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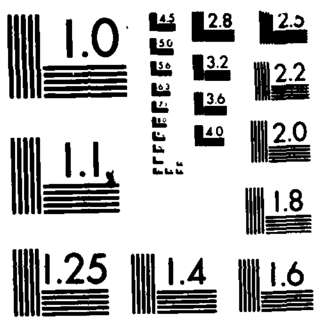

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MRC Technical Summary Report #2651

A DIRECT APPROACH TO THE VILLARCEAU CIRCLES OF A TORUS

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ABSTRACT

Let $T(\Gamma,c)$ denote a torus enveloped by a sphere of radius c whose center describes a circle Γ of radius a, $(c < a)$. We call Γ the central circle of the torus $T(\Gamma,c)$.

Let $T(\Gamma,c_1)$ and $T(\Gamma',c_1')$ denote two tori satisfying the following conditions:

1. The central circles Γ and Γ' have equal radii, both = a.

2. The two tari $T(\Gamma,c_1)$ and $T(\Gamma',c_1')$ are linked, like the two consecutive elements of a chain. This requires that

$$a > c_1 + c_1'.$$

It is shown that the two tori $T(\Gamma,c_1)$ and $T(\Gamma',c_1')$ can be placed in such a position that they are tangent to each other along a closed curve γ (which is not a circle) without any gaps between the tori. The simple closed curve γ is not a circle.

It is also shown that this property is equivalent to the circles on the torus discovered by the astronomer Yvon Villarceau in 1848.

The paper is to appear in the Dutch periodical "Simon Stevin".

AMS(MOS) Subject Classification: 51-01, 01-A55

Key Words: The torus, Helicoidal motion of a circle.

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SIGNIFICANCE AND EXPLANATION

Let T and T' be two tori which are linked like the two consecutive elements of a chain. Moreover ~~we~~ assume that T and T' have central circles of equal radii. By central circle of a torus ~~we~~ mean the locus of the center of the sphere of constant radius which envelopes the torus.

It is shown that the linked tori can be so placed that they are tangent to each other along simple closed curve γ which is not a circle. In this mutually tangent position there is no gap between the two tori T and T'.

It is shown that the above property is equivalent to the (slanting) circles of a torus discovered by Yvon Villarceau in 1848.

The responsibility for the wording and views expressed in this descriptive summary lies with MRC, and not with the author of this report.

A DIRECT APPROACH TO THE VILLARCEAU CIRCLES OF A TORUS

I. J. Schoenberg

A well known result states that if we cut a horizontal torus T by a slanting plane which is tangent in two (hyperbolic) points of the surface of T, we obtain as section two equal circles called the Villarceau circles of T*. A direct approach to these circles is described in the present note.

Fig. 1 shows the horizontal circle $\Gamma = ABC$, of center O and radius a, where AOC is a diameter and $\angle AOB = 90^\circ$. We are also given c such that

(1) $$a > c > 0,$$

and we define $b > 0$ by

(2) $$b^2 + c^2 = a^2.$$

On OC we mark the point O' so that $OO' = c$, and draw $O'D" \perp O'O$ such that $D \in \Gamma$. Evidently, by (2) we have $O'D = b$. At D we erect the vertical line DE and choose E so that $DE = c$. From (2) we conclude that $O'E = a$.

We now perform in succession on Γ the following two rigid motions:

1. We translate Γ along OC by the amount c, which moves O to O', and B to F, where $O'F = a$. Call Γ_1 the new position of Γ.

2. We rotate the circle Γ_1 around AC by the angle $\angle FO'E = \alpha$, which moves F to E. From the triangle $O'DE$ we see that

(3) $$\cos\alpha = \frac{b}{a}, \quad \sin\alpha = \frac{c}{a}.$$

* See [1], [2, pp. 63-68] and [3].

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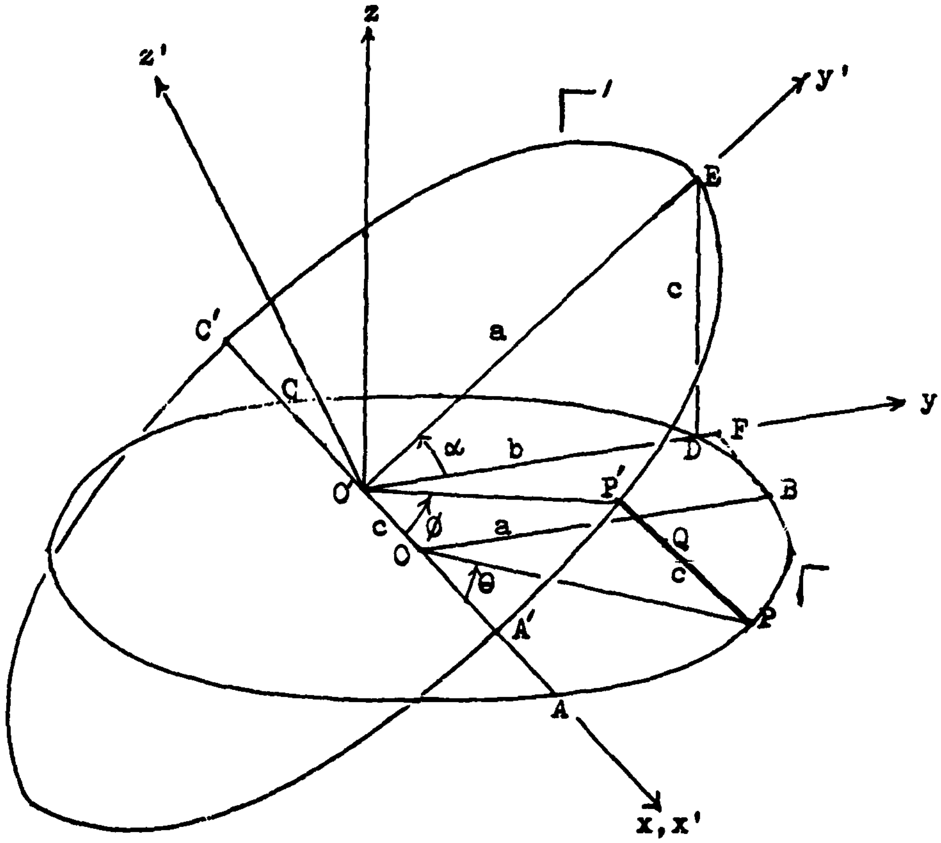

Figure 1.

<u>We denote by</u> $\Gamma' = A'EC'$ <u>the final position of</u> Γ. Notice that by our construction the three segments AA', DE, CC' satisfy

$$AA' = DE = CC' = c,$$

and that they are common normals of Γ and Γ'. It seems remarkable that <u>all</u> common normals of Γ and Γ' are of constant length = c. This property and further details of the relationship between Γ and Γ' are described by the following theorem.

<u>Theorem</u> 1. <u>On</u> Γ <u>and</u> Γ' <u>we select points</u> P <u>and</u> P', <u>respectively, such that</u>

(4) $$\theta = \angle AOP, \ \phi = \angle A'O'P', \quad (0 < \theta, \phi < 2\pi),$$

and we use these notations throughout. There is a 1-1 *mapping*

(5) $$V: P \to P', \quad (P \in \Gamma, P' \in \Gamma'),$$

of Γ *onto* Γ' *defined by the equations*

(6) $$\cos \phi = \frac{a \cos \theta + c}{a + c \cos \theta}, \quad \sin \phi = \frac{b \sin \theta}{a + c \cos \theta},$$

the inverse mappin V^{-1} *being given by*

(7) $$\cos \theta = \frac{a \cos \phi - c}{a - c \cos \phi}, \quad \sin \theta = \frac{b \sin \phi}{a - c \cos \phi}.$$

The properties of $P' = VP$ *are as follows:*

(8) $$PP' = c \quad \textit{for all} \quad P \in \Gamma.$$

(9) *The vector* $\overrightarrow{PP'}$ *is perpendicular to the tangents at* P *and* P' *of* Γ *and* Γ', *respectively*.

(10) *If* $p \in \Gamma$, $p' \in \Gamma'$ *are two points such that* $p' \neq Vp$, *then* $pp' > c$.

See the Concluding Remarks at the end of this note. We say that the circles Γ and Γ' form a *Villarceau pair of circles*.

Notice that the relationship between Γ and Γ' is *symmetric*, a fact already seen because the right-hand sides of (7) are obtained from those of (6) by replacing c by $-c$.

Proof of Theorem 1. In the cartesian axes $O'xyz$ of Fig. 1 we have

(11) $$P: \quad x = c + a \cos \theta, \quad y = a \sin \theta, \quad z = 0.$$

We now rotate $O'xyz$ about $O'x$ by the angle α obtaining $O'x'y'z'$ $(O'x' = O'x)$. From P': $x' = a \cos \phi$, $y' = a \sin \phi$, $z' = 0$, we find in $O'xyz$ the coordinates

$$P': \begin{array}{l} x = x' \\ y = y' \cos \alpha - z' \sin \alpha \\ z = y' \sin \alpha + z' \cos \alpha. \end{array}$$

Using (3) these become

$$(12) \qquad P': \begin{array}{l} x = a \cos \phi \\ y = b \sin \phi \\ z = c \sin \phi. \end{array}$$

In terms of (11) and (12) the equation (8) becomes

$$(PP')^2 = (c + a \cos \theta - a \cos \phi)^2 + (a \sin \theta - b \sin \phi)^2 + c^2 \sin^2 \phi$$

which easily gives

$$(13) \qquad (PP')^2 = c^2 + 2a(a + c(\cos \theta - \cos \phi) - a \cos \theta \cos \phi - b \sin \theta \sin \phi).$$

Therefore the property (8) is equivalent to the equation

$$(14) \qquad a + c(\cos \theta - \cos \phi) - a \cos \theta \cos \phi - b \sin \theta \sin \phi = 0.$$

This is the equation between θ and ϕ which we want to solve for ϕ. Doing this directly would be awkward as it would introduce complicated expressions and an extreneous solution. However, it becomes easy if we express and establish at the same time the properties (9):

The components of the vector $\overrightarrow{PP'}$ are, by (11) and (12),

$$(15) \qquad \overrightarrow{PP'} = (a \cos \phi - c - a \cos \theta,\ b \sin \phi - a \sin \theta,\ c \sin \phi),$$

while

$$\frac{dP}{d\theta} = (-a \sin \theta, a \cos \theta, 0)$$

$$\frac{dP'}{d\phi} = (-a \sin \phi, b \cos \phi, c \cos \phi).$$

Using inner products, the orthogonality properties

$$\overrightarrow{PP'} \perp \frac{dP}{d\theta}, \quad \overrightarrow{PP'} \perp \frac{dP'}{d\phi} \tag{16}$$

easily reduce, using (3), to

$$c \sin \theta - a \cos \phi \sin \theta + b \sin \phi \cos \theta = 0 \tag{17}$$

and

$$c \sin \phi + a \cos \theta \sin \phi - b \cos \phi \sin \theta = 0, \tag{18}$$

respectively. If we rearrange (14) and (17) as linear functions of $\cos \phi$ and $\sin \phi$ we find the quivalent equations

$$\begin{aligned} (c + a \cos \theta)\cos \phi + b \sin \theta \sin \phi &= a + c \cos \theta \\ a \sin \theta \cos \phi - b \cos \theta \sin \phi &= c \sin \theta. \end{aligned} \tag{19}$$

Solving (19) for $\cos \phi$ and $\sin \phi$ we readily find that

$$\cos \phi = \frac{a \cos \theta + c}{a + c \cos \theta}, \quad \sin \phi = \frac{b \sin \theta}{a + c \cos \theta} \tag{20}$$

and the solution (6) is established: We easily verify that the right sides of (20) are the coordinates of a point on the unit circle. It is also immediate that (20) satisfy identically the third equation (18). Solving the equations (20) for $\cos \theta$ and $\sin \theta$, we find the equations (7). Notice that we have established (6), (7), (8), and (9).

Proof of (10). Let the value ϕ^* in (12) produce the point $P^{*\prime}$. By (13) and (14) we are to show the following:

If

(21) $$\phi^* \neq \phi, \quad (0 < \emptyset < 2\pi),$$

then

(22) $$F(\theta,\phi^*) = a + c(\cos\theta - \cos\phi^*) - a\cos\theta\cos\phi^* - b\sin\theta\sin\phi^* > 0.$$

A proof follows from the following facts:

1. $F(\theta,\phi^*)$ is a first order trigonometric polynomial in ϕ^* which vanishes for $\phi^* = \emptyset$, by (14).

2. By (9) $\phi^* = \phi$ must be a double zero of $F(\theta, \phi^*)$ and therefore $F(\theta, \phi^*) > 0$ for ϕ^* sufficiently close to ϕ. Since such a polynomial may have in its period at most two zeros, (22) is established.

Concluding Remarks. 1. From (8) and (9) we see that Γ' is a Villarceau circle of the torus T having Γ as a central circle, and also that Γ is a Villarceau circle of the torus T' having Γ' as central circle.

2. Let us denote by $T(\Gamma,c)$ a torus enveloped by a sphere of radius c whose center describes the circle Γ.

From Fig. 1 we already know that the common normal PP' of Γ and Γ', is of constant length with $PP' = c$. We divide PP' into two parts with

$$PQ = c_1,\ QP' = c_1',\ c_1 \text{ and } c_1' \text{ constants},\ c_1 + c'_1 = c,$$

and consider the two tori

(23) $$T(\Gamma,c_1) \text{ and } T(\Gamma',c_1').$$

I claim: The two tori (23) are tangent to each other along all the points of a closed curve γ described by the point Q.

Proof: PQ is a normal of $T(\Gamma, c_1)$, while QP' is a normal of $T(\Gamma', c_1')$. It follows that the tori (23) are both tangent to the plane perpendicular to PP' at the point Q.

Let

$$(24) \qquad \tilde{T} = T(\tilde{\Gamma}, c_1) \quad \text{and} \quad \tilde{T}' = T(\tilde{\Gamma}', c_1'),$$

be two tori that are linked, like two consecutive rings of a chain; we also assume these tori to have equal central circles of radii = a. The diameter of the "hole" of T is $= 2(a - c_1)$. Since T' passes through this hole, we must have $2c_1' < 2(a - c_1)$, or

$$a > c_1 + c_1'.$$

Corollary 1. We can so place the linked tori (24) that they are tangent to each other along a simple closed curve γ which is not a circle.

Proof: We set $c = c_1 + c_1'$ and identify the tori (24) with those of (23), and place them in the position of Fig. 1. As already mentioned, the tori (24) are now tangent to each other along the curve γ described by the point Q with $PQ = c_1$, $P'Q = c_1'$.

Notice that if we let $c_1 \to c$ and $c_1' \to 0$, then the torus $T(\Gamma', c_1')$ becomes a Villarceau circle of $T(\Gamma, c)$.

Conversely, Corollary 1 imples Villarceau's property, or V-property of tori: If Q is a point of contact of the tori, then their normals $QP = c_1$ and $QP' = c_1'$ fit together to give the constant shortest distance $PP' = c$ between the circles Γ and Γ'. Therefore Γ' is a Villarceau circle of the torus $T(\Gamma, c)$.

A pair of linked tori T and T', made of wood or lucite, having equal central circles can be used to exhibit to the "man on the street" their tangency in an appropriate position. This seems easier than showing the intersection of the torus by a bi-tangent plane.

For a different approach to the Villarceau circles see O. Bottema's note [1].

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20. Abstract (cont.)

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