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ON THE IDENTIFICATION OF A
STOCHASTIC RESPONSE MODEL IN ACTIVE SONAR*

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20 Cont. $Z(t) = \int G(t,x)dL(x)$ where L is a Lévy process (possibly non-stationary) which describes the joint distribution of the point sources and their intensities. The problem consists of identifying the model most closely compatible with the experimental results.

ON THE IDENTIFICATION OF A
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Denis de Brucq

Abstract

The detection of a marine target with the aid of an active sonar necessitates a probabilistic model of the phenomena, and detection improves according to how well the model takes account of the various random and deterministic physical phenomena. The signal wave $Z(t)$ received from a point source x of the target is described by a deterministic scalar or vector function $G(t,x)$. Random reflection is characterized by a scalar multiplying factor $C(x)$, randomly distributed according to a mixture of central Gaussian laws. Also considered is the model consisting of a linear process of the type $Z(t) = \int G(t,x)dL(x)$ where L is a Lévy process (possibly non-stationary) which describes the joint distribution of the point sources and their intensities. The problem consists of identifying the model most closely compatible with the experimental results.

Introduction

The better the model of the reflected sonar wave from a marine target the more efficient is its detection. The model must be true and precise

It is essential to understand the physics of the phenomena in order to specify the mathematical model. The parameters of this model must be determined from experimental results and this procedure must lead to formulae exploitable by computer. More precisely the statistical law of the observations is obtained by means of its characteristic function.

This work extends a more theoretical study "Identification of a class of filtered Poisson processes" by the author and Antonio Gualtierroti.

The target consists of points shining under the effect of the incident wave from the active sonar; certain points reflect sufficiently in the direction of the receivers.

Paragraph I is devoted to an elementary description of the physical phenomena. A geometrical (ray) approximation is used to calculate the phase changes in the emitted wave (supposed sinusoidal). Thus we find a response

(I.1) $P(t) = C(x)G(t,x)$ received at instant t from the point x of the target. The number N of point sources, their positions x in the target, and the coefficients $C(x)$ are all random. A process with distribution $\sum_i C(x_i)\delta_{x_i}(x)$ (a Lévy process) may describe these random phenomena. We suppose that effects due to the different sources are additive. This corresponds to the hypothesis of small movements in mechanics.

Paragraph II defines the probability laws of N , X and C . Three distinct situations are described as are the experimental results which determine the most realistic and the most tractable model.

In Paragraph III the variance and the characteristic function of the observation are treated case by case. Finally the identification of the

probability laws is presented in the last paragraph. It seems that the function $G(t,x)$ may not be statistically estimated from the observations alone; a knowledge of the physics of the phenomena must furnish the function $G(t,x)$, otherwise some very restrictive hypotheses on the nature of G would have to be imposed.

The first step of the identification is to specify the deterministic function $G(t,x)$ using physical considerations.

Secondly it must be established from the normalized observations whether the law is symmetric. In the non-symmetric case only the model no. 2 may be applied.

If symmetric (the most plausible case) the characteristic function of $v=u^2$ is tested to see if it is the Laplace transform of a probability--if so, models 1 and 3 may be used (cf. IV.12 or IV.14). If in the more restricted case the second characteristic function of $v=u^2$ is the Laplace transform of a probability (cf. IV.13) then the second model may be used with a symmetric Lévy process composed of spherically invariants defined by the probability $\lambda.\gamma$ (see formula III.9).

Proposition IV.3 gives a criterion for compatibility of a given model with the experimental results and to know whether a moment method or a rapid Fourier transform method may be used to find $\lambda.\gamma$.

I. Description of the phenomena

The submarine (target) is represented by a segment $M = [-l, +l]$ whose centre is at a distance R from the active sonar S . In the plane defined by the sonar and the target, the disposition is as in Figure 1.

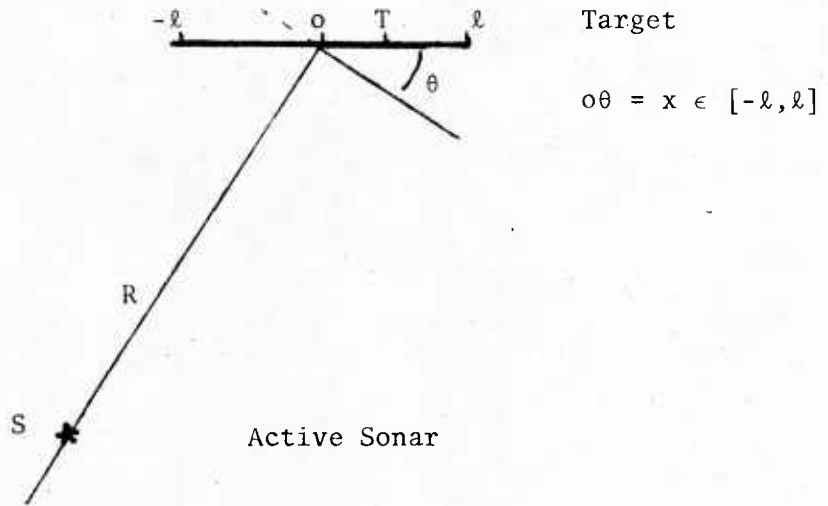


Figure 1.

The distance R between sonar and target is supposed to be large with respect to the dimension l and if $\frac{l}{R} \ll 1$ the emitted acoustic wave is supposed plane. (An approximation by spherical waves would modify the deterministic function $G(t, x)$ but not the probabilistic modelling of the phenomena).

With this approximation, let us follow the emitted wave until it is received again.

Emission: $P(t) = P_0 \cos(\omega t + \phi)$, where ω = angular velocity
 ϕ = phase at $t=0$

At point T of the target:

$$P(t) = P_0 \frac{e^{-\gamma(R + x \sin \theta)}}{(R + x \sin \theta)} \cos \left[\omega \left(t - \frac{R + x \sin \theta}{c} \right) + \phi \right]$$

Distance SN has been approximated by $R + x \sin \theta$.

Energy losses involve a factor $e^{-\gamma}$ per unit length.

The celerity is denoted c .

Reemission at point T:

Only a part of the energy is retransmitted in the direction S. We introduce a random multiplicative factor $C(x)$: thus

$$P(t) = P_0 C(x) \frac{e^{-\gamma(R + x \sin \theta)}}{(R + x \sin \theta)} \cos \left[\omega \left(t - \frac{R + x \sin \theta}{c} \right) + \phi \right]$$

Reception at point S:

The wave emitted from point T returns to points

$$P(t) = P_0 C(x) \frac{e^{-2\gamma(R + x \sin \theta)}}{(R + x \sin \theta)^2} \cos \left[\omega \left(t - \frac{2(R + x \sin \theta)}{c} \right) + \phi \right]$$

For simplicity the receivers are at the same point S as the emitters.

This simplified analysis introduces the function $C(t,x)$ defined by the equations

$$(I.1) \quad P(t) = C(x)G(t,x)$$

$$(I.2) \quad G(t,x) = P_0 \frac{e^{-2\gamma(R + x \sin \theta)}}{(R + x \sin \theta)^2} \cos \left[\omega t - 2\omega x \sin \theta - \frac{2\omega R}{c} + \phi \right]$$

It is possible to introduce a function $G(1-\tau)$ which necessitates new approximations (cf. Westcott). Again for $\frac{l}{R} \ll 1 \Leftrightarrow x \sin \theta \ll R$, and putting

$\phi_1 = \phi - \frac{2\omega R}{c}$ we have

$$G(t,x) \approx P_0 \frac{e^{-2\gamma R}}{R^2} \cos \left[\omega t - \frac{2\omega x \sin \theta}{c} + \phi_1 \right]$$

The position x of T may be replaced by a temporal equivalent by putting

$t(x) = \frac{2x \sin \theta}{c}$, from which

$$G(t,n) = G(t-t(x)) \approx P_0 \frac{e^{-2\gamma R}}{R^2} \cos \left[\omega(t-t(x)) + \phi_1 \right]$$

The sources instead of being situated at points x of the target would be at dates $\tau = t(x)$. These last approximations do not in fact seem to be particularly useful in what follows.

II. Probabilistic model

Certain points T_j at positions x_j of the target emit in direction $T_j S$. We suppose the effects of these different points to be additive and we take the model

$$(II.1) \quad Z(t) = \sum_j C(x_j) G(t, x_j) = \int G(t, x) dL(x)$$

where the process L is defined on the target (that is for $x \in M = [-l, +l]$).

$$(II.2) \quad L(x) \triangleq \sum_j C(x_j) \delta_{x_j}(x).$$

Stationarity of L has not been supposed a priori.

Case no. 1.

Each observation at S corresponds to one bright point T at x on the target. The distribution of x is given by a probability $\nu(dx)$. In this case

$$(II.3) \quad Z(t) = C(x) G(t, x) .$$

It remains to specify the law of C when x is given. So as not to be limited to a simple centred Gaussian law, we may suppose C to be spherically invariant with associated probability $\nu(x, da)$. Thus

$$(II.4) \quad E(\exp iu C(x)) \triangleq \int_{\mathbf{R}^+} e^{-u^2 a^2 / 2} \mu(x, da) .$$

More precisely ν is a transition probability. In this way the law of C is centred and symmetric but the distribution $\mu(x, \cdot)$ of the standard deviations a of the mixture of centred Gaussian laws depends on the position x of the bright point of the target.

Case no. 2.

By giving the process with distribution

$$(II.5) \quad L = \sum_j C(x_j) \delta_{x_j}$$

we fix both the laws of the x_j and of the C . Let us take for L a second-order

Lévy process (cf. de Brucq, part H).

We introduce a positive bounded measure (x,a) on $M \times \mathbb{R}_+$ satisfying the inequality
$$\int_{M \times \mathbb{R}_+} a^2 \mu(dx, da) < \infty.$$

By hypothesis, for all square integrable functions ϕ (i.e., $\int_{M \times \mathbb{R}_+} \phi(x)^2 a^2 \mu(dx, da) < \infty$) the second characteristic function of the random variable

$$(II.6) \quad \langle L, \phi \rangle \triangleq \int \phi(x) dL(x) \quad \text{is given by}$$

$$(II.7) \quad \Psi_\psi = \log E(\exp i\langle L, \phi \rangle) = \int (e^{ia\phi(x)} - 1 - ia\phi(x)) d\mu(x, a).$$

The stationary case corresponds to a product measure of the type $d\mu(x, a) = dx \otimes \mu(da)$, and in the Poisson case $d\mu(x, a) = dx \otimes \lambda \delta_1(da)$.

Another particular case is to take the bright points x_j according to a Poisson law of intensity $\lambda(dx)$ on $M = [-\ell, +\ell]$ and the amplitude C according to a spherically invariant law with associated measure $\gamma(x, dc)$: the Lévy process is here restricted to a spherically invariant mixture. Thus

$L = \sum_j C(x_j) \delta x_j$ must be a Lévy process with

$$\Psi_\phi = \int (e^{ia\phi(x)} - 1 - ia\phi(x)) d\mu(x, a) = \int (e^{-(c^2 \phi(x)^2)/2} - 1) \lambda(dx) \gamma(x, da)$$

where the last expression results from a direct evaluation of the second characteristic function (of de Brucq and Gualtierotti, p.13). Replacing ϕ by $u\phi$ (where $u \in \mathbb{R}$) and observing that $\int (e^{i\alpha\theta} - 1 - i\alpha\theta) \frac{1}{\sqrt{2\pi}} e^{-\theta^2/2} d\theta = e^{-\alpha^2/2} - 1$, it follows that

$$\begin{aligned} & \int (e^{-(u^2 c^2 \phi(x)^2)/2} - 1) \lambda(dx) \gamma(x, dc) \\ &= \int \lambda(dx) \gamma(x, dc) \left[\int \{ e^{iuc\phi(x)\theta} - 1 - iuc\phi(x)\theta \} \frac{1}{\sqrt{2\pi}} e^{-\theta^2/2} d\theta \right] \\ &= \int \{ e^{iu(c\theta)\phi(x)} - 1 - iu(c\theta)\phi(x) \} \lambda(dx) \frac{1}{\sqrt{2\pi}} e^{-\theta^2/2} d\theta \gamma(x, dc). \end{aligned}$$

Making the change of variables $(c, \theta) \rightarrow a \triangleq c\theta$, and putting $\mu(x, da)$ for the

image of the transition probability $\frac{1}{\sqrt{2\pi}} e^{-\theta^2/2} d\theta \gamma(x, dc)$ under the change of variables, we have for a test function ϕ

$$(II.8) \quad \begin{aligned} \Psi_{\phi}(u) &= \int (e^{-u^2 c^2 \phi(x)^2 / 2} - 1) \lambda(dx) \gamma(x, dc) \\ &= \int \{e^{iua\phi(x)} - 1 - iua\phi(x)\} \lambda(dx) \mu(x, da). \end{aligned}$$

This second characteristic function $\Psi_{\phi}(u)$ is remarkable in that it is real and depends only on u^2 . In this case Ψ is invariant when u is replaced by $-u$; so also is the characteristic function:

$$E(e^{iu\langle L, \phi \rangle}) = \exp \Psi(u) = \exp \Psi(-u) = E(e^{-iu\langle L, \phi \rangle}).$$

The moments of odd order, where they exist, are therefore zero. If this last fact is the only hypothesis imposed on the process L , we have for all test functions ϕ and for all $u \in \mathbb{C}$

$$\begin{aligned} \Psi_{\phi}(u) &= \int \{e^{iua\phi(x)} - 1 - iua\phi(x)\} d\mu(a, x) \\ &= \int \{e^{-iua\phi(x)} - 1 + iua\phi(x)\} d\mu(a, x) = \Psi_{\phi}(-u) \\ &= \int \{e^{iu(-a)\phi(x)} - 1 - iu(-a)\phi(x)\} d\mu(a, x). \end{aligned}$$

In this case the probability μ and the image of μ when a is replaced by $-a$ lead to the same second characteristic function. Taking the average of these two probabilities we can therefore suppose that μ is invariant for the change of variables $a \rightarrow -a$: the probability μ is symmetric in a . In this case we talk of a symmetric Lévy process.

Case no. 3

We introduce here a generalization of Gaussian measures which we can talk of as Spherically Invariant Measure.

Supposing that $\int |G(t, x)|^2 a^2 \nu(dx) \mu(da) < \infty$ we give a sense to the expression

$$(II.9) \quad Z(t) = \int_{[-\ell, +\ell]} G(t, x) C(dx) .$$

Thus, the points x of the target have a continuous instead of a discrete random effect.

We introduce, for every function ϕ of $L^2(M, M, \nu)$, a random variable $C(\phi)$, centred with variance

$$(II.10) \quad E(C(\phi)^2) = \int_M \phi^2 \nu(dx) \int_{R^+} a^2 \mu(da)$$

and with characteristic function

$$(II.11) \quad E(\exp iC(\phi)) = \int_{R^+} \exp \left(\frac{-a^2 \int_M \phi(x)^2 \nu(dx)}{2} \right) \mu(da) .$$

In the application $M = [-\ell, +\ell]$ and $M = \underline{[-\ell, +\ell]}$, so that

$$(II.12) \quad Z(t) = C[G(t, x)] = \int_{[-\ell, +\ell]} G(t, x) C(dx) ,$$

the process $\phi \in L^2(\underline{[-\ell, +\ell]}, \underline{[-\ell, +\ell]}, \nu(dx)) \rightarrow C(\phi)$ is the spherically invariant measure; the various points T of the target reemit in a correlated way according to a Gaussian law with random variance a^2 .

The various receivers at point S give an observation $Z(t)$ which may be a scalar or a vector. In general the random numerical values are time-averages.

We may write

$$(II.13) \quad \langle Z, f \rangle = \int Z(t) f(t) dt \quad \text{where } f \text{ is a weighting function. The}$$

observation Z thus becomes a linear process $f \rightarrow \langle Z, f \rangle = \sum_i C(x_j) \int G(t, x_j) f(t) dt;$

the parameter f may be for example a square-integrable function

$(\int |f(t)|^2 dt < \infty)$. The Hilbert space of the parameters f will be called

H , in what follows.

III. Variance and characteristic function of the process Z

Formally the various cases correspond to the expression

$$(III.1) \quad \langle Z, f \rangle = \sum_j C(x_j) \int G(x_j, t) f(t) dt.$$

We will establish the variance (III.2) and the characteristic function (III.3), taking the probabilistic model of paragraph II.

$$(III.2) \quad \sigma^2(f) = E(\langle Z, f \rangle^2)$$

$$(III.3) \quad \Phi(f) = E(\exp i \langle Z, f \rangle)$$

Due to the linearity of the process Z, we have for all $u \in \mathbb{R}$: $uf \in H$ and

$$E(\exp iu \langle Z, f \rangle) = E(\exp i \langle Z, uf \rangle)$$

We suppose the function $x \mapsto \langle G, f \rangle = \langle G(t, x), f(t) \rangle$ to be measurable in x , and square-integrable with respect to the measures associated with x .

Case no. 1.

$$(III.4) \quad \begin{aligned} \sigma^2(f) &= E \left[\left(\int C(X) f(t) dt \right)^2 \right] = \int \nu(dx) E[C(X)^2 | X=x] \langle G(t, x), f(t) \rangle^2 \\ &= \int \nu(dx) \mu(x, da) a^2 \langle G, f \rangle^2. \end{aligned}$$

$$\begin{aligned} \Phi(f) &= E \left[\exp i C(X) \int G(t, X) f(t) dt \right] \\ &= \int \nu(dx) E \left[\exp i C(X) \int G(t, X) f(t) dt | X=x \right] \\ &= \int \nu(dx) \mu(x, da) \exp^{-a^2 \langle G, f \rangle^2 / 2}. \end{aligned}$$

Putting $\mu(dx, da) = \nu(dx) \mu(x, da)$ and writing uf for f

$$(III.5) \quad \Phi(f) = \int \mu(dx, da) \exp \frac{-u^2 a^2 \langle G, f \rangle^2}{2}.$$

We have used the hypothesis of a random variable $C(X)$ spherically invariant with probability $\mu(x, da)$ when $X=x$.

Case no. 2.

By hypothesis, the observation Z for the test function f is written

$$\langle Z, f \rangle = \int \langle G(t, x), f(t) \rangle dL(x) \text{ and we suppose that } \int \langle G(t, x), f(t) \rangle^2 a^2 d\mu(x, a) < \infty.$$

It follows that

(III.6) $\sigma^2(f) = \int \langle G(t,x), f(t) \rangle^2 a^2 d\mu(x,a)$, from the second order expansion of the second characteristic function

$$(III.7) \quad \Psi(\langle Z, f \rangle) = \int (e^{ia\langle G, f \rangle} - 1 - ia\langle G, f \rangle) d\mu(x,a).$$

This expression (III.7) is not essentially changed in the special case where μ is symmetric with respect to the variable a . Nevertheless, it is possible to carry out the integration in a on the interval $[0, +\infty]$ instead of on $[-\infty, +\infty]$.

A more important special case is the spherically invariant case. According to (II.8), the second characteristic function is then

$$\begin{aligned} \Psi(\langle Z, f \rangle) &= \int e^{ia\langle G, f \rangle} - 1 - ia\langle G, f \rangle d\mu(x,a) \\ &= \int (e^{-c^2\langle G, f \rangle^2} - 1) \lambda(dx) \gamma(x,dc), \end{aligned}$$

from which we obtain, replacing f by uf

$$\Psi_{\langle Z, f \rangle}(u) = \int (e^{-(u^2 c^2 \langle G, f \rangle^2)/2} - 1) \lambda(dx) \gamma(x,dc).$$

Case no. 3.

The calculation is now carried out putting $(x) \stackrel{\Delta}{=} \langle G(t,x), f(t) \rangle$ in equations (II.10) and (II.11). We suppose that the function $x \mapsto \langle G(t,x), f(t) \rangle$ is square-integrable on $[-l, +l]$. Hence $\int \langle G(t,x), f(t) \rangle C(dx) = C[\langle G(t,x), f(t) \rangle]$ is a centred spherically invariant random variable with associated probability $\mu(da)$:

$$\begin{aligned} (III.10) \quad \sigma^2(f) &\stackrel{\Delta}{=} E \left[\left(\int \langle G, f \rangle C(dx) \right)^2 \right] = \int a^2 \langle G, f \rangle^2 v(dx) \mu(da) \\ \Phi(f) &\stackrel{\Delta}{=} E \left[\exp i \int \langle G, f \rangle C(dx) \right] = \int \exp \left[-a^2/2 \int \langle G, f \rangle^2 v(dx) \right] \mu(da) \\ \text{and } \Phi_f(u) &= E \left[\exp iu \langle Z, f \rangle \right] = \int \exp \left[-\frac{u^2 a^2}{2} \int \langle G, f \rangle^2 v(dx) \right] \mu(da). \end{aligned}$$

IV. Identification.

It is remarkable that the three cases considered lead to the same covariance. For all $f, g \in H$ we have

(IV.1) $\Gamma(f, g) = E(\langle Z, f \rangle \langle Z, g \rangle) = \int \mu(dx, da) a^2 \langle G(t, x), f(t) \rangle \langle G(t, x), g(t) \rangle$
 (for the case no. 3, $\mu(dx, da) = \nu(dx)\mu(da)$. If for all $x \in [-\ell, +\ell]$ there exists a vector $f \in H$ such that $\langle G(t, x), f(t) \rangle \stackrel{\Delta}{=} \int G(t, x) f(t) dt = 0$, then $\langle Z, f \rangle = 0$ almost surely, for $E(\langle Z, f \rangle^2) = 0$. Thus for

$$G(t, x) = P_0 \frac{e^{-2\gamma(R + x \sin \theta)}}{(R + x \sin \theta)^2} \cos \left[\omega t - \frac{2\omega x \sin \theta}{c} - \frac{2\omega R}{c} + \phi \right]$$

every function f which satisfies

$$\int \cos(\omega t) f(t) dt = \int \sin(\omega t) f(t) dt = 0$$

also satisfies the condition and so $\langle Z, f \rangle = 0$.

Hypotheses of a physical origin on the function $G(t, x)$ may thus be tested in this way.

The physical analysis of the phenomena establishes the form of the function $G(t, x)$. This function does not appear to me to be identifiable from the covariance Γ , unless supplementary hypotheses are added.

The problem is to identify the most precise model compatible with the experimental results.

To fix our ideas, let us start with Case no. 2, for which the odd moments are not a priori zero.

Recall the second characteristic function

$$(III.7) \quad \Psi(f) = \int [\exp ia \langle G, f \rangle - 1 - ia \langle G, f \rangle] d\mu(x, a).$$

Considering, for an $u \in \mathbb{R}$, the function uf , this leads to

$$(IV.2) \quad \Psi_f(u) \stackrel{\Delta}{=} \log E(\exp iu \langle Z, f \rangle) = \int [\exp iua \langle G, f \rangle - 1 - iua \langle G, f \rangle] d\mu(a, x).$$

The function $u \rightarrow \Psi(u)$ can thus be established statistically for a certain

number of points u_1, u_2, \dots, u_r . Making the change of variables

(IV.3) $b \triangleq a \langle G(t,x), f(t) \rangle$, and writing σ_f for the measure image of μ ,

$$(IV.4) \quad \Psi_f(u) = \int_{-\infty}^{\infty} \sigma_f(db) [\exp iub - 1 - iub]$$

For $b = 0$, the expression $\exp iub - 1 - iub$ is zero, and so $\sigma_f(\{0\})$ is undefined.

From a theoretical point of view, for $u \in]0, \infty[$ we have

$$+\Psi_f''(u) = -\int_{-\infty}^{\infty} b^2 \sigma_f(db) \exp iub.$$

The measure $b^2 \sigma_f(db)$ is determined by its Fourier transform $\Psi_f''(u)$ which we suppose known. Thus the measure is defined except for $\{0\}$.

From a practical point of view, $\sigma_f(db)$ may be determined by the method of moments from a series expansion in terms of u (of Akheizer, p.22). Let us calculate the moments:

$$(IV.5) \quad \Psi_f(u) = \sum_{n=2}^{\infty} \frac{(iu)^n}{n!} \int_{-\infty}^{\infty} b^n \sigma_f(db)$$

The estimation of the various moments of $\langle Z, f \rangle$ permits a limited series expansion of $\Psi_f(u)$ in terms of u . Since $\Psi(u) = \log \phi$ is the second characteristic function of $\langle Z, f \rangle$, the equation

$$(IV.6) \quad \Psi'(u) \phi(u) = \phi'(u)$$

gives us a recurrence relation for calculating the terms of the expansion of Ψ . Let

$$(IV.7) \quad \phi(u) \triangleq 1 + \frac{i u}{1!} m_1 + \frac{(i u)^2}{2!} m_2 + \dots + \frac{(i u)^\ell}{\ell!} m_\ell + \dots$$

$$(IV.8) \quad \Psi(u) \triangleq \frac{i u}{1!} K_1 + \frac{(i u)^2}{2!} K_2 + \dots + \frac{(i u)^\ell}{\ell!} K_\ell + \dots$$

In this case the derivatives ϕ' and Ψ' are easily calculated and the equation $\Psi'(u)\phi(u) = \phi'(u)$ leads to

$$\sum_{p=1}^{\infty} \frac{i^p u^{p-1}}{(p-1)!} K_p \sum_{q=0}^{\infty} \frac{i^q u^q}{q!} m_q = \sum_{\ell=1}^{\infty} \frac{i^\ell u^{\ell-1}}{(\ell-1)!} m_\ell$$

or indeed to the equation for the coefficients of powers of u

$$\sum_{p+q=\ell} \frac{K_p^{mq}}{(p-1)!q!} = \frac{m_\ell}{(\ell-1)!} \quad \text{from which}$$

$$m_\ell = \sum_{p=1}^{\ell} c_{p-1}^{\ell-1} K_p^{m_{\ell-p}} \quad (c_r^n \text{ is a binomial coefficient})$$

We deduce the recurrence relation for the K_ℓ , $\ell=1,2,\dots$

$$(IV.9) \quad x_\ell = m_\ell - \sum_{p=1}^{\ell-1} c_{\ell-1}^{p-1} K_p^{m_{\ell-p}} .$$

Thus the moments m_1, m_2, \dots, m_ℓ of $\langle z, f \rangle$ allow us to calculate the coefficients K_1, K_2, \dots, K_ℓ of $\Psi(u)$. The series expansion of the two sides of equation

$$(IV.4) \quad \Psi_f(u) = \int_{-\infty}^{\infty} (\exp iub - 1 - iub) \sigma_f(db)$$

leads to the equations

$$K_0 = 0$$

$$K_1 = 0 = \int_{-\infty}^{\infty} b \sigma_f(db) \quad \text{by hypothesis}$$

$$K_2 = 1 = \int_{-\infty}^{\infty} b^2 \sigma_f(db) \quad \text{by normalization}$$

\vdots

$$K_\ell = \int_{-\infty}^{\infty} b^\ell \sigma_f(db) .$$

Proposition IV.1: Assuming z arises from a Lévy process, the matrix $m(i,j) \stackrel{\Delta}{=} K_{i+j}$; $i,j=1,2,\dots,n$ must necessarily be of positive type.

Proof: In fact for all complex numbers c_1, c_2, \dots, c_n , we must have

$$\sum_{i,j} \bar{c}_i c_j m(i,j) = \int_{-\infty}^{\infty} \left| \sum_{i=1}^n c_i b^i \right|^2 \sigma_f(b) \geq 0 . \quad \text{The series expansion may seem}$$

a complicated detour: in fact it is possible, starting from the equation

$$(IV.4) \quad \Psi = \log E(e^{iu\langle z, f \rangle}) = \int_{-\infty}^{\infty} \left(\frac{\exp iub - 1 - iub}{b^2} \right) b^2 \sigma_f(db)$$

to make direct estimates of $E(e^{iu_j \langle z, f \rangle})$ and hence also of $\Psi_f(u_j)$ and its derivatives $\Psi_f''(u_j)$, $j=1,2,\dots,r$. Since the observation $\langle z, f \rangle$ is by hypothesis normalized (i.e., $E\langle z, f \rangle = 0$, $E\langle z, f \rangle^2 = 1$) we obtain $K_1=0$ from (IV.9) and we may seek a probability $b^2 \sigma$ on $\mathbb{R} \setminus \{0\}$ satisfying $\int b^n b^2 \sigma_f(db) = K(n+2)$.

With these supplementary hypotheses, $\int_{-\infty}^{\infty} \exp(iub) b^2 \sigma_f(db) = -\Psi_f''(u)$ and the characteristic function of $b^2 \sigma_f$ is known for values u_1, u_2, \dots, u_r .

We have proved

Proposition IV.2: Assuming z is due to a Lévy process, and for all f , the function $-\Psi_f''(u)$, $u \in \mathbb{R}$ must necessarily be a characteristic function.

In this case $b^2 \sigma_f(db)$ plays the role of a spectral measure to be identified from the "complex covariance" $-\Psi_f''(u)$! If we suppose also that σ_f is absolutely continuous with respect to Lebesgue measure, then

$$b^2 \frac{d\sigma_f}{db}(b) = \frac{1}{2\pi} \int_{-\infty}^{\infty} \exp iub [-\Psi_f''(u)] du.$$

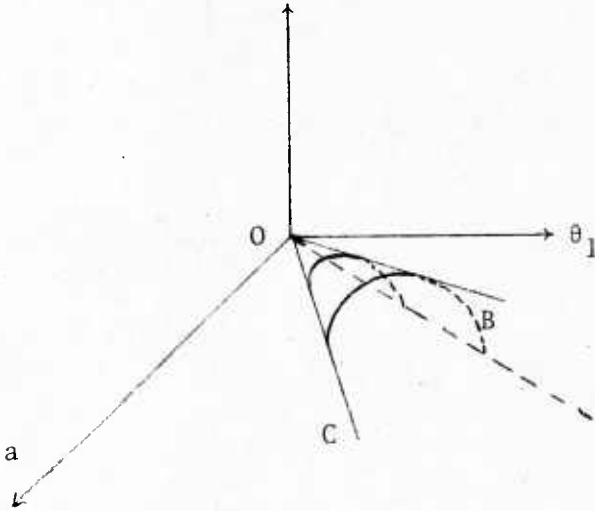
We have to calculate the Fourier inverse transform of the function $-\Psi''(u)$ known at u_1, u_2, \dots, u_r . Rapid Fourier transform methods may be applied.

We return to the general case of a normalized vector observation $\langle z, f_1 \rangle$, $\langle z, f_2 \rangle, \dots, \langle z, f_n \rangle$ associated with the n test functions f_1, f_2, \dots, f_n . The second characteristic function, for u_1, u_2, \dots, u_n ,

$$\begin{aligned} \Psi(u_1, u_2, \dots, u_n) &\stackrel{\Delta}{=} \log E[\exp i(u_1 \langle G, f_1 \rangle + \dots + u_n \langle G, f_n \rangle)] \\ &= \int \{ \exp ia(u_1 \langle G, f_1 \rangle + \dots + u_n \langle G, f_n \rangle) - 1 - ia(u_1 \langle G, f_1 \rangle + \dots + u_n \langle G, f_n \rangle) \} \mu(dx, da) \\ &= \int \{ \exp ia(u_1 \theta_1 + \dots + u_n \theta_n) - 1 - ia(u_1 \theta_1 + \dots + u_n \theta_n) \} \nu(d\theta, da) \end{aligned}$$

after the change of variable $x \rightarrow \theta_j \stackrel{\Delta}{=} \langle G(t, x), f_j(t) \rangle$, $j=1,2,\dots,n$ and writing ν for the probability image of μ .

The probability ν is singular relative to the variables $\theta_j, j=1,2,\dots,n$, and is situated on the curve $B: x \rightarrow \theta_j(x)$; thus only the cone $C(O,B)$ with vertex O and section B carries the probability .



The cone $C(O,B)$ may not be determined from the function $\Psi(u_1, u_2, \dots, u_n)$; on the contrary, the deterministic physics of the phenomena must be used to define the functions $f_j, j=1,2,\dots,n$, and the response G . Hence the functions $fx \rightarrow G(t,x), f_j(t)$ are evaluated and we deduce the curve B and the cone (O,B) on which the probability ν must lie.

Let us further restrict the measures, in order to arrive at the spherically invariant laws.

Firstly, it is necessary that the distributions be symmetric about 0, i.e. the odd moments are zero. The equation $\Psi(u) = \log\phi(u)$ shows that for replacement of u by $-u$, the invariance of ϕ is equivalent to that of Ψ . By expanding the formula (III.8) $\Psi_{\langle G, f \rangle}(u) = \sum_{n=1}^{\infty} (-1)^n \frac{u^{2n}}{(2n)!} 2 \int a^{2n} \langle G, f \rangle^{2n} d\mu(x, a)$

and by comparing with (IV.8) the moments satisfy $\int a^{2n} \langle G, f \rangle^{2n} d\mu(x, a) = \frac{1}{2} K_{2n}$ and proposition IV does not lead to a new condition on the model whereas in the spherically invariant case new conditions appear. Using now formula (III.9)

$$\Psi_{\langle Z, f \rangle}(u) = \sum_{n=1}^{\infty} (-1)^n \frac{1}{2^n} \frac{u^{2n}}{n!} \int c^{2n} \langle G, f \rangle^{2n} \lambda(dx) \gamma(x, dc) \text{ and comparing with (IV.8)}$$

we have

$$(IV.10) \quad \int c^{2n} \langle Z, f \rangle^{2n} \lambda(dx) \gamma(x, dc) = \frac{2^n n!}{(2n)!} K_{2n} = \frac{K_{2n}}{(2n-1)(2n-3)\dots 3 \cdot 1}$$

Proposition IV.3 For a symmetric Levy process to be restricted to the spherically invariant case, the coefficients $\frac{K_{2n}}{(2n-1)(2n-3)\dots 3 \cdot 1}$ (instead of K_{2n})

must be the even moments of a measure for all $n \in \mathbb{N}$.

Restricting again by supposing that the measure μ is of product type, i.e. in the spherically invariant case $\gamma(x,dc) = \gamma(dc)$. In this case

$$\int c^{2n} \langle G, f \rangle^{2n} \lambda(dx) \gamma(x,dc) = \int c^{2n} \gamma(dc) \int \langle G, f \rangle^{2n} \lambda(dx) = \frac{2^n n!}{(2n)!} K_{2n} .$$

The two measures γ and λ are to be identified, and this problem is clearly undetermined.

If we suppose that the function $x \longmapsto \langle a(tcx), f(t) \rangle$ is known and limiting ourselves to a stationary Poisson law, intensity λdx , it follows that

$$(IV.11) \quad \lambda \int c^{2n} \gamma(dc) = \frac{2^n n!}{(2n)!} K_{2n} \frac{1}{\int \langle G, f \rangle^{2n} dx} ; \text{ this case is treated in}$$

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We return to the three cases considered earlier (sect. III). In (III.5), by changing variables $u \longmapsto v = u^2$; $(a, x) \longmapsto a \langle G, f \rangle$ it follows that

$$(IV.12) \quad E(\exp in \langle Z, f \rangle) = \int \exp(-vb) \nu(db), \text{ where } \nu \text{ is the image of } \mu .$$

Similarly in (III.9) we put $v \triangleq u^2$ and $b \triangleq c^2 \langle G, f \rangle$.

$$(IV.13) \quad \log E(\exp in \langle Z, f \rangle) = \int [\exp(-vb) - 1] \nu(db) \text{ with } \nu = \text{measure image of } \lambda \cdot \gamma .$$

Also in (III.5) putting $v \triangleq u^2$ and $b \triangleq \frac{a^2}{2} \int \langle G, f \rangle^2 \nu(dx)$,

$$(IV.14) \quad E(\exp in \langle Z, f \rangle) = \int_0^\infty \exp(-vb) \nu(db) .$$

Thus in all cases the problem is to identify ν from the Laplace transform known for a finite number of points v_1, v_2, \dots, v_n .

By a judicious choice of points u_1, u_2, \dots, u_n we are returned to the problem of moments:

Take u_1 arbitrary; and u_j according to $u_j^2 = v_j = jv_1 = ju_1^2$. Thus

$$E(\exp in j \langle Z, f \rangle) = \int [\exp -v_1 b]^j \nu(db) . \text{ The change of variables } b \longmapsto c = \exp(-v_1 b) \text{ leads to the moment problem for the probability image } \sigma: E(\exp in j \langle Z, f \rangle) = \int c^j \sigma(dc) .$$

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