 \sin^2 \omega}\, d\lambda_1(\omega) &= \lim_{\omega \to \beta^-} \sigma(\omega) - \lim_{\omega \to \alpha^+} \sigma(\omega) \\ &\quad + \lim_{\omega \to \omega_0^-} \sigma(\omega) - \lim_{\omega \to \omega_0^+} \sigma(\omega) \\ &= \lim_{\omega \to \beta^-} \sigma(\omega) - \lim_{\omega \to \alpha^+} \sigma(\omega) \\ &\quad + \pi(1-|\xi|_3^2)^{-1/2}.\end{aligned} \tag{46}$$

Recall that $\theta \in [0,\pi]$, by (29); we consider each of the possible positions of θ, as follows:

(a) If $\theta = 0$, then (41) becomes, using (45) and (43),

$$I = (1-|\xi|_3^2)^{-1/2} \cdot 2\left\{ \lim_{\omega \to \frac{3\pi}{2}^-} \sigma(\omega) - \lim_{\omega \to \frac{\pi}{2}^+} \sigma(\omega) \right\} = 2\pi(1-|\xi|_3^2)^{-1}. \quad (47)$$

(b) If $\theta = \pi$, then, each interval $(-\pi/2, \pi/2)$ and $(3\pi/2, 5\pi/2)$ containing no odd-integral multiple of $\pi/2$, just as in case (i), we can again apply (45) and (43) in (41) to obtain

$$\begin{aligned} I = (1-|\xi|_3^2)^{-1/2} \cdot &\left\{ \lim_{\omega \to \frac{\pi}{2}^-} \sigma(\omega) - \lim_{\omega \to -\frac{\pi}{2}^+} \sigma(\omega) \right. \\ &\left. + \lim_{\omega \to \frac{5\pi}{2}^-} \sigma(\omega) - \lim_{\omega \to \frac{3\pi}{2}^+} \sigma(\omega) \right\} \qquad (48) \\ = 2\pi(1-|\xi|_3^2)^{-1}. \end{aligned}$$

(c) If $0 < \theta < \pi$, then it is easy to see that $[\pi/2 -\theta, 3\pi/2 -\theta]$ and $[\pi/2 +\theta, 3\pi/2 +\theta]$ each contain precisely one odd-integral multiple of $\pi/2$ ($\pi/2$ in the former, $3\pi/2$ in the latter), so we apply (46) in (41):

$$\begin{aligned} I = (1-|\xi|_3^2)^{-1} \cdot \{ &\tan^{-1} ((1-|\xi|_3^2)^{1/2} \tan (3\pi/2 -\theta)) \\ &-\tan^{-1} ((1-|\xi|_3^2)^{1/2} \tan (\pi/2 -\theta)) \\ &+\tan^{-1} ((1-|\xi|_3^2)^{1/2} \tan (3\pi/2 +\theta)) \\ &-\tan^{-1} ((1-|\xi|_3^2)^{1/2} \tan (\pi/2 +\theta)) +2\pi \} \end{aligned}$$

$$= 2\pi(1-|\xi|_3^2)^{-1}, \tag{49}$$

since $\tan(3\pi/2 - \theta) = -\tan(3\pi/2 + \theta)$, $\tan(\pi/2 - \theta) = -\tan(\pi/2 + \theta)$, and $\tan^{-1}$ is an odd function on $\mathbb{R}$.

Thus, (15) has been proven.

As noted, the proof of the lemma is now complete. $\square$.

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