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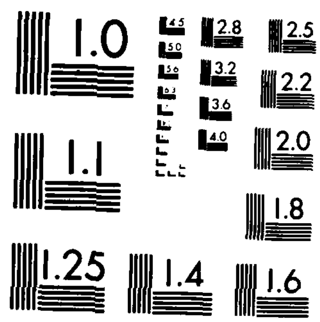

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A REMARK ON SEMILINEAR PERTURBATIONS OF ABSTRACT PARABOLIC EQUATIONS

Michael G. Crandall

Mathematics Research Center
University of Wisconsin—Madison
610 Walnut Street
Madison, Wisconsin 53705

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ABSTRACT

The abstract semilinear evolution problem $u' = Au + f(u)$, $u(0) = \varphi$, is considered in which A generates an analytic semigroup in a Banach space X and $f:X_\alpha \to X$, where $0 < \alpha < 1$ and X_α is the domain of a fractional power $(\lambda I - A)^\alpha$ with the graph topology, satisfies mild regularity conditions. It is shown that the solution $u(t,\varphi)$ is continuous from its domain of definition in $[0,\infty)\times X_\alpha$ into X_1 for $t > 0$, while $tu'(t,\varphi)$ is continous from its domain into X_α for $0 < t$. Higher differentiability of f provides continuity of $t^n u^{(n)}(t,\varphi)$ into X_α for higher n.

AMS (MOS) Subject Classifications: 35K15, 34G20, 47H20, 35K30

Key Words: regularity, semilinear parabolic equations, continuous dependence, initial-value problems, nonlinear evolution

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SIGNIFICANCE AND EXPLANATION

Abstract results are given which apply to semilinear parabolic equations. Simple examples are provided by initial-value problems for semilinear equations like $u_t = \Delta u + f(u)$. As applied to this situation, the results show that for suitable f and $t > 0$ the solution $u(t,\phi)$ of the problem with the initial-value ϕ varies continuously together with its space derivatives as ϕ is varied in a weaker sense - e.g. small perturbations of ϕ in some L^p-norms cause small changes in $u(t,\phi)$ in the space of continuously differentiable functions.

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A REMARK ON SEMILINEAR PERTURBATIONS OF ABSTRACT PARABOLIC EQUATIONS

Michael G. Crandall

I. Introduction

We are interested in the abstract semilinear Cauchy problem (CP)

$$\text{(CP)} \quad \begin{cases} \text{(E)} \quad \dfrac{du}{dt} = Au + f(u) \\ \text{(IC)} \quad u(0) = \varphi \end{cases}$$

consisting of the equation (E) and the initial condition (IC). In (E), A is a linear operator in a Banach space X and f is a nonlinear mapping. We will assume that A satisfies:

(A) A is the infinitesimal generator of an analytic semigroup T(t), $t > 0$, in X.

To describe the conditions on f, let (A) hold and λ be sufficiently large so that the fractional powers $(\lambda I - A)^{\alpha}$, $0 < \alpha$, are defined. Let X_{α} be $D((\lambda I - A)^{\alpha})$ with the graph topology and $B(X_{\alpha}, X)$ denotes the bounded linear maps from X_{α} to X. We will assume that:

$$\text{(f)} \begin{cases} 0 < \alpha < 1,\ f:X_{\alpha} \to X \text{ and there is a mapping } f':X_{\alpha} \to B(X_{\alpha}, X) \text{ such that} \\ \qquad \lim\limits_{\gamma \to 0} \dfrac{f(x+\gamma y) - f(x)}{\gamma} = f'(x)y \quad \text{for } x,\ y \in X_{\alpha} \\ \text{and } X_{\alpha} \times X_{\alpha} \ni (x,y) \to f'(x)y \in X \text{ is continuous.} \end{cases}$$

When (f) holds, then f is locally Lipschitz continuous and for each $\varphi \in X_{\alpha}$

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there is a unique maximally defined function $u(t,\varphi)$, $0 \le t < T_\varphi$ which is continuous in t into X_α and satisfies:

$$(1) \qquad u(t,\varphi) = T(t)\varphi + \int_0^t T(t-s)f(u(s,\varphi))ds, \ 0 \le t < T_\varphi .$$

Let D_E be the domain of existence given by

$$D_E = \{(t,\varphi) : \varphi \in X_\alpha \text{ and } 0 \le t < T_\varphi\}.$$

It is known that T_φ is lower semicontinuous as a function of φ in X_α, so D_E is open in $[0,\infty)\times X_\alpha$. We let $D_E^0 = \{(t,\varphi) \in D_E : t > 0\}$ denote the interior of D_E. We will prove the following simple result:

<u>Theorem 1</u>. Let A and (f) hold, and $\varphi \in X_\alpha$. Then $D_E^0 \ni (t,\varphi) \to u(t,\varphi)$ is continuous from $(0,\infty)\times X_\alpha$ into X_1. Moreover, if $v(t,\varphi) = tu'(t,\varphi)$ for $t > 0$ and $v(0,\varphi) = 0$, then $D_E \ni (t,\varphi) \to v(t,\varphi)$, is continuous into X_α.

Problems of the form (CP) in which (A) and (f) are satisfied arise in many applications. We are assuming the reader is familiar with the basic facts about *semilinear perturbations of generators* of analytic semigroups as exposed in, e.g., [3] or [4] (see also [2]). Theorem 1 is nearly contained in the results presented in [3], but it does not quite seem to fit the precise hypotheses of these results. However, it is a simple variant. We discuss it here because the proof is simple and perhaps of some interest while the statement is relatively clean and perhaps convenient. As a simple example, observe that if $n:\mathbb{R} \to \mathbb{R}$ has a bounded continuous derivative and $n(0) = 0$, then the corresponding substitution operator $N(h)(s) = n(h(s))$ acting in an L^p space, $1 \le p < \infty$, will have the property (f) if $X = L^p$ and $\alpha = 0$, but N may not be Frechet differentiable. If n is as above and $n(0) = 0$, then Theorem 1 applies to the pure Cauchy problem for $u_t = \Delta u + n(u)$ in $\mathbb{R}^m$ in each space $L^p(\mathbb{R}^m)$ for $1 \le p < \infty$, since Δ generates an analytic semigroup in these spaces (see [4]). This is a somewhat more interesting statement than one might make for the corresponding boundary value problem on a bounded domain, since compactness properties are stronger then.

II. The Proof of Theorem 1.

We begin with a lemma. To formulate it let

$$F_T = \{g \in C[0,T:X] \text{ such that } tg(t) \in C^1[0,T:X]\} \quad ,$$

where $C[0,T:X]$ is the space of continuous functions from $[0,T]$ into X and $tg(t) \in C^1[0,T:X]$ means $g \in C^1((0,T]:X)$ and $\lim_{t \downarrow 0} (tg'(t) + g(t)) = \ell$ exists. If also $g \in C[0,T:X]$, then ℓ must be $g(0)$ and $tg'(t) \to 0$ as $t \downarrow 0$.

<u>Lemma 1</u>. Let A satisfy (A), $\varphi \in X$, $T > 0$ and $g \in F_T$. Let

$$(3) \qquad u(t) = T(t)\varphi + \int_0^t T(t-s)g(s)ds = T(t)\varphi + \int_0^t T(s)g(t-s)ds, \quad 0 \le t \le T.$$

Then $u \in F_T$ and

$$(4) \qquad tu'(t) = tAT(t)\varphi + \int_0^t \big(T(t-s)sg'(s) + (T(t-s) + (t-s)AT(t-s))g(s)\big)ds.$$

If also $0 \le \alpha \le 1$ and $\varphi \in X_\alpha$, then $u \in C[0,T:X_\alpha]$ and $tu \in C^1[0,T:X_\alpha]$

<u>Proof</u>. It is clear that $u \in C^1((0,T]:X)$. Set

$$(5) \qquad u_\varepsilon(t) = T(t)\varphi + \int_0^t T(s)g(t-s+\varepsilon)ds, \quad 0 \le t \le T-\varepsilon .$$

Then

$$\begin{aligned} tu_\varepsilon'(t) &= tAT(t)\varphi + tT(t)g(\varepsilon) + \int_0^t T(s)tg'(t-s+\varepsilon)ds \\ (6) \qquad &= tAT(t)\varphi + tT(t)g(\varepsilon) + \int_0^t T(s)(t-s)g'(t-s+\varepsilon)ds \\ &\quad + \int_0^t T(s)sg'(t-s+\varepsilon)ds \quad . \end{aligned}$$

Moreover

$$\begin{aligned} \int_0^t T(s)sg'(t-s+\varepsilon)ds &= \int_0^t (sT(s))\left(-\frac{d}{ds} g(t-s+\varepsilon\right)ds \\ (7) \qquad &= -tT(t)g(\varepsilon) + \int_0^t \left(\frac{d}{ds}(sT(s))\right)g(t-s+\varepsilon)ds \\ &= -tT(t)g(\varepsilon) + \int_0^t (T(s) + sAT(s))g(t-s+\varepsilon)ds \quad . \end{aligned}$$

This calculation is justified by the fact that $(sT(s))' = T(s) + sAT(s)$ is continuous for $s > 0$ and tends strongly to the identity as $s \downarrow 0$ by (A). From (6) and (7) follows

$$
(8) \quad \begin{aligned} tu_\varepsilon'(t) &= tAT(t)\varphi + \int_0^t \big((T(s)(t-s)g'(t-s+\varepsilon) + (T(s)+sAT(s))g(t-s+\varepsilon)\big)ds \\ &= tAT(t)\varphi + \int_0^t \big(T(t-s)sg'(s+\varepsilon) + (T(t-s)+(t-s)AT(t-s))g(s+\varepsilon)\big)ds. \end{aligned}
$$

Since $g(s+\varepsilon) \to g(s)$ and $sg'(s+\varepsilon) \to sg'(s)$ boundedly as $\varepsilon \downarrow 0$, we conclude that (4) holds. The final assertion is immediate from (3), (4).

Remark. Another way to say $g \in F_T$ is to say there is an $h \in C[0,T:X]$ such that

$$g(t) = \frac{1}{t}\int_0^t h(s)ds \text{ for } 0 < t \leq T.$$

If we abreviate this relation by $g = P(h)$, then (4) still holds whenever (3) holds, $g = P(h)$ and both h and $P(\|h\|) \in L^1[0,T:X]$. This can be proved from Lemma 1 and an approximation argument.

To continue with the Theorem, it is enough to work locally. Fix $\varphi_0 \in X$. Then there are numbers $r, a > 0$ such that for $\|\varphi - \varphi_0\|_\alpha \leq r$ (where $\| \ \|_\alpha$ is the norm of X_α) such that the iteration process

$$
(9) \quad \begin{aligned} &u_0(t,\varphi) \equiv \varphi \text{ and} \\ &u_{n+1}(t,\varphi) = T(t)\varphi + \int_0^t T(t-s)f(u_n(s,\varphi))ds \text{ for } n = 0,1,.., \end{aligned}
$$

corresponds to a strict contraction (uniformly for $\|\varphi - \varphi_0\|_\alpha \leq r$) mapping of a suitable subset of $C([0,a]:X_\alpha)$ into itself, and $u_n(t,\varphi) \to u(t,\varphi)$ in $C([0,a]:X_\alpha)$ uniformly in φ. We also choose r and a so that $f'(u_n(t,\varphi))$ remains bounded in $B(X_\alpha,X)$. By (f), (9) and Lemma 1, $u_n \in F_a$ and $v_n = tu_n'$ satisfies

$$v_0 \equiv 0 \quad \text{and}$$

$$(10) \qquad v_{n+1}(t,\varphi) = tAT(t)\varphi + \int_0^t \Big(T(t-s)f'(u_n(s,\varphi))v_n(s,\varphi) + (T(t-s) + (t-s)AT(t-s))f(u_n(s,\varphi))\Big)ds \; .$$

Fix φ, $\|\varphi - \varphi_0\|_\alpha < r$. Using (f), we have that $f'(u_n(s,\varphi)) \to f'(u(s,\varphi))$ strongly and boundedly while $f(u_n(s,\varphi)) \to f(u(s,\varphi))$ in X uniformly on [0,a]. It now follows from (10) that $\{v_n\}$ is convergent in X_α uniformly on [0,a] to a limit v which is the unique solution of

$$(11) \qquad v(t,\varphi) = tAT(t)\varphi + \int_0^t \Big(T(t-s)f'(u(s,\varphi))v(s,\varphi) + (T(t-s) + (t-s)AT(t-s))f(u(s,\varphi))\Big)ds \; .$$

To see this, let K_n denote the operator defined by (10) as it acts on v_n and K be the operator of (11) acting on v. Then K_n and K have a Lipschitz constant uniformly less than one as operators in $C[0,a:X_\alpha]$ if a is small and $K_n v$ tends to Kv, so v_n tends to v as claimed. From this convergence we deduce that $tu'(t,\varphi) = v(t,\varphi)$. But then $u \in F_a$. From (11) and the continuous dependence of u on φ in $C[0,a:X_\alpha]$ follows the continuous dependence on φ of v in $C[0,a:X_\alpha]$ in the standard way. Finally, $Au(t,\varphi) = u'(t,\varphi) - f(u(t,\varphi))$, which shows that u is continuous into X_1. The proof is complete.

Let us indicate how the result goes in the case of higher differentiability by an interesting device. The formula used above can be motivated the following way: If $u' = Au + g$ and (formally) $v = tu'$, then $v' = tu'' + u' = t(Au' + g)' + Au + g = Av + Au + (tg)'$. That is

$$(12) \qquad \begin{aligned} u' &= Au + g \\ v' &= Av + Au + (tg)' \; . \end{aligned}$$

It was remarked in [1] that (A) holds if and only if the operator $\mathbf{A}$ given by

$$D(\mathbf{A}) = D(A)\times D(A), \; \mathbf{A}(x,y) = (Ax, Ax + Ay)$$

is the generator of a strongly continous semigroup $\mathbf{T}(t)$ on $X\times X$. Moreover,

$$\mathbf{T}(t)(x,y) = (T(t)x,\ tAT(t)x + T(t)y) \quad .$$

Thus the variation of parameters formula for (12) leads one to (4) immediately. If $g = f(u)$, then $tf(u)' = f'(u)(tu') = f'(u)v$ and we have (11).

Now consider the case in which f has two derivatives, that is we assume that f satisfies (f) and $f':X_\alpha \to B(X_\alpha,X)$ also has a strong Gateaux derivative $f'':X_\alpha \to B(X_\alpha,B(X_\alpha,X))$, satisfying the continuity requirement

(f') $\qquad X_\alpha\times X_\alpha\times X_\alpha \ni (x,y,z) \to f''(x)(y,z)$ is continuous,

where we are writing $Q(y,z)$ for the value of Qy at z if $Q \in B(X_\alpha,B(X_\alpha,X))$. In this case, we claim that the mapping $(t,\varphi) \to u(t,\varphi)$ is continuous from D_E^0 into X_1 as before and, in addition, $(t,\varphi) \to t^2u''(t,\varphi)$ is defined on D_E^0 and extends to a continuous mapping on D_E into X_α. To proceed, let u and $v = tu'$ be as in the discussion before (12) and put $w = tv' = t^2u'' + tu'$. The equation for w is

$$w' = Aw + 2Av + Au + (t(tg)')$$

which we append to the system (12). The operator $\mathbf{A}$ in X^3 given by

$$\mathbf{A}(x,y,z) = (Ax, Ax + Ay, Ax + 2Ay + Az)$$

on $D(A)^3$ generates a strongly continuous semigroup $\mathbf{T}(t)$ given by

$$\mathbf{T}(t)(x,y,z)=(T(t)x, tAT(t)x+T(t)y, t^2A^2T(t)x+tAT(t)x+2tAT(t)y+T(t)z).$$

Thus if u is the solution of (CP), we deduce from the formal equation for $w = t(tu')'$ and the variation of parameters formula that one should have

$$w(t,\varphi) = tAT(t)\varphi + t^2A^2T(t)\varphi + \int_0^t (3(t-s)A + (t-s)^2A^2)T(t-s)f(u(s,\varphi))ds +$$

$$+ \int_0^t \Big((2(t-s)A+I)T(t-s)f'(u(s,\varphi)v(s,\varphi) +$$

$$+T(t-s)(f''(u(s,\varphi))(v,v)(s,\varphi) + f'(u(s,\varphi)w(s,\varphi))\Big)ds,$$

and the proof proceeds along the previous lines. Likewise, higher differentiability of f provides continuity of $t^n u^{(n)}$ into X_α for higher n.

Remark. Of course, the proofs above show that $v(t,\varphi)$ in the first case and $w(t,\varphi)$ in the second are continuous into X_η for $0 \leq \eta < 1$ if for $t > 0$. This follows form the continuity already proved and the formulas.

Remark. In the situation in which f depends on t, $f:[0,T]\times X_\alpha \to X$, then the results remain true if

$$[0,T]\times X_\alpha \times X_\alpha \ni \to (f'(t,x)y,\ f_t(t,x)),$$

with the obvious meaning, is continuous.

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