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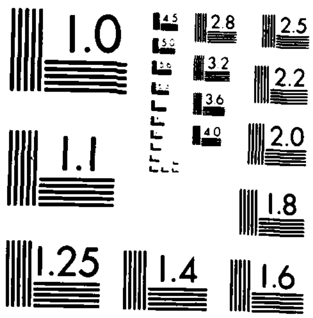

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MRC Technical Summary Report #2656

A FREE BOUNDARY PROBLEM FOR DEGENERATE QUASILINEAR PARABOLIC EQUATIONS

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March 1984

(Received October 18, 1983)

Sponsored by

U. S. Army Research Office
P. O. Box 12211
Research Triangle Park
North Carolina 27709

UNIVERSITY OF WISCONSIN-MADISON
MATHEMATICS RESEARCH CENTER

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ABSTRACT

In this paper, we consider a new kind of free boundary problem related to the investigation of the structure of discontinuous solutions of degenerate quasilinear parabolic equations. A thorough treatment is given for the following special cases:

$$\frac{\partial u}{\partial t} = \frac{\partial^2 u^m}{\partial x^2} \quad (m > 1) ,$$

$$u\big|_{x=\lambda(t)} = 0 ,$$

$$\frac{\partial u^m}{\partial x}\Big|_{x=\lambda(t)} = \alpha\lambda'(t) , \qquad (\alpha < 0, \beta > 0 - \text{const.})$$

$$u\big|_{t=0} = \beta ,$$

which can be reduced to a problem in ordinary differential equations with a certain singularity.

AMS (MOS) Subject Classifications: 35K60, 35K65

Key Words: Degenerate quasilinear parabolic equations, Free boundary problem, Discontinuous solutions, Jump conditions, Existence and uniqueness, Structure of solutions

Work Unit Number 1 (Applied Analysis)

Sponsored by the United States Army under Contract No. DAAG29-80-C-0041.

SIGNIFICANCE AND EXPLANATION

This paper is concerned with a free boundary problem which arises in the study of a class of degenerate parabolic equations. A thorough treatment is given for a special case which can be reduced to a problem in ordinary differential equations upon introduction of the appropriate similarity variable. Beyond its inherent interest, the solvability of the resulting problem establishes that an analysis by Vol'pert and Hudjaev of jump conditions satisfied by solutions of degenerate parabolic equations is not correct in general.

The responsibility for the wording and views expressed in this descriptive summary lies with MRC, and not with the author of this report.

A FREE BOUNDARY PROBLEM FOR DEGENERATE QUASILINEAR PARABOLIC EQUATIONS

Zhuoqun Wu

§1. Introduction

In this paper, we are concerned with a free boundary problem for quasilinear equations of the form

$$\frac{\partial u}{\partial t} = \frac{\partial^2 A(u)}{\partial x^2} \tag{1.1}$$

with

$$A'(u) = a(u) \geq 0 .$$

Our problem arises, in particular, from the investigation of the structure of discontinuous solutions of equation (1.1). It is shown in [1] that discontinuity occurs in a generalized solution only if there exists an interval (u_1,u_2) such that

$$a(u) = 0 \quad \text{for} \quad u \in (u_1,u_2) .$$

Things are simple if the initial data

$$u|_{t=0} = u_0(x) \tag{1.2}$$

happen to either fall in an interval where $a(u) > 0$ or fall in an interval where $a(u) \equiv 0$. What we need to investigate is the case that $a(u_0(x)) > 0$ for some x and $a(u_0(x)) = 0$ for some other x. The following situation seems to be typical:

$$\begin{aligned} &a(u) = 0 \quad \text{for} \quad u < 0 \quad \text{and} \quad a(u) > 0 \quad \text{for} \quad u > 0 \\ &u_0(x) < 0 \quad \text{for} \quad x < 0 \quad \text{and} \quad u_0(x) > 0 \quad \text{for} \quad x > 0 . \end{aligned} \tag{1.3}$$

In this case, one could conjecture that the corresponding generalized solution u would have a line of discontinuity $x = \lambda(t)$ with $\lambda'(t) < 0$ starting from $(0,0)$. The question is: how does the discontinuity emerge and develop? To get an answer, we need to

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make essential use of the jump conditions which the generalized solutions satisfy. In [1], two jump conditions are presented, which are

$$a(u) = 0 \quad \text{for} \quad u \in I(u^-,u^+) \tag{1.4}$$

and

$$[\operatorname{sgn}(u^+ - k) - \operatorname{sgn}(u^- - k)](\bar{u} - k)\nu_t \le 0 \quad \text{for} \quad k \in R \tag{1.5}$$

for (1.1), where $\bar{u} = \frac{u^- + u^+}{2}$, u^- and u^+ denote the approximate limits at the points of jump, (ν_t,ν_x) the normal to the set of points of jump and $I(u^-,u^+)$ the interval with endpoints u^- and u^+. In particular, (1.5) implies

$$(u^+ - u^-)\nu_t = 0 \quad \text{or} \quad \nu_t = 0 \ , \tag{1.6}$$

which means that all of the normals to the set of points of jump are parallel to the x-axis. Hence the line of discontinuity starting from (0,0), if any, should be a straight line perpendicular to the x-axis. However, this assertion is not true in general; things are not really so simple. The wrong assertion comes from the incorrect jump condition (1.5). The correct form of this condition should be

$$[\operatorname{sgn}(u^+ - k) - \operatorname{sgn}(u^- - k)][(\bar{u} - k)\nu_t - \overline{\frac{\partial A(u)}{\partial x}}\,\nu_x] \le 0 \quad \text{for} \quad k \in R \tag{1.7}$$

and

$$(u^+ - u^-)\nu_t - [(\frac{\partial A(u)}{\partial x})^+ - (\frac{\partial A(u)}{\partial x})^-]\nu_x = 0$$

or

$$(u^+ - u^-)\frac{dx}{dt} + [(\frac{\partial A(u)}{\partial x})^+ - (\frac{\partial A(u)}{\partial x})^-] = 0 \tag{1.8}$$

where $\frac{dx}{dt}$ denotes the slope of the tangent to the set of points of jump.

Suppose (1.3) holds and $x = \lambda(t)$, where $\lambda'(t) < 0$, is the line of discontinuity of the corresponding generalized solution $u(t,x)$. Then, since $u_0(x) < 0$ $(x < 0)$ and $a(u) = 0$ $(u < 0)$, we first have

$$u(t,x) = u_0(x) \quad \text{for} \quad x < \lambda(t) \ .$$

Secondly, from (1.4) it follows that

$$u^+|_{x=\lambda(t)} = 0 \ ,$$

and hence, noticing that $\left(\frac{\partial A(u)}{\partial x}\right)^{-}\Big|_{x=\lambda(t)} = 0$, from (1.8) we get

$$-u_0(\lambda(t))\lambda'(t) + \left(\frac{\partial A(u)}{\partial x}\right)^{+}\Big|_{x=\lambda(t)} = 0 .$$

Thus $(u(t,x),\lambda(t))$ is a solution of the free boundary problem for (1.1) with boundary conditions

$$u|_{x=\lambda(t)} = 0 , \tag{1.9}$$

$$\frac{\partial A(u)}{\partial x}\Big|_{x=\lambda(t)} = u_0(\lambda(t))\lambda'(t) , \tag{1.10}$$

and the initial condition

$$u|_{t=0} = u_0(x) \quad \text{for} \quad x > 0 . \tag{1.11}$$

We will prove the inverse in §2, namely, if $(u(t,x),\lambda(t))$ is a solution of the free boundary (1.1), (1.9), (1.10), (1.11), then we can immediately obtain a generalized solution of the initial value problem (1.1), (1.2) with $x = \lambda(t)$ as the line of discontinuity.

The rest of this paper (§3-§5) is devoted to a detailed study of the special case:

$$\begin{aligned} &A(u) = 0 \quad \text{for} \quad u < 0 \quad \text{and} \quad A(u) = u^m \quad \text{for} \quad u > 0 , \\ &u_0(x) = \alpha < 0 \quad \text{for} \quad x < 0 \quad \text{and} \quad u_0(x) = \beta > 0 \quad \text{for} \quad x > 0 , \end{aligned} \tag{1.12}$$

where $m > 1$ and α,β are constants. The possibility, in this case, of reducing to a problem in ordinary differential equations simplifies matters. The main results are included in the theorem in §5.

By the way, since the proof of the uniqueness theorem in [1] is based on jump conditions (for (1.1), they are (1.4), (1.5)) and one of them is not true in general, a revision of the proof is required. We will derive the correct form of the jump condition and complete the proof of uniqueness in another paper [2].

The author would like to thank Professor Michael Crandall for several interesting and helpful discussions during the preparation of this paper.

§2. Generalized Solutions of (1.1), (1.2)

Suppose $A(u) \in C^1(R)$, $u_0(x) \in L^\infty(R) \cap C^1(R\backslash\{0\})$.

Proposition 1. If $(u(t,x),\lambda(t))$ is a solution of the free boundary problem (1.1), (1.9), (1.10), (1.11), then the function

$$w(t,x) = \begin{cases} u(t,x) & \text{for } x > \lambda(t) \\ u_0(x) & \text{for } x < \lambda(t) \end{cases}$$

is a generalized solution of the initial value problem (1.1), (1.2) with $x = \lambda(t)$ as the line of discontinuity.

By a solution of the free boundary problem (1.1), (1.9), (1.10), (1.11), we mean a pair of functions $(u(t,x),\lambda(t))$, such that

(2.1) $u \in L^\infty(G) \cap C(\bar{G}\backslash\{(0,0)\}) \cap C^{1,2}(G)$, $G = \{(t,x);\ 0 < t < T,\ x > \lambda(t)\}$,

(2.2) $\iint_D |\frac{\partial u}{\partial t}|dtdx$, $\iint_D |\frac{\partial u}{\partial x}|dtdx < +\infty$ for any bounded domain D with $\bar{D} \subset \bar{G} - \{t = 0\}$,

(2.3) $$\frac{\partial A(u)}{\partial x} \in C(\bar{G}\backslash\{(0,0)\}) ,$$

(2.4) $$\lambda(t) \in C[0,T] \cap C^1(0,T] ,$$

(2.5) $$\lambda'(t) < 0$$

and (1.1), (1.9), (1.10), (1.11) are satisfied. Here $C^{1,2}(G)$ denotes the class of functions which are continuous on G with their first order derivative with respect to t and their first and second order derivatives with respect to x.

Proof. We need to check all of the conditions in the definition of generalized solutions given in [1].

Since $u \in L^\infty(G)$, $u_0 \in L^\infty(R)$, we have $w \in L^\infty(Q)$, where $Q = (0,T) \times R$. The assumption $u_0 \in C^1(R\backslash\{0\})$ and (2.2) imply that $w \in BV(Q)$. By $BV(Q)$, we mean the class of integrable functions on Q of locally bounded variation.

To prove $\frac{\partial R(w)}{\partial x} \in L^2_{loc}(Q)$ where $R(u) = \int_0^u \gamma(\tau)d\tau$ and $\gamma(\tau) = a(\tau)^{\frac{1}{2}}$, we multiply (1.1) by ϕu with $\phi \geq 0$ and $\phi \in C_0^\infty(Q)$ and get

$$\frac{1}{2}\frac{\partial \phi u^2}{\partial t} + \phi a(u)\left(\frac{\partial u}{\partial x}\right)^2 = \frac{1}{2}\frac{\partial \phi}{\partial t}u^2 + \frac{\partial}{\partial x}\left(\phi u \frac{\partial A(u)}{\partial x}\right) - \frac{\partial \phi}{\partial x} u \frac{\partial A(u)}{\partial x} .$$

Integrating this relation over $G_\varepsilon = \{(t,x) \in G;\ x > \lambda(t) + \varepsilon,\ \varepsilon > 0\}$ (notice that u need not be differentiable up to $L : x = \lambda(t)$) yields

$$\frac{1}{2}\int_{\lambda(T)+\varepsilon}^{\infty} \phi u^2\Big|_{t=T} dx - \frac{1}{2}\int_{\lambda(T)+\varepsilon}^{\varepsilon} \phi u^2\Big|_{x=\lambda(t)+\varepsilon} dx - \frac{1}{2}\int_{\varepsilon}^{\infty} \phi u^2\Big|_{t=0} dx + \iint_{G_\varepsilon} \phi a(u)\left(\frac{\partial u}{\partial x}\right)^2 dtdx$$

$$(2.6)\qquad = \frac{1}{2}\iint_{G_\varepsilon} \frac{\partial \phi}{\partial t} u^2 dtdx - \int_0^T \phi u \frac{\partial A(u)}{\partial x}\Big|_{x=\lambda(t)+\varepsilon} dt - \iint_{G_\varepsilon} \frac{\partial \phi}{\partial x} u \frac{\partial A(u)}{\partial x} dtdx .$$

Since $\phi|_{t=0,T} = 0$, the first and third term of (2.6) are zero. By virtue of (1.9), (1.10), the second and sixth term tend to zero as $\varepsilon \to 0$. The fifth and seventh term are bounded uniformly in ε. Thus, letting $\varepsilon \to 0$, from (2.6) we get

$$(2.7)\qquad \iint_G \phi a(u)\left(\frac{\partial u}{\partial x}\right)^2 dtdx < +\infty .$$

Denote

$$g = \begin{cases} \gamma(u)\,\dfrac{\partial u}{\partial x} & \text{for } x > \lambda(t) \\ 0 & \text{for } x < \lambda(t) . \end{cases}$$

Then by (2.7), $g \in L^2_{loc}(\Omega)$.

Clearly, for any $\phi \in C_0^\infty(\Omega)$,

$$\iint_{|x-\lambda(t)|\geq\varepsilon} \phi \frac{\partial R(w)}{\partial x} dtdx = \iint_{|x-\lambda(t)|\geq\varepsilon} \phi g dtdx \to \iint_\Omega \phi g dtdx \quad \text{as} \quad \varepsilon \to 0 .$$

Integrating by parts gives

$$\iint_{|x-\lambda(t)|<\varepsilon} \phi \frac{\partial R(w)}{\partial x} dtdx = -\iint_{|x-\lambda(t)|<\varepsilon} \frac{\partial \phi}{\partial x} R(w) dtdx$$

$$+ \int_{x=\lambda(t)+\varepsilon} \phi R(w) dt - \int_{x=\lambda(t)-\varepsilon} \phi R(w) dt \to 0 \quad \text{as} \quad \varepsilon \to 0 .$$

So

$$\iint_Q \phi \frac{\partial R(w)}{\partial x}\, dtdx = \iint_{|x-\lambda(t)|<\varepsilon} + \iint_{|x-\lambda(t)|\geq\varepsilon} \phi \frac{\partial R(w)}{\partial x}\, dtdx + \iint_Q \phi g dtdx \ .$$

This means that $\frac{\partial R(w)}{\partial x} = g \in L^2_{loc}(Q)$.

Now we prove that w satisfies the integral inequality

$$\text{(2.8)} \qquad J(w,k,\phi) = \iint_Q \operatorname{sgn}(w-k)\left[(w-k)\frac{\partial\phi}{\partial t} - \frac{\partial A(w)}{\partial x}\frac{\partial\phi}{\partial x}\right]dtdx \geq 0 \quad \text{for} \quad k \in R,$$
$$\phi \in C_0^\infty(Q),\ \phi \geq 0 \ .$$

Divide $J(w,k,\phi)$ into two parts:

$$\text{(2.9)} \qquad \begin{aligned} J(w,k,\phi) &= \iint_{x<\lambda(t)} \operatorname{sgn}(u_0-k)(u_0-k)\frac{\partial\phi}{\partial t}\, dtdx + \iint_{x>\lambda(t)} \operatorname{sgn}(u-k)\left[(u-k)\frac{\partial\phi}{\partial t} - \frac{\partial A(u)}{\partial x}\frac{\partial\phi}{\partial x}\right]dtdx \\ &= J_1(u_0,k,\phi) + J_2(u,k,\phi) \ . \end{aligned}$$

Obviously

$$\text{(2.10)} \qquad \begin{aligned} J_1(u_0,k,\phi) &= \int_{-\infty}^{\lambda(T)} |u_0-k|dx \int_0^T \frac{\partial\phi}{\partial t}\, dt + \int_{\lambda(T)}^0 |u_0-k|dx \int_0^{\lambda^{-1}(x)} \frac{\partial\phi}{\partial t}\, dt \\ &= \int_{\lambda(T)}^0 |u_0-k|\phi(\lambda^{-1}(x),x)dx = -\int_0^T |u_0-k|\phi|_{x=\lambda(t)}\lambda'(t)dt \ , \end{aligned}$$

$$\text{(2.11)} \qquad J_2(u,k,\phi) = \lim_{\eta\to 0+} J_{2,\eta}(u,k,\phi)$$

where

$$J_{2,\eta}(u,k,\phi) = \iint_{x>\lambda(t)} \operatorname{sgn}_\eta(u-k)\left[(u-k)\frac{\partial\phi}{\partial t} - \frac{\partial A(u)}{\partial x}\frac{\partial\phi}{\partial x}\right]dtdx$$

$$\operatorname{sgn}_\eta(\tau) = \begin{cases} 1 & \text{for} \quad \tau > \eta \\ \frac{\tau}{\eta} & \text{for} \quad |\tau| \leq \eta \\ -1 & \text{for} \quad \tau < -\eta \ . \end{cases}$$

We have

$$\begin{aligned} &\operatorname{sgn}_\eta(u-k)\left[(u-k)\frac{\partial\phi}{\partial t} - \frac{\partial A(u)}{\partial x}\frac{\partial\phi}{\partial x}\right] \\ &= \frac{\partial}{\partial t}(\operatorname{sgn}_\eta(u-k)(u-k)\phi) - \operatorname{sgn}_\eta(u-k)\frac{\partial u}{\partial t}\phi - \operatorname{sgn}'_\eta(u-k)\frac{\partial u}{\partial t}(u-k)\phi \end{aligned}$$

$$-\frac{\partial}{\partial x}\left(\operatorname{sgn}_\eta(u-k)\frac{\partial A(u)}{\partial x}\phi\right)+\operatorname{sgn}_\eta(u-k)\frac{\partial^2 A(u)}{\partial x^2}+\operatorname{sgn}_\eta'(u-k)\phi(u)\left(\frac{\partial u}{\partial x}\right)^2 .$$

Since u satisfies (1.1) whenever $x > \lambda(t)$, we have

$$J_{2,\eta}(u,k,\phi) \geq \iint_{x>\lambda(t)} \left[\frac{\partial}{\partial t}\left(\operatorname{sgn}_\eta(u-k)(u-k)\phi\right)-\frac{\partial}{\partial x}\left(\operatorname{sgn}_\eta(u-k)\frac{\partial A(u)}{\partial x}\phi\right)\right]dtdx$$

$$-\iint_{x>\lambda(t)} \operatorname{sgn}_\eta'(u-k)\frac{\partial u}{\partial t}(u-k)\phi dtdx$$

$$=\int_0^T \operatorname{sgn}_\eta(u-k)(u-k)\phi\big|_{x=\lambda(t)}\lambda'(t)dt+\int_0^T \operatorname{sgn}_\eta(u-k)\frac{\partial A(u)}{\partial x}\phi\big|_{x=\lambda(t)}dt$$

$$-\iint_{x>\lambda(t)} \operatorname{sgn}_\eta'(u-k)\frac{\partial u}{\partial t}(u-k)\phi dtdx$$

$$=\int_0^T \operatorname{sgn}_\eta(k)k\phi\big|_{x=\lambda(t)}\lambda'(t)dt-\int_0^T \operatorname{sgn}_\eta(k)u_0\big|_{x=\lambda(t)}\lambda'(t)dt$$

$$-\iint_{x>\lambda(t)} \operatorname{sgn}_\eta'(u-k)\frac{\partial u}{\partial t}(u-k)\phi dtdx .$$

For any fixed $\varepsilon > 0$, since $u \in C^1(G_\varepsilon)$, we have

$$\iint_{x>\lambda(t)+\varepsilon} \operatorname{sgn}_\eta'(u-k)\frac{\partial u}{\partial t}(u-k)\phi dtdx \to 0 \quad \text{as} \quad \eta \to 0 .$$

By (2.2),

$$\left|\iint_{\lambda(t)<x<\lambda(t)+\varepsilon} \operatorname{sgn}_\eta'(u-k)\frac{\partial u}{\partial t}(u-k)\phi dtdx\right| \leq \iint_{\lambda(t)<x<\lambda(t)+\varepsilon} \phi\left|\frac{\partial u}{\partial t}\right|dtdx \to 0 \quad \text{as} \quad \varepsilon \to 0 .$$

Thus

$$\iint_{x>\lambda(t)} \operatorname{sgn}_\eta'(u-k)\frac{\partial u}{\partial t}(u-k)\phi dtdx \to 0 \quad \text{as} \quad \eta \to 0$$

and hence

$$\lim_{\eta\to 0+} J_{2,\eta}(u,k,\phi) \geq \int_0^T |k|\phi\big|_{x=\lambda(t)}\lambda'(t)dt-\int_0^T \operatorname{sgn}(k)u_0\big|_{x=\lambda(t)}\lambda'(t)dt . \tag{2.12}$$

Combining (2.9)-(2.12) gives

$$J(w,k,\phi) \geq \int_0^T (-|u_0 - k| + |k| - \mathrm{sgn}(k) \cdot u_0)\phi|_{x=\lambda(t)}\lambda'(t)dt \geq 0$$

since $\lambda'(t) < 0$. This proves (2.8) and the proof of Proposition is complete.

§3. Estimates on the Similarity Solution

From now on we consider the special case (1.12). In this case, the free boundary problem becomes

$$\frac{\partial u}{\partial t} = \frac{\partial^2 u^m}{\partial x^2} , \tag{3.1}$$

$$u|_{x=\lambda(t)} = 0 , \tag{3.2}$$

$$\left.\frac{\partial u^m}{\partial x}\right|_{x=\lambda(t)} = \alpha\lambda'(t) , \tag{3.3}$$

$$u|_{t=0} = \beta . \tag{3.4}$$

We seek solutions of the form

$$u = u(\xi), \quad \xi = \frac{x}{t^{1/2}} . \tag{3.5}$$

Then we arrive at a free boundary problem for an ordinary differential equation, namely:

$$(u^m)'' = -\frac{1}{2}\xi u' ,$$

$$u|_{\xi=\xi_0} = 0 ,$$

$$(u^m)|_{\xi=\xi_0} = \frac{\alpha}{2}\xi_0 ,$$

$$u|_{\xi=\infty} = \beta .$$

Let

$$v = u^m . \tag{3.6}$$

The problem for $v(\xi)$ is

$$v'' = -\frac{\xi}{2m} v^{\frac{1}{m}-1} v' , \tag{3.7}$$

$$v|_{\xi=\xi_0} = 0 , \tag{3.8}$$

(3.9) $$v'|_{\xi=\xi_0} = \frac{\alpha}{2}\xi_0 ,$$

(3.10) $$v|_{\xi=\infty} = \beta^m .$$

We seek solutions $v \in C^1[\xi_0,\infty) \cap C^2(\xi_0,\infty)$, which are positive for $\xi > \xi_0$. We will first solve (3.7), (3.8), (3.9) for any fixed $\xi_0 < 0$ and then choose ξ_0 such that the solution satisfies (3.10) as well.

Integrating (3.7) and using (3.8), (3.9) we get

(3.11) $$v'(\xi) = \frac{\alpha\xi_0}{2}\exp\Big(-\frac{1}{2m}\int_{\xi_0}^{\xi} \tau v^{\frac{1}{m}-1}(\tau)d\tau\Big) ,$$

(3.12) $$v(\xi) = \frac{\alpha\xi_0}{2}\int_{\xi_0}^{\xi} \exp\Big(-\frac{1}{2m}\int_{\xi_0}^{\tau} s v^{\frac{1}{m}-1}(s)ds\Big)d\tau .$$

If we write (3.7) as

$$v'' = -\frac{3}{2}\Big(v^{\frac{1}{m}}\Big)' = -\frac{1}{2}\Big(\xi v^{\frac{1}{m}}\Big)' + \frac{1}{2}v^{\frac{1}{m}} ,$$

integrate and use (3.8), (3.9), then we get

(3.13) $$v'(\xi) = \frac{\alpha\xi_0}{2} - \frac{1}{2}\xi v^{\frac{1}{m}}(\xi) + \frac{1}{2}\int_{\xi_0}^{\xi} v^{\frac{1}{m}}(\tau)d\tau ,$$

(3.14) $$v(\xi) = \frac{\alpha\xi_0}{2}(\xi-\xi_0) + \frac{1}{2}\int_{\xi_0}^{\xi}(\xi-2\tau)v^{\frac{1}{m}}(\tau)d\tau .$$

In order to solve (3.7), (3.8), (3.9), it suffices to solve the integral equation (3.12) or (3.14) - to find a solution $v \in C[\xi_0,\infty)$ which is positive for $\xi > \xi_0$. We remark that the integrand in (3.14) is not singular at $v = 0$ in contrast to (3.12).

Before we discuss the solvability of (3.7), (3.8), (3.9), we first derive some estimates on the solutions.

From (3.11) we see that v' is strictly increasing for $\xi_0 < \xi < 0$. In particular, $v' > \frac{\alpha\xi_0}{2}$ for $\xi_0 < \xi < 0$ and hence

$$(3.15) \qquad v(\xi) > \frac{\alpha\xi_0}{2}(\xi - \xi_0) \quad \text{for} \quad \xi_0 < \xi < 0 .$$

Using (3.11), (3.15) we further obtain

$$v'(\xi) < \frac{\alpha\xi_0}{2} \exp\left[-\frac{1}{2m}\int_{\xi_0}^{\xi} \tau\left(\frac{\alpha\xi_0}{2}\right)^{\frac{1}{m}-1}(\tau - \xi_0)^{\frac{1}{m}-1} d\tau\right]$$

$$< \frac{\alpha\xi_0}{2} \exp\left[\frac{1}{2m}\left(\frac{|\alpha|}{2}\right)^{\frac{1}{m}-1} |\xi_0|^{\frac{1}{m}} \int_{\xi_0}^{\xi} (\tau - \xi_0)^{\frac{1}{m}-1} d\tau\right]$$

$$< \frac{\alpha\xi_0}{2} \exp\left[\frac{1}{2}\left(\frac{|\alpha|}{2}\right)^{\frac{1}{m}-1} |\xi_0|^{\frac{2}{m}}\right] \quad \text{for} \quad \xi_0 < \xi < 0 ,$$

or

$$(3.16) \qquad v'(\xi) < \frac{\alpha R\xi_0}{2} \quad \text{for} \quad \xi_0 < \xi < 0 ,$$

where

$$(3.17) \qquad R = \exp\left[\frac{1}{2}\left(\frac{|\alpha|}{2}\right)^{\frac{1}{m}-1} |\xi_0|^{\frac{2}{m}}\right] .$$

Hence

$$(3.18) \qquad v(\xi) < \frac{\alpha R\xi_0}{2}(\xi - \xi_0) \quad \text{for} \quad \xi_0 < \xi < 0 .$$

Combining (3.18) with (3.15) gives

$$(3.19) \qquad \frac{\alpha\xi_0}{2}(\xi - \xi_0) < v(\xi) < \frac{\alpha R\xi_0}{2}(\xi - \xi_0) \quad \text{for} \quad \xi_0 < \xi < 0 .$$

Next we prove that

$$(3.20) \qquad v(\xi) < \alpha M\xi_0$$

whenever $v(\xi)$ exists. Here M is a constant such that

$$(3.21) \qquad \frac{R|\xi_0|}{2} + \left(Rm^{\frac{1}{2}}|\alpha|^{\frac{m-1}{2m}}|\xi_0|^{\frac{m-1}{2m}}\int_0^{\infty}\exp(-\eta^2)d\eta\right)M^{\frac{m-1}{2m}} < \frac{M}{2} .$$

Let $[\xi_0,\xi_1)$ be the maximal existence interval of $v(\xi)$. If $\xi_1 < 0$, then (3.18) implies (3.20). Suppose $\xi_1 > 0$ and there exists $\xi_2 > 0$, $\xi_2 < \xi_1$ such that

(3.22) $\qquad v(\xi) < \alpha M\xi_0$ for $0 \le \xi < \xi_2$ and $v(\xi_2) = \alpha M\xi_0$.

Then, we divide the right hand side of (3.12) into two parts:

$$v(\xi) = \frac{\alpha\xi_0}{2}\int_{\xi_0}^{0} \exp\Big(-\frac{1}{2m}\int_{\xi_0}^{\tau} sv^{\frac{1}{m}-1}(s)ds\Big)d\tau + \frac{\alpha\xi_0}{2}\int_{0}^{\xi} \exp\Big(-\frac{1}{2m}\int_{\xi_0}^{\tau} sv^{\frac{1}{m}-1}(s)ds\Big)d\tau$$

$$= I_1 + I_2$$

and estimate each of them. Using (3.15) we get

$$I_1 = \frac{\alpha\xi_0}{2}\int_{\xi_0}^{0} \exp\Big(-\frac{1}{2m}\int_{\xi_0}^{\tau} sv^{\frac{1}{m}-1}(s)ds\Big)d\tau \le \alpha\xi_0\,\frac{R|\xi_0|}{2} .$$

Using (3.15) and the first part of (3.22) we get for $0 \le \xi < \xi_2$,

$$I_2 = \frac{\alpha\xi_0}{2}\int_{0}^{\xi} \exp\Big(-\frac{1}{2m}\int_{\xi_0}^{\tau} sv^{\frac{1}{m}-1}(s)ds\Big)d\tau$$

$$= \frac{\alpha\xi_0}{2}\int_{0}^{\xi} \exp\Big(-\frac{1}{2m}\int_{\xi_0}^{0} sv^{\frac{1}{m}-1}(s)ds\Big)\exp\Big(-\frac{1}{2m}\int_{0}^{\tau} sv^{\frac{1}{m}-1}(s)ds\Big)d\tau$$

$$\le \frac{\alpha\xi_0}{2}\exp\Big[\frac{1}{2}\Big(\frac{|\alpha|}{2}\Big)^{\frac{1}{m}-1}|\xi_0|^{\frac{2}{m}}\Big]\int_{0}^{\xi} \exp\Big(-\frac{1}{2m}\int_{0}^{\tau} sv^{\frac{1}{m}-1}(s)ds\Big)d\tau$$

$$\le \frac{\alpha\xi_0}{2}R\int_{0}^{\xi} \exp\Big(-\frac{1}{2m}\int_{0}^{\tau} s(\alpha M\xi_0)^{\frac{1}{m}-1}ds\Big)d\tau$$

$$= \frac{\alpha\xi_0}{2}R\int_{0}^{\xi} \exp\Big[-\frac{1}{4m}(\alpha M\xi_0)^{\frac{1}{m}-1}\tau^2\Big]d\tau$$

$$= \frac{\alpha\xi_0}{2}R\gamma^{-1}\int_{0}^{\gamma\xi} \exp(-s^2)ds \qquad \Big(\gamma = \Big[\frac{1}{4m}(\alpha M\xi_0)^{\frac{1}{m}-1}\Big]^{\frac{1}{2}}\Big)$$

$$\le \alpha\xi_0\Big(Rm^{\frac{1}{2}}|\alpha|^{\frac{m-1}{2m}}|\xi_0|^{\frac{m-1}{2m}}\int_{0}^{\infty} \exp(-s^2)ds \cdot M^{\frac{m-1}{2m}}\Big) .$$

Hence, by the choice of M (see (3.21))

$$v(\xi_2) < \alpha\xi_0\left(\frac{R|\xi_0|}{2} + Rm^{\frac{1}{2}}|\alpha|^{\frac{m-1}{2m}}|\xi_0|^{\frac{m-1}{2m}}\int_0^\infty \exp(-s^2)ds \cdot M^{\frac{m-1}{2m}}\right) < \frac{M}{2}\alpha\xi_0 ,$$

which contradicts the second part of (3.22).

Furthermore, using (3.11), (3.15) and (3.20) we get

(3.23) $$v'(\xi) < \frac{\alpha\xi_0}{2} R \exp\left[-\frac{1}{4m}(\alpha M\xi_0)^{\frac{1}{m}-1}\xi^2\right] \quad \text{for} \quad 0 < \xi < \xi_1 .$$

§4. Existence for (3.7), (3.8), (3.9)

Now we prove

<u>Proposition 2</u>. The problem (3.7), (3.8), (3.9) has a solution $v \in C^1[\xi_0,\infty) \cap C^2(\xi_0,\infty)$ and $v(\xi) > 0$ for $\xi > \xi_0$. Solutions of (3.7), (3.8), (3.9) are unique and depend continuously on $\xi_0 < 0$ and $\alpha < 0$.

<u>Proof</u>. First we prove the existence. By virtue of (3.20), (3.16) and (3.23), it suffices to prove that (3.7), (3.8), (3.9) has a solution on $\xi_0 \le \xi \le 0$ which is positive except at $\xi = \xi_0$.

Denote by M the class of functions $v \in C[\xi_0,0]$ such that

$$0 < v(\xi) \le N \quad \text{for} \quad \xi > \xi_0$$

where N is a constant satisfying

$$\frac{1}{2}\,(|\alpha| + 3N^{\frac{1}{m}})|\xi_0|^2 \le N \;.$$

Clearly, M is a closed convex subset of $C[\xi_0,0]$.

Consider the operator

$$w = Tv = \frac{\alpha\xi_0}{2}\,(\xi - \xi_0) + \frac{1}{2}\int_{\xi_0}^{\xi} (\xi - 2\tau)v^{\frac{1}{m}}(\tau)d\tau \;.$$

For any $v \in M$, $w = Tv \in C[\xi_0,0]$ and $w = Tv > 0$ except at $\xi = \xi_0$. Moreover

$$w \le \frac{|\alpha|}{2}\,|\xi_0|^2 + \frac{3}{2}\,|\xi_0|^2 N^{\frac{1}{m}} \le N \;.$$

So $w \in M$. It is easy to see that TM is compact and T is continuous on M. Thus, Schauder's fixed point theorem gives the existence of solutions to (3.14) and hence to (3.7), (3.8), (3.9).

To prove the uniqueness of solutions, let v_1, v_2 be two solutions of (3.7), (3.8), (3.9). It suffices to prove that $v = v_1 - v_2 = 0$ in a neighborhood of ξ_0, say $\xi_0 \le \xi \le \xi_0 + \delta$ with $\delta > 0$ small.

From (3.14) we have

$$v(\xi) = \frac{1}{2}\int_{\xi_0}^{\xi} (\xi - 2\tau)(v_1^{\frac{1}{m}}(\tau) - v_2^{\frac{1}{m}}(\tau))d\tau$$

$$= \frac{1}{2}\int_{\xi_0}^{\xi} (\xi - 2\tau) \cdot \frac{1}{m} \tilde{v}^{\frac{1}{m}-1}(\tau)v(\tau)d\tau$$

where $\tilde{v}(\tau)$ is a certain point between $v_1(\tau)$ and $v_2(\tau)$ and hence from (3.15)

$$\tilde{v}(\tau) > \frac{\alpha\xi_0}{2}(\tau - \xi_0) .$$

Thus

$$|v(\xi)| < \frac{3}{2}|\xi_0| \int_{\xi_0}^{\xi} \frac{1}{m}(\frac{\alpha\xi_0}{2})^{\frac{1}{m}-1}(\tau - \xi_0)^{\frac{1}{m}-1}|v(\tau)|d\tau$$

$$< \frac{3}{2}(\frac{|\alpha|}{2})^{\frac{1}{m}-1}|\xi_0|^{\frac{1}{m}}\delta^{\frac{1}{m}} \max_{\xi_0<\xi<\xi_0+\delta}|v(\xi)| \quad \text{for} \quad \xi_0 \le \xi \le \xi_0 + \delta .$$

It is impossible for δ small enough, unless $v \equiv 0$ on $\xi_0 \le \xi \le \xi_0 + \delta$.

Finally, we prove the continuous dependence of solutions upon $\xi_0 < 0$, and $\alpha < 0$. Denote by $v(\xi;\xi_0,\alpha)$ the solution of (3.7), (3.8), (3.9). Let $\xi_0 < 0$, $\alpha < 0$ be fixed. What we want to prove is that for any $\xi_1 > \xi_0$ and $\varepsilon > 0$, there exists a constant $\delta > 0$ such that

$$|v(\xi;\xi_0',\alpha') - v(\xi;\xi_0,\alpha)| < \varepsilon, \ |v'(\xi;\xi_0',\alpha') - v'(\xi;\xi_0,\alpha_0)| < \varepsilon$$

whenever $|\xi_0' - \xi_0| < \delta$, $|\alpha' - \alpha| < \delta$ and $\max(\xi_0',\xi_0) \le \xi \le \xi_1$.

Suppose it is not the case. Then there exists $\varepsilon_0 > 0$ and $\xi_0^{(n)},\alpha^{(n)},\xi^{(n)}$ such that $\xi_0^{(n)} \to \xi_0$, $\alpha^{(n)} \to \alpha$ $(n \to \infty)$, $\max(\xi_0^{(n)},\xi_0) \le \xi^{(n)} \le \xi_1$, and $|v(\xi^{(n)};\xi_0^{(n)},\alpha^{(n)}) - v(\xi^{(n)};\xi_0,\alpha_0)| \ge \varepsilon$ or $|v'(\xi^{(n)};\xi_0^{(n)},\alpha^{(n)}) - v'(\xi^{(n)};\xi_0,\alpha_0)| \ge \varepsilon$.

Suppose, for example, the case is the former:

$$(4.1) \qquad |v(\xi^{(n)};\xi_0^{(n)},\alpha^{(n)}) - v(\xi^{(n)};\xi_0,\alpha)| \ge \varepsilon .$$

We have

$$(4.2)\qquad \begin{aligned} v'(\xi;\xi_0^{(n)},\alpha^{(n)}) &= \frac{\alpha^{(n)}\xi_0^{(n)}}{2} - \frac{1}{2}\,\xi v^{\frac{1}{m}}(\xi;\xi_0^{(n)},\alpha^{(n)}) + \frac{1}{2}\int_{\xi_0^{(n)}}^{\xi} v^{\frac{1}{m}}(\tau;\xi_0^{(n)},\alpha^{(n)})d\tau \\ v(\xi;\xi_0^{(n)},\alpha^{(n)}) &= \frac{\alpha^{(n)}\xi_0^{(n)}}{2}(\xi - \xi_0^{(n)}) + \frac{1}{2}\int_{\xi_0^{(n)}}^{\xi} (\xi - 2\tau)v^{\frac{1}{m}}(\xi;\xi_0^{(n)},\alpha^{(n)})d\tau \ . \end{aligned}$$

From (3.20), (3.16), (3.23) we see that $v(\xi;\xi_0',\alpha')$ and $v'(\xi;\xi_0',\alpha')$ are bounded uniformly in (ξ',α') in a small neighborhood of (ξ_0,α) and hence from (4.2), it is easy to see that $\{v(\xi;\xi_0^{(n)},\alpha^{(n)})\}$ and $\{v'(\xi;\xi_0^{(n)},\alpha^{(n)})\}$ are equicontinuous. Thus $\{v(\xi;\xi_0^{(n)},\alpha^{(n)})\}$ and $\{v'(\xi;\xi_0^{(n)},\alpha^{(n)})\}$ have subsequences converging uniformly. Suppose they are $\{v(\xi;\xi_0^{(n)},\alpha^{(n)})\}$ and $\{v'(\xi;\xi_0^{(n)},\alpha^{(n)})\}$ themselves and

$$(4.3)\qquad \lim_{n\to\infty} v(\xi;\xi_0^{(n)},\alpha^{(n)}) = w(\xi), \quad \lim_{n\to\infty} v'(\xi;\xi_0^{(n)},\alpha^{(n)}) = \tilde{w}(\xi) \ .$$

From (4.2), (4.3) we get

$$\begin{aligned} \tilde{w}(\xi) &= \frac{\alpha\xi_0}{2} - \frac{1}{2}\,\xi w^{\frac{1}{m}}(\xi) + \frac{1}{2}\int_{\xi_0}^{\xi} w^{\frac{1}{m}}(\tau)d\tau \ , \\ w(\xi) &= \frac{\alpha\xi_0}{2}(\xi - \xi_0) + \frac{1}{2}\int_{\xi_0}^{\xi} (\xi - 2\tau)w^{\frac{1}{m}}(\tau)d\tau \end{aligned}$$

whence $w(\xi)$ is a solution of (3.7)-(3.9) and $\tilde{w}(\xi) = w'(\xi)$. By uniqueness, $w(\xi) \equiv v(\xi;\xi_0,\alpha)$.

On the other hand, we may suppose that $\{\xi^{(n)}\}$ converges:

$$\lim_{n\to\infty} \xi^{(n)} = \tilde{\xi} \ .$$

Letting $n \to \infty$ in (4.1), from the uniform convergence we get

$$|w(\tilde{\xi}) - v(\tilde{\xi};\xi_0,\alpha)| \geq \varepsilon \ .$$

The contradiction means that what we want to prove is true. The proof of Proposition 2 is complete.

Remark. So far we have proved that for any $\xi_0 < 0$ and $\alpha < 0$, there exists a function $v \in C^1[\xi_0,\infty) \cap C^2(\xi_0,\infty)$ with $v(\xi) > 0$ for $\xi > \xi_0$ satisfying (3.7), (3.8), (3.9). Since $v(\xi)$ is bounded and increasing, $\lim_{\xi\to\infty} v(\xi)$ exists and is positive. Let

$$\beta_0^m = \lim_{\xi\to\infty} v(\xi) ,$$

$$u = v^{\frac{1}{m}} .$$

Then it is clear that $u(\frac{x}{t^{1/2}})$ and $\xi_0 t^{\frac{1}{2}}$ satisfy (2.1)-(2.5) and hence, by Proposition 1,

$$w = \begin{cases} u(\frac{x}{t^{1/2}}) & \text{for } x > \xi_0 t^{\frac{1}{2}} \\ \alpha & \text{for } x < \xi_0 t^{\frac{1}{2}} \end{cases} ,$$

is a generalized solution of the initial value problem

$$\frac{\partial u}{\partial t} = \frac{\partial^2 A(u)}{\partial x^2} ,$$

$$u|_{t=0} \equiv \begin{cases} \beta_0 & \text{for } x > 0 \\ \alpha & \text{for } x < 0 , \end{cases}$$

where $A(u) = \max(|u|^{m-1}u,0)$. The fact that this generalized solution w has a line of discontinuity $x = \xi_0 t^{\frac{1}{2}}$ shows, in particular, that (1.6) is incorrect.

§5. Existence and Uniqueness for the Free Boundary Problem

Now we return to the free boundary problem (3.7)-(3.10) and state our main results in the following:

<u>Theorem</u>. The free boundary problem (3.7)-(3.10) has a unique solution $(v(\xi),\xi_0)$ with $\xi_0 < 0$ and $v(\xi) \in C^1[\xi_0,\infty) \cap C^2(\xi_0,\infty)$. Moreover, if we denote $(v(\xi),\xi_0)$ by $(v(\xi;\alpha,\beta),\xi_0(\alpha,\beta))$ to express explicitly the dependence on α and β, then $v(\xi;\alpha,\beta)$ and $-\xi_0(\alpha,\beta)$ are strictly decreasing in α and strictly increasing in β.

<u>Proof</u>. Denote by $v(\xi;\xi_0)$ the solution of (3.7)-(3.9) for the moment. For any $\xi_0 < 0$, the limit

$$\lim_{\xi\to\infty} v(\xi;\xi_0) = \gamma(\xi_0)$$

exists. Let

$$E = \{\xi_0 < 0;\ \gamma(\xi_0) < \beta^m\} .$$

From (3.20), (3.21), it is clear that E is nonempty. Let

$$\bar{\xi}_0 = \inf E .$$

We want to prove

$$\lim_{\xi\to+\infty} v(\xi;\bar{\xi}_0) = \gamma(\bar{\xi}_0) = \beta^m . \tag{5.1}$$

Suppose $\gamma(\bar{\xi}_0) \neq \beta^m$. Denote

$$\delta = |\gamma(\bar{\xi}_0) - \beta^m| . \tag{5.2}$$

From (3.21), (3.23), it is clear that we can choose positive constants c_1 and c_2 such that

$$0 < v'(\xi;(\xi_0) < c_1\exp(-c_2\xi^2) \quad \text{for} \quad \xi > 0 \tag{5.3}$$

for $\xi_0 \in [\frac{3\bar{\xi}_0}{2}, \frac{\bar{\xi}_0}{2}]$. Choose ξ_1 such that

$$c_1 \int_{\xi_1}^{\infty} \exp(-c_2\xi^2)d\xi < \frac{\delta}{4} . \tag{5.4}$$

By the continuous dependence of solutions of (3.7)-(3.9) upon ξ_0, there exists $\eta \in (0, |\bar{\xi}_0|/2)$ such that

(5.5) $$|v(\xi_1;\xi_0) - v(\xi_1;\bar{\xi}_0)| < \frac{\delta}{4}$$

whenever $\xi \in [\bar{\xi}_0 - \eta, \bar{\xi}_0 + \eta]$.

Now for $\xi_0 \in [\bar{\xi}_0 - \eta, \bar{\xi}_0 + \eta]$, integrating (5.3) over (ξ_1, ∞) and using (5.4) we get

$$0 < \gamma(\xi_0) - v(\xi_1;\xi_0) < \frac{\delta}{4} ;$$

in particular,

$$0 < \gamma(\bar{\xi}_0) - v(\xi_1;\bar{\xi}_0) < \frac{\delta}{4} .$$

Hence

$$|\gamma(\xi_0) - \gamma(\bar{\xi}_0)| < |v(\xi_1;\xi_0) - v(\xi_1;\bar{\xi}_0)| + \frac{\delta}{4}$$

and by (5.5)

(5.6) $$|\gamma(\xi_0) - \lambda(\bar{\xi}_0)| < \frac{\delta}{2} .$$

If $\gamma(\bar{\xi}_0) = \beta^m - \delta$, then (5.2) and (5.6) imply

$$\gamma(\xi_0) < \beta^m - \frac{\delta}{2}$$

for $\xi_0 \in [\bar{\xi}_0 - \eta, \bar{\xi}_0 + \eta]$. This means that $[\bar{\xi}_0 - \eta, \bar{\xi}_0 + \eta] \subset E$, which contradicts the definition of $\bar{\xi}_0$.

If $\gamma(\bar{\xi}_0) = \beta^m + \delta$, then (5.2) and (5.6) imply

$$\gamma(\xi_0) > \beta^m + \frac{\delta}{2}$$

for $\xi_0 \in [\bar{\xi}_0 - \eta, \bar{\xi}_0 + \eta]$. This means that no point on $[\bar{\xi}_0 - \eta, \bar{\xi}_0 + \eta]$ is in E, which also contradicts the definition of $\bar{\xi}_0$.

Therefore (5.1) holds and the existence is proved.

Now we prove the uniqueness. Suppose (3.7)-(3.10) has two solutions $(v_1(\xi), \xi_{0,1})$ and $(v_2(\xi), \xi_{0,2})$ and $\xi_{0,1} \neq \xi_{0,2}$. (If $\xi_{0,1} = \xi_{0,2}$ then, by Proposition 2, $v_1(\xi) \equiv v_2(\xi)$). Then from the relation

(5.7) $$v'(\xi) = v'(\eta_0)\exp\left(-\frac{1}{2m}\int_{\eta_0}^{\xi} \tau v^{\frac{1}{m}-1}(\tau)d\tau\right) ,$$

which holds for any solution $v(\xi)$ of (3.7), it is easily seen that the curves $v = v_1(\xi)$ and $v = v_2(\xi)$ cannot intersect at any point (η_0, v_0) with $\eta_0 > 0$. In fact, if

$$v_1(\eta_0) = v_2(\eta_0) = v_0 ,$$

then we must have

$$v_1'(\eta_0) \neq v_2'(\eta_0) ;$$

otherwise the standard uniqueness theorem would imply that $v_1(\xi) \equiv v_2(\xi)$. Suppose

$$v_1'(\eta_0) > v_2'(\eta_0) .$$

Then we have, in a right neighborhood of η_0,

(5.8) $$v_1(\xi) > v_2(\xi) .$$

Note that (5.7) implies

(5.9) $$v_1'(\xi) > v_2'(\xi)$$

whenever (5.8) holds. Thus (5.9) would be true for all $\xi > \eta_0$ and hence

$$\lim_{\xi\to\infty} v_1(\xi) > \lim_{\xi\to\infty} v_2(\xi) ,$$

which is impossible.

In addition, a similar argument shows that if, for example,

(5.10) $$v_1(0) > v_2(0) ,$$

then

(5.11) $$v_1'(0) < v_2'(0) .$$

Let $\xi(v)$ be the inverse function of $v(\xi)$, $v(\xi)$ being a solution of (3.7)-(3.10) which is positive except at $\xi = \xi_0$. Then from (3.7)-(3.9), we obtain

(5.12) $$\xi'' = \frac{1}{2^m} v^{\frac{1}{m}-1} \xi\xi' ,$$

(5.13) $$\xi|_{v=0} = \xi_0 ,$$

(5.14) $$\xi'|_{v=0} = \frac{2}{\alpha\xi_0} .$$

If we denote the inverse functions of $v_1(\xi)$ and $v_2(\xi)$ by $\xi_1(v)$ and $\xi_2(v)$, then from (5.10) we can assert that

(5.15)
$$\xi_{0,1} < \xi_{0,2} ,$$

otherwise there exists v_0, $0 < v_0 < v_2(0)$ such that

$$\xi_1(v) > \xi_2(v) \quad \text{for} \quad 0 < v < v_0, \quad \xi_1(v_0) = \xi_2(v_0) ,$$

and hence, noticing that

$$\xi_1'(0) = \frac{2}{\alpha\xi_{0,1}} > \frac{2}{\alpha\xi_{0,2}} = \xi_2'(0) ,$$

we may conclude that $\xi(v) = \xi_1(v) - \xi_2(v)$ achieves its maximum at a certain point in $(0,v_0)$ where

$$\xi > 0, \; \xi' = 0, \; \xi'' < 0 .$$

On the other hand, from (5.12)

$$\xi'' = \xi_1'' - \xi_2'' = \frac{1}{2m} v^{\frac{1}{m}-1} (\xi_1\xi_1' - \xi_2\xi_2')$$

$$= \frac{1}{2m} v^{\frac{1}{m}-1} (\xi_1 - \xi_2)\xi_1' + \frac{1}{2m} v^{\frac{1}{m}-1} \xi_2(\xi_1' - \xi_2')$$

$$= \frac{1}{2m} v^{\frac{1}{m}-1} \xi\xi_1' + \frac{1}{2m} v^{\frac{1}{m}-1} \xi_2\xi' > 0 ,$$

since $\xi_1' > 0$, $\xi > 0$, $\xi' = 0$. The contradiction proves (5.15).

By the way, what we have just proved is that the curves $v = v_1(\xi)$ and $v = v_2(\xi)$ cannot intersect at a point in the halfplane $\xi < 0$.

Now we consider the function $\xi(v) = \xi_1(v) - \xi_2(v)$ again on the interval $[0,v_2(0)]$. We have

$$\xi_1'(0) < \xi_2'(0)$$

and from (5.10), (5.11)

$$\xi_1'(v_2(0)) > \xi_1'(v_1(0)) = \frac{1}{v_1'(0)} > \frac{1}{v_2'(0)} = \xi_2'(v_2(0)) .$$

Therefore the function $\xi(v) = \xi_1(v) - \xi_2(v)$ on $[0,v_2(0)]$ would achieve its minimum at some point in $(0,v_2(0))$, which is impossible. This completes the proof of uniqueness.

The rest of the theorem can be proved by a similar argument.

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SECURITY CLASSIFICATION OF THIS PAGE (When Data Entered)

REPORT DOCUMENTATION PAGE		READ INSTRUCTIONS BEFORE COMPLETING FORM
1. REPORT NUMBER 2656	2. GOVT ACCESSION NO. AD-A141608	3. RECIPIENT'S CATALOG NUMBER
4. TITLE (and Subtitle) A FREE BOUNDARY PROBLEM FOR DEGENERATE QUASILINEAR PARABOLIC EQUATIONS		5. TYPE OF REPORT & PERIOD COVERED Summary Report - no specific reporting period
		6. PERFORMING ORG. REPORT NUMBER
7. AUTHOR(s) Zhuoqun Wu		8. CONTRACT OR GRANT NUMBER(s) DAAG29-80-C-0041
9. PERFORMING ORGANIZATION NAME AND ADDRESS Mathematics Research Center, University of Wisconsin 610 Walnut Street Madison, Wisconsin 53706		10. PROGRAM ELEMENT, PROJECT, TASK AREA & WORK UNIT NUMBERS Work Unit Number 1 - Applied Analysis
11. CONTROLLING OFFICE NAME AND ADDRESS U. S. Army Research Office P.O. Box 12211 Research Triangle Park, North Carolina 27709		12. REPORT DATE March 1984
		13. NUMBER OF PAGES 22
14. MONITORING AGENCY NAME & ADDRESS(if different from Controlling Office)		15. SECURITY CLASS. (of this report) UNCLASSIFIED
		15a. DECLASSIFICATION/DOWNGRADING SCHEDULE

16. DISTRIBUTION STATEMENT (of this Report)

Approved for public release; distribution unlimited.

17. DISTRIBUTION STATEMENT (of the abstract entered in Block 20, if different from Report)

18. SUPPLEMENTARY NOTES

19. KEY WORDS (Continue on reverse side if necessary and identify by block number)

Degenerate quasilinear parabolic equations
Free boundary problem
Discontinuous solutions
Jump conditions
Existence and uniqueness
Structure of solutions

20. ABSTRACT (Continue on reverse side if necessary and identify by block number)

In this paper, we consider a new kind of free boundary problem related to the investigation of the structure of discontinuous solutions of degenerate quasilinear parabolic equations. A thorough treatment is given for the following special cases:

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20. **ABSTRACT** - cont'd.

$$\frac{\partial u}{\partial t} = \frac{\partial^2 u^m}{\partial x^2} \quad (m > 1) \; ,$$

$$u\Big|_{x=\lambda(t)} = 0 \; ,$$

$$\frac{\partial u^m}{\partial x}\Big|_{x=\lambda(t)} = \alpha\lambda'(t) \; ,$$

$$(\alpha < 0,\ \beta > 0 - \text{const.})$$

$$u\Big|_{t=0} = \beta \; ,$$

which can be reduced to a problem in ordinary differential equations with a certain singularity.

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