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MULTIPLE LIMIT POINT BIFURCATION FOR FREDHOLM OPERATORS OF NON-ZERO INDEX

R. R. Huilgol

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ABSTRACT

Using the method of embedding a given equation into a 'higher dimensional problem', examples of multiple limit point solutions emanating from a single limit point are constructed within the context of algebraic and ordinary differential equations. In both instances, the linearized problem around the limit point is assumed to lead to non-self-adjoint operators.

AMS (MOS) Subject Classifications: 58E07, 76A05 Key Words: Fredholm operators, non-zero index, bifurcation, limit point. Work Unit Number 1 - Applied Analysis

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SIGNIFICANCE AND EXPLANATION

In order to solve equations of the form

$$G(u,\lambda) = 0$$

at a limit point where $u = u(\lambda)$ ceases to be a single valued function of λ , and $du/d\lambda$ becomes unbounded, it has been a common practice in recent years to imbed the equation $G(u,\lambda) = 0$ into a 'higher dimensional' problem so that the Jacobian of the enlarged system is non-singular at the limit point. This method permits one to seek single valued solutions for $u = u(\varepsilon)$ and $\lambda = \lambda(\varepsilon)$ in terms of a new parameter ε . Despite this, it is common to speak of the solution curve passing through the limit point as exhibiting limit point bifurcation, because u is a double-valued function of λ .

Limit point bifurcations arise in numerous areas of mechanics, such as the buckling of a shallow clamped shell under uniform pressure, the flow of a non-Newtonian fluid in between rotating disks and in chemical reactor problems.

In the present paper, the possibility of two curves passing through a single limit point is discussed and is shown to depend on the dimension of the null space of the Jacobian of $G(u,\lambda)$ at the limit point, as well as the dimension of the null space of the adjoint of the Jacobian. With this method it becomes possible to study the existence of multiple limit point bifurcations in mechanics and engineering. Finally, because the theory is developed to be applicable to abstract equations, it is cast in terms of the 'Jacobian' being a Fredholm operator and its index is non-zero because the dimensions of the null spaces of the operator and its adjoint are not equal.

The responsibility for the wording and views expressed in this descriptive summary lies with MRC, and not with the author of this report.

MULTIPLE LIMIT POINT BIFURCATION FOR FREDHOLM OPERATORS OF NON-ZERO INDEX

1. INTRODUCTION

Let X and Y be real Banach spaces and $\lambda \in R$ be a scalar. Consider a map G : X × R + Y and let us seek solutions of

$$G(u,\lambda) = 0 \tag{1}$$

in the vicinity of a known singular solution $(u^{\circ}, \lambda^{\circ}) \in X \times R$. Assuming G to be sufficiently smooth, we shall seek a solution "arc" $(u(\varepsilon), \lambda(\varepsilon))$, with $(u(0), \lambda(0)) \equiv (u^{\circ}, \lambda^{\circ})$, depending on ε in $|\varepsilon| < \varepsilon_{o}$, in the neighborhood of $(u^{\circ}, \lambda^{\circ})$.

If we expand $G(u,\lambda)$ about (u^{0},λ^{0}) we find that

$$G(u,\lambda) = G_{u}^{o}(u - u^{o}) + G^{o}(\lambda - \lambda^{o}) + \frac{1}{2}G_{uu}^{o}(u - u^{o})(u - u^{o}) + G_{u\lambda}^{o}(u - u^{o})(\lambda - \lambda^{o}) + \frac{1}{2}G_{\lambda\lambda}^{o}(\lambda - \lambda^{o})(\lambda - \lambda^{o}) + \text{higher order terms}$$

$$(2)$$

In (2), $G_u^o = \frac{\partial G}{\partial u} (u^o, \lambda^o)$, and the other operators are similarly defined.

Let G_u^o be a Fredholm operator¹ with a null space $N(G_u^o)$, of dimension $m \ge 2$, and G_λ^o be such that it does not lie in the range $R(G_u^o)$ of G_u^o , i.e.,

$$\mathbf{g}_{\lambda}^{\mathbf{o}} \notin \mathbf{R}(\mathbf{g}_{\mathbf{u}}^{\mathbf{o}})$$
 . (3.)

Now, because G_U^O is Fredholm,

$$X = N(G_n^0) \oplus X_1 , \qquad (4)$$

$$\mathbf{Y} = \mathcal{R}(\mathbf{G}_{11}^{\mathbf{O}}) \oplus \mathbf{Y}_{1} \quad , \tag{5}$$

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If G_{U}^{O} is bounded, the argument that follows is unaffected. If G_{U}^{O} is unbounded, its domain which is dense in X can be turned into a Banach space via the graph norm, and one can proceed with requisite modifications. Details are left to the reader.

where Y_1 has the same dimension as that of the null space of the adjoint $(G_u^0)^*$ of G_u^0 . If we seek small solutions to $G(u,\lambda) = 0$ when $G(u,\lambda)$ has the expansion (2), it is clear that we are led to:

$$(u - u^{\circ})(\varepsilon) = \varepsilon \phi + \varepsilon^2 w(\varepsilon)$$
, (6)

$$(\lambda - \lambda^{\circ})(\varepsilon) = \varepsilon^{2} \zeta(\varepsilon)$$
, (7)

where $\phi \in N(G_u^{\circ})$, $w \in X_1$, $\zeta \in \mathbb{R}$. As (7) implies, on every solution arc bifurcating from $(u^{\circ}, \lambda^{\circ})$, $d\lambda/d\epsilon = 0$ at $\epsilon = 0$ [1,2].

Following Keller [1], and Decker and Keller [2], Huilgol [3] showed that the solutions (6) - (7) are of the "multiple limit point bifurcation" type provided G_{λ}^{O} is bounded and $R(G_{\mu}^{O}) \in R(G_{\lambda}^{O}) = Y$, (8)

i.e.,

$$R(G_{\lambda}^{O}) = Y_{1} \qquad (9)$$

Now, a glance at equations (2) - (3) will show that because G_{λ}^{o} : R + Y₁, one has dim Y₁ = 1; moreover, dim Y₁ = dim $N((G_{u}^{o})^{+})$ and thus the Fredholm index of G_{u}^{o} is m - 1 \neq 0, since m > 2.

In order to establish (6) - (7), one can fix $\phi \in N(G_{U}^{O})$ and define a new map $g: \mathbb{R} \times X_{1} \times \mathbb{R} + Y$ through

Clearly, g(0,0,0) = 0 because $\phi \in N(G_n^{\circ})$.

Since (8) holds, by the implicit function theorem [4], the equation $g(\varepsilon, v, \xi) = 0$ has a unique solution $v = v(\varepsilon)$, $\xi = \xi(\varepsilon)$, for $|\varepsilon| < \varepsilon_o$, such that v(0) = 0, $\xi(0) = 0$. Using the smoothness of G, one can write $v(\varepsilon) = \varepsilon w(\varepsilon)$, $\xi(\varepsilon) = \varepsilon \zeta(\varepsilon)$ and recover (6) -(7).

Now, there are a number of situations as in Hopf bifurcation (Weber [5]), or in the example below when it is computationally attractive to add an extra condition to (10) - (11). For instance, if $X = \mathbb{R}^p$, we know that we can choose v to be orthogonal to ϕ .

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More generally, if G_u^O is Fredholm then there is always a linear operator (e.g., a projection operator) $C^* : X + R^m$ such that $C^* : X_1 + \{0\}$ and C^* is an isomorphism between $N(G_u^O)$ and R^m , since dim $N(G_u^O) \sim m$.

To incorporate such a C^* , it is essential to seek the solution of $G(u,\lambda) = 0$ about (u^0,λ^0) by imbedding the original problem into a "higher dimensional" one. That is to say, we solve a second equation $g_1(\varepsilon,v,\xi) = 0$, where $g_1 : \mathbb{R} \times X_1 \times \mathbb{R} + Y$, and g_1 is defined, for a fixed $\phi \in N(G_u^0)$, through

$$\begin{cases} \frac{1}{\varepsilon} G(u^{0} + \varepsilon \phi + \varepsilon v, \lambda^{0} + \varepsilon \xi), \varepsilon \neq 0 , \qquad (12) \end{cases}$$

$$\left(C^{*}v, \varepsilon \neq 0, \right)$$
(13)

$$g_{1}(\varepsilon, v, \xi) = \begin{cases} \\ \int G_{u}^{0}(\phi + v) + G_{\lambda}^{0}\xi, \ \varepsilon = 0 \end{cases},$$
(14)

$$\begin{cases} c^* v, \varepsilon = 0 \end{cases} . \tag{15}$$

Clearly, $g_1(0,0,0) = 0$ and in order to employ the implicit function theorem so that $g_1(\varepsilon, v, \xi) = 0$ has a unique solution $(v(\varepsilon), \xi(\varepsilon))$ with v(0) = 0, $\xi(0) = 0$, one needs to establish that the "linear operator"

$$A \equiv \begin{bmatrix} G_{u}^{o} & G_{\lambda}^{o} \\ \\ G^{*} & 0 \end{bmatrix}$$
(16)

is non-singular. Note that A arises because one can write (14) - (15) symbolically as:

$$g_{1}(0,v,\xi) = A\begin{bmatrix}v\\\xi\end{bmatrix}.$$
 (17)

It was proved in Theorem I of [3] that A is non-singular, provided

- (i) $R(G_u^o) \oplus R(G_\lambda^o) = Y$,
- (ii) $N(c^{*}) = X_{1}, R(c^{*}) = R^{m}$,
- (iii) dim $N(G_u^0) = m$,

(iv) $N(c^*) \cap N(G_{ij}^0) = \{0\}$.

The above idea has obvious extensions to the case where $\lambda \in \mathbb{R}^k$ in (1) - see Theorem II of [3]. This is not pursued here.

Now, on putting $v(\varepsilon) = \varepsilon w(\varepsilon)$, $\xi(\varepsilon) = \varepsilon \zeta(\varepsilon)$ in (6) - (7), we can write $G(u,\lambda) = 0$ in (2) as:

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$$A\begin{bmatrix} \mathbf{w}\\ \mathbf{z}\end{bmatrix} \equiv \begin{bmatrix} \mathbf{G}_{\mathbf{u}}^{\mathbf{o}} & \mathbf{G}_{\boldsymbol{\lambda}}^{\mathbf{o}}\\ \mathbf{c}^{*} & \mathbf{0} \end{bmatrix} \begin{bmatrix} \mathbf{w}\\ \mathbf{z}\end{bmatrix} = -\begin{bmatrix} \frac{1}{2} \mathbf{G}_{\mathbf{u}\mathbf{u}}^{\mathbf{o}}\phi\phi + O(\varepsilon)\\ 0 \end{bmatrix}$$
(18)

where (6) and (7) are used. Since dim $N(G_{u}^{O}) > 2$, the possibility exists that more than one solution pair (w, ζ) can be found by solving (18) when we let ϕ be a different linear combination of $\{\phi^{1}, \ldots, \phi^{m}\}$ which span $N(G_{u}^{O})$.

In this paper, using this idea, we construct two examples of multiple limit point bifurcation, when $\lambda \in R$.

2. ALGEBRAIC EQUATIONS

Firstly, let us consider the equations

$$G_{i}(\underline{u},\lambda) = \sum_{j=1}^{3} \alpha_{ij}(\underline{u}_{j} - \underline{u}_{j}^{0}) + \psi_{i}(\lambda - \lambda^{0}) + \frac{1}{2} \sum_{j,k=1}^{3} \beta_{ijk}(\underline{u}_{j} - \underline{u}_{j}^{0})(\underline{u}_{k} - \underline{u}_{k}^{0})$$
(19)

+ higher order terms = 0, i = 1,2 ,

where a_{i+} is the 2 × 3 matrix:

$$\alpha_{\underline{ij}} = \frac{\partial G_{\underline{i}}}{\partial u_{\underline{j}}} (\underline{\mu}^{\circ}, \lambda^{\circ}) , \qquad (20)$$

 $\lambda \in \mathbb{R}, \psi_i, i = 1, 2$, are the components of the vector ψ spanning $N(\underline{\alpha}^*)$ with $\underline{\alpha}^*$ being the adjoint of $\underline{\alpha}$. Clearly, $\psi \notin R(\underline{\alpha})$. Now, let $\underline{\alpha}$ have a null space of dimension 2, spanned by $\underline{\phi}^1$ and $\underline{\phi}^2$ respectively. Then, dim $N(\underline{\alpha}^*) = 1$. Now we can rewrite (19), by appealing to (18), as:

$$\begin{pmatrix} \alpha_{11} & \alpha_{12} & \alpha_{13} & | & \psi_1 \\ \alpha_{21} & \alpha_{22} & \alpha_{23} & | & \psi_2 \\ & & & & & \\ & & & & & \\ \hline & & & & & \\ \bullet_{1}^{1} & \phi_{2}^{1} & \phi_{3}^{1} & | & 0 \\ \phi_{1}^{2} & \phi_{2}^{2} & \phi_{3}^{2} & | & 0 \\ \end{pmatrix} \begin{pmatrix} w_1 \\ w_2 \\ w_2 \\ w_3 \\ \zeta \end{pmatrix} = - \begin{pmatrix} \frac{1}{2} & \frac{3}{2} & \beta_{1jk} \phi_{j} \phi_{k} + O(\varepsilon) \\ & \frac{1}{2} & \frac{3}{2} & \beta_{2jk} \phi_{j} \phi_{k} + O(\varepsilon) \\ & & & \\ 0 \\ & & & 0 \\ & & & 0 \\ \end{pmatrix}$$
 (21)

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In (21), on the left side, we have employed the fact that $\frac{1}{2}^{1}$ and $\frac{1}{2}^{2}^{2}$ are orthogonal to y. Thus the lower 2 × 3 matrix represents C^{*}. On the right side of (21), we have taken $\frac{1}{2} \in N(\frac{1}{2})$ to stand for $\frac{1}{2}^{1}$ or $\frac{1}{2}^{2}$ or a linear combination of both.

Since rank $\mathfrak{g} = 1$, the row $[\alpha_{11} \alpha_{12} \alpha_{13}]$ is a multiple of $[\alpha_{21} \alpha_{22} \alpha_{23}]$, with a factor Y, say. Provided $\Sigma(\beta_{1jk} - \gamma\beta_{2jk})\phi_j\phi_k) \neq 0$ we get a solution $\zeta(0) \neq 0$. This follows from the fact that rank $\begin{bmatrix} \alpha_{11} \alpha_{12} \alpha_{13} \psi_1 \\ \alpha_{21} \alpha_{22} \alpha_{23} \psi_2 \end{bmatrix} = 2$ and thus $\psi_1 \neq \gamma\psi_2$.

Now we can have two distinct solutions for $\zeta(0)$, leading to multiple limit point bifurcation, provided (say)

$$\int_{j,k=1}^{3} {}^{(\beta}_{1jk} - {}^{\gamma}_{\beta}_{2jk}) \phi_{j}^{1} \phi_{k}^{1} \neq \int_{j,k=1}^{3} {}^{(\beta}_{1jk} - {}^{\gamma}_{\beta}_{2jk}) \phi_{j}^{2} \phi_{k}^{2} .$$
(22)

As a specific example let

$$\alpha_{ij} = \begin{bmatrix} 1 & 1 & 1 \\ 1 & 1 & 1 \\ 1 & 1 & 1 \end{bmatrix} .$$
 (23)

Then N(g) is spanned by:

$$\mathbf{\dot{x}}^{1} = \begin{bmatrix} 1\\ -1\\ 0 \end{bmatrix}, \quad \mathbf{\dot{x}}^{2} = \begin{bmatrix} 1\\ 1\\ -2 \end{bmatrix}, \quad (24)$$

and the domain X of g has the decomposition:

$$X \equiv R^{3} = N(\underline{a}) \quad \oplus \quad X_{1},$$

$$X_{1} = \text{span} \quad \left\{ \begin{bmatrix} 1\\1\\1\\1 \end{bmatrix} \right\} \quad . \tag{25}$$

Similarly, since $g: R^3 + R^2$, the range space Y has the decomposition,

$$\mathbf{Y} \equiv \mathbf{R}^2 = \mathbf{\hat{R}}(\underline{\alpha}) \oplus \mathbf{Y}_1 \quad , \tag{26}$$

where $R(\underline{a})$ and \underline{Y}_1 are described through

$$R(g) = \text{span} \left\{ \begin{bmatrix} 1 \\ 1 \end{bmatrix} \right\}$$
, $Y_1 = \text{span} \left\{ \begin{bmatrix} 1 \\ -1 \end{bmatrix} \right\}$, dim $Y_1 = 1$. (27)

Moreover, $\underline{\psi} = \begin{bmatrix} 1 \\ -1 \end{bmatrix}$ spans $N(\underline{\alpha}^*)$ as well and does not lie in $R(\underline{\alpha})$.

For simplicity, let us choose

 $\begin{cases} \beta_{111} = 1, \ \beta_{213} = 2, \\ \\ \text{other} \ \beta_{1jk} = 0 \end{cases}$ (28)

Then, if we take $\phi = \phi^1$ in (21) we get at $\varepsilon = 0$:

$$\begin{bmatrix} 1 & 1 & 1 & 1 \\ 1 & 1 & 1 & -1 \\ & & & \\ 1 & -1 & 0 & 0 \\ 1 & -1 & -2 & 0 \end{bmatrix} \begin{bmatrix} w_1^0 \\ w_2^0 \\ w_3^0 \\ \zeta^0 \end{bmatrix} = - \begin{bmatrix} \frac{1}{2} \\ 0 \\ 0 \\ 0 \end{bmatrix}$$
 (29)

Now, the 4×4 matrix on the left side is non-singular¹ and we get the unique solutions:

$$\begin{aligned} \rho_1^{o} &= w_2^{o} = w_3^{o} = -\frac{1}{12} \\ & \zeta_1^{o} &= -\frac{1}{4} \end{aligned}$$
 (30)

Thus, in this case, the solution to (19) is of the form

$$(\underline{u} - \underline{u}^{\circ})(\varepsilon) = \varepsilon \underline{t}^{1} + \varepsilon^{2} \underline{u}^{\circ} + \mathcal{O}(\varepsilon^{3}),$$

$$(\lambda - \lambda^{\circ})(\varepsilon) = \varepsilon^{2} \zeta^{\circ} + \mathcal{O}(\varepsilon^{3}),$$
(31)

where w_i^0 , i = 1,2,3, and ζ^0 are given by (30). Had we employed g^2 in (21), we would get a solution set similar to (31), except that g^1 is replaced by g^2 and the new values

The usefulness of incorporating C^* in (12) - (15) becomes clear here. For, without it, we have 2 equations in (19) for w_1 , w_2 , w_3 and ζ . Of course, the first three are interrelated. The operator C^* makes this explicit as well as leading to a square matrix A, which is invertible.

of w^{O} and ζ^{O} are:

$$\begin{cases} w_1^{\circ} = w_2^{\circ} = w_3^{\circ} = \frac{1}{4} \\ \zeta^{\circ} = -\frac{5}{4} \end{cases}$$
(32)

Hence we have constructed a pair of algebraic equations yielding multiple limit point bifurcating solutions.

Now, Decker and Keller [2], while assuming $\lambda \in \mathbb{R}$, require G_u^O to have a Fredholm index zero. Therefore, their theory is inapplicable to the example chosen here since dim $N(\underline{a}) \neq \dim N(\underline{a}^{\bullet})$, while the example follows from the theorem proved in [3].

It seems to be the simplest example of multiple limit point bifurcation, since this type of bifurcation requires [2] dim $N(G_{ij}^{O}) > 2$.

3. A DIFFERENTIAL EQUATION

Once an example such as in (19), (23) - (28) has been constructed, it is clear how to generate others. To be specific, let $u^{0} = u^{0}(x)$ belong to $X \equiv L_{2}(0, 2\pi)$. Now, consider the differential equation for $u \in X$:

$$G(u,\lambda) = \frac{d^2}{dx^2} (u - u^0) + (u - u^0) + (\lambda - 1)\psi +$$

$$+ f(u - u^0, \lambda - 1, \frac{d}{dx} (u - u^0)) = 0 \quad .$$
(33)

In (33), $f(0,0,0) \approx 0$ and f is not linear in its arguments and is as smooth as one desires. The boundary conditions associated with (33) are taken to be:

$$u(0) = u(2\pi)$$
 (34)

Clearly, $(u^{0}, 1)$ is a solution of (33) ~ (34) if $u^{0} \in X$ meets (34).

Now, the adjoint boundary conditions are:

$$u(0) = u(2\pi) = 0, u'(0) = u'(2\pi)$$
 (35)

Hence, if we let $G_u^0 = \frac{d^2}{dx^2} + 1$, with the boundary conditions as given by (34), then

$$N(G_{ij}^{O}) = \text{span} \{ \sin x, \cos x \} .$$
(36)

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However, the adjoint of G_u^0 has the form $(G_u^0)^* = \frac{d^2}{dx^2} + 1$, with (35) as its boundary conditions, and hence

$$N((G_{ij}^{O})^{*}) = \operatorname{span} \{\sin x\}$$
(37)

leading to the index of G_{u}^{O} being 1. Now, while the domain $\mathcal{P}(G)$ of G contains u C X meeting (34) we have yet to specify the range R(G) of G. To this end, set Y = X\{cos x}; then we put R(G) = Y. Finally, let the function ψ be:

$$y(\mathbf{x}) = \sin \mathbf{x} \quad . \tag{38}$$

It is now clear that (33) is similar to (19) and, therefore, multiple limit point bifurcations arise from $(u^0, 1)$ provided (cf. (22)):

$$f(\phi_1, \lambda - 1, \frac{d}{dx} \phi_1) \neq f(\phi_2, \lambda - 1, \frac{d}{dx} \phi_2)$$
(39)

where $\phi_1(x) = \sin x$, $\phi_2(x) = \cos x$.

4. CONCLUDING REMARKS

As mentioned in [2], there are numerous examples of limit point or turn j int bifurcation in the literature, especially in mechanics. In this literature, it is said that there are <u>two</u> solutions near a turning point because in the usual $(|u|,\lambda)$ diagram near $(u^{\circ},\lambda^{\circ})$, there are two values for |u|, for each $\lambda \neq \lambda^{\circ}$. In the approach adopted here, following [2,3], such a solution is assumed to represent a <u>single</u> limit point solution for, in terms of ε , the solution curve through $(u^{\circ},\lambda^{\circ})$ has a unique representation. To be specific, if $\varepsilon = 0$ corresponds to $(u^{\circ},\lambda^{\circ})$, then the values of $\varepsilon < 0$ and $\varepsilon > 0$ correspond to distinct values of (u,λ) , respectively, around $(u^{\circ},\lambda^{\circ})$; or, we have a smooth solution are through the limit point.

What has been done in the present paper is to construct examples of double solution arcs through the limit point. In the spirit of [2,3], these are called <u>multiple limit</u> <u>point hifurcations</u>, and are believed to be the first known examples of this kind in the literature.

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